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On the Bounds of the Radio Numbers of Stacked-Book Graph

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Abstract: A Stacked-book graph $G_{m,n}$ results from the Cartesian product of a star graph S_m and path P_n, where m and n are the orders of S_m and P_n respectively. A radio labeling problem of a simple and connected graph, G, involves a non-negative integer function $f: V(G) \to \mathbb{Z}^+$ on the vertex set $V(G)$ of G, such that for all $u, v \in V(G)$, $|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$, where $\text{diam}(G)$ is the diameter of G and $d(u, v)$ is the shortest distance between u and v. Suppose that f_{min} and f_{max} are the respective least and largest values of f on $V(G)$, then, span$f$, the absolute difference of f_{min} and f_{max}, is the span of f while the radio number $rn(G)$ of G is the least value of span$f$ over all the possible radio labels on $V(G)$. In this paper, we obtain the radio number for the stacked-book graph $G_{m,n}$ where $m \geq 4$ and n is even, and obtain bounds for $m = 3$ which improves existing upper and lower bounds for $G_{m,n}$ where $m = 3$.

Key Words: Radio number, Cartesian product of graphs, stacked-book graph.

AMS(2010): 05C15, 05C12.

§1. Introduction

The graph G considered in this paper is simple and undirected. The vertex and edge sets of G are $V(G)$ and $E(G)$. For $e = uv \in E(G)$, e connects two vertices u and v while $d(u, v)$ is the distance between u, v and $\text{diam}(G)$ is the diameter of G. Radio number labeling problem, which is mostly applied in frequency assignment for signal transmission, where it mitigates the problems of signal interference. It was first suggested in 1980 by Hale [6].

Let f be a non negative integer function on $V(G)$ such that the radio labeling condition, $|f(u) - f(v)| \geq \text{diam}G + 1 - d(u, v)$ is satisfied for every pair $u, v \in V(G)$. The span of f, span$f$, is the difference between f_{min} and f_{max}, the minimum and the maximum radio label on G respectively. Thus the smallest possible value of span$f$ is the radio number, $rn(G)$, of G. The radio labeling condition guarantees that every vertex on G has unique radio label. Therefore, $rn(G) \geq |V(G)| - 1$ is trivially true. However, establishing the radio number of graphs has proved to be quite tedious. Even so, such numbers have been completely determined for some

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graphs. Liu and Zhu [10] showed that for path, P_n, $n \geq 3$,

$$rn(P_n) = \begin{cases} 2k(k-1)+1 & \text{if } n = 2k; \\ 2k^2+2 & \text{if } n = 2k+1. \end{cases}$$

This improves results in [5] and [4] by Chatrand, et. al. where the upper and lower bounds for the same class of graph are obtained. Furthermore, Liu and Xie [9], found the radio number for the square of a path, P_n^2 as:

$$rn(P_n^2) = \begin{cases} k^2+2 & \text{if } n \equiv 1 (\text{mod } 4), \text{n} \geq 9; \\ k^2+1 & \text{if otherwise.} \end{cases}$$

Similar results are obtained in [8] for square of cycles. Jiang [7] completely solved the radio number problem for the grid graph ($P_m \Box P_n$), where for $m, n > 2$, it is noted that $rn(P_m \Box P_n) = \frac{mn^2+nm^2-n}{2} - mn - m + 2$, for m-odd and n even. Saha and Panigrahi [12] and Ajayi and Adefokun [1] obtained results on the radio numbers of Cartesian products of two cycles (toroidal grid) and of path and star graph (stacked-book graph) respectively. In the case of stacked-book graph $G = S_n \Box P_m$, $rn(G) \leq n^2 m + 1$, which the authors noted is not tight. Recent results on radio number include those on middle graph of path [2], trees, [3] and edge-joint graphs [11].

In this paper, for even positive integer n, we consider the stacked-book graph $G_{m,n}$ and derive the $rn(G_{m,n})$ for the case $m \geq 4$. Furthermore, new lower and upper bounds of the number are obtained for $m = 3$, which improve similar results in [1].

§2. Preliminaries

Let S_m be a star of order $m \geq 3$ and for each vertex $v_i \in V(S_m)$, $2 \leq i \leq m$, v_i is adjacent to v_1, the center vertex of S_m. Also, let P_n be a path such that $|V(P_n)| = n$. The Graph $G_{m,n} = S_m \Box P_n$, is obtained by the Cartesian product of S_m and P_n. The vertex set $V(G_{m,n})$ is the Cartesian product $V(S_m) \times V(P_n)$, such that for any $u_i v_j \in V(G_{m,n})$, then, $u_i \in V(S_m)$ and $v_j \in E(P_n)$. For $E(G_{m,n})$, $u_i v_j \ u_k v_l$ is contained in $E(G_{m,n})$ for $u_i v_j$, $u_k v_l \in V(G_{m,n})$, then either $u_i = u_k$ and $v_j v_l \in E(P_m)$ or $u_i u_k \in E(S_m)$ and $v_j = v_l$. Geometrically, $V(G_{m,n})$ contains n number of S_m stars, namely $S_{m(1)}, S_{m(2)}, \cdots, S_{m(n)}$, such that for every pair $v_i \in S_{m(i)}$ and $v_{i+1} \in S_{m(i+1)}$, $v_i v_{i+1} \in E(G_{m,n})$. These are, in fact, the only type of edges on $G_{m,n}$ apart from those on its S_m stars. This geometry fetched $G_{m,n}$ the name *stacked-book* graph.

Remark 2.1 It is easy to see that $diam(G_{m,n}) = n+1$, being the number of edges from $u_i v_1 \to u_1 v_1 \to u_1 v_2 \to \cdots \to u_1 v_n \to u_j v_n$, where $i \neq j$.

Remark 2.2 For convenience, we write $u_i v_j$ as $u_{i,j}$ in certain cases and $u_{i,j} u_{k,l}$ is the edge induced by $u_i v_j$ and $u_k v_l$.

Definition 2.1 *Let* $G_{m,n} = S_m \Box P_n$*. The vertex set* $V_{(i)} \subset V(G_{m,n})$ *is the set of vertices on* *star* $S_{m(i)}$, *defined by the set* $\{u_1v_i, u_2v_i, \cdots, u_mv_i\}$.

We introduce the following definition:

Definition 2.2 *Let* $G_{m,n} = S_m\square P_n$. *Then, the pair* $\left\{S_{m(i)}, S_{m(i+\frac{n}{2})}\right\}$ *is a subgraph* $G(i) \subseteq G_{m,n}$ *induced by* $V_{(i)}$ *and* $V_{(i+\frac{n}{2})}$.

Remark 2.3 The maximum number of $G(i)$ subgraph in a $G_{m,n}$ graph, n even, is $\frac{n}{2}$ and the $diam(G(i)) = \frac{n}{2} + 2$.

Remark 2.4 Let $\left\{V_{(i)}, V_{(i+\frac{n}{2})}\right\}$ induce $G(i)$, such that $V_{(i)} = \{u_{1,i}, u_{2,i}, \cdots, u_{m,i}\}$ and $V_{(i+\frac{n}{2})} = \left\{u_{1,\frac{i+n}{2}}, u_{2,\frac{i+n}{2}}, \cdots, u_{m,\frac{i+n}{2}}\right\}$. Then, for $u \in V_{(i)}$, $v \in V_{(i+\frac{n}{2})}$ and $d(u, v) = p$, where $p \in \left\{\frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2\right\}$ and for $u_{k,i}, v_{t,i+\frac{n}{2}}$,

$$p = \begin{cases} \frac{n}{2} & \text{if } k = t; \\ \frac{n}{2} + 1 & \text{if } t = 1, k \neq t; \\ \frac{n}{2} + 2 & \text{if } t \neq 1, k \neq 1, k \neq t. \end{cases}$$

§3. Results

In this section, we investigate the radio number of stacked-book graphs and obtain the exact radio number for $G_{m,n}$, for $m \geq 4$, n even.

Lemma 3.1 *Let* S_m *be a star on* $G_{m,n}$ *and* f, *a radio label function on* $G_{m,n}$. *Then spanf on* S_m *is* $n(m-1) + 1$.

Proof Let the center vertex of S_m be v_1 and let $f(v_1)$ be the radio label on v_1. There exists some $v_2 \in V(S_m)$ such that $d(v_1, v_2) = 1$. Therefore, by the definition, $f(v_2) \geq f(v_1) + n + 1$. Suppose that $k \notin \{1, 2\}$, then $d(v_2, v_k) = 2$, for all $v_k \in V(S_m)$. Thus, without loss of generality, suppose that v_m is the last vertex on $V(S_m)$, then $f(v_m) \geq f(v_0) + (n+1) + n(m-2)$ and the claim follows. $\square$

Remark 3.1 It is easy to confirm that given a star S_m with center vertex v_1, if for a positive integer α, $rn(S_m) = \alpha$, then either $f(v_1)$ is f_{min} or f_{max}.

Now we establish lower bound for $G(i)$.

Lemma 3.2 *Let* $G(i)$ *be a subgraph of* $G_{m,n}$ *and let* f *be the radio label on* $V(G_{m,n})$. *Then,* $rn(G(i)) \geq f(v_1) + mn - \frac{n}{2} + 2$, *where* v_1 *is the center vertex of* $S_{m(i+\frac{n}{2})}$.

Proof Let $S_{m(i)}$ and $S_{m(j)}$ be the stars on $G(i) \subset G_{m,n}$, where $j = i + \frac{n}{2}$. By Lemma 3.1, $f(v_m) = f(v_1) + mn - n + 1$, with $f(v_m) = \max\{f(v_t) : v_t \in V(S_m(j))\}$, and v_1 the center of $S_{m(j)}$. Now, let u_1 be the center vertex of $S_{m(i)}$. It is clear that $d(u_1, v_1) = \frac{n+2}{2}$. Thus, $f(u_1) \geq f(v_1) + mn - n + 1 + \frac{n+2}{2} = f(v_1) + mn - \frac{n}{2} + 2$.

Claim. For optimal radio labeling of $G(i)$, maximum label on $S_{m(i)}$ is at least $f(u_1)$.

Consider some $u_m \in V(S_m)$, such that $m \neq 1$ and $d(u_m, v_m) = \frac{n}{2} + 2$. Then $f(u_m) = f(v_1) + mn - \frac{n}{2} + 1$. By Lemma 3.1, the span$f$ of f for a star S_m is $mn - n + 1$. Now, $f(u_m) - (mn - n + 1) = f(v_i) + \frac{n}{2}$. Thus, by Remark 3.1, $f(u_1) = f(v_1 + \frac{n}{2})$. This is a contradiction, considering that $d(u_1, v_1) = \frac{n}{2}$. □

Lemma 3.3 *Let* $G^+(i) \subset G_{m,n}$ *be* $G(i) \cup w_1$*, where* w_1 *is the center vertex of* $S_{m(j+1)}$ *and let* f *be a radio labeling on* $G_{m,n}$*, where* n *is even. Then, the span*f *of* f *on* $G^+(i) \geq mn + 3$.

Proof Let u_1 be the center vertex of $S_{m(i)}$. It can be verified that $d(u_1, w_1) = \frac{n}{2}$. By the proof of Lemma 3.1, $f(u_1) \geq f(v_1) + mn - \frac{n}{2} + 2$, where v_1 is the center vertex of $S_{m(j)}$. Thus by definition, $f(w_1) \geq f(v_1) + mn - \frac{n}{2} + 2 + \frac{n+2}{2} = f(v_1) + mn + 3$. Since $f(v_1)$ is the minimum label on $G(i)$, the result follows. □

Now we present the lower bound for stacked-book graph $G_{m,n}$, where n is an even integer and $m \geq 3$.

Theorem 3.1 *Let* $G = G_{m,n}$ *be a stacked-book graph with* $m \geq 3$ *and* n *an even integer. Furthermore, let* f *be the radio labeling on* G*. Then,* $rn(G) \geq \frac{mn^2}{2} + n - 1$.

Proof From the definition of $G(i)$, graph $G_{m,n}$ contains $\frac{n}{2}$ number of $G(i)$ subgraphs. Likewise, it can be seen that $G_{m,n}$ contains $\frac{n-1}{2}$ number of $G^+(i)$ subgraphs. Now, let $G(\frac{n}{2})$, induced by $S_{m(\frac{n}{2})}$ and $S_{m(n)}$ be the last $G(i)$ subgraphs on $G_{m,n}$ and $G^+(1), G^+(2), \cdots, G^+(\frac{n-1}{2})$ be the $\frac{n-1}{2}$ number of $G^+(i)$ graphs. By the earlier result, if $f(v_1) = 0$, then $rn(G_{m,n}) \geq \left(\frac{n-1}{2}\right)(mn + 3) + mn - \frac{n}{2} + 2 = \frac{mn^2}{2} + n - 1$. □

In what follows, we examine the upper bound of the stacked-book graph $G_{m,n}$.

Lemma 3.4 *Let* $G(i)$ *be a subgraph of* $G_{m,n}$ *induced by* $\left\{V_{(i)}, V_{(i+\frac{n}{2})}\right\}$*. Then for any pair* $v \in V_{(i)}$ *and* $u \in V_{(i+\frac{n}{2})}$*, such that* $d(u, v) \geq \frac{n}{2} + 1$*,* $|f(v) - f(u)| \geq \frac{n}{2}$.

Proof Let $u = u_{k,i} \in V_{(i)}$ and $v = u_{t,i+\frac{n}{2}} \in V_{(i+\frac{n}{2})}$. Since $d(u, v) > \frac{n}{2}$, then by Remark 2.4, $k \neq t$. Suppose that neither u nor v is the center vertex of their respective stars $S_{m(i)}$ and $S_{m(i+\frac{n}{2})}$. Then, $d(u, v) = diam(G(i))$. Now, let the radio label on u and v be $f(u)$ and $f(v)$ respectively. Suppose, without loss of generality, that $f(v) > f(u)$. Then $f(v) \geq f(u) + diam(G_{m,n}) + 1 - diam(G(i))$, which implies that

$$f(v) \quad \geq \quad f(u) + \frac{n}{2}.$$

This implies that $f(v) - f(u) \geq \frac{n}{2}$. Similarly, if $f(u) \geq f(v)$, then $f(u) - f(v) \geq \frac{n}{2}$ and thus, the claim follows. □

The following remarks can be confirmed by applying similar methods as in the proof of Lemma 3.4.

Remark 3.2 Suppose that either of u, v in Lemma 3.4, say u, is such that for any $u' \in V_{(i)}$,

$uu' \in E(S_{m(i)})$. Then $d(u,u') = \frac{n}{2}+1$ and $|f(u)-f(v)| \geq \frac{n}{2}+1$.

Remark 3.3 Let $u, u' \in V_{(i)}$. If $d(u,u')=1$, then $|f(u)-f(u')| \geq n+1$ and $|f(u)-f(u')| \geq n$ for $d(u,u')=2$.

Theorem 3.2 *Let $m>3$ be odd and $G(i) \subseteq G_{m,n}$, be induced by $\left\{V_{(i)}, V_{(i+\frac{n}{2})}\right\}$. then, $rn(G(i)) \leq f(v_1)+mn-\frac{n}{2}+2$, where v_1 is the center star $S_{m(1+\frac{n}{2})}$.*

Proof Let $V_{(i)} = \{u_{1,i}, u_{2,i}, \cdots, u_{m,i}\}$ and $V_{(t)} = \{u_{1,t}, u_{2,t}, \cdots, u_{m,t}\}$, where $t = i+\frac{n}{2}$. For $r \in [1,m]$, set $u_{r,i} \in V_{(i)}$ as α_r and $u_{r,t} \in V_{(t)}$ as β_r. From earlier remark, $d(\beta_r, \alpha_r) \in \left\{\frac{n}{2}+1, \frac{n}{2}+2\right\}$ for $r \neq s$. Now, for every pair α_s, β_r, where $\alpha_s \in V_{(i)}$, and $\beta_r \in V_{(t)}$, let $r \neq s$ except otherwise stated. Let α_1 and β_1 be the respective centers of the stars $S_{m(i)}$ and $S_{m(t)}$ induced by $V_{(i)}$ and $V_{(t)}$ and let the radio label on β_1 be $f(\beta_1)$ such that $f(\beta_1)=\min\{f(\beta_i): 1 \leq i \leq m\}$. Since β_1 is the center of $S_{m(t)}$, then given $\alpha_2 \in V_{(i)}$, $d(\beta_1, \alpha_2) = \frac{n}{2}+1$. Now set $p = diam(G_{m,n})+1-d(\beta_1,\alpha_r)$, $r \neq 1$. Hence, $p = n+2-(\frac{n}{2}+1) = \frac{n}{2}+1$. Suppose that $\alpha_j \in V_{(i)}$ and $\beta_k \in V_{(t)}$, such that $1 \neq j \neq k \neq 1$. Then, $d(\alpha_j, \beta_k) = \frac{n}{2}+2$. So we set $q = diam(G_{m,n})+1-d(\alpha_j,\beta_k) = \frac{n}{2}$. For $f(\beta_1)$ and some $\alpha_2 \in V_{(i)}$, $f(\alpha_2) = f(\beta_1)+p$. Also, for $\beta_3 \in V_{(t)}$, $f(\beta_3) = f(\alpha_2)+q = f(\beta_1)+p+q$ and $f(\alpha_4) = f(\beta_1)+2q+p$. We continue to label the vertices on both $V_{(i)}$ and $V_{(t)}$ alternatively based on the last value attained. Therefore, for m odd,

$$\begin{aligned} f(\beta_m) &= f(\alpha_{m-1})+\frac{n}{2} \\ &= f(\beta_1)+(m-2)q+p. \end{aligned}$$

It can be seen that there does not exist $\alpha_d \in V_{(i)}$, such that $d>m$. So, we reverse the order of labeling, such that for β_m, α_3, $f(\alpha_3) = f(\beta_m)+q = f(\beta_1)+(m-2)q+2p$. Also, for the pair α_3, β_2, $f(\beta_2) = f(\beta_1)+(m-2)q+2q+p$. This continues until we reach the pair α_m, β_{m-1}, and obtain

$$f(\alpha_{m-1}) = f(\beta_1)+(2m-3)q+p.$$

Finally, we consider the pair β_{m-1} and α_1. Since α_1 is the center of $S_{(i)}$, then $d(\alpha_1, \beta_{m-1}) = \frac{n}{2}+1$ and hence,

$$\begin{aligned} f(\alpha_1) &= f(\alpha_{m-1})+p \\ &= f(\beta_1)+(2m-3)q+2p \\ &= f(\beta_1)+mn-\frac{n}{2}+2. \end{aligned}$$

Hence, $rn(G(i)) \leq f(v_1)+mn-\frac{n}{2}+2$, where m is odd and n even. $\Box$

Next, we directly apply Theorem 3.2 to get results following.

Lemma 3.5 *Let $\bar{G}(i)$ be induced by $\left\{S_{m(i)}, S_{m(i+\frac{n}{2})}, \gamma_1\right\}$, where γ_1 is the center of star $S_{m(i+\frac{n}{2}+1)}$, induced by $V_{(i+\frac{n}{2}+1)}$. Then, $f(\gamma_1) \leq f(\beta_1)+mn+3$.*

Proof For α_1 and β_1 centers of stars $S_{(i)}$ and $S_{(i+\frac{n}{2})}$ respectively, let $f(\alpha_1) = f(\beta_1) + mn - \frac{n}{2} + 2$, as established in Theorem 3.2. Then, $d(\alpha_1, \gamma_1) = \frac{n}{2} + 1$. Therefore,

$$\begin{aligned} f(\gamma_1) &= f(\alpha_1) + p \\ &= f(\beta_1) + mn + 3. \end{aligned}$$

This completes the proof. □

Now, for β_1, the center of $S_{m(1+\frac{n}{2})}$, induced by $V_{(1+\frac{n}{2})}$. By setting $f(\beta_1) = 0$, we establish an upper bound for the radio number of a stacked-book graph $G_{m,n}$ in the next results.

Theorem 3.3 *For a graph $G_{m,n}$ with m odd and n even, $rn(G_{m,n}) \leq \frac{mn^2}{2} + n - 1$.*

Proof Let $\left\{v_{1(1)}, v_{2(1)}, v_{3(1)}, \cdots, v_{n(1)}\right\}$ be the set of the respective centers of stars $S_{m(1)}$, $S_{m(2)}$, $S_{m(3)}$, $\cdots$, $S_{m(n)}$ in $G_{m,n}$. Also, suppose that $f(v_{\frac{n}{2}+1(1)}) = 0$. From the Lemma 3.5, $f(v_{\frac{n}{2}+2(1)}) = mn+3$; $f(v_{\frac{n}{2}+3(1)}) = 2(mn+3)$ and so on. In the end, $f(v_{n(1)}) = (\frac{n}{2}-1)(mn+3)$. Also, let $v_{n-\frac{n}{2}(1)} = v_{\frac{n}{2}(1)}$ be the center of $S_{m(\frac{n}{2})} \subset G_{m,n}$ and let $S_{m(\frac{n}{2})}, S_{m(n)}$ induce the graph $G(\frac{n}{2}) \subset G_{m,n}$. By Theorem 3.2,

$$\begin{aligned} rn\left(G\left(\frac{n}{2}\right)\right) &\leq f(v_{n(1)}) + mn - \frac{n}{2} + 2 \\ &\leq \frac{mn^2}{2} + n - 1. \end{aligned}$$

This completes the proof. □

Theorem 3.4 *Let m, n be even. Then $rn(G_{m,n}) \leq \frac{mn^2}{2} + n - 1$.*

Proof The proof follows similar argument and technique as in Theorems 3.2, 3.3 and Lemma 3.5. □

Notice that Theorems 3.1, 3.3 and 3.4 establish the radio number of $G_{m,n}$, where $m \geq 4$ and n is even, as recapped in the next theorem.

Theorem 3.5 *Let $G_{m,n}$ be a stacked-book graph with $m \geq 4$ and n even, then, $rn(G_{m,n}) = \frac{mn^2}{2} + n - 1$.*

Next we consider the case where $m = 3$. First we present a result that is equivalent to Theorem 3.2 with respect to $m = 3$.

Theorem 3.6 *Let $G_{3,n}$ be a stacked-book graph, where n is even. Suppose that the pair $\left\{S_{3(i)}, S_{3(i+\frac{n}{2})}\right\}$ form a subgraph $G(i)$ of $G_{3,n}$. Then, $rn(G(i)) \leq f(u_1) + \frac{5n}{2} + 3$, where u_1 is the center vertex of $S_{3(i+\frac{n}{2})}$.*

Proof Let $V_{(i)} = \{v_1, v_2, v_3\}$ and $V_{(i+\frac{n}{2})} = \{u_1, u_2, u_3\}$ where $V_{(i)}$ and $V_{(i+\frac{n}{2})}$ are vertex sets of stars $S_{3(i)}$ and $S_{3(i+\frac{n}{2})}$ in $G_{3,n}$ respectively. Also, let v_1 and u_i be the respective center vertices of $S_{3(i)}$ and $S_{3(i+\frac{n}{2})}$. From earlier remark, $d(v_1, u_1) = \frac{n}{2} + 1$. Suppose that $f(u_1)$, the radio label of u_1 is the smallest possible radio label on $G(i)$, then,

$$\begin{aligned} f(v_2) &= f(v_1) + diam(G_{3,n}) + 1 - d(v_1, u_1) \\ &= f(u_i) + \frac{n}{2} + 1. \end{aligned}$$

For v_2, u_3, $d(v_2, u_3) = \frac{n}{2} + 2$, $f(u_3) = f(u_1) + n + 1$; For u_3, v_1, $d(u_3, v_1) = \frac{n}{2} + 1$, $f(v_1) = f(u_1) + \frac{3n}{2} + 2$; For v_1, u_2, $d(v_1, u_2) = \frac{n}{2} + 1$, $f(u_2) = f(u_1) + 2n + 3$ and finally, for the pair v_3, u_2, $d(v_3, u_3) = \frac{n}{2} + 2$ and $f(u_3) = f(u_1) + \frac{5n}{2} + 3$. Hence, $rn(G(i)) \leq f(u_1) + \frac{5n}{2} + 3$.□

Next, we obtain the following result.

Lemma 3.6 *Let* κ_1 *be the center of star* $S_{3(i+\frac{n}{2})+1} \subseteq G_{3,n}$ *and let* $\bar{H}(1)$ *be a subgraph of* $G_{(3,m)}$ *induced by* $\left\{S_{3(i)}, S_{3(i+\frac{n}{2}),\kappa_1}\right\}$. *Then* $f(\kappa_1) \leq 3n + 1$.

Proof The vertex with the maximum value of radio label in Theorem 3.6 is u_3. Let us adopt this, with $f(u_3) = f(u_1) + \frac{5n}{2} + 3$. Now, $d(u_3, \kappa_1) = \frac{n}{2} + 2$. Therefore, $f(\kappa_1) = f(u_1) + 3n + 3$. This completes the proof. □

In the final result here, we set $f(u_1) = 0$, for u_i, the center of star $S_{3(1+\frac{n}{2})}$.

Theorem 3.7 *Let* n *be an even positive integer. Then,* $rn(G_{3,n}) \leq \frac{3n^2}{2} + n$.

Proof The proof follows similar technique adopted in Theorem 3.4. □

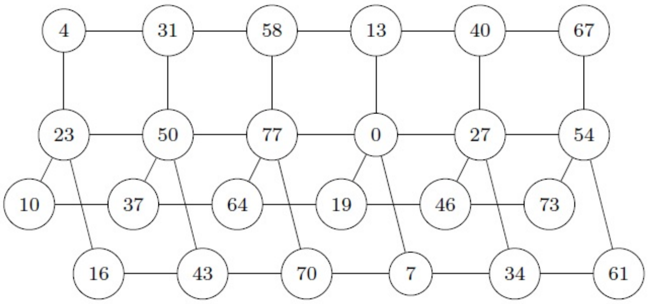

Figure 1. A $G_{4,6}$ graph with $rn(G_{4,6}) \leq 77$.

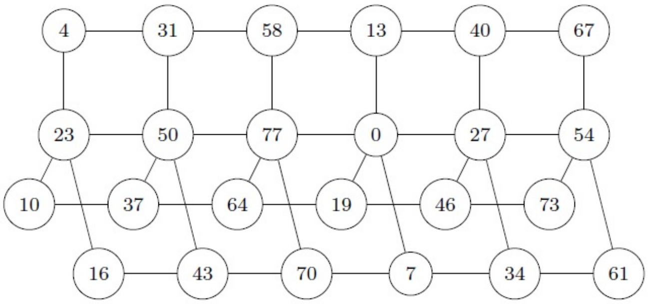

Figure 1. A $G_{3,6}$ graph with $rn(G_{3,6}) \leq 60$.

Notice that the radio numberings for $G_{4,6}$ and $G_{3,6}$ are shown in Figures 1 and 2. They demonstrate that that the radio numbers of the graphs can not be more than 77 and 60 respectively.

§4. Conclusion

It is noteworthy to look at some of the results in [7]. A $G_{3,n}$ is a $3 \times n$ grid. By [7], it is seen that $rn(G_{3,6}) = 59$, which is better than the result in Figure 2 above by 1. But this is still a considerable improvement compared with a upper bound of 109 suggested in [1]. In establishing the upper bound for $G_{3,n}$, it is observed that the number of the pair $u, v \in V(G_{3,n})$ for which $d(u,v) = \frac{diam(G_{3,n})+1}{2}$ is more than the case where $d(u,v) = \frac{n}{2}$ in each of the segments of radio labeling of the stacked-graph. However, the reverse proves to be the case in $G_{m,n}$, $m \geq 4$.

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