*International J.Math. Combin. Vol.4(2022), 50-59*

# Optimality of the Generalization New Class of Caputo-Katugampola Fractional Optimal Control Problems

Moataz Abbas Holel and Sameer Qasim Hasan

(Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq)

E-mail: moataz.abbas92@gmail.com

**Abstract**: In this work, a generalization of a new class of fractional order optimal control problems with Caputo-Katugampola derivative of order $\alpha, \beta \in (0,1)$ and $\rho > 0$ was studied and considers the final time $t_f$ is free. The necessary optimality conditions with Lagrange multipliers $\lambda(t) \in \mathbb{R}$ of fractional order optimal control problems were derived in case $t \in [A, t_f]$ and $a < A$. The formula for the integral by parts has been proven for the left Caputo-Katugampola fractional derivative that contributes to the finding and deriving the necessary optimality conditions.

**Key Words**: Fractional calculus, Caputo-Katugampola derivative, necessary optimality conditions, hamiltonian system.

**AMS(2010)**: 49C20.

## §1. Introduction

Fractional calculus contributes to many important aspects and fields of life such as science, engineering and physical applications, since the fractional order models gives an accurate description of non-linear and complex natural systems than integer order models , which prompted researchers to interest in studying these systems in advanced methods and in more than one way.

We mention some of its applications with optimal fractional control (FOCP) that are subject to dynamic constraints with the objective function problems, chaotic systems [1], bioscience [2], conformable to the FOCP [3], aerospace [4], economic [5] and so on [6-8]. The reader can refer to the books [9-12] for more details on fractional calculus.

Agrawal O. P. [13] is using Riemann-Liouville (R-LFD) to provide a general formulation and find an approximate solution for a class of (FOCPs).

Study of the necessary and sufficient optimality conditions for fractional optimal control problems for one-dimensional with Caputo fractional derivative by Pooseh S., et al. [14] and for system with JiHuan He's fractional derivative by Sayevand K., et al. [15] and for composition(FOCPs) by QasimHasan S., and Abbas Holel M. [16].

New types of fractional operators were introduced by U. Katugampola [17], These are done by generalizing the (R-L) and Hadamard fractional integrals, same author, introduce a

---

[1] Received November 15, 2022, Accepted December 9, 2022.

new fractional derivative [18], and see [19]. Almeida R., et al. [20] proves the existence and uniqueness theorem by using the Caputo-Katugampola derivative for a fractional Cauchy type problem.A definition has been given with some properties of generalized fractional derivatives by Jarad F., et al. [21].

This paper aims to study a generalization of a new class of (FOCPs) with left (C-KD) of order $\alpha, \beta \in (0,1)$ and $\rho > 0$,let $\mathcal{F}$, $g$ are two differentiable functions with domain $[a, +\infty) \times \mathbb{R}^2$ and $\Psi : [a, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a differentiable function. And considering $(\mathcal{R}_1, \mathcal{R}_2) \neq (0,0)$, $x_A$ is a fixed real number $t \in [A, t_f]$ and $a < A$, as follows:

$$minimize \;\; \mathbf{J}(x, u, t_f) = \int_A^{t_f} \mathcal{F}\left(x(t), u(t), t\right) dt + \Psi\left(t_f, x(t_f)\right), \tag{1}$$

subject to dynamic control system

$$\mathcal{R}_1 {}^{CK}_{\;a}D_t^{\alpha,\rho} x(t) + \mathcal{R}_2 {}^{CK}_{\;a}D_t^{\beta,\rho} x(t) = g\left(x(t), u(t), t\right), \tag{2}$$

$$x(A) = x_A. \tag{3}$$

This paper contains four sections: in Section 2, preliminaries and prove integration by parts formula for (C-KFD). The necessary optimality conditions are studied for a generalization class of (C-K FOCPs) in details in Section 3. The conclusions are introduced in Section 4.

## §2. Preliminaries

The basic definitions of (FDs) and (FIs) are presented with proof of theorems are used in work.

**Definition** 2.1([18, 20]) *Let $\alpha > 0$, $\rho > 0$ and an interval $[a, b]$ of $\mathbb{R}$, where $0 < a < b$ The left and right (R-KFI) and (R-KFD) of a function $f \in L^1([a, b])$ are defined by*

$${}^{RK}_{\;a}D_t^{-\alpha,\rho} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{4}$$

$${}^{RK}_{\;t}D_b^{-\alpha,\rho} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{5}$$

$${}^{RK}_{\;a}D_t^{\alpha,\rho} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\alpha} f(\tau) d\tau, \tag{6}$$

$${}^{RK}_{\;t}D_b^{\alpha,\rho} f(t) = \frac{-\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_t^b \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^\alpha} f(\tau) d\tau, \tag{7}$$

**Definition** 2.2([18, 20]) *Let $\alpha \in (0,1)$, $\rho > 0$ and an interval $[a, b]$ of $\mathbb{R}$, where $0 < a < b$. The left and right (C-KFD) are defined by*

$${}^{CK}_{\;a}D_t^{\alpha,\rho} f(t) = {}^{RK}_{\;a}D_t^{\alpha,\rho} [f(t) - f(a)] = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\alpha} [f(\tau) - f(a)] d\tau, \tag{8}$$

$$ {}^{CK}_{\;t}D^{\alpha,\rho}_{b}f(t) = {}^{RK}_{\;t}D^{\alpha,\rho}_{b}[f(t)-f(b)] = \frac{-\rho^{\alpha}}{\Gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_{t}^{b}\frac{\tau^{\rho-1}}{(\tau^{\rho}-t^{\rho})^{\alpha}}[f(\tau)-f(b)]d\tau. \tag{9} $$

**Theorem** 2.1 *Let $\alpha \in (0,1)$, $\rho > 0$, then left and right (C-KFD) of a function $f \in \mathbb{C}^1[a,b]$ is given by*

$$ {}^{CK}_{\;a}D^{\alpha,\rho}_{t}f(t) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_{a}^{t}(t^{\rho}-\tau^{\rho})^{-\alpha}f'(\tau)d\tau, \tag{10} $$

$$ {}^{CK}_{\;t}D^{\alpha,\rho}_{b}f(t) = \frac{-\rho^{\alpha}}{\Gamma(1-\alpha)}\int_{t}^{b}(\tau^{\rho}-t^{\rho})^{-\alpha}f'(\tau)d\tau. \tag{11} $$

*Proof* We are proving the left (C-KFD) in (10) using Definition 2.2 in Eq. (8) and let

$$ {}^{CK}_{\;a}D^{\alpha,\rho}_{t}f(t) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_{a}^{t}udv. \tag{12} $$

Now, using integration by part, derive relative to and substitute the result of Eq. (12), to get

$$ \begin{aligned} {}^{CK}_{\;a}D^{\alpha,\rho}_{t}f(t) &= \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}t^{1-\rho}\left[(1-\alpha)(\rho t^{\rho-1})\int_{a}^{t}\frac{1}{\rho(1-\alpha)}(t^{\rho}-\tau^{\rho})^{-\alpha}f'(\tau)d\tau\right] \\ &= \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_{a}^{t}(t^{\rho}-\tau^{\rho})^{-\alpha}f'(\tau)d\tau. \end{aligned} $$

This completes the proof. □

**Theorem** 2.2 *Let $f(t) \in C[a,b]$ and $g(t) \in C^1[a,b]$ be two functions and $\alpha \in (0,1)$, $\rho > 0$. Then*

$$ \int_{a}^{b} f(t)\cdot {}^{CK}_{\;a}D^{\alpha,\rho}_{t}g(t)dt = \int_{a}^{b}\left(g(t)t^{\rho-1}\right){}^{RK}_{\;t}D^{\alpha,\rho}_{b}(t^{1-\rho}f(t))dt + \Big[g(t)\,{}^{RK}_{\;t}D^{-(1-\alpha,\rho)}_{b}(t^{1-\rho}f(t))\Big]_{t=a}^{t=b}. $$

*Proof* By using Theorem 2.1 to obtain:

$$ \int_{a}^{b} f(t)\cdot {}^{CK}_{\;a}D^{\alpha,\rho}_{t}g(t)dt = \int_{a}^{b} f(t)\left[\frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_{a}^{t}(t^{\rho}-\tau^{\rho})^{-\alpha}\frac{d}{d\tau}g(\tau)d\tau\right]dt. \tag{13} $$

By using the Dirichlet's formula for Eq. (13), to get

$$ \int_{a}^{b} f(t)\cdot {}^{CK}_{\;a}D^{\alpha,\rho}_{t}g(t)dt = \int_{a}^{b}\frac{d}{dt}g(t)\left[\frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_{t}^{b}(\tau^{\rho}-t^{\rho})^{-\alpha}\frac{\tau^{1-\rho}f(\tau)}{\tau^{1-\rho}}d\tau\right]dt. \tag{14} $$

Using definition of the right (R-KFI) of $\left(t^{1-\rho}f(t)\right)$ with order $(1-\alpha,\rho)$ in Eq. (14) to get

$$ \int_{a}^{b} f(t)\cdot {}^{CK}_{\;a}D^{-(1-\alpha,\rho)}_{t}g(t)dt = \int_{a}^{b}\frac{d}{dt}g(t)\,{}^{RK}_{\;t}D^{\alpha,\rho}_{b}(t^{1-\rho}f(t))dt = \int_{a}^{b}\frac{d}{dt}g(t)h(t)dt, $$

where $h(t) = {}^{RK}_{\;t}D^{-(1-\alpha,\rho)}_{b}\left(t^{1-\rho}f(t)\right)$.

Now, using integration by parts of above equation to obtain:

$$= \left[ \begin{array}{c} g(t). \\ {}_{t}^{RK}D_{b}^{-(1-\alpha,\rho)} \left(t^{1-\rho} f(t)\right) \end{array} \right]_{t=a}^{t=b} - \int_a^b \left(g(t) t^{1-\rho}\right) (-1) \left[ \begin{array}{c} \frac{-\rho^{\alpha}}{\Gamma(1-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right) \cdot \\ \int_t^b \left(\tau^{\rho} - t^{\rho}\right)^{-\alpha} \frac{\tau^{1-\rho} f(\tau)}{\tau^{1-\rho}} d\tau \end{array} \right] dt. \tag{15}$$

Using definition of right (R-KFD) of $\left(t^{1-\rho} f(t)\right)$ with order $(1-\alpha, \rho)$ in Eq. (7) to get

$$\int_a^b f(t) \cdot {}_{a}^{CK}D_{t}^{\alpha,\rho} g(t) dt = \int_a^b \left(g(t) t^{1-\rho}\right) \cdot {}_{t}^{RK}D_{b}^{\alpha,\rho} \left(t^{1-\rho} f(t)\right) dt + \left[ g(t) {}_{t}^{RK}D_{b}^{-(1-\alpha,\rho)} \left(t^{1-\rho} f(t)\right) \right]_{t=a}^{t=b}.$$

## §3. Studying the Necessary Optimality Conditions of the Generalization a Class of (C-K)FOCPs

We study and derive the necessary optimality conditions the left (C-KFD) of order $\alpha, \beta \in (0,1)$, $\rho > 0$ in the following theorem.

**Theorem** 3.1 *If $(x, u, t_f)$ is a minimizer of (1) under the sum of two (C-K) FD of dynamic constraint (2), and the condition (3), then a function $\lambda(t) \in \mathbb{R}$, for $(x, u, \lambda)$ satisfies:*

**1.Hamiltonian system**

$$\begin{cases} t^{\rho-1} \left[ \mathcal{R}_1 {}_{t}^{RK}D_{t_f}^{\alpha,\rho} \left(t^{1-\rho} \lambda(t)\right) + \mathcal{R}_2 {}_{t}^{RK}D_{t_f}^{\beta,\rho} \left(t^{1-\rho} \lambda(t)\right) \right] = \frac{\partial H}{\partial x} \left(x(t), u(t), \lambda(t), t\right), \\ \mathcal{R}_1 {}_{a}^{RK}D_{t}^{\alpha,\rho} x(t) + \mathcal{R}_2 {}_{a}^{RK}D_{t}^{\beta,\rho} x(t) = \frac{\partial H}{\partial \lambda} \left(x(t), u(t), \lambda(t), t\right), \quad for all t \in [A, t_f], \end{cases} \tag{16}$$

and

$$\begin{cases} {}_{t}^{RK}D_{t_f}^{\alpha,\rho} \left(t^{1-\rho} \lambda(t)\right) - {}_{t}^{RK}D_{A}^{\alpha,\rho} \left(t^{1-\rho} \lambda(t)\right) = 0, \\ {}_{t}^{RK}D_{t_f}^{\beta,\rho} \left(t^{1-\rho} \lambda(t)\right) - {}_{t}^{RK}D_{A}^{\beta,\rho} \left(t^{1-\rho} \lambda(t)\right) = 0, \end{cases} \quad for\ all\ t \in [a, A]$$

**2.The stationary condition**

$$\frac{\partial \mathcal{H}}{\partial u} \left(x(t), u(t), \lambda(t), t\right) = 0, \quad for\ all\ t \in [A, t_f]. \tag{17}$$

**3.The transversality conditions**

$$\left[ \begin{array}{c} \mathcal{H} \left(x(t), u(t), \lambda(t), t\right) + \mathcal{R}_1 x'(t) {}_{t}^{RK}D_{t_f}^{-(1-\alpha,\rho)} t^{1-\rho} \lambda(t) \\ -\mathcal{R}_2 x'(t) {}_{t}^{RK}D_{t_f}^{-(1-\beta,\rho)} t^{1-\rho} \lambda(t) - \mathcal{R}_1 \lambda(t) {}_{a}^{CK}D_{t}^{\alpha,\rho} x(t) \\ -\mathcal{R}_2 \lambda(t) {}_{a}^{CK}D_{t}^{\beta,\rho} x(t) + \frac{\partial \Psi}{\partial t} \left(t, x(t)\right) \end{array} \right]_{t=t_f} = 0. \tag{18}$$

$$\left[ \mathcal{R}_1 {}_{t}^{RK}D_{t_f}^{-(1-\alpha,\rho)} t^{1-\rho} \lambda(t) + \mathcal{R}_2 {}_{t}^{RK}D_{t_f}^{-(1-\beta,\rho)} t^{1-\rho} \lambda(t) - \frac{\partial \Psi}{\partial x} \left(t, x(t)\right) \right]_{t=t_f} = 0, \tag{19}$$

$$\begin{cases} {}_{a}^{RK}D_{t_f}^{-(1-\alpha,\rho)} a^{1-\rho} \lambda(a) - {}_{a}^{RK}D_{A}^{-(1-\alpha,\rho)} a^{1-\rho} \lambda(a) = 0, \\ {}_{a}^{RK}D_{t_f}^{-(1-\beta,\rho)} a^{1-\rho} \lambda(a) - {}_{a}^{RK}D_{A}^{-(1-\beta,\rho)} a^{1-\rho} \lambda(a) = 0. \end{cases}$$

*Proof* Constructing the problem as minimizing and by define the Hamiltonian function $\mathcal{H}\left(x(t), u(t), \lambda(t)\right)$, as follows:

$$\mathcal{H}\left(x(t), u(t), \lambda(t), t\right) = \mathcal{F}\left(x(t), u(t)t\right) + \lambda(t) g\left(x(t), u(t), t\right), \tag{20}$$

$$\mathbf{J}^{*}\left(x, u, t_f, \lambda\right) = \int_{A}^{t_f} \left[ \begin{array}{c} \mathcal{H}\left(x(t), u(t), \lambda(t), t\right) \\ -\lambda(t)\left\{\mathcal{R}_1 {}^{CK}_{\ a}D^{\alpha,\rho}_{t} x(t) + \mathcal{R}_2 {}^{CK}_{\ a}D^{\beta,\rho}_{t} x(t)\right\} \end{array} \right] dt + \Psi\left(t_f, x\left(t_f\right)\right). \tag{21}$$

Using variations $x + \delta x$, $\lambda + \delta\ \lambda u + \delta u$, $, t_f + \delta t_f$ and $(\delta \mathbf{J}^{*} = 0)$, we conclude that

$$\begin{aligned} 0 \ &= \int_{A}^{t_f} \left[ \begin{array}{c} \delta\mathcal{H}\left(x(t), u(t), \lambda(t), t\right) \\ -\delta\lambda(t) \left\{ \begin{array}{c} \mathcal{R}_1 {}^{CK}_{\ a}D^{\alpha,\rho}_{t} x(t) \\ +\mathcal{R}_2 {}^{CK}_{\ a}D^{\beta,\rho}_{t} x(t) \end{array} \right\} \end{array} \right] dt \\ &\quad +\delta t_f \left[ \begin{array}{c} \mathcal{H}\left(x(t), u(t), \lambda(t), t\right) \\ -\lambda(t) \left\{ \begin{array}{c} \mathcal{R}_1 {}^{CK}_{\ a}D^{\alpha,\rho}_{t} x(t) \\ +\mathcal{R}_2 {}^{CK}_{\ a}D^{\beta,\rho}_{t} x(t) \end{array} \right\} \end{array} \right]_{t=t_f} + \delta\Psi\left(t_f, x\left(t_f\right)\right). \end{aligned} \tag{22}$$

Thus,

$$\begin{aligned} &= \int_{A}^{t_f} \left[ \begin{array}{c} \frac{\partial\mathcal{H}}{\partial x(t)}\delta x(t) + \frac{\partial\mathcal{H}}{\partial u(t)}\delta u(t) + \frac{\partial\mathcal{H}}{\partial\lambda(t)}\delta\lambda(t) \\ -\delta\lambda(t)\left\{\mathcal{R}_1 {}^{CK}_{\ a}D^{\alpha,\rho}_{t} x(t) + \mathcal{R}_2 {}^{CK}_{\ a}D^{\beta,\rho}_{t} x(t)\right\} \\ -\mathcal{R}_1\lambda(t) {}^{CK}_{\ a}D^{\alpha,\rho}_{t}\delta x(t) - \mathcal{R}_2\lambda(t) {}^{CK}_{\ a}D^{\beta,\rho}_{t}\delta x(t) \end{array} \right] dt \\ &\quad + \delta t_f \left[ \begin{array}{c} \mathcal{H}\left(x(t), u(t), \lambda(t), t\right) \\ -\lambda(t)\left\{\mathcal{R}_1 {}^{CK}_{\ a}D^{\alpha,\rho}_{t} x(t) + \mathcal{R}_2 {}^{CK}_{\ a}D^{\beta,\rho}_{t} x(t)\right\} \end{array} \right]_{t=t_f} \\ &\quad + \frac{\partial\Psi}{\partial t}\left(t_f, x\left(t_f\right)\right)\delta t_f + \frac{\partial\Psi}{\partial x}\left(t_f, x\left(t_f\right)\right)\left(x'\left(t_f\right)\delta t_f + \delta x\left(t_f\right)\right). \end{aligned} \tag{23}$$

Now, integration by part the relation in Eq. (23) by using Theorem 2.2 in the form

$$\begin{aligned} &= \int_{A}^{t_f} \lambda(t) {}^{CK}_{a}D^{\alpha,\rho}_{t} \delta x(t) dt \\ &= \int_{a}^{t_f} \lambda(t) {}^{CK}_{a}D^{\alpha,\rho}_{t} \delta x(t) dt - \int_{a}^{A} \lambda(t) {}^{CK}_{a}D^{\alpha,\rho}_{t} \delta x(t) dt \\ &= \int_{a}^{t_f} \delta x(t) t^{\rho-1} {}^{RK}_{\ t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\lambda(t)\right) dt + \left[\delta x(t) {}^{RK}_{\ t}D^{-(1-\alpha,\rho)}_{t_f} t^{1-\rho}\lambda(t)\right]_{t=a}^{t=t_f} \\ &\quad - \int\limits_{a}^{A} \delta x(t) t^{\rho-1} {}^{RK}_{\ t}D^{\alpha,\rho}_{A}\left(t^{1-\rho}\lambda t\right) dt - \left[\delta x(t) {}^{RK}_{\ t}D^{-(1-\alpha,\rho)}_{A} t^{1-\rho}\lambda(t)\right]_{t=a}^{t=A} \\ &= \int_{a}^{A} \delta x(t) t^{\rho-1} {}^{RK}_{\ t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\lambda(t)\right) dt + \int_{A}^{t_f} \delta x(t) t^{\rho-1} {}^{RK}_{\ t}D^{\alpha,\rho}_{t_f}\left(t^{1-\rho}\lambda(t)\right) dt \end{aligned}$$

$$
\begin{aligned}
&+\left[\delta x(t)\,{}_{t}^{RK}D_{t_f}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)\right]_{t=a}^{t=t_f}\\
&-\int_a^A \delta x(t)t^{\rho-1}\,{}_{t}^{RK}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)dt-\left[\delta x(t)\,{}_{t}^{RK}D_{A}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)\right]_{t=a}^{t=A}\\
={}&\int_a^A \delta x(t)t^{\rho-1}\left[{}_{t}^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}_{t}^{RK}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&+\int_A^{t_f} \delta x(t)t^{\rho-1}\,{}_{t}^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)dt\\
&+\left[\delta x(t)\,{}_{t}^{RK}D_{t_f}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)\right]_{t=a}^{t=t_f}-\left[\delta x(t)\,{}_{t}^{RK}D_{A}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)\right]_{t=a}^{t=A}\\
={}&\int_a^A \delta x(t)t^{\rho-1}\left[{}_{t}^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}_{t}^{RK}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&+\int_A^{t_f} \delta x(t)t^{\rho-1}\,{}_{t}^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)dt\\
&+\left[\delta x(t)\,{}_{t}^{RK}D_{t_f}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)\right]_{t=t_f}-\left[\delta x(t)\,{}_{t}^{RK}D_{t_f}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)\right]_{t=a}\\
&-\left[\delta x(t)\,{}_{t}^{RK}D_{A}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)\right]_{t=A}-\left[\delta x(t)\,{}_{t}^{RK}D_{A}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)\right]_{t=a}.
\end{aligned}
$$

Since $(\delta x(A)=0)$, we get

$$
\begin{aligned}
={}&\int_a^A \delta x(t)t^{\rho-1}\left[{}_{t}^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}_{t}^{RK}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&+\int_A^{t_f} \delta x(t)t^{\rho-1}\,{}_{t}^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)dt+\delta x(t_f)\left[{}_{t}^{RK}D_{t_f}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)\right]_{t=t_f}\\
&-\delta x(a)\left[{}_{a}^{RK}D_{t_f}^{-(1-\alpha,\rho)}a^{1-\rho}\lambda(a)-{}_{a}^{RK}D_{A}^{-(1-\alpha,\rho)}a^{1-\rho}\lambda(a)\right].
\end{aligned}
\tag{24}
$$

Also, in the same way we have

$$
\begin{aligned}
={}&\int_a^A \delta x(t)t^{\rho-1}\left[{}_{t}^{RK}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}_{t}^{RK}D_{A}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&+\int_A^{t_f} \delta x(t)t^{\rho-1}\,{}_{t}^{RK}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)dt+\delta x(t_f)\left[{}_{t}^{RK}D_{t_f}^{-(1-\beta,\rho)}t^{1-\rho}\lambda(t)\right]_{t=t_f}\\
&-\delta x(a)\left[{}_{a}^{RK}D_{t_f}^{-(1-\beta,\rho)}a^{1-\rho}\lambda(a)-{}_{a}^{RK}D_{A}^{-(1-\beta,\rho)}a^{1-\rho}\lambda(a)\right].
\end{aligned}
\tag{25}
$$

Substitute the results of Eq. (24) and Eq. (25), into Eq. (23) as follows

$$
\begin{aligned}
={}&\int_A^{t_f}\left[\begin{array}{c}
\frac{\partial\mathcal{H}}{\partial x(t)}\delta x(t)+\frac{\partial\mathcal{H}}{\partial u(t)}\delta u(t)+\frac{\partial\mathcal{H}}{\partial\lambda(t)}\delta\lambda(t)\\
-\delta\lambda(t)\left\{\mathcal{R}_1\,{}_{a}^{CK}D_{t}^{\alpha,\rho}x(t)+\mathcal{R}_2\,{}_{a}^{CK}D_{t}^{\beta,\rho}x(t)\right\}\\
-\delta x(t)t^{\rho-1}\left[\mathcal{R}_1\,{}_{t}^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)+\mathcal{R}_2\,{}_{t}^{RK}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]
\end{array}\right]dt\\
&-\mathcal{R}_1\int_a^A \delta x(t)t^{\rho-1}\left[{}_{t}^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}_{t}^{RK}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&-\mathcal{R}_2\int_a^A \delta x(t)t^{\rho-1}\left[{}_{t}^{RK}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}_{t}^{RK}D_{A}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt
\end{aligned}
$$

$$
\begin{aligned}
&-\delta x\left(t_f\right)\left[\mathcal{R}_1\left[{}^{RK}_{\ t}D_{t_f}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)\right]_{t=t_f}+\mathcal{R}_2\left[{}^{RK}_{\ t}D_{t_f}^{-(1-\beta,\rho)}t^{1-\rho}\lambda(t)\right]_{t=t_f}\right]\\
&+\mathcal{R}_1\delta x(a)\left[{}^{RK}_{\ a}D_{t_f}^{-(1-\alpha,\rho)}a^{1-\rho}\lambda(a)-{}^{RK}_{\ a}D_{A}^{-(1-\alpha,\rho)}a^{1-\rho}\lambda(a)\right]\\
&+\mathcal{R}_2\delta x(a)\left[{}^{RK}_{\ a}D_{t_f}^{-(1-\beta,\rho)}a^{1-\rho}\lambda(a)-{}^{RK}_{\ a}D_{A}^{-(1-\beta,\rho)}a^{1-\rho}\lambda(a)\right]\\
&+\delta t_f\left[\begin{array}{c}\mathcal{H}\left(x(t),u(t),\lambda(t),t\right)\\ -\lambda(t)\left\{\mathcal{R}_1{}^{CK}_{\ a}D_{t}^{\alpha,\rho}x(t)+\mathcal{R}_2{}^{CK}_{\ a}D_{t}^{\beta,\rho}x(t)\right\}\end{array}\right]_{t=t_f}\\
&+\frac{\partial\Psi}{\partial t}\left(t_f,x\left(t_f\right)\right)\delta t_f+\frac{\partial\Psi}{\partial x}\left(t_f,x\left(t_f\right)\right)\left(x'\left(t_f\right)\delta t_f+\delta x\left(t_f\right)\right),
\end{aligned}
$$

Thus,

$$
\begin{aligned}
0=&\int_A^{t_f}\left[\begin{array}{c}\delta x(t)\left(\frac{\partial\mathcal{H}}{\partial x(t)}-t^{\rho-1}\left[\mathcal{R}_1{}^{RK}_{\ t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)+\mathcal{R}_2{}^{RK}_{\ t}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]\right)\\ +\delta u(t)\frac{\partial\mathcal{H}}{\partial u(t)}+\delta\lambda(t)\left(\frac{\partial\mathcal{H}}{\partial\lambda(t)}-\mathcal{R}_1{}^{CK}_{\ a}D_{t}^{\alpha,\rho}x(t)-\mathcal{R}_2{}^{CK}_{\ a}D_{t}^{\beta,\rho}x(t)\right)\end{array}\right]dt\\
&-\mathcal{R}_1\int_a^A\delta x(t)t^{\rho-1}\left[{}^{RK}_{\ t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}^{RK}_{\ t}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&-\mathcal{R}_2\int_a^A\delta x(t)t^{\rho-1}\left[{}^{RK}_{\ t}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}^{RK}_{\ t}D_{A}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&-\delta x\left(t_f\right)\left[\mathcal{R}_1{}^{RK}_{\ t}D_{t_f}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)+\mathcal{R}_2{}^{RK}_{\ t}D_{t_f}^{-(1-\beta,\rho)}t^{1-\rho}\lambda(t)-\frac{\partial\Psi}{\partial x}\left(t,x(t)\right)\right]_{t=t_f}\\
&+\delta t_f\left[\begin{array}{c}\mathcal{H}\left(x(t),u(t),\lambda(t),t\right)-\lambda(t)\left\{\mathcal{R}_1{}^{CK}_{\ a}D_{t}^{\alpha,\rho}x(t)+\mathcal{R}_2{}^{CK}_{\ a}D_{t}^{\beta,\rho}x(t)\right\}\\ +\frac{\partial\Psi}{\partial t}\left(t,x(t)\right)+\frac{\partial\Psi}{\partial x}\left(t,x(t)\right)x'(t)\end{array}\right]_{t=t_f}\\
&+\mathcal{R}_1\delta x(a)\left[{}^{RK}_{\ a}D_{t_f}^{-(1-\alpha,\rho)}a^{1-\rho}\lambda(a)-{}^{RK}_{\ a}D_{A}^{-(1-\alpha,\rho)}a^{1-\rho}\lambda(a)\right]\\
&+\mathcal{R}_2\delta x(a)\left[{}^{RK}_{\ a}D_{t_f}^{-(1-\beta,\rho)}a^{1-\rho}\lambda(a)-{}^{RK}_{\ a}D_{A}^{-(1-\beta,\rho)}a^{1-\rho}\lambda(a)\right].
\end{aligned}
\tag{26}
$$

Now, we rewrite the transversality conditions in Eq. (26) by using the Taylor series for $f=(x+\delta x)$ about the point $x=t_f$ can be given as

$$
\begin{aligned}
&(x+\delta x)\left(t_f+\delta t_f\right)=x+\delta x)\left(t_f\right)+x'\left(t_f\right)\delta t_f+\mathbf{O}\left(\delta t_f^2\right),\\
&\underbrace{(x+\delta x)\left(t_f+\delta t_f\right)-x\left(t_f\right)}-\delta x\left(t_f\right)=x'\left(t_f\right)\delta t_f+\mathbf{O}\left(\delta t_f^2\right)\\
&x\left(t_f\right)-\delta x\left(t_f\right)=x'\left(t_f\right)\delta t_f+\mathbf{O}\left(\delta t_f^2\right),
\end{aligned}
$$

$$\delta x\left(t_f\right)-\delta x_{t_f}=-x'\left(t_f\right)\delta t_f+\mathbf{O}\left(\delta t_f^2\right), \tag{27}$$

$$\delta x\left(t_f\right)=\delta x_{t_f}-x'\left(t_f\right)\delta t_f+\mathbf{O}\left(\delta t_f^2\right), \tag{28}$$

where $\delta x_{t_f}=(x+\delta x)\left(t_f+\delta t_f\right)-x\left(t_f\right)$ and $\lim_{\gamma\to\infty}\frac{\mathbf{O}(\gamma)}{\gamma}$ is finite.

Substitute the Eq. (27) and Eq. (28), into Eq. (26), we have

$$
\begin{aligned}
&= \int_{A}^{t_f} \left[ \begin{array}{c} \delta x(t) \left( \frac{\partial \mathcal{H}}{\partial x(t)} - t^{\rho-1} \left[ \mathcal{R}_1 {}^{RK}_{\;t}D^{\alpha,\rho}_{t_f} \left( t^{1-\rho}\lambda(t) \right) + \mathcal{R}_2 {}^{RK}_{\;t}D^{\beta,\rho}_{t_f} \left( t^{1-\rho}\lambda(t) \right) \right] \right) \\ + \delta u(t) \frac{\partial \mathcal{H}}{\partial u(t)} + \delta\lambda(t) \left( \frac{\partial \mathcal{H}}{\partial \lambda(t)} - \mathcal{R}_1 {}^{CK}_{\;a}D^{\alpha,\rho}_{t} x(t) - \mathcal{R}_2 {}^{CK}_{\;a}D^{\beta,\rho}_{t} x(t) \right) \end{array} \right] dt \\
&- \mathcal{R}_1 \int_{a}^{A} \delta x(t) t^{\rho-1} \left[ {}^{RK}_{\;t}D^{\alpha,\rho}_{t_f} \left( t^{1-\rho}\lambda(t) \right) - {}^{RK}_{\;t}D^{\alpha,\rho}_{A} \left( t^{1-\rho}\lambda(t) \right) \right] dt \\
&- \mathcal{R}_2 \int_{a}^{A} \delta x(t) t^{\rho-1} \left[ {}^{RK}_{\;t}D^{\beta,\rho}_{t_f} \left( t^{1-\rho}\lambda(t) \right) - {}^{RK}_{\;t}D^{\beta,\rho}_{A} \left( t^{1-\rho}\lambda(t) \right) \right] dt \\
&\left[ \begin{array}{c} -\mathcal{R}_1 \left( \delta x_t - x'(t)\delta t \right) {}^{RK}_{\;t}D^{-(1-\alpha,\rho)}_{t_f} t^{1-\rho}\lambda(t) \\ -\mathcal{R}_2 \left( \delta x_t - x'(t)\delta t \right) {}^{RK}_{\;t}D^{-(1-\beta,\rho)}_{b} t^{1-\rho}\lambda(t) \\ + \delta x(t) \frac{\partial \Psi}{\partial x}(t, x(t)) + \delta t \mathcal{H}(x(t), u(t), \lambda(t), t) \\ -\mathcal{R}_1 \delta t \lambda(t) {}^{CK}_{\;a}D^{\alpha,\rho}_{t} x(t) - \mathcal{R}_2 \delta t \lambda(t) {}^{CK}_{\;a}D^{\beta,\rho}_{t} x(t) \\ + \delta t \frac{\partial \Psi}{\partial x}(t, x(t)) + \frac{\partial \Psi}{\partial x}(t, x(t)) \left( \delta x_t - \delta x(t) \right) \end{array} \right]_{t=t_f} \\
&+ \mathcal{R}_1 \delta x(a) \left[ {}^{RK}_{\;a}D^{-(1-\alpha,\rho)}_{t_f} a^{1-\rho}\lambda(a) - {}^{RK}_{\;a}D^{-(1-\alpha,\rho)}_{A} a^{1-\rho}\lambda(a) \right] \\
&+ \mathcal{R}_2 \delta x(a) \left[ {}^{RK}_{\;a}D^{-(1-\beta,\rho)}_{t_f} a^{1-\rho}\lambda(a) - {}^{RK}_{\;a}D^{-(1-\beta,\rho)}_{A} a^{1-\rho}\lambda(a) \right] + \mathbf{O}\left( \delta t_f^2 \right) = 0,
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&= \int_{A}^{t_f} \left[ \begin{array}{c} \delta x(t) \left( \frac{\partial \mathcal{H}}{\partial x(t)} - t^{\rho-1} \left[ \mathcal{R}_1 {}^{RK}_{\;t}D^{\alpha,\rho}_{t_f} \left( t^{1-\rho}\lambda(t) \right) + \mathcal{R}_2 {}^{RK}_{\;t}D^{\beta,\rho}_{t_f} \left( t^{1-\rho}\lambda(t) \right) \right] \right) \\ + \delta u(t) \frac{\partial \mathcal{H}}{\partial u(t)} + \delta\lambda(t) \left( \frac{\partial \mathcal{H}}{\partial \lambda(t)} - \mathcal{R}_1 {}^{CK}_{\;a}D^{\alpha,\rho}_{t} x(t) - \mathcal{R}_2 {}^{CK}_{\;a}D^{\beta,\rho}_{t} x(t) \right) \end{array} \right] dt \\
&-\mathcal{R}_1 \int_{a}^{A} \delta x(t) t^{\rho-1} \left[ {}^{RK}_{\;t}D^{\alpha,\rho}_{t_f} \left( t^{1-\rho}\lambda(t) \right) - {}^{RK}_{\;t}D^{\alpha,\rho}_{A} \left( t^{1-\rho}\lambda(t) \right) \right] dt \\
&-\mathcal{R}_2 \int_{a}^{A} \delta x(t) t^{\rho-1} \left[ {}^{RK}_{\;t}D^{\beta,\rho}_{t_f} \left( t^{1-\rho}\lambda(t) \right) - {}^{RK}_{\;t}D^{\beta,\rho}_{A} \left( t^{1-\rho}\lambda(t) \right) \right] dt \\
&+\delta t_f \left[ \begin{array}{c} \mathcal{H}(x(t), u(t), \lambda(t), t) + \mathcal{R}_1 x'(t) {}^{RK}_{\;t}D^{-(1-\alpha,\rho)}_{t_f} t^{1-\rho}\lambda(t) \\ + \mathcal{R}_2 x'(t) {}^{RK}_{\;t}D^{-(1-\beta,\rho)}_{b} t^{1-\rho}\lambda(t) - \mathcal{R}_1 \lambda(t) {}^{CK}_{\;a}D^{\alpha,\rho}_{t} x(t) \\ -\mathcal{R}_2 \lambda(t) {}^{CK}_{\;a}D^{\beta,\rho}_{t} x(t) + \frac{\partial \Psi}{\partial t}(t, x(t)) \end{array} \right]_{t=t_f} \\
&-\delta x_{t_f} \left[ \mathcal{R}_1 {}^{RK}_{\;t}D^{-(1-\alpha,\rho)}_{t_f} t^{1-\rho}\lambda(t) + \mathcal{R}_2 {}^{RK}_{\;t}D^{-(1-\beta,\rho)}_{t_f} t^{1-\rho}\lambda(t) - \frac{\partial \Psi}{\partial x}(t, x(t)) \right]_{t=t_f} \\
&+\mathcal{R}_1 \delta x(a) \left[ {}^{RK}_{\;a}D^{-(1-\alpha,\rho)}_{t_f} a^{1-\rho}\lambda(a) - {}^{RK}_{\;a}D^{-(1-\alpha,\rho)}_{A} a^{1-\rho}\lambda(a) \right] \\
&+\mathcal{R}_2 \delta x(a) \left[ {}^{RK}_{\;a}D^{-(1-\beta,\rho)}_{t_f} a^{1-\rho}\lambda(a) - {}^{RK}_{\;a}D^{-(1-\beta,\rho)}_{A} a^{1-\rho}\lambda(a) \right] + \mathbf{O}\left( \delta t_f^2 \right) = 0. \qquad (29)
\end{aligned}
$$

Since the variation functions were chosen arbitrarily, we get the necessary optimality conditions from Eq. (29) for sum two C-KFOCPs. □

## §4. Conclusion

In this paper, a new system for the generalization a class of (FOCPs) with Caputo-Katugampola derivativesin the case where the lower bound of the integral of $J$ is greater than of a of ${}^{CK}_{\ a}D_t^{\alpha,\rho}x(t) + {}^{CK}_{\ a}D_T^{\beta,\rho}x(t)$ has been studied and derived. We are assuming that the end time $t_f$ free. The necessary optimality conditions for the system are obtained when $\alpha, \beta \in (0,1)$ and $\rho > 0$ and $a \in \mathbb{R}$ and consist of a Hamiltonian system, stationary condition and transversality conditions,which contributes to solving non-linear dynamical control systems with FDs to obtain approximate solutions for state and control variables with the help of the proposed numerical methods.

## References

[1] Ghasemi S., Nazemi A., Tajik R., & Mortezaee M. (2021), On fractional optimal control problems with an application in fractional chaotic systems using a Legendre collocation-optimization technique, *Transactions of the Institute of Measurement and Control*, 43(6), 1268-1285.

[2] Ameen I. G., & Ali H. M. (2020), Application of fractional optimal control problems on some mathematical bioscience, In *Advanced Applications of Fractional Differential Operators to Science and Technology* (pp. 41-56). IGI Global.

[3] Chiranjeevi T., & Biswas R. K. (2022), Application of conformable fractional differential transform method for fractional optimal control problems. *IFAC-PapersOnLine*, 55(1), 643-648.

[4] Zhu, J., Trlat, E., & Cerf, M. (2017), Geometric optimal control and applications to aerospace, *Pacific Journal of Mathematics for Industry*, 9(1), 1-41.

[5] Lin Z, Wang H., Modeling and application of fractional-order economic growth model with time delay, *Fractal Fract.*, 5(3)(2021), 1-18. Congress on Applied Sciences, (pp. 101-120).

[6] Heydari MH, Razzaghi M., A new class of orthonormal basis functions: application for fractional optimal control problems, *Int J. Syst. Sci.*, 53(2)(2022), 240-252.

[7] Luo D, Wang JR, Fekan M., Applying fractional calculus to analyze economic growth modelling, *JAMSI*, 14(1)(2018), 25-36.

[8] Heydari MH, Razzaghi M., Piecewise Chebyshev cardinal functions: Application for constrained fractional optimal control problems, *Chaos Solitons Fractals*, 150(2021), 1-11.

[9] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New York, 1993.

[10] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[11] J. Sabatier, O.P. Agrawal, J.A. Tenreiro Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, 2007.

[12] G. Samko A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[13] Agrawal OP., A general formulation and solution scheme for fractional optimal control problems, *Nonlinear Dyn.*, 38(1)(2004), 323-337.

[14] Pooseh S., Almeida R., & Torres D. F. (2013), Fractional order optimal control problems with free terminal time, *arXiv:* 1302.1717.

[15] Sayevand K., & Rostami M. (2018), Fractional optimal control problems: optimality conditions and numerical solution, *IMA Journal of Mathematical Control and Information*, 35(1), 123-148.

[16] QasimHasan S., & Abbas Holel M.(2018), Solution of some types for composition fractional order differential equations corresponding to optimal control problems, *Journal of Control Science and Engineering*, 2018.

[17] Katugampola UN., New approach to a generalized fractional integral, *Appl. Math. Comput.*, 218(3)(2011), 860-865.

[18] Katugampola UN., A new approach to generalized fractional derivatives, *Bull. Math. Anal. Appl.*, 6(4)(2014), 1-15.

[19] Almutairi O, Kiliman A., New generalized Hermite-Hadamard inequality and related integral inequalities involving Katugampola type fractional integrals, *Symmetry*, 12(4)(2020), 1-14.

[20] Almeida R., Malinowska A. B., & Odzijewicz T.(2016), Fractional differential equations with dependence on the Caputo-Katugampola derivative, *Journal of Computational and Nonlinear Dynamics*, 11(6).

[21] Jarad F., Abdeljawad T., & Baleanu D.(2017), On the generalized fractional derivatives and their Caputo modification, *J.Nonlinear Sciences and Applications*, Vol.10, No.5, 2607–2619.