International J.Math. Combin. Vol.4(2024), 9-19

On the Klein Cubic Threefold in $PG(4,2)$

A. Altıntaş, A. Bayar, S. Ekmekçi and Z. Akça

(Eskişehir Osmangazi University, Faculty of Science, Dept. of Math. and Comp. Sci., 26480 Eskişehir, Türkiye)

E-mail: zakca@ogu.edu.tr

Abstract: This paper investigates the structural properties of the Klein cubic threefold $\mathcal{F}$ in $4-$dimensional projective space over the finite field $GF(2)$. We focus on the intersection properties of the lines and the planes with $\mathcal{F}$ in $PG(4,2)$. Notably, it is identified six spreads, each containing five lines in $\mathcal{F}$. Additionally, two distinct affine plane models are presented by using the tangent planes of $\mathcal{F}$. Furthermore, it is shown that Desargues' theorem does not hold in $\mathcal{F}$.

Key Words: Klein cubic threefold, projective spaces, spread, Galois field.

AMS(2010): 51E20, 51E30.

§1. Introduction

Cubic surfaces have been extensively studied in algebraic geometry and have applications in fields such as those of the computer graphics, physics and engineering. One notable early example is the non-singular Klein cubic threefold studied by Klein in 1879, [11]. The classification of non-singular cubic surfaces, particularly over finite fields, remains a significant area of research. For instance, it has been shown that a non-singular cubic surface over the field $GF(2)$ can have 15, 9, 5, 3, 2, 1, or 0 lines in [5,6,10]. In [13], Rosati further demonstrated that when q is odd, the number of lines must be one of 27, 15, 9, 7, 5, 3, 2, 1, or 0. In the 1960s, Hirschfeld initiated a program to classify cubic surfaces with 27 lines over finite fields, [8]. This work is a substantial contribution to this problem. Some examples of the nonsingular cubic surfaces were given in [9].

In order to classify projective spaces, tools from Veronesean embedding and quadric theory were used [1]-[4], [7], [12].

In this paper, we delve into the structural properties of the Klein cubic threefold $\mathcal{F}$ situated in $4-$dimensional projective space over the finite field $GF(2)$. Our investigation primarily focuses on the intricate intersection properties of lines and planes with $\mathcal{F}$. Notably, every point on the Klein cubic threefold is identified as an Eckardt point, marking a fundamental characteristic of $\mathcal{F}$.

Utilizing Schläfli labeling, we systematically notate the 15 lines comprising $\mathcal{F}$. Through our analysis, we observe that each line intersects six others while remaining skew to eight additional

[1]Received August 25,2024, Accepted October 22,2024.

lines. Moreover, $\mathcal{F}$ has six spreads, each consisting of five lines.

An important finding in our study is the determination of the point $(1,1,1,1,1)$, which is the nucleus of $\mathcal{F}$. We meticulously examine the tangent planes to $\mathcal{F}$ and present two distinct affine plane models based on their specific properties.

Furthermore, we rigorously demonstrate that Desargues' theorem, a cornerstone of projective geometry, does not hold in $\mathcal{F}$. This observation underscores the unique geometric and algebraic characteristics that distinguish $\mathcal{F}$.

Throughout this paper, we aim to provide a comprehensive exploration of these structural properties, offering insights into the rich interplay between algebraic geometry and combinatorial structure in the context of cubic threefold $PG(4,2)$.

§2. Preliminaries

Let $GF(q)$ denote Galois field of order $q = p^k$ where p is a prime. If any $(n+1)-$dimensional vector space V, the $n-$dimensional projective space $PG(n,q)$ over $GF(q)$ is the set of all subspaces of V distinct from the trivial subspaces. $1-$dimensional subspaces are called the points of $PG(n,K)$, $2-$dimensional subspaces are called the (projective) lines and $3-$dimensional ones are called (projective) planes. We remark that by going from a vector space to the associated projective space, the dimension drops by one unit. Hence an $(n+1)-$dimensional vector space V gives rise to an $n-$dimensional projective space $PG(n,K)$ [3]-[7].

This is, the points in projective space $PG(n,q)$ are defined by equivalence classes of nonzero vectors in the vector space V.

For example, for $4-$dimensional vector space, the associated $3-$dimensional projective space would be where points are represented by equivalence classes of vectors (w,x,y,z), reducing the dimension by one unit.

The $4-$dimensional projective space $PG(4,q)$ over $GF(q)$ contains $q^4+q^3+q^2+q+1$ points and $PG(4,2)$ has 31 points. The points of $PG(4,2)$ are respectively listed as follows:

$P_1(0,0,0,0,1)$, $P_2(0,0,0,1,0)$, $P_3(0,0,1,0,0)$, $P_4(0,0,1,0,1)$, $P_5(0,0,1,1,1)$,
$P_6(0,1,0,0,0)$, $P_7(0,1,0,0,1)$, $P_8(0,1,0,1,0)$, $P_9(0,1,1,1,0)$, $P_{10}(1,0,0,0,0)$,
$P_{11}(1,0,0,1,0)$, $P_{12}(1,0,0,1,1)$, $P_{13}(1,0,1,0,0)$, $P_{14}(1,1,0,0,1)$, $P_{15}(1,1,1,0,0)$,
$P_{16}(0,0,0,1,1)$, $P_{17}(0,0,1,1,0)$, $P_{18}(0,1,1,0,0)$, $P_{19}(1,0,0,0,1)$, $P_{20}(1,1,0,0,0)$,
$P_{21}(0,1,0,1,1)$, $P_{22}(0,1,1,0,1)$, $P_{23}(1,0,1,0,1)$, $P_{24}(1,0,1,1,0)$, $P_{25}(1,1,0,1,0)$,
$P_{26}(0,1,1,1,1)$, $P_{27}(1,0,1,1,1)$, $P_{28}(1,1,0,1,1)$, $P_{29}(1,1,1,0,1)$, $P_{30}(1,1,1,1,0)$,
$P_{31}(1,1,1,1,1)$.

The Klein cubic threefold $\mathcal{F}$ is given by the equation

$$\mathcal{F} : x^2y + y^2z + z^2v + v^2w + w^2x = 0,$$

where x, y, z, v and w represent the coordinates of a point (v,w,x,y,z) in the projective space

$PG(4,q)$ over $GF(q)$. In algebraic geometry, a cubic threefold is a hypersurface of degree 3 in 4$-$dimensional projective space. This Klein cubic threefold $\mathcal{F}$ over the field $F(q)$ is the zero set of a homogeneous cubic equation in five variables over $GF(q)$.

2.1. Klein Cubic Threefold $\mathcal{F}$ with 15 Lines

We use the combinatorial definition where a line is considered as a subsets of points on $\mathcal{F}$.

Proposition 2.1 *Every Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$ contains 15 points and 15 lines. Every line has three points on it.*

Proof In $PG(4,2)$, a point is denoted by $P(a_0,a_1,a_{2,}a_3,a_4)$. A line through the points $P(a_0,a_1,a_2,a_3,a_4)$ and $P(b_0,b_1,b_{2,}b_3,b_4)$ is denoted by

$$l = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \end{bmatrix}.$$

Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$. $\mathcal{F}$ can be identified with a set of the points P_i satisfying the equation $x^2y + y^2z + z^2v + v^2w + w^2x = 0$ in $PG(4,2)$ such that $P_1(0,0,0,0,1)$, $P_2(0,0,0,1,0)$, $P_3(0,0,1,0,0)$, $P_4(0,0,1,0,1)$, $P_5(0,0,1,1,1)$, $P_6(0,1,0,0,0)$, $P_7(0,1,0,0,1)$, $P_8(0,1,0,1,0)$, $P_9(0,1,1,1,0)$, $P_{10}(1,0,0,0,0)$, $P_{11}(1,0,0,1,0)$, $P_{12}(1,0,0,1,1)$, $P_{13}(1,0,1,0,0)$, $P_{14}(1,1,0,0,1)$ and $P_{15}(1,1,1,0,0)$. The incidence relation on $\mathcal{F}$ over the field $GF(2)$ is given as the following table: We define the incidence matrix $A = [a_{ij}]$ of $\mathcal{F}$ such that $a_{ij} = 1$, where $i, j \in \{1, 2, \cdots, 15\}$ if and only if the line l_i is incident to the point P_j and 0 otherwise; here the rows represent the lines and the columns the points of $\mathcal{F}$. This matrix represents the incidence relation between the points P_j and the lines l_i of the Klein cubic threefold over the field $GF(2)$ in Table 1.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
l_1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0
l_2	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0
l_3	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0
l_4	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0
l_5	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0
l_6	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0
l_7	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0
l_8	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0
l_9	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1
l_{10}	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0
l_{11}	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0
l_{12}	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1
l_{13}	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0
l_{14}	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0
l_{15}	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1

Table 1. Incidence relation between the points and the lines of $\mathcal{F}$

This completes the proof. □

Proposition 2.2 *Every point of Klein cubic threefold over the field $GF(2)$ is an Eckardt point, and the three Eckardt points lie on a line on the cubic threefold $\mathcal{F}$.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$ of characteristic 2. Every point of Klein cubic threefold lies on three lines in $\mathcal{F}$. So, the number of Eckardt points of Klein cubic threefold over the field $GF(2)$ is 15. Eckardt points with their coordinates as the point of concurrency of three labeled lines can be seen in Table 1. Also, the three Eckardt points lie on a line on the cubic threefold $\mathcal{F}$. For example, P_1, P_3 and P_4 lie on the line l_1 of $\mathcal{F}$. □

Proposition 2.3 *Each line in $\mathcal{F}$ intersects exactly six others and is skew to the remaining eight lines in $\mathcal{F}$.*

Proof Consider the subset of 15 lines of the Klein cubic threefold over the field $GF(2)$ labeled according to the Schläfli notation. The intersection properties of these lines indicate that each line intersects exactly six others and is skew to the remaining eight. The intersection table for these 15 lines is provided in Table 2, where intersections are marked and non-intersecting lines are represented by 0.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
l_1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0
l_2	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0
l_3	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0
l_4	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0
l_5	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0
l_6	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0
l_7	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0
l_8	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0
l_9	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1
l_{10}	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0
l_{11}	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0
l_{12}	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1
l_{13}	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0
l_{14}	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0
l_{15}	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1

Table 2. Pairwise intersection table of the 15 lines of $\mathcal{F}$

This completes the proof. □

§3. Classifications the Lines of the Klein Cubic Threefold $\mathcal{F}$ Modulo 2

3.1. Line Spreads of $\mathcal{F}$

A (crisp) $k-$spread, or simply spread of the projective geometry $PG(n, K)$ is a partition of the point set of $PG(n, K)$ into $k-$spaces, for some k, $1 \leq k \leq n-1$. We now give line spreads of Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$.

Proposition 3.1 *Every Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$ has* 6 *spreads with five* *lines. Moreover, every line of the threefold $\mathcal{F}$ belongs to exactly two spreads.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$. Let S_i be the set of lines spreads of Klein cubic threefold $\mathcal{F}$. $1-$spread S_i of $\mathcal{F}$ must be five lines because of a partition of the point set of $\mathcal{F}$. It is easily obtained that there are six line spreads of $\mathcal{F}$ from Table 1.

$l_1 = \{P_1, P_3, P_4\}$	$l_6 = \{P_2, P_{10}, P_{11}\}$	$l_{11} = \{P_5, P_{12}, P_{13}\}$
$l_2 = \{P_1, P_6, P_7\}$	$l_7 = \{P_3, P_8, P_9\}$	$l_{12} = \{P_6, P_{13}, P_{15}\}$
$l_3 = \{P_1, P_{11}, P_{12}\}$	$l_8 = \{P_3, P_{10}, P_{13}\}$	$l_{13} = \{P_7, P_{10}, P_{14}\}$
$l_4 = \{P_2, P_4, P_5\}$	$l_9 = \{P_4, P_{14}, P_{15}\}$	$l_{14} = \{P_8, P_{12}, P_{14}\}$
$l_5 = \{P_2, P_6, P_8\}$	$l_{10} = \{P_5, P_7, P_9\}$	$l_{15} = \{P_9, P_{11}, P_{15}\}$.

Table 3. Lines spreads of Klein cubic threefold $\mathcal{F}$

and lines spreads of Klein cubic threefold $\mathcal{F}$ are S_i, $i = 1, 2, \cdots, 6$, where

$$S_1 = \{l_1, l_5, l_{11}, l_{13}, l_{15}\}, \quad S_2 = \{l_1, l_6, l_{10}, l_{12}, l_{14}\}, \quad S_3 = \{l_2, l_4, l_8, l_{14}, l_{15}\},$$
$$S_4 = \{l_2, l_6, l_7, l_9, l_{11}\}, \quad S_5 = \{l_3, l_4, l_7, l_{12}, l_{13}\}, \quad S_6 = \{l_3, l_5, l_8, l_9, l_{10}\}.$$

From Table 3, two different spreads of $\mathcal{F}$ have a common line. □

3.2. Skew-Tangent-Secant Lines to $\mathcal{F}$

The study of lines in relation to the Klein cubic threefold $\mathcal{F}$ reveals geometric properties and relationships. The classification of lines as skew, tangent, or secant provides insight into the structure and interaction of $\mathcal{F}$ within the projective space $PG(4,2)$. The following theorem presents detailed characteristics of these lines in relation to $\mathcal{F}$.

Theorem 3.2 *Let $\mathcal{F}$ be a Klein cubic threefold with exactly 15 lines. Then,*

(1) *There are four tangent lines and eight secant lines passing through any point in $\mathcal{F}$;*

(2) *$PG(4,2)$ has exactly 20 lines not intersecting with $\mathcal{F}$, 60 tangent lines to $\mathcal{F}$, and 60 secant lines to $\mathcal{F}$;*

(3) *Four lines of the lines passing through any point in $PG(4,2)\backslash\mathcal{F}$ intersect with $\mathcal{F}$ at two points; seven lines of them intersect with $\mathcal{F}$ at one point; and four lines of them intersect with $\mathcal{F}$ at no point;*

(4) *Eight lines of the lines passing through any point on $\mathcal{F}$ intersect with $\mathcal{F}$ at two points; four lines of them intersect with $\mathcal{F}$ at one point; and three lines of them intersect with $\mathcal{F}$ at three points.*

Proof (1) Let $\mathcal{F}$ be the Klein cubic threefold with exactly 15 lines. Since the incidence relation between the points and the lines of $\mathcal{F}$, there are three points on any line and three lines passing through any point P_i, $i = 1, 2, \cdots, 15$ in $\mathcal{F}$. Also, three of the 15 lines in $PG(4,2)$ passing through any point of $\mathcal{F}$ belong to $\mathcal{F}$. There are seven points of $\mathcal{F}$ on these three lines, and there are eight points of $\mathcal{F}$ apart from these lines. Therefore, 8 lines passing through any point P_i, $i = 1, 2, \cdots, 15$ of $\mathcal{F}$ are formed secant lines. Thus, the remaining 4 lines passing through any point P_i of $\mathcal{F}$ are tangent lines.

(2) Since the number of tangent lines passing through each of the 15 points on $\mathcal{F}$ is 4, the total number of tangent lines to $\mathcal{F}$ in $PG(4,2)$ is 60. In addition, since the number of secant lines passing through a point on F is 8, the total number of secant lines is calculated $\frac{15.8}{2} = 60$.

(3) Let the points of $PG(4,2)$ be denoted by their indices, and the lines of $PG(4,2)$ denoted by the points on them. There exist 15 lines passing through any point in $PG(4,2)$. It is easily seen that four lines of them do not intersect with $\mathcal{F}$, four lines of them intersect with the $\mathcal{F}$ at two points, and seven lines of them intersect with $\mathcal{F}$ at one point. The lines in $PG(4,2)$ through the points not on $\mathcal{F}$ are listed according to the intersection points with $\mathcal{F}$ in the following tables.

16-20-28	17-19-27	18-19-29	19-26-30	21-24-29
16-18-26	17-20-30	18-24-25	20-22-23	22-24-28
16-23-24	17-21-22	18-27-28	20-26-27	22-25-27
16-29-30	17-28-29	19-21-25	21-23-30	23-25-26

Table 4. Lines not intersecting with $\mathcal{F}$

16- 4 -17	17- 12 -23	19- 5 -24	21- 3 -26	23- 8 -31	26- 10 -31
16- 6 -21	17- 15 -25	19- 7 -20	21- 13 -31	23- 9 -28	26- 11 -29
16- 9 -22	17- 14 -31	19- 8 -28	21- 15 -27	24- 1 -27	26- 13 -28
16- 11 -19	18- 1 -22	19- 9 -31	21- 10 -28	24- 6 -30	27- 6 -31
16- 13 -27	18- 5 -21	19- 15 -22	22- 2 -26	24- 7 -31	27- 7 -30
16- 14 -25	18- 11 -30	20- 2 -25	22- 11 -31	24- 14 -26	27- 8 -29
16- 15 -31	18- 12 -31	20- 4 -29	22- 10 -29	25- 1 -28	28- 3 -31
17- 7 -26	18- 13 -20	20- 5 -31	22- 12 -30	25- 3 -30	28- 4 -30
17- 8 -18	18- 14 -23	20- 9 -24	23- 2 -27	25- 4 -31	29- 2 -31
17- 10 -24	19- 3 -23	20- 12 -21	23- 6 -29	25- 5 -29	30- 1 -31

Table 5. Tangent lines to $\mathcal{F}$

16- 1 -2	18- 4 -7	21- 1 -8	23- 5 -11	26- 1 -9	28- 6 -12
16- 3 -5	18- 10 -15	21- 2 -7	23- 7 -15	26- 4 -8	28- 7 -11
16- 7 -8	19- 1 -10	21- 4 -9	24- 2 -13	26- 5 -6	29- 1 -15
16- 10 -12	19- 2 -12	21- 11 -14	24- 3 -11	26- 12 -15	29- 3 -14
17- 1 -5	19- 4 -13	22- 3 -7	24- 4 -12	27- 3 -12	29- 9 -12
17- 2 -3	19- 6 -14	22- 4 -6	24- 8 -15	27- 4 -11	29- 7 -13
17- 6 -9	20- 1 -14	22- 5 -8	25- 11 -6	27- 9 -14	30- 2 -15
17- 11 -13	20- 3 -15	22- 13 -14	25- 7 -12	27- 5 -10	30- 5 -14
18- 2 -9	20- 6 -10	23- 1 -13	25- 8 -10	28- 2 -14	30- 8 -13
18- 3 -6	20- 8 -11	23- 4 -10	25- 9 -13	28- 5 -15	30- 9 -10

Table 6. Secant lines to $\mathcal{F}$

(4) It is easily seen that eight lines of the lines passing through any point on $\mathcal{F}$ intersect with $\mathcal{F}$ at two points from Table 6; four lines of them intersect with $\mathcal{F}$ at one point from Table 5; and three lines of them intersect with $\mathcal{F}$ at three points from Table 3. □

Let B_n be a set of $q^{n-1}+q^{n-2}+\cdots+q+1$ points, not all on a hyperplane in the $n-$dimensional projective space $PG(n,q)$ over the Galois field $GF(q)$, $n \geq 2$. A point not in B_n is called a nucleus of B_n if every line through it meets B_n (exactly once, of course). The set of all nuclei of B_n is denoted by $N(B_n)$.

Proposition 3.3 *The point $P_{31}=(1,1,1,1,1)$ not on $\mathcal{F}$ in $PG(4,2)$ is nucleus of $\mathcal{F}$.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$ of characteristic 2. The projective space $PG(4,2)$ contains 31 points and 135 lines. There are 15 lines through every point. 15 points of these 31 points are on $\mathcal{F}$ and these points are labeled P_i, $i=1,\cdots,15$. Consider the point $P_{31}=(1,1,1,1,1)$ in $PG(4,2)$ not satisfying the equation

$$x^2y+y^2z+z^2v+v^2w+w^2x=0$$

and every line passing through the point P_{31} is a tangent line of $\mathcal{F}$ from Table 6. So, P_{31} is a nucleus of $\mathcal{F}$. □

§4. Geometric Structures Associated with the Klein Cubic Threefold $\mathcal{F}$

In this section, we investigate well-known geometric structures such as the Fano plane, affine plane, and Desargues configuration associated with the Klein cubic threefold $\mathcal{F}$.

First of all, we show that F does not include any projective plane. Then, we determine the planes that are tangent to $\mathcal{F}$. We give two different affine plane models with these tangent planes. Finally, we show that the Desarg theorem is not valid in $\mathcal{F}$.

Proposition 4.1 *There is no any projective plane in Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$. If there is a projective plane in Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$,then there are seven points and seven lines such that three points on any line and three lines passing through any point in this projective plane in $\mathcal{F}$. It is well known that seven points of the projective plane on three lines passing through any point. But the remaining 4 lines of the projective plane are secant lines from Table 7 in $PG(4,2)$. So, there is no any projective plane in Klein cubic threefold $\mathcal{F}$. □

Proposition 4.2 *Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$ in $PG(4,2)$. There is a single tangent projective plane at every point of the Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$. Let the points of $\mathcal{F}$ be shown with their indices. Table 7 shows the tangent projective planes π_i at the points P_i,

$i = 1, 2, \cdots, 15$, to $\mathcal{F}$ over the field $GF(2)$.

$$
\begin{aligned}
\pi_1 &= \{\{1,18,22\},\{1,24,27\},\{1,25,28\},\{18,27,28\},\{22,24,28\},\{22,25,27\},\{18,24,25\}\}\\
\pi_2 &= \{\{2,20,25\},\{2,22,26\},\{2,23,27\},\{22,25,27\},\{20,26,27\},\{23,25,26\},\{20,22,23\}\}\\
\pi_3 &= \{\{3,19,23\},\{3,21,26\},\{3,25,30\},\{23,25,26\},\{19,21,25\},\{19,26,30\},\{21,23,30\}\}\\
\pi_4 &= \{\{4,16,17\},\{4,20,29\},\{4,20,28\},\{16,29,30\},\{17,20,30\},\{17,28,29\},\{16,20,28\}\}\\
\pi_5 &= \{\{5,18,21\},\{5,19,24\},\{5,25,29\},\{18,24,25\},\{19,21,25\},\{21,24,29\},\{18,19,29\}\}\\
\pi_6 &= \{\{6,16,21\},\{6,23,29\},\{6,24,30\},\{16,23,24\},\{16,29,30\},\{21,24,29\},\{21,23,30\}\}\\
\pi_7 &= \{\{7,17,26\},\{7,19,20\},\{7,27,30\},\{17,20,30\},\{19,26,30\},\{20,26,27\},\{17,19,27\}\}\\
\pi_8 &= \{\{8,17,18\},\{8,19,28\},\{8,27,29\},\{17,28,29\},\{18,19,29\},\{18,27,28\},\{17,19,27\}\}\\
\pi_9 &= \{\{9,16,22\},\{9,20,24\},\{9,23,28\},\{16,23,24\},\{20,22,23\},\{22,24,28\},\{16,20,28\}\}\\
\pi_{10} &= \{\{10,17,24\},\{10,21,28\},\{10,22,29\},\{17,28,29\},\{21,24,29\},\{22,24,28\},\{17,21,22\}\}\\
\pi_{11} &= \{\{11,16,19\},\{11,18,30\},\{11,26,29\},\{16,29,30\},\{18,19,29\},\{19,26,30\},\{16,18,26\}\}\\
\pi_{12} &= \{\{12,17,23\},\{12,20,21\},\{12,22,30\},\{17,21,22\},\{20,22,23\},\{21,23,30\},\{17,20,30\}\}\\
\pi_{13} &= \{\{13,16,27\},\{13,18,20\},\{13,26,28\},\{16,20,28\},\{18,27,28\},\{16,18,26\},\{20,26,27\}\}\\
\pi_{14} &= \{\{14,16,25\},\{14,18,23\},\{14,24,26\},\{16,23,24\},\{18,24,25\},\{23,25,26\},\{16,18,26\}\}\\
\pi_{15} &= \{\{15,17,25\},\{15,21,27\},\{15,19,22\},\{17,19,27\},\{19,21,25\},\{22,25,27\},\{17,21,22\}\}
\end{aligned}
$$

Table 7. Tangent projective planes to $\mathcal{F}$

This completes the proof. □

The following results are obtained from the Table 7.

Corollary 4.3 (i) *Each tangent plane of the surface* $\mathcal{F}$ *contains three tangent lines passing through the tangent point on* $\mathcal{F}$ *and four lines not intersecting the surface* $\mathcal{F}$.

(ii) *Three tangent planes of* $\mathcal{F}$ *intersect along a line not intersecting with* $\mathcal{F}$.

(iii) *The nucleus of* $\mathcal{F}$ *is not on any tangent planes of* $\mathcal{F}$ *in* $PG(4,2)$.

(iv) *There are four tangent lines any point in* $\mathcal{F}$.

Theorem 4.4 *Three of the planes passing through the nucleus of* $\mathcal{F}$ *intersect along a line with* $\mathcal{F}$*, and four of them intersect with* $\mathcal{F}$ *at two points.*

Proof The planes D_i, $i = 1, \cdots, 7$, passing through the nucleus of $\mathcal{F}$ can be listed as

$$
\begin{aligned}
D_1 &= \{\{1,6,7\},\{1,24,27\},\{1,30,31\},\{6,24,30\},\{6,37,31\},\{7,27,30\},\{7,24,31\}\},\\
D_2 &= \{\{1,3,4\},\{1,25,28\},\{1,30,31\},\{3,25,30\},\{3,28,31\},\{4,28,30\},\{4,25,31\}\},\\
D_3 &= \{\{1,11,12\},\{1,18,22\},\{1,30,31\},\{11,18,30\},\{11,22,31\},\{12,22,30\},\{12,18,31\}\},\\
D_4 &= \{\{1,5,17\},\{1,14,20\},\{1,30,31\},\{5,14,30\},\{5,20,31\},\{14,17,31\},\{17,20,30\}\},\\
D_5 &= \{\{1,9,26\},\{1,10,19\},\{1,30,31\},\{9,10,30\},\{9,19,31\},\{10,26,31\},\{19,26,30\}\},\\
D_6 &= \{\{1,8,21\},\{1,13,23\},\{1,30,31\},\{8,13,30\},\{8,23,31\},\{13,21,31\},\{21,23,30\}\},\\
D_7 &= \{\{1,2,16\},\{1,15,29\},\{1,30,31\},\{2,15,30\},\{2,29,31\},\{15,16,31\},\{16,29,30\}\}.
\end{aligned}
$$

It is easily seen that the planes D_1, D_2, and D_3 intersect along a line with $\mathcal{F}$, and the others intersect with $\mathcal{F}$ at two points. □

Theorem 4.5 *There are six-tangent planes of* $\mathcal{F}$ *passing through any point not on* $\mathcal{F}$ *in* $PG(4,2)$*. Moreover, these planes form four different plane bundles that intersect a line three by three, so that two different bundles have a common tangent plane.*

Proof It is seen from Table 7 that any point P_i, $P_i \in \{16, \cdots, 30\}$ not on $\mathcal{F}$ in $PG(4,2)$ is contained in six-tangent planes. For example, the point $(0, 1, 1, 0, 1)$ not on $\mathcal{F}$ is on the tangent planes π_1, π_2, π_9, π_{10}, π_{12},and π_{15}. The triplets of different tangent planes passing through a common line are $\{\pi_1,\ \pi_2,\ \pi_{10}\}$, $\{\pi_1, \pi_{12}, \pi_{15}\}$, $\{\pi_2,\ \pi_9,\ \pi_{12}\}$ and $\{\pi_{10}, \pi_{12}, \pi_{15}\}$. □

Theorem 4.6 *An affine plane in $PG(4, 2)$ can be formed with six tangent planes that passes through a point outside the surface $\mathcal{F}$.*

(1) *Let each of the triplets of different tangent planes tangent to the surface from a point outside the surface in $PG(4, 2)$ passing through a common line be called a point and each of the tangent planes be called a line. The incidence relation between a point and a line means that each point is on the three tangent planes (three lines) that form it;*

(2) *Let each of the triplets of different tangent planes that are tangent to the surface from a point outside the surface in $PG(4, 2)$ not containing a common line be a point and each of the tangent planes be a line. The incidence relation means that every point is on the tangent planes that form it.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$. An affine plane is a collection of points and lines in space that follow the following fairly sensible rules:

($A1$) *Given any two points, there is a unique line joining any two points.*

($A2$) *Given a point P and a line L not containing P , there is a unique line that contains P and does not intersect L.*

($A3$) *There are four points, no three of which are collinear.*

Since every tangent plane in only two triplets of different tangent planes, A1 is satisfied. Since a tangent plane is in only two triplets and not in two triplets of different tangent planes, A2 is satisfied. Since there are only four triplets, and any three of these have not common tangent plane, A3 is satisfied. □

Example 4.7 The six tangent planes passing through the point $(0, 1, 1, 0, 1)$, which is not on the surface $\mathcal{F}$ are π_1, π_2, π_9, π_{10}, π_{12} and π_{15}. The triplets of different tangent planes passing through a common line are

$$\{\pi_1, \pi_2, \pi_{10}\},\ \{\pi_1, \pi_{12}, \pi_{15}\},\ \{\pi_2, \pi_9, \pi_{12}\} \text{ and } \{\pi_{10}, \pi_{12}, \pi_{15}\}.$$

If we define respectively the points set, line set and the incidence relation as:

the point set

$$\mathcal{P} = \{\{\pi_1, \pi_2, \pi_{10}\}, \{\pi_1, \pi_{12}, \pi_{15}\}, \{\pi_2, \pi_9, \pi_{12}\}, \{\pi_{10}, \pi_{12}, \pi_{15}\}\},$$

the line set

$$\mathcal{L} = \{\pi_1, \pi_2, \pi_9, \pi_{10}, \pi_{12}, \pi_{15}\},$$

the incidence relation

$$I: \text{ the point } \{\pi_i, \pi_j, \pi_k\}\ I \text{ the lines } \pi_i,\ \pi_j,\ \text{ and } \pi_k$$

then, the structure $(\mathcal{P}, \mathcal{L}, I)$ is an affine plane of order 2.

Example 4.8 The six tangent planes passing through the point 22, which is not on the surface $\mathcal{F}$ are π_1, π_2, π_9, π_{10}, π_{12} and π_{15}. The triplets of different tangent planes not containing a common line are

$$\{\pi_1, \pi_2, \pi_9\},\ \{\pi_1, \pi_{10}, \pi_{15}\},\ \{\pi_2, \pi_{12}, \pi_{15}\} \text{ and } \{\pi_9, \pi_{10}, \pi_{12}\}.$$

If we define respectively the points set, line set and the incidence relation as:

the point set

$$\mathcal{P}'= \{\{\pi_1, \pi_2, \pi_9\}, \{\pi_1, \pi_{10}, \pi_{15}\}, \{\pi_2, \pi_{12}, \pi_{15}\}, \{\pi_9, \pi_{10}, \pi_{12}\}\},$$

the line set

$$\mathcal{L}'= \{\pi_1, \pi_2, \pi_9, \pi_{10}, \pi_{12}, \pi_{15}\},$$

the incidence relation

$$I' : \text{ the point } \{\pi_i, \pi_j, \pi_k\}\ I' \text{ the lines } \pi_i, \pi_j, \text{and } \pi_k,$$

then, the structure $(\mathcal{P}',\mathcal{L}',I')$ is an affine plane of order 2.

Proposition 4.9 *If two triangles are in perspective centrally in Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$, then they are not in perspective from an axis in $PG(4,2)\backslash\mathcal{F}$.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$. Denote the three vertices of one triangle by P_3, P_6 and P_{11} and those of the other by P_4, P_7 and P_{12}. Central perspectivity means that the three lines P_3P_4, P_6P_7, and $P_{11}P_{12}$ are concurrent at the point P_1 called the center of perspectivity. Axial perspectivity means that lines P_3P_6 and P_4, P_7 meet in the point P_{18}, lines P_3P_{11} and P_4P_{12} meet in a second point P_{24} and lines P_6P_{11} and P_7P_{12} meet in a third point P_{25} and that these three points all lie on a common line called the axis of perspectivity. This axis $\{P_{18}, P_{24}, P_{25}\}$ of perspectivity is not on $\mathcal{F}$. The line joining the three collinear points of intersection of the extensions of corresponding sides in perspective triangles is not intersecting with $\mathcal{F}$. □

§5. Conclusion

This study has delved into the structural properties of the cubic threefold $\mathcal{F}$ over the field $GF(2)$ in $4-$dimensional projective space, focusing particularly on its intersection properties with lines and planes. The analysis has revealed the presence of six spreads, each composed of five lines, offering insights into the combinatorial structure of $\mathcal{F}$. Exploring tangent planes to $\mathcal{F}$ has led to the formulation of two distinct affine plane models, highlighting its geometric versatility. Additionally, the non-validity of Desargues' theorem within $\mathcal{F}$ underscores its departure from classical projective geometry norms. This research contributes to advancing our understanding of cubic threefolds over finite fields, pointing towards further investigations into their algebraic and geometric intricacies.

References

[1] Z. Akça, A. Bayar, S. Ekmekçi, R. Kaya, T. J A, and H. Van Maldeghem, Generalized Veronesean embeddings of projective spaces Part II the lax case, *Ars Combinatoria*, 103: 65–80, 2012.

[2] Z. Akça, A. Bayar, S. Ekmekçi, R. Kaya, H. Van Maldeghem, *Bazıgeometrilerin sonlu projektif uzaylara gömülmeleri üzerine*, Tübitak Proje No: 108T340, 2010.

[3] Z. Akça and A. Altıntaş, Fuzzy counterpart of Klein quadric, *International Electronic Journal of Geometry*, 16(2): 680–688,2023.

[4] A. Bayar, Z. Akça, and S. Ekmekçi, On embedding the projective plane PG(2,4) to the projective space PG(4,4), *New Trends in Mathematical Sciences*, 10(4): 142–150, Dec. 2022.

[5] A. Betten, F. Karaoğlu, Cubic surfaces over small finite fields, *Designs, Codes and Cryptography*, 87 (4): 931–953, 2019.

[6] L.E., Dickson, Projective classification of cubic surfaces modulo 2, *Annals of Mathematics*, 16: 139–157, 1915.

[7] S. Ekmekçi, A. Bayar, Z. Akça, On the projective planes in projective space PG(4,4), *Erciyes Üniversitesi Fen Bilimleri Enstitüsü Fen Bilimleri Dergisi*, Vol. 38, No. 3, pp. 519–524, Dec. 2022.

[8] Hirschfeld JWP, Classical configurations over finite fields. I. The double- six and the cubic surface with 27 lines, *Rendiconti di Matematica e delle sue Applicazioni*, 26 (5): 115–152, 1967.

[9] F., Karaoğlu, Nonsingular cubic surfaces over $\mathbb{F}_{2^k}$, *Turk. J. Math.*, 45: 2492 – 2510, 2021.

[10] F., Karaoglu, A., Smooth cubic surfaces with 15 lines, *Applicable Algebra in Engineering, Communication and Computing*, 33: 823–853, 2022.

[11] F. Klein, Uber die Ausflsung Gleichungen siebenten und achten Grades, *Math. Ann.*, 15: 251-282, 1879.

[12] H. J. Kroll, R. Vincenti, Partitions of the Klein quadric, *Electronic Notes in Discrete Mathematics*, 26(1): 151-157, 2006.

[13] L.A. Rosati, L'equazione delle 27 rette della superficie cubica generale in un corpo finito, *Bollettino dell'Unione Matematica Italiana*, 12(3): 612-626, 1957.