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On the Complexity of Some Classes of Circulant Graphs and Chebyshev Polynomials

S. N. Daoud[1,2], Muhammad Kamran Siddiqui[3]

[1].Department of Mathematics
Faculty of Science, Taibah University, Al-Madinah 41411, Saudi Arabia

[2]. Department of Mathematics and Computer Science
Faculty of Science, Menoufia University, Shebin El Kom 32511, Egypt

[3].Department of Mathematics
COMSATS University Islamabad, Lahore Campus, 54000, Pakistan

E-mail: salamadaoud@gmail.com, kamransiddiqui75@gmail.com

Abstract: Deriving closed formulae of the number of spanning trees for various graphs has attracted the attention of a lot of researchers. In this paper we derive simple and explicit formulas for the number of spanning trees in many classes of circulant graphs using the properties of Chebyshev polynomials. Deriving closed formulae of the number of spanning trees for various graphs has attracted the attention of a lot of researchers. In this paper we derive simple and explicit formulas for the number of spanning trees in many classes of circulant graphs using the properties of Chebyshev polynomials.

Key Words: Number of spanning trees, circulant graphs, Chebyshev polynomials.

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§1. Introduction

The number of spanning trees $\tau(G)$ in a graph G (networks) is an important invariant. We call $\tau(G)$ the complexity of G. The evaluation of this number is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. In this work we consider finite undirected graph with no loops or multiple edges. Let G be such a graph of n vertices. A spanning tree for a graph G is a subgraph of G that is a tree and contains all vertices of $\tau(G)$. The number of spanning trees of G, is the total number of distinct spanning subgraphs of G that are trees. A classic result of Kirchhoff [**?**] can be used to determine the number $\tau(G)$ for $G(V, E)$. Let $V = \{v_1, v_2, \cdots, v_n\}$. The Kirchhoff matrix H is defined as $n \times n$ characteristic matrix $H = D - A$, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G. Then the matrix $H - [a_{ij}]$ is defined as follows:

(1) a_{ij}, when v_i and v_j are adjacent and $i \neq j$;
(*ii*) a_{ij}, is equal to the degree of vertex v_i if $i = j$;
(*iii*) $a_{ij} = 0$, otherwise. All of co-factors of H are equal to $\tau(G)$.

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There are more than one method for calculating $\tau(G)$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$ denote the eigenvalues of matrix of a p point graph. It can be easily shown that $\mu_p = 0$. Kelmans and Chelnokov [2] proved that The formula for the number of spanning trees in a d-regular graph can be expressed as

$$\tau(G) = \frac{1}{p}\Big[\prod_{k=1}^{p-1}(d-\mu_k)\Big]$$

where $\mu_0 = d, \mu_1, \mu_2, \cdots, \mu_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. Many works have conceived techniques to derive the number of spanning tree of a graph can be found at [3-12]. The circulant graphs are an important class of graphs. Among other applications, they are used in the design of local area networks, see [13-19].

Let $1 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_k \leq \frac{n}{2}$, where n and $a_i (i = 1, 2, \ldots, k)$ are positive integers. An undirected circulant graph $C_n(a_1, a_2, a_3, \cdots, a_k)$ is a regular graph whose set of vertices is $V = \{0, 1, 2, [dots, n-1\}$ and whose set of edges is $E = \{i, i+a_i (\mathrm{mod}\, n)/i = 0, 1, 2, \cdots, n-1, j = 1, 2, \cdots, k\}$. If $a_k \leq \frac{n}{2}$, then $C_n(a_1, a_2, a_3, \cdots, a_k)$ is a $2k$-regular graph; if $a_k = \frac{n}{2}$, then it is a $2k - 1$-regular one, see Nikolopoulos [20] and Papadopoulos [21]. The well known formula $\tau(C_n(1, 2)) = nF_n^2$, where F_n is the n^{th} Fibonacci number, see Kleiman, and Golden [22]. We have obtained another proof for this formula in Theorem 3.3. The formulas of $\tau(C_{2n}(1, n))$, $\tau(C_{3n}(1, n))$, $\tau(C_{4n}(1, n))$ can be found in Yuanping, et. al.[23].

§2. Chebyshev Polynomial

In this section we introduce some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations. We begin from their definitions, see Yuanping, et. al.[24]. Let $A_n(x)$ be $n \times n$ matrix such that:

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & \ldots & \ldots \\ -1 & 2x & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2x & -1 \\ \vdots & \ddots & 0 & -1 & 2x \end{pmatrix}$$

where all other elements are zeros.

Further, we recall that the Chebyshev polynomials of the first kind are defined by:

$$T_n(x) = \cos(n \arccos). \tag{1}$$

The Chebyshev polynomials of the second kind are defined by

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \arccos)}{\sin(\arccos)}. \tag{2}$$

It is easily verified that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0. \tag{3}$$

It can then be shown from this recursion that by expanding one gets

$$U_n(x) = det(A_n(x)), \quad n \geq 1. \tag{4}$$

Furthermore by using standard methods for solving the recursion (3), one obtains the explicit formula

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\Big[\big(x+\sqrt{x^2-1}\big)^{n+1} - \big(x-\sqrt{x^2-1}\big)^{n+1}\Big], \quad n \geq 1, \tag{5}$$

where the identity is true for all complex (except at $x = \pm 1$, where the function can be taken as the limit). The definition of easily yields its zeros and it can therefore be verified that

$$U_{n-1}(x) = 2^{n-1}\Big[\prod_{j=1}^{n-1}(x - \cos\frac{j\pi}{n})\Big]. \tag{6}$$

One further notes that

$$U_{n-1}(-x) = (-1)^{n-1}U_{n-1}(x). \tag{7}$$

These two results yield another formula for $U_n(x)$

$$U_{n-1}^2(x) = 4^{n-1}\Big[\prod_{j=1}^{n-1}(x^2 - \cos^2\frac{j\pi}{n})\Big]. \tag{8}$$

Finally, a simple manipulation of the above formula yields the following formula (9), which is extremely useful to us latter:

$$U_{n-1}^2\left(\sqrt{\frac{x+2}{4}}\right) = \Big[\prod_{j=1}^{n-1}(x - 2\cos\frac{j\pi}{n})\Big] \tag{9}$$

Furthermore, one can show that

$$U_{n-1}^2(x) = \frac{1}{2(1-x^2)}\big[1 - T_{2n}\big] = \frac{1}{2(1-x^2)}\big[1 - T_{2n}(2-2x^2)\big] \tag{10}$$

and

$$T_n(x) = \frac{1}{2}\Big[\big(x+\sqrt{x^2-1}\big)^n + \big(x-\sqrt{x^2-1}\big)^n\Big]. \tag{11}$$

§3. Main Results

In our main results, i.e., Theorems 3.1 - 3.6 we use the following conclusion.

Lemma 3.1([25]) *The Kirchhoff matrix of the circulant graph* $C_n(s_1, s_2, s_3, \cdots, s_k)$ *has* n

eigenvalues, namely: 0 and the value $2k-\varepsilon^{-s_1 j}-\cdots-\varepsilon^{-s_k j}-\varepsilon^{-s_1 j}-\cdots-\varepsilon^{-s_k j}$ *with* $\varepsilon=e^{\frac{2\pi j}{n}}$ *for any* $j=\{1,2,\cdots,n-1\}$

Corollary 3.2 *For the circulant graph* $C_n(s_1,s_2,s_3,\cdots,s_k)$,

$$\begin{aligned}\tau(C_n(s_1,s_2,s_3,\cdots,s_k)) &= \frac{1}{n}\left[\prod_{j=1}^{n-1}\left(2k-\varepsilon^{-s_1 j}-\cdots-\varepsilon^{-s_k j}-\varepsilon^{-s_1 j}-\cdots-\varepsilon^{-s_k j}\right)\right]\\ &= \frac{1}{n}\left[\prod_{j=1}^{n-1}\left(\sum_{i=1}^{k}(2-2\cos\frac{js_i\pi}{n})\right)\right].\end{aligned}$$

Proof The proof follows immediately from Lemma 3.1. □

Theorem 3.3 *For the spanning trees of* C_{12n} *with three jumps* $1, 2n, 3n$ *, we have:*

$$\begin{aligned}\tau(C_{12n}(1,2n,3n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{4n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{4n}-1\right]^2\\ &\times\frac{n}{12}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2\\ &\times\frac{n}{12}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{2n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{2n}\right]^2\\ &\times\frac{n}{12}\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2\\ &\times\frac{n}{12}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}-1\right]^2\end{aligned}$$

Proof Let $\varepsilon=e^{\frac{2\pi i}{12n}}$. Applying Lemma 3.1, we have

$$\begin{aligned}\tau(C_{12n}(n,2n,3n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-\varepsilon^{-j}-\varepsilon^{-2nj}-\varepsilon^{-3nj}-\varepsilon^{j}-\varepsilon^{2nj}-\varepsilon^{3nj}\right)\\ &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{4\pi jn}{12n}-2\cos\frac{6\pi jn}{12n}\right)\\ &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{12n}\prod_{j=1,2\nmid j}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)\\
&\quad\times\prod_{j=1,2|j}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}\right)\\
&= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)\\
&\quad\times\prod_{j=1}^{12n-1}\frac{6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}}{6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}}
\end{aligned}
$$

If we put $j=2j'$ in the second term for some integer j' we get

$$
\begin{aligned}
\tau(C_{12n}(n,2n,3n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)\\
&\quad\times\prod_{j=1}^{6n-1}\frac{6-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-2\cos\pi j}{6-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}}\\
&= \frac{1}{12n}\prod_{j=1,2\nmid j,3\nmid j}^{12n-1}\left(5-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1,2|j,3|j}^{12n-1}\left(4-2\cos\frac{2\pi j}{12n}\right)\\
&\quad\times\prod_{j=1,2|j,3|j}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1,2|j,3|j}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)\\
&\quad\times\frac{\prod_{j=1,2\nmid j}^{6n-1}\left(8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}\right)\prod_{j=1,2\nmid j}^{6n-1}\left(4-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}\right)}{\prod_{j=1}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}\right)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\tau(C_{12n}(n,2n,3n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(5-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{2n-1}\frac{\left(5-2\cos\frac{2\pi j}{2n}\right)\left(4-2\cos\frac{2\pi j}{2n}\right)}{\left(8-2\cos\frac{2\pi j}{2n}\right)\left(7-2\cos\frac{2\pi j}{2n}\right)}\\
&\quad\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}}{5-2\cos\frac{2\pi j}{6n}}\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{5-2\cos\frac{2\pi j}{4n}}\\
&\quad\times\frac{\prod_{j=1}^{6n-1}\left(8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}\right)\prod_{j=1}^{3n-1}\left(4-2\cos\frac{2\pi j}{3n}-2\cos\frac{4\pi j}{3}\right)}{\prod_{j=1}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}\right)\prod_{j=1}^{3n-1}\left(8-2\cos\frac{2\pi j}{3n}-2\cos\frac{4\pi j}{3}\right)}.
\end{aligned}
$$

So, we get that

$$\begin{aligned}\tau(C_{12n}(n,2n,3n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(5-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{2n-1}\frac{\left(5-2\cos\frac{2\pi j}{2n}\right)\left(4-2\cos\frac{2\pi j}{2n}\right)}{\left(8-2\cos\frac{2\pi j}{2n}\right)\left(7-2\cos\frac{2\pi j}{2n}\right)}\\ &\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}}{5-2\cos\frac{2\pi j}{6n}}\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{5-2\cos\frac{2\pi j}{4n}}\\ &\times\frac{\prod\limits_{j=1,3|j}^{6n-1}\left(9-2\cos\frac{2\pi j}{6n}\right)\prod\limits_{j=1,3|j}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}\right)}{\prod\limits_{j=1,3|j}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod\limits_{j=1,3|j}^{6n-1}\left(4-2\cos\frac{2\pi j}{6n}\right)}\\ &\times\frac{\prod\limits_{j=1}^{3n-1}\left(6-2\cos\frac{2\pi j}{3n}-4\cos^2\frac{2\pi j}{3}\right)}{\prod\limits_{j=1}^{3n-1}\left(10-2\cos\frac{2\pi j}{3n}-4\cos^2\frac{2\pi j}{3}\right)},\end{aligned}$$

i.e.,

$$\begin{aligned}\tau(C_{12n}(n,2n,3n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(5-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{2n-1}\frac{\left(5-2\cos\frac{2\pi j}{2n}\right)\left(4-2\cos\frac{2\pi j}{2n}\right)}{\left(8-2\cos\frac{2\pi j}{2n}\right)\left(7-2\cos\frac{2\pi j}{2n}\right)}\\ &\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}}{5-2\cos\frac{2\pi j}{6n}}\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{5-2\cos\frac{2\pi j}{4n}}\\ &\times\frac{\prod\limits_{j=1}^{6n-1}\left(9-2\cos\frac{2\pi j}{6n}\right)\prod\limits_{j=1}^{6n-1}\frac{\left(6-2\cos\frac{2\pi j}{6n}\right)}{\left(9-2\cos\frac{2\pi j}{6n}\right)}}{\prod\limits_{j=1}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod\limits_{j=1}^{6n-1}\frac{\left(4-2\cos\frac{2\pi j}{6n}\right)}{\left(7-2\cos\frac{2\pi j}{6n}\right)}}\\ &\times\frac{\prod\limits_{j=1,3|j}^{3n-1}\left(5-2\cos\frac{2\pi j}{3n}\right)\prod\limits_{j=1,3|j}^{3n-1}\left(2-2\cos\frac{2\pi j}{3n}\right)}{\prod\limits_{j=1,3|j}^{3n-1}\left(9-2\cos\frac{2\pi j}{3n}\right)\prod\limits_{j=1,3|j}^{3n-1}\left(6-2\cos\frac{2\pi j}{3n}\right)}\end{aligned}$$

Furthermore,

$$\begin{aligned}\tau(C_{12n}(n,2n,3n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(5-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{2n-1}\frac{\left(5-2\cos\frac{2\pi j}{2n}\right)\left(4-2\cos\frac{2\pi j}{2n}\right)}{\left(8-2\cos\frac{2\pi j}{2n}\right)\left(7-2\cos\frac{2\pi j}{2n}\right)}\\ &\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}}{5-2\cos\frac{2\pi j}{6n}}\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{5-2\cos\frac{2\pi j}{4n}}\end{aligned}$$

$$\times\frac{\prod_{j=1}^{6n-1}\left(9-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{\left(6-2\cos\frac{2\pi j}{6n}\right)}{\left(9-2\cos\frac{2\pi j}{6n}\right)}}{\prod_{j=1}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{\left(4-2\cos\frac{2\pi j}{6n}\right)}{\left(7-2\cos\frac{2\pi j}{6n}\right)}}$$

$$\times\frac{\prod_{j=1}^{3n-1}\left(5-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1}^{3n-1}\frac{\left(2-2\cos\frac{2\pi j}{3n}\right)}{\left(5-2\cos\frac{2\pi j}{3n}\right)}}{\prod_{j=1}^{3n-1}\left(9-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1}^{3n-1}\frac{\left(6-2\cos\frac{2\pi j}{3n}\right)}{\left(9-2\cos\frac{2\pi j}{3n}\right)}}.$$

We get that

$$\begin{aligned}\tau(C_{12n}(n,2n,3n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(5-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{2n-1}\frac{\left(5-2\cos\frac{2\pi j}{2n}\right)\left(4-2\cos\frac{2\pi j}{2n}\right)}{\left(8-2\cos\frac{2\pi j}{2n}\right)\left(7-2\cos\frac{2\pi j}{2n}\right)}\\
&\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}}{5-2\cos\frac{2\pi j}{6n}}\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{5-2\cos\frac{2\pi j}{4n}}\\
&\times\frac{\prod_{j=1}^{6n-1}\left(9-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{\left(6-2\cos\frac{2\pi j}{6n}\right)}{\left(9-2\cos\frac{2\pi j}{6n}\right)}}{\prod_{j=1}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{\left(4-2\cos\frac{2\pi j}{6n}\right)}{\left(7-2\cos\frac{2\pi j}{6n}\right)}}\\
&\times\frac{\prod_{j=1}^{3n-1}\left(5-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1}^{n-1}\frac{\left(2-2\cos\frac{2\pi j}{n}\right)}{\left(5-2\cos\frac{2\pi j}{n}\right)}}{\prod_{j=1}^{3n-1}\left(9-2\cos\frac{2\pi j}{n}\right)\prod_{j=1}^{n-1}\frac{\left(6-2\cos\frac{2\pi j}{n}\right)}{\left(9-2\cos\frac{2\pi j}{n}\right)}}.\end{aligned}$$

Thus,

$$\begin{aligned}\tau(C_{12n}(n,2n,3n)) &= \frac{1}{12n}\times U_{12n-1}^2\left(\sqrt{\frac{7}{4}}\right)\times\frac{U_{2n-1}^2\left(\sqrt{\frac{7}{4}}\right)\times U_{2n-1}^2\left(\sqrt{\frac{3}{2}}\right)}{U_{2n-1}^2\left(\frac{3}{2}\right)\times U_{2n-1}^2\left(\sqrt{\frac{5}{2}}\right)}\\
&\times\frac{U_{4n-1}^2\left(\frac{3}{2}\right)\times U_{6n-1}^2\left(\sqrt{\frac{5}{2}}\right)}{U_{6n-1}^2\left(\sqrt{\frac{7}{2}}\right)\times U_{6n-1}^2\left(\sqrt{\frac{7}{2}}\right)}\\
&\times\frac{U_{6n-1}^2\left(\sqrt{\frac{11}{4}}\right)\times U_{2n-1}^2\left(\frac{3}{2}\right)\times U_{2n-1}^2\left(\sqrt{2}\right)\times U_{3n-1}^2\left(\frac{7}{4}\right)\times U_{n-1}^2\left(\frac{11}{4}\right)n^2}{U_{6n-1}^2\left(\frac{3}{2}\right)\times U_{2n-1}^2\left(\sqrt{\frac{3}{2}}\right)\times U_{n-1}^2\left(\sqrt{\frac{7}{4}}\right)\times U_{3n-1}^2\left(\sqrt{\frac{11}{4}}\right)\times U_{n-1}^2\left(\sqrt{2}\right)},\end{aligned}$$

which implies that

$$\tau(C_{12n}(n,2n,3n)) = \frac{n}{12}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{4n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{4n}-1\right]^2$$

$$\times\frac{n}{12}\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2$$
$$\times\frac{n}{12}\Big[\Big(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\Big)^{2n}+\Big(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\Big)^{2n}\Big]^2$$
$$\times\frac{n}{12}\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2$$
$$\times\frac{n}{12}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}-1\right]^2,$$

where (6), (8) , (9) and (10) are used to derive the last two steps. □

Theorem 3.4 *For the spanning trees of C_{6n} with three jumps $1, 3n, 6n$, we have*

$$\begin{aligned}\tau(C_{12n}(1,3n,6n)) &= \frac{3n}{4}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{6n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{6n}\right]^2\\ &\quad\times\frac{3n}{4}\left[\left(\sqrt{2}+1\right)^{3n}+\left(\sqrt{2}-1\right)^{3n}\right]^2\end{aligned}$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Applying Lemma 3.1, we have

$$\begin{aligned}\tau(C_{12n}(1,3n,6n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-\varepsilon^{-j}-\varepsilon^{-3nj}-\varepsilon^{-6nj}-\varepsilon^{j}-\varepsilon^{3nj}-\varepsilon^{6nj}\right)\\ &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{6\pi jn}{12n}-2\cos\frac{12\pi jn}{12n}\right)\\ &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{2}-2\cos\pi j\right)\\ &= \frac{1}{12n}\prod_{j=1,2\nmid j}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1,2\mid j}^{12n-1}\left(4-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{2}\right).\end{aligned}$$

Whence,

$$\begin{aligned}\tau(C_{12n}(1,3n,6n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{6n-1}\frac{4-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{2}}{8-2\cos\frac{2\pi j}{6n}}\\ &= \frac{1}{12n}\frac{\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)}{\prod_{j=1}^{6n-1}\left(8-2\cos\frac{2\pi j}{6n}\right)}\times\prod_{j=1,2\nmid j}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1,2\mid j}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}\right).\end{aligned}$$

Thus,

$$\begin{aligned}
\tau(C_{12n}(1,3n,6n)) &= \frac{1}{12n}\frac{\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)}{\prod_{j=1}^{6n-1}\left(8-2\cos\frac{2\pi j}{6n}\right)}\prod_{j=1}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}\right)\\
&\quad\times\prod_{j=1}^{6n-1}\frac{2-2\cos\frac{2\pi j}{6n}}{6-2\cos\frac{2\pi j}{6n}}\\
&= \frac{1}{12n}\frac{\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)}{\prod_{j=1}^{6n-1}\left(8-2\cos\frac{2\pi j}{6n}\right)}\prod_{j=1}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}\right)\\
&\quad\times\prod_{j=1}^{3n-1}\frac{2-2\cos\frac{2\pi j}{3n}}{6-2\cos\frac{2\pi j}{3n}},
\end{aligned}$$

which implies that,

$$\begin{aligned}
\tau(C_{12n}(1,3n,6n)) &= \frac{1}{12n}\times\frac{U_{12n-1}^2\left(\sqrt{\frac{5}{2}}\right)\times(3n)^2\times U_{6n-1}^2(\sqrt{2})}{U_{6n-1}^2\left(\sqrt{\frac{5}{2}}\right)\times U_{3n-1}^2(\sqrt{2})}\\
&= \frac{3n}{4}\times\frac{U_{12n-1}^2\left(\sqrt{\frac{5}{2}}\right)\times U_{6n-1}^2(\sqrt{2})}{U_{6n-1}^2\left(\sqrt{\frac{5}{2}}\right)\times U_{3n-1}^2(\sqrt{2})},
\end{aligned}$$

i.e.,

$$\begin{aligned}
\tau(C_{12n}(1,3n,6n)) &= \frac{3n}{4}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{6n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{6n}\right]^2\\
&\quad\times\frac{3n}{4}\left[\left(\sqrt{2}+1\right)^{3n}+\left(\sqrt{2}-1\right)^{3n}\right]^2,
\end{aligned}$$

where (6), (8) , (9) and (10) are used to derive the last two steps. □

Theorem 3.5 *For the spanning trees of* C_{12n} *with three jumps* $1, 3n, 4n$ *, we have:*

$$\begin{aligned}
\tau(C_{12n}(1,3n,4n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}}\right)^{4n}+\left(\sqrt{\frac{9}{4}}-\sqrt{\frac{5}{4}}\right)^{4n}-1\right]^2\\
&\quad\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2
\end{aligned}$$

$$\times\left[\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{2n}+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)^{2n}\right]^{2}$$

$$\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^{2}$$

$$\times\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}-1\right]^{2}$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Applying Lemma 3.1, we have

$$\begin{aligned}
\tau(C_{12n}(1,3n,4n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-\varepsilon^{-j}-\varepsilon^{-3nj}-\varepsilon^{-4nj}-\varepsilon^{j}-\varepsilon^{3nj}-\varepsilon^{4nj}\right)\\
&= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}\right)\\
&= \frac{1}{12n}\prod_{j=1,2\nmid j}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)\\
&\times \prod_{j=1,2\mid j}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}\right)\\
&= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)\\
&\quad\times\prod_{j=1}^{6n-1}\frac{6-2\cos\frac{2\pi j}{6n}-2\cos\pi j-2\cos\frac{4\pi j}{3}}{6-2\cos\frac{2\pi j}{6n}-2\cos\frac{4\pi j}{3}}\\
&= \frac{1}{12n}\prod_{j=1,3\nmid j}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1,3\mid j}^{12n-1}\left(4-2\cos\frac{2\pi j}{12n}\right)\\
&\quad\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}-2\cos\pi j-2\cos^{2}\frac{2\pi j}{3}}{\prod_{j=1}^{6n-1}8-2\cos\frac{2\pi j}{6n}-2\cos^{2}\frac{2\pi j}{3}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\tau(C_{12n}(1,3n,4n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{4n-1}\frac{4-2\cos\frac{2\pi j}{4n}}{7-2\cos\frac{2\pi j}{4n}}\\
&\times\frac{\prod_{j=1,2\nmid j}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}-4\cos^{2}\frac{2\pi j}{3}\right)\prod_{j=1,2\mid j}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}-4\cos^{2}\frac{2\pi j}{3}\right)}{\prod_{j=1,3\nmid j}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1,3\mid j}^{6n-1}\left(4-2\cos\frac{2\pi j}{6n}\right)}.
\end{aligned}$$

We therefore get that

$$\tau(C_{12n}(1,3n,4n)) = \frac{1}{12n}\prod_{j=1}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{4n-1}\frac{4-2\cos\frac{2\pi j}{4n}}{7-2\cos\frac{2\pi j}{4n}} \times\frac{\prod_{j=1}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}-4\cos^2\frac{2\pi j}{3}\right)\prod_{j=1}^{3n-1}\frac{6-2\cos\frac{2\pi j}{3n}-4\cos^2\frac{2\pi j}{3}}{10-2\cos\frac{2\pi j}{3n}-4\cos^2\frac{2\pi j}{3}}}{\prod_{j=1}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{4-2\cos\frac{2\pi j}{2n}}{7-2\cos\frac{2\pi j}{2n}}},$$

i.e.,

$$\tau(C_{12n}(1,3n,4n)) = \frac{1}{12n}\prod_{j=1}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{4n-1}\frac{4-2\cos\frac{2\pi j}{4n}}{7-2\cos\frac{2\pi j}{4n}} \times\frac{\prod_{j=1,3\nmid j}^{6n-1}\left(9-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1,3\mid j}^{6n-1}\left(6-2\cos\frac{2\pi j}{6n}\right)}{\prod_{j=1}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{4-2\cos\frac{2\pi j}{2n}}{7-2\cos\frac{2\pi j}{2n}}} \times\frac{\prod_{j=1,3\nmid j}^{6n-1}\left(5-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1,3\mid j}^{6n-1}\left(7-2\cos\frac{2\pi j}{3n}\right)}{\prod_{j=1,3\nmid j}^{6n-1}\left(9-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1,3\mid j}^{6n-1}\left(6-2\cos\frac{2\pi j}{3n}\right)},$$

which implies that

$$\tau(C_{12n}(1,3n,4n)) = \frac{1}{12n}\prod_{j=1}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1}^{4n-1}\frac{4-2\cos\frac{2\pi j}{4n}}{7-2\cos\frac{2\pi j}{4n}} \times\frac{\prod_{j=1}^{6n-1}\left(9-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{6n-1}\frac{6-2\cos\frac{2\pi j}{6n}}{9-2\cos\frac{2\pi j}{6n}}}{\prod_{j=1}^{6n-1}\left(7-2\cos\frac{2\pi j}{6n}\right)\prod_{j=1}^{2n-1}\frac{4-2\cos\frac{2\pi j}{2n}}{7-2\cos\frac{2\pi j}{2n}}} \times\frac{\prod_{j=1}^{3n-1}\left(5-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1}^{3n-1}\frac{7-2\cos\frac{2\pi j}{3n}}{5-2\cos\frac{2\pi j}{3n}}}{\prod_{j=1}^{3n-1}\left(9-2\cos\frac{2\pi j}{3n}\right)\prod_{j=1}^{n-1}\frac{6-2\cos\frac{2\pi j}{n}}{9-2\cos\frac{2\pi j}{n}}}.$$

Thus,

$$\tau(C_{12n}(1,3n,4n)) = \frac{1}{12n}\times\frac{\frac{U_{4n-1}^2\left(\sqrt{\frac{3}{2}}\right)\times U_{12n-1}^2\left(\frac{3}{2}\right)}{U_{4n-1}^2\left(\sqrt{\frac{3}{2}}\right)}}{\frac{U_{2n-1}^2\left(\sqrt{\frac{3}{2}}\right)\times U_{6n-1}^2\left(\frac{3}{2}\right)}{U_{2n-1}^2\left(\sqrt{\frac{3}{2}}\right)}}$$

$$\times \frac{\frac{U_{6n-1}^{2}\left(\sqrt{\frac{11}{4}}\right)\times U_{2n-1}^{2}\left(\sqrt{2}\right)}{U_{2n-1}^{2}\left(\sqrt{\frac{11}{4}}\right)}\times\frac{n^{2}\times U_{3n-1}^{2}\left(\sqrt{\frac{7}{4}}\right)}{U_{3n-1}^{2}\left(\sqrt{\frac{11}{4}}\right)}}{\frac{U_{n-1}^{2}\left(\sqrt{2}\right)\times U_{3n-1}^{2}\left(\frac{11}{4}\right)}{U_{n-1}^{2}\left(\sqrt{\frac{11}{4}}\right)}}.$$

We have

$$\begin{aligned}\tau(C_{12n}(1,3n,4n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}}\right)^{4n}+\left(\sqrt{\frac{9}{4}}-\sqrt{\frac{5}{4}}\right)^{4n}-1\right]^{2}\\ &\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^{2}\\ &\times\left[\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)^{2n}+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)^{2n}\right]^{2}\\ &\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^{2}\\ &\times\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}-1\right]^{2},\end{aligned}$$

where (6), (8) , (9) and (10) are used to derive the last two steps. □

Theorem 3.6 *For the spanning trees of C_{12n} with three jumps $1, 2n, 3n, 6n$, we have*

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,6n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{4n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{4n}-1\right]^{2}\\ &\times\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}+1\right]^{2}\\ &\times\left[\left(\sqrt{\frac{13}{4}}+\sqrt{\frac{9}{4}}\right)^{4n}+\left(\sqrt{\frac{13}{4}}-\sqrt{\frac{9}{4}}\right)^{4n}+1\right]^{2}\\ &\times\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{2n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{2n}\right]^{2}\\ &\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^{2}\\ &\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^{2}.\end{aligned}$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Applying Lemma 3.1, we get the required result. □

Theorem 3.7 *For the spanning trees of C_{12n} with four jumps $1, 2n, 3n, 4n$, we have*

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{6n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{6n}\right]^2\\ &\times\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{2n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{2n}+1\right]^2\\ &\times\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{2n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{2n}-1\right]^2\\ &\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2.\end{aligned}$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Applying Lemma 3.1, we have

$$\begin{aligned}&\tau(C_{12n}(1,2n,3n,4n))\\ &=\frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-\varepsilon^{-j}-\varepsilon^{-2nj}-\varepsilon^{-3nj}-\varepsilon^{-4nj}-\varepsilon^{j}-\varepsilon^{2nj}-\varepsilon^{3nj}-\varepsilon^{4nj}\right)\\ &=\frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi jn}{12n}-2\cos\frac{6\pi jn}{12n}-2\cos\frac{8\pi jn}{12n}\right)\\ &=\frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}\right)\\ &=\frac{1}{12n}\prod_{j=1,2\nmid j}^{12n-1}\left(8-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}\right)\\ &\times\prod_{j=1,2\mid j}^{12n-1}\left(8-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}\right)\\ &=\frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}\right)\\ &\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{\pi j}{3}-2\cos\frac{\pi j}{2}-2\cos\frac{2\pi j}{3}}{8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-2\cos\frac{4\pi j}{3}}.\end{aligned}$$

Thus,

$$\tau(C_{12n}(1,2n,3n,4n)) = \frac{1}{12n}\prod_{j=1,3\nmid j}^{12n-1}\left(9-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)$$

$$\times \prod_{j=1,3|j}^{12n-1}\left(6-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)$$

$$\times \frac{\prod_{j=1}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}-2\cos\pi j\right)}{\prod_{j=1}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}\right)},$$

i.e.,

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n)) &= \frac{1}{12n}\prod_{j=}^{12n-1}\left(9-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)\\ &\times \prod_{j=1}^{4n-1}\frac{6-2\cos\frac{2\pi j}{4n}-2\cos\frac{\pi j}{3}}{9-2\cos\frac{2\pi j}{4n}-2\cos\frac{\pi j}{3}}\\ &\times \frac{\prod_{j=1}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}-2\cos\pi j\right)}{\prod_{j=1}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}\right)}.\end{aligned}$$

We get that

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(9-2\cos\frac{2\pi j}{12n}-2\cos\frac{\pi j}{3}\right)\\ &\times \prod_{j=1}^{4n-1}\frac{6-2\cos\frac{2\pi j}{4n}-2\cos\frac{\pi j}{3}}{9-2\cos\frac{2\pi j}{4n}-2\cos\frac{\pi j}{3}}\\ &\times \frac{\prod_{j=1,2\nmid j}^{6n-1}\left(12-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}\right)}{\prod_{j=1,3\nmid j}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}\right)}\\ &\times \frac{\prod_{j=1,2|j}^{6n-1}\left(8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}\right)}{\prod_{j=1,3|j}^{6n-1}\left(4-2\cos\frac{2\pi j}{6n}\right)},\end{aligned}$$

which implies that

$$\tau(C_{12n}(1,2n,3n,4n)) = \frac{1}{12n}\prod_{j=1,2\nmid j,3\nmid j}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1,2|j,3|j}^{12n-1}\left(7-2\cos\frac{2\pi j}{12n}\right)$$

$$\times \prod_{j=1,2\nmid j,3|j}^{12n-1}\left(11-2\cos\frac{2\pi j}{12n}\right)\prod_{j=1,2\nmid j,3|j}^{12n-1}\left(10-2\cos\frac{2\pi j}{12n}\right)$$

$$\times \frac{\prod_{j=1,2\nmid j}^{4n-1}\left(8-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1,2|j}^{4n-1}\left(4-2\cos\frac{2\pi j}{4n}\right)}{\prod_{j=1,2\nmid j}^{4n-1}\left(11-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1,2|j}^{4n-1}\left(7-2\cos\frac{2\pi j}{4n}\right)}$$

$$\times \frac{\prod_{j=1}^{6n-1}\left(12-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}\right)}{\prod_{j=1}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}\right)}$$

$$\times \frac{\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}{12-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}}{\prod_{j=1}^{6n-1}\frac{4-2\cos\frac{2\pi j}{6n}}{10-2\cos\frac{2\pi j}{6n}}}.$$

Thus,

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)\\ &\times \prod_{j=1}^{12n-1}\frac{\left(8-2\cos\frac{2\pi j}{12n}\right)\left(7-2\cos\frac{2\pi j}{12n}\right)}{\left(11-2\cos\frac{2\pi j}{12n}\right)\left(10-2\cos\frac{2\pi j}{12n}\right)}\\ &\times \prod_{j=1}^{12n-1}\frac{10-2\cos\frac{2\pi j}{12n}]}{8-2\cos\frac{2\pi j}{12n}}\times\prod_{j=1}^{12n-1}\frac{11-2\cos\frac{2\pi j}{12n}}{8-2\cos\frac{2\pi j}{12n}}\\ &\times \frac{\prod_{j=1}^{4n-1}\left(8-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1}^{4n-1}\frac{4-2\cos\frac{2\pi j}{4n}}{8-2\cos\frac{2\pi j}{4n}}}{\prod_{j=1}^{4n-1}\left(11-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{11-2\cos\frac{2\pi j}{4n}}}\\ &\times \frac{\prod_{j=1}^{6n-1}\left(12-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}\right)}{\prod_{j=1}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}\right)}\\ &\times \frac{\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}{12-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}}{\prod_{j=1}^{6n-1}\frac{4-2\cos\frac{2\pi j}{6n}}{10-2\cos\frac{2\pi j}{6n}}}.\end{aligned}$$

We get that

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n)) &= \frac{1}{12n}\prod_{j=1}^{12n-1}\left(8-2\cos\frac{2\pi j}{12n}\right)\\ &\times\prod_{j=1}^{12n-1}\frac{\left(8-2\cos\frac{2\pi j}{12n}\right)\left(7-2\cos\frac{2\pi j}{12n}\right)}{\left(11-2\cos\frac{2\pi j}{12n}\right)\left(10-2\cos\frac{2\pi j}{12n}\right)}\\ &\times\prod_{j=1}^{12n-1}\frac{10-2\cos\frac{2\pi j}{12n}}{8-2\cos\frac{2\pi j}{12n}}\prod_{j=1}^{12n-1}\frac{11-2\cos\frac{2\pi j}{12n}}{8-2\cos\frac{2\pi j}{12n}}\\ &\times\frac{\prod_{j=1}^{4n-1}\left(8-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1}^{4n-1}\frac{4-2\cos\frac{2\pi j}{4n}}{8-2\cos\frac{2\pi j}{4n}}}{\prod_{j=1}^{4n-1}\left(11-2\cos\frac{2\pi j}{4n}\right)\prod_{j=1}^{4n-1}\frac{7-2\cos\frac{2\pi j}{4n}}{11-2\cos\frac{2\pi j}{4n}}}\\ &\times\frac{\prod_{j=1}^{6n-1}\left(12-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}\right)}{\prod_{j=1}^{6n-1}\left(10-2\cos\frac{2\pi j}{6n}\right)}\\ &\times\frac{\prod_{j=1}^{2n-1}\frac{8-2\cos\frac{2\pi j}{2n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}{12-2\cos\frac{2\pi j}{2n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}}{\prod_{j=1}^{3n-1}\frac{4-2\cos\frac{2\pi j}{3n}}{10-2\cos\frac{2\pi j}{3n}}},\end{aligned}$$

i.e.,

$$\begin{aligned}&\tau(C_{12n}(1,2n,3n,4n))=\frac{1}{12n}\\ &\times\frac{U^2_{12n-1}\left(\sqrt{\frac{11}{4}}\right)\times U^2_{2n-1}\left(\sqrt{\frac{11}{4}}\right)\times U^2_{2n-1}\left(\sqrt{\frac{7}{2}}\right)\times U^2_{4n-1}\left(\sqrt{\frac{7}{2}}\right)\times U^2_{6n-1}\left(\sqrt{\frac{13}{4}}\right)}{U^2_{2n-1}\left(\sqrt{\frac{7}{2}}\right)\times U^2_{2n-1}\left(\sqrt{\frac{13}{4}}\right)\times U^2_{4n-1}\left(\sqrt{\frac{11}{4}}\right)\times U^2_{6n-1}\left(\sqrt{\frac{11}{4}}\right)}\\ &\times\frac{U^2_{6n-1}\left(\sqrt{\frac{11}{4}}\right)\times U^2_{2n-1}\left(\sqrt{2}\right)\times U^2_{3n-1}\left(\sqrt{\frac{7}{4}}\right)\times U^2_{n-1}\left(\sqrt{\frac{11}{4}}\right)\times U^2_{2n-1}\left(\sqrt{\frac{11}{4}}\right)}{U^2_{2n-1}\left(\sqrt{\frac{11}{4}}\right)\times U^2_{n-1}\left(\sqrt{\frac{7}{4}}\right)\times U^2_{3n-1}\left(\sqrt{\frac{11}{4}}\right)\times U^2_{n-1}\left(\sqrt{2}\right)\times U^2_{6n-1}\left(\sqrt{\frac{11}{4}}\right)}\end{aligned}$$

So, we have

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{6n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{6n}\right]^2\\ &\times\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{2n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{2n}+1\right]^2\end{aligned}$$

$$\times\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{2n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{2n}-1\right]^2$$
$$\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2,$$

where (6), (8), (9) and (10) are used to derive the last two steps. □

Theorem 3.8 *For the spanning trees of C_{6n} with four jumps $1, 3n, 4n, 6n$, we have*

$$\begin{aligned}\tau(C_{12n}(1,3n,4n,6n)) &= \frac{n}{12}\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{8n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{8n}+1\right]^2\\ &\times\left[\left(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}}\right)^{2n}+\left(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}}\right)^{2n}-1\right]^2\\ &\times\left[\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{5}{2}}\right)^{2n}+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{5}{2}}\right)^{2n}\right]^2\\ &\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2\\ &\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2\\ &\times\left[\left(\sqrt{\frac{13}{4}}+\sqrt{\frac{9}{4}}\right)^{2n}+\left(\sqrt{\frac{13}{4}}-\sqrt{\frac{9}{4}}\right)^{2n}+1\right]^2\end{aligned}$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Apply Lemma 3.1, we get the required result. □

Theorem 3.9 *For the spanning trees of C_{6n} with four jumps $1, 2n, 3n, 4n, 6n$, we have*

$$\begin{aligned}\tau(C_{12n}(1,2n,3n,4n,6n)) &= \frac{n}{12}\times\left[\left(\sqrt{2}+1\right)^{n}+\left(\sqrt{2}-1\right)^{n}\right]^2\\ &\times\left[\left(\sqrt{2}+\sqrt{3}\right)^{n}+\left(\sqrt{2}-\sqrt{3}\right)^{n}-1\right]^2\\ &\times\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{2n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{2n}+1\right]^2\\ &\times\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{2n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{2n}-1\right]^2\\ &\times\left[\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}}\right)^{2n}+\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)^{2n}\right]^2\end{aligned}$$

$$\times \left[\left(\sqrt{\frac{7}{2}} + \sqrt{\frac{5}{2}} \right)^{4n} + \left(\sqrt{\frac{7}{2}} - \sqrt{\frac{5}{2}} \right)^{4n} + 1 \right]^2$$

Proof Let $\varepsilon = e^{\frac{2\pi i}{12n}}$. Applying Lemma 3.1, We get the required result. □

§4. Conclusions

The number of spanning trees in graphs (networks) is an important invariant. The evaluation of this number is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory we prove our results in Section 3.

References

[1] G. G. Kirchhoff, Uber die Auflosung der Gleichungen, auf welche man be ider Untersuchung der Linearen Verteilung galvanischer Storme gefuhrt wird, *Ann. Phys. Chem.*, 72 (1847), 497-508.

[2] A. K. Kelmans and V. M. Chelnokov, A certain polynomials of a graph and graphs with an extermal number of trees, *J. Comb. Theory* (B), 16 (1974), 197-214.

[3] S. N. Daoud, The deletion- contraction method for counting the number of spanning trees of graphs, *European Journal of Physical Plus*, Vol.130, 10(2015), 1-14.

[4] S. N. Daoud, Number of spanning trees in different product of complete and complete tripartite graphs, *Ars Combinatoria*, Vol.139 (2018), 85-103.

[5] S. N. Daoud, Complexity of graphs generated by wheel graph and their asymptotic limits, *Journal of the Egyptian Math. Soc.*, Vol.25, 4(2017), 424-433.

[6] Jia-Bao Liu and S. N. Daoud, Complexity of some of pyramid graphs created from a gear graph, *Symmetry*, 2018, 10, 689; doi:10.3390/sym10120689.

[7] S. N. Daoud, Complexity of some families of cycle related graphs, *Journal of Taibah University for Science*, 11 (2017), 205-228.

[8] S. N. Daoud, Complexity of some special named graphs and Chebyshev polynomials, *International Journal Applied Mathematical and Statistics*, Ceser Pub. Vol.32, 2(2013), 77-84.

[9] S. N. Daoud, Generating formulas of the number of spanning trees of some special graphs, *Eur. Phys. J. Plus*, Vol. 129 (2014), 1-14.

[10] S. N. Daoud, Number of spanning trees of different products of complete and complete bipartite graphs, *Journal of Mathematical Problems in Engineering*, Hindawi Publishing Corporation Volume 2014, Article ID 965105, 23 pages.

[11] S. N. Daoud, On a class of some pyramid graphs and Chebyshev polynomials, *Journal of Mathematical Problems in Engineering*, Hindawi Publishing Corporation Volume 2013,

Article ID 820549, 11 pages.

[12] Jia-Bao Liu and S. N. Daoud, Number of spanning trees in the sequence of some graphs, *Complexity*, Hindawi Publ. Corp. Volume 2019, Article ID 4271783, 22. pages 2019.

[13] A. K. Kelmans, Connectivity of probabilistic networks, *Auto Remote Contr.*, 3(1967), 444-460.

[14] R. S. Wilkov, Analysis and design of reliable computer networks, *IEEE Trans. Commun. Syst.*, CS11 (1963), 230-247.

[15] J. Louis, Asymptotics for the number of spanning trees in circulant graphs and degenerating d-dimensional discrete tori, *Annals of Combinatorics*, 19(3) (2015), 513-543.

[16] Chen Xiebin, The numbers of spanning trees in undirected circulant graphs, *J. Zhangzhou Teachers College*, 13(4) (2000), 1-6.

[17] Chen Xiebin, Qiuying Lin, Fuji Zhang, The number of spanning trees in odd valent circulant graphs, *Discrete Math.*, 282(1) (2004) 69-79.

[18] X. Yong, Talip and Acenjian, The number of spanning trees of the cubic cycle C3n and the quadruple cycle C4n, *Discrete Math.*, 169 (1997), 293-298.

[19] Zhang Yuanping, Yong Xuerong, M.J.Golin, The number of spanning trees in circulant graphs, *Discrete. Math.*, 223(1) (2000) 337-350.

[20] D. Nikolopoulos and C. Papadopoulos, The number of spanning trees in complements of quasi-threshold graphs, *J. Graphs and Combin.*, 20 (2004), 383-397.

[21] D. Nikolopoulos and C. Papadopoulos, On the number of spanning trees of multi-star related graph, *Inform. Process. Lett.*, 65 (1998), 183-188.

[22] D. J. Kleiman and B. Golden, Counting trees in a certain class of graphs, *Amer Math. Mon.*, 82 (1975), 40-44.

[23] Z.Yuanping, Y. Xuerong and J. Mordecai, Chebyshev polynomials and spanning trees formulas for circulant and related graphs, *Discrete Math.*, 298 (2005), 334-364.

[24] Z.Yuanping, Y.Xuerong and J. Mordecai, The number of spanning trees in circulant graphs, *Discrete Math.*, 223 (2000), 337-350.

[25] N. Biggs, *Algebraic Graph Theory* (Second ed.), Cambridge University Press, London, (1993).