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PMC-Labeling of Certain Tree Related Graphs and Prism of Wheel Graph

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Abstract: The graph $G = (V, E)$ consists of p vertices and q edges. Let

$$\rho = \begin{cases} \frac{p}{2}, & p \text{ is even} \\ \frac{p-1}{2}, & p \text{ is odd,} \end{cases}$$

and $\Gamma = \{\pm 1, \pm 2, \ldots, \pm \rho\}$. Consider a function $\Lambda : V \to \Gamma$ that allocates unique labels from Γ to the various vertices of V when p is even and allocates a unique labels in Γ to $p-1$ vertices of V, repeating a label for the remaining one vertex when p is odd. Then, the labeling as mentioned above is called a pair mean cordial labeling (PMC-labeling) if for every edge uv of G, there is a labeling

$$\begin{cases} \dfrac{\Lambda(u) + \Lambda(v)}{2} & \text{if } \Lambda(\text{u}) + \Lambda(\text{v}) \text{ is even,} \\ \dfrac{\Lambda(u) + \Lambda(v) + 1}{2} & \text{if } \Lambda(\text{u}) + \Lambda(\text{v}) \text{ is odd} \end{cases}$$

such that $|\bar{\mathbb{S}}_{\Lambda_1} - \bar{\mathbb{S}}_{\Lambda_1^c}| \leq 1$, and a Smarandachely PMC-labeling if $|\bar{\mathbb{S}}_{\Lambda_1} - \bar{\mathbb{S}}_{\Lambda_1^c}| \geq 2$, where $\bar{\mathbb{S}}_{\Lambda_1}$ and $\bar{\mathbb{S}}_{\Lambda_1^c}$ are denoted the number of edges labelled with 1 and the number of edges not labelled with 1, respectively. A graph G that has a pair mean cordial labeling is called a pair mean cordial graph (PMC-graph). In this research paper, we prove the existences of the PMC-labeling of some tree related graphs like the X-tree, Y-tree, prism of wheel graph, subdivision of bistar graph and coconut tree.

Key Words: X-tree, Y-tree, prism of wheel graph, subdivision of bistar graph, coconut tree.

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§1. Introduction

All graphs considered in this paper are simple, finite and undirected. Let $G = (V(G), E(G))$ be a graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges where $V(G)$ and $E(G)$ denote the vertex

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set and edge set of a graph G. For all terminologies and notations of graph theory, we refer the book of Harary [4]. The idea of graceful labeling was introduced by Rosa in [12]. Further results on the radio number of trees were explored in [1]. Janani and Ramachandran [5] have worked on relatively prime edge labeling of graphs. Sunitha and Sheriba [14] have been investigated the Gaussian tribonacci R-graceful labeling of the path, comb, coconut tree, regular caterpillar graph, Bistar graph and subdivision of Bistar graph. Zeen [15] proved that the existence of edge δ-graceful labeling for some cyclic related graphs like wheel graph, prism graph, double wheel graph, prism of the wheel graph, gear graph, closed helm, butterfly graph, alternate triangular cycle and friendship graph. The concept of cordial labeling was introduced by Cahit in [2]. Some new families of 3-equitable prime cordial graphs were discussed in [13]. Product cordial graph in the context of some graph operations on gear graph have been investigated in [7]. Prajapati et al. [8] have studied the SD-prime cordial labeling of K_4-snake and related graphs. For a dynamic survey on graph labeling, we follow the book of Gallian [3]. Also we have introduced a PMC-labeling in [9] and the PMC-labeling behavior of more graphs like web graph, jewel graph, sun flower graph, flower graph, tadpole graph, dumbbell graph, umbrella graph, butterfly graph, jelly fish, triangular book graph, quadrilateral book graph, triangular snake, alternate triangular snake, quadrilateral snake and alternate quadrilateral snake have been investigated in [9-11]. In this paper, we investigate the PMC-labeling behavior of some tree related graphs like the X-tree, Y-tree, prism of wheel graph, subdivision of bistar graph and coconut tree.

§2. **Preliminaries**

We present a few fundamental definitions that are essential for the upcoming section.

Definition 2.1([14]) *The coconut tree* $CT_{m,n}$ *is a graph obtained by connecting the center vertex of* $K_{1,n}$ *with a pendant vertex of the path* P_m.

Definition 2.2([14]) *The bistar graph* $B_{m,n}$ *is obtained from* K_2 *by attaching* m *pendant edges to one end of* K_2 *and* n *pendant edges to the other end of* K_2.

Definition 2.3([14]) *The subdivision of bistar graph* $S(B_{m,n})$ *is obtained by subdividing each edges of a bistar graph* $B_{m,n}$.

Definition 2.4([5]) *The Y-tree* Y_n *is a tree of three paths with exactly three vertices of degree one, one vertex of degree three and other vertices of degree two.*

Definition 2.5([5]) *The X-tree* X_n *is a tree of four paths with exactly four vertices of degree one, one vertex of degree four and other vertices of degree two.*

Definition 2.6([15]) *For* $n \geq 3$, *let* $\{u_0, u_1, u_2, \ldots, u_n\}$ *be the vertices of the wheel graph* W_n *with hub vertex* u_0 *and* $\{v_0, v_1, v_2, \ldots, v_n\}$ *be the vertices of* $W_n^{'}$ *a copy of the wheel graph* W_n *with hub vertex* v_0. *The prism of the wheel graph* W_n, PW_n *is obtained by joining* u_0 *of* W_n *to the corresponding vertex* v_0 *of* $W_n^{'}$ *and each* u_i *of* W_n *to the corresponding vertex* v_i *of* $W_n^{'}$ *for all* $i = 1, 2, \ldots, n$. *ie.,* $PW_n = K_2 \times W_n$.

§3. Main Results

Theorem 3.1 *The X-tree X_n is a PMC-graph for all n.*

Proof The vertex set and edge set of the X-tree X_n denoted by $V(X_n) = \{u_0, u_i, v_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(X_n) = \{u_0u_1, u_0v_1, u_0x_1, u_0y_1, u_iu_{i+1}, v_iv_{i+1}, x_ix_{i+1}, y_iy_{i+1} \mid 1 \leq i \leq n-1\}$ respectively. Then, X_n has $4n$ edges and $4n+1$ vertices. Let $\Lambda(u_0) = 2$, $\Lambda(x_1) = -1$, $\Lambda(x_3) = -n-1$ and $\Lambda(y_n) = 1$. We have consider the two cases.

Case 1. n is odd.

First, assign the labels $-1, -2, \cdots, \frac{-n-1}{2}$ and $3, 4, \cdots, \frac{n+3}{2}$ respectively according to the vertices $u_1, u_3, \ldots, u_n$ and $u_2, u_4, \cdots, u_{n-1}$. Then, assign the labels $\frac{n+5}{2}, \frac{n+7}{2}, \ldots, n+2$ and $\frac{-n-3}{2}, \frac{-n-5}{2}, \ldots, -n$ to the vertices $v_1, v_3, \cdots, v_n$ and $v_2, v_4, \ldots, v_{n-1}$ respectively. Also, assign the labels $-n-2, -n-3, \cdots, \frac{-3n-1}{2}$ and $n+3, n+4, \ldots, \frac{3n+1}{2}$ corresponding to the vertices $x_2, x_4, \cdots, x_{n-1}$ and $x_5, x_7, \ldots, x_n$. Assign the labels $\frac{-3n-3}{2}, \frac{-3n-5}{2}, \cdots, -2n$ and $\frac{3n+3}{2}, \frac{3n+5}{2}, \cdots, 2n$ to the vertices $y_1, y_3, \cdots, y_n$ and $y_2, y_4, \cdots, y_{n-1}$ respectively.

Case 2. n is odd.

Assign the labels $-1, -2, \cdots, \frac{-n}{2}$ and $3, 4, \cdots, \frac{n+4}{2}$ according to the vertices $u_1, u_3, \cdots, u_{n-1}$ and $u_2, u_4, \cdots, u_n$. Then, assign the labels $\frac{-n-2}{2}, \frac{-n-4}{2}, \cdots, -n$ and $\frac{n+6}{2}, \frac{n+8}{2}, \cdots, n+2$ to the vertices $v_1, v_3, \cdots, v_{n-1}$ and $v_2, v_4, \cdots, v_n$ respectively. Consequently, assign the labels $-n-2, -n-3, \cdots, \frac{-3n-2}{2}$ and $n+3, n+4, \cdots, \frac{3n}{2}$ corresponding to the vertices $x_2, x_4, \cdots, x_n$ and $x_5, x_7, \cdots, x_{n-1}$. Finally, assign the labels $\frac{3n+2}{2}, \frac{3n+4}{2}, \cdots, 2n$ and $\frac{-3n-4}{2}, \frac{-3n-6}{2}, \cdots, -2n$ to the vertices $y_1, y_3, \cdots, y_{n-1}$ and $y_2, y_4, \cdots, y_{n-2}$ respectively. In both cases, we have $\bar{\mathbb{S}}_{\Lambda_1} = 2n = \bar{\mathbb{S}}_{\Lambda_1^c}$ and the proof is complete. $\square$

Example 3.2 A PMC-labeling of the X-tree X_4 is given in Figure 1.

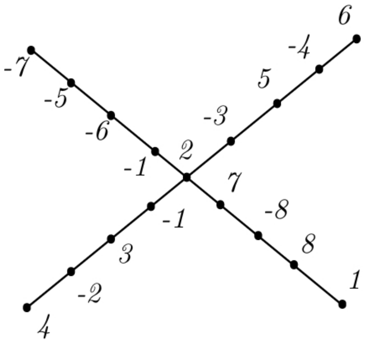

Figure 1

Theorem 3.3 *The Y-tree Y_n is a PMC-graph for all n.*

Proof The vertex set and edge set of the Y-tree Y_n are denoted by $V(Y_n) = \{u_0, u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(Y_n) = \{u_0u_1, u_0v_1, u_0x_1, u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1} \mid 1 \leq i \leq n-1\}$ respectively. Then, Y_n has $3n$ edges and $3n+1$ vertices. Let $\Lambda(u_0) = 2$ and $\Lambda(w_n) = 1$. We have consider the two cases.

Case 1. n is odd.

First, assign the labels to the vertices $u_i, 1 \leq i \leq n$ as in case (i) of Theorem 3.1. Then, assign the labels $\frac{-n-3}{2}, \frac{-n-5}{2}, \cdots, -n-1$ and $\frac{n+5}{2}, \frac{n+7}{2}, \cdots, n+1$ to the vertices $v_1, v_3, \cdots, v_n$ and $v_2, v_4, \cdots, v_{n-1}$ respectively. Subsequently, assign the labels $-n-2, -n-3, \cdots, \frac{-3n-1}{2}$ and $n+2, n+3, \cdots, \frac{3n+1}{2}$ corresponding to the vertices $w_1, w_3, \cdots, w_{n-2}$ and $w_2, w_4, \cdots, w_{n-1}$. Hence $\bar{\mathbb{S}}_{\Lambda_1} = \frac{3n-1}{2}$ and $\bar{\mathbb{S}}_{\Lambda_1^c} = \frac{3n+1}{2}$.

Case 2. n is even.

Next, assign the labels to the vertices $u_i, 1 \leq i \leq n$ as in Case 1 of Theorem 3.1. So, assign the labels $\frac{-n-2}{2}, \frac{-n-4}{2}, \cdots, -n$ and $\frac{n+4}{2}, \frac{n+6}{2}, \cdots, n+1$ to the vertices $v_1, v_3, \cdots, v_{n-1}$ and $v_2, v_4, \cdots, v_n$ respectively. Label the vertex w_1 by $-n-1$. Consequently, assign the labels $-n-2, -n-3, \cdots, \frac{-3n}{2}$ and $n+2, n+3, \cdots, \frac{3n}{2}$ corresponding to the vertices $w_2, w_4, \cdots, w_{n-2}$ and $w_3, w_5, \cdots, w_{n-1}$. Thus, $\bar{\mathbb{S}}_{\Lambda_1} = \frac{3n}{2} = \bar{\mathbb{S}}_{\Lambda_1^c}$. □

Example 3.4 A PMC-labeling of the Y-tree Y_5 is given in Figure 2.

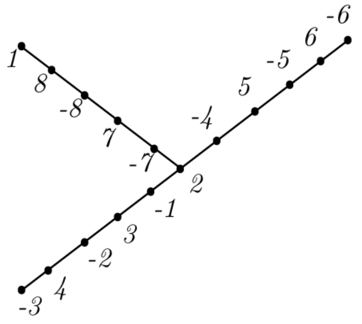

Figure 2

Theorem 3.5 *The prism of wheel graph PW_n is not PMC-graph for all $n \geq 3$.*

Proof Let us consider the prism of wheel graph PW_n, $n \geq 3$. Let $V(PW_n) = \{u_0, v_0, u_i, v_i \mid 1 \leq i \leq n\}$ and $E(PW_n) = \{u_0v_0, u_0u_i, u_0v_i, u_iv_i \mid 1 \leq i \leq n\} \cup \{u_iu_{i+1}, u_nu_1, v_iv_{i+1}, v_nv_1 \mid 1 \leq i \leq n-1\}$ denote, respectively, the vertex set and edge set of the prism of wheel graph PW_n. Then, PW_n has $5n+1$ edges and $2n+2$ vertices. Suppose that the prism of wheel graph PW_n is a PMC-graph. We have the maximum possible number of edges designated with a label 1 is $2n-1$. Consequently, the minimum number of edges that are not designated with a label 1 is $3n+2$. Therefore, $\bar{\mathbb{S}}_{\Lambda_1^c} - \bar{\mathbb{S}}_{\Lambda_1} \geq n+3 \geq 6 > 1$, a contradiction arises. □

Theorem 3.6 *The subdivision $S(B_{m,n})$ of bistar graph $B_{m,n}$ is a PMC-graph for all m and n.*

Proof Let $V(S(B_{m,n})) = \{u_0, u_i, x_i, x_0, v_0, v_j, y_j \mid 1 \leq i \leq m \,\text{and}\, 1 \leq j \leq n\}$ and $E(S(B_{m,n})) = \{u_0x_i, x_iu_i, u_0x_0, x_0v_0, v_0y_j, y_jv_j \mid 1 \leq i \leq m \,\text{and}\, 1 \leq j \leq n\}$ denote, respectively, the vertex set and edge set of the subdivision of bistar graph $S(B_{m,n})$. Then, $S(B_{m,n})$ has

$2m+2n+2$ edges and $2m+2n+3$ vertices. Let $\Lambda(u_0)=2, \Lambda(v_0)=-m-n-1$ and $\Lambda(x_0)=1$. Next we assign the labels $2,3,\ldots,m+1$ and $-1,-2,\cdots,-m$ to the vertices $u_1,u_2,\cdots,u_m$ and $x_1,x_2,\cdots,x_m$ respectively. Consequently, assign the labels $-m-1,-m-2,\cdots,-m-n$ and $m+2,m+3,\cdots,m+n+1$ corresponding to the vertices $v_1,v_2,\cdots,v_n$ and $y_1,y_2,\cdots,y_n$. Thus, $\bar{\mathbb{S}}_{\Lambda_1}=m+n+1=\bar{\mathbb{S}}_{\Lambda_1^c}$. □

Example 3.7 A PMC-labeling of the subdivision of bistar graph $S(B_{3,4})$ is given in Figure 3.

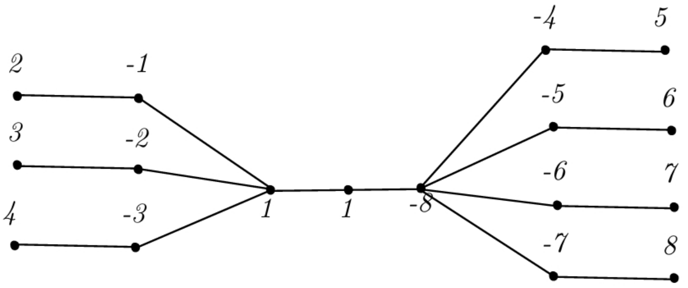

Figure 3

Theorem 3.8 *The coconut tree $CT(m,n)$ is a PMC-graph for every m,n with $|m-n|\le 3$.*

Proof Let $V(CT(m,n))=\{u_i,v_j \mid 1\le i\le m, 1\le j\le n\}$ and $E(CT(m,n))=\{u_iu_{i+1},u_nv_j : 1\le i\le m-1, 1\le j\le n\}$. Clearly, the coconut tree $CT(m,n)$ has $m+n-1$ edges and $m+n$ vertices.

Case 1. $|m-n|=0$.

Then, $m=n$. We have to show that $CT(m,n)$ is a PMC-graph.

Subcase 1.1 m is odd.

Let us assign the labels $2,3,\cdots,\frac{m+3}{2}$ and $-1,-2,\cdots,\frac{-m+1}{2}$ according to the vertices $u_1,u_3,\cdots,u_m$ and $u_2,u_4,\cdots,u_{m-1}$. Next, assign labels $\frac{-m-1}{2},\frac{-m-3}{2},\cdots,-m$ and $\frac{m+5}{2},\frac{m+7}{2},\cdots,m$ corresponding to the vertices $v_1,v_2,\cdots,v_{\frac{m+1}{2}}$ and $v_{\frac{m+3}{2}},v_{\frac{m+5}{2}},\cdots,v_{m-1}$. Label the vertex v_m by 1.

Subcase 1.2 m is even.

If $m=2$, define $\Lambda(u_1)=2$, $\Lambda(u_2)=-1$, $\Lambda(v_1)=-2$ and $\Lambda(v_2)=1$. Therefore, $\bar{\mathbb{S}}_{\Lambda_1}=1$ and $\bar{\mathbb{S}}_{\Lambda_1^c}=2$. If $m>2$, we assign the labels $2,3,\cdots,\frac{m+2}{2}$ and $-1,-2,\cdots,\frac{-m}{2}$ to the vertices $u_1,u_3,\cdots,u_{m-1}$ and $u_2,u_4,\cdots,u_m$ respectively. Further, assign the labels $\frac{-m-2}{2},\frac{-m-4}{2},\cdots,-m$ and $\frac{m+4}{2},\frac{m+6}{2},\cdots,m$ corresponding to the vertices $v_1,v_2,\cdots,v_{\frac{m}{2}}$ and $v_{\frac{m+2}{2}},v_{\frac{m+4}{2}},\cdots,v_{m-1}$. Label the vertex v_m by 1. In each cases, $\bar{\mathbb{S}}_{\Lambda_1}=m$ and $\bar{\mathbb{S}}_{\Lambda_1^c}=m-1$.

Case 2. $|m-n|=1$.

Then, $m-n=1$ or $m-n=-1$.

Subcase 2.1 $m-n=1$.

Then we have to prove that $CT(m, m-1)$ is a PMC-graph. Define $\Lambda(u_1) = 1$. If m is odd, we assign the labels $2, 3, \cdots, \frac{m+1}{2}$ and $-1, -2, \cdots, \frac{-m+1}{2}$ to the vertices $u_2, u_4, \cdots, u_{m-1}$ and $u_3, u_5, \cdots, u_m$ respectively. Now, assign the labels $\frac{-m-1}{2}, \frac{-m-3}{2}, \cdots, -m+1$ and $\frac{m+3}{2}, \frac{m+5}{2}, \cdots, m-1$ corresponding to the vertices $v_1, v_2, \cdots, v_{\frac{m-1}{2}}$ and $v_{\frac{m+1}{2}}, v_{\frac{m+3}{2}}, \cdots, v_{m-2}$. Then label the vertex v_{m-1} by 2. If m is even, let us assign the labels $2, 3, \cdots, \frac{m+2}{2}$ and $-1, -2, \cdots, \frac{-m+2}{2}$ to the vertices $u_2, u_4, \cdots, u_m$ and $u_3, u_5, \cdots, u_{m-1}$ respectively. Also, we assign the labels $\frac{-m}{2}, \frac{-m-2}{2}, \cdots, -m+1$ and $\frac{m+4}{2}, \frac{m+6}{2}, \cdots, m-1$ corresponding to the vertices $v_1, v_2, \cdots, v_{\frac{m}{2}}$ and $v_{\frac{m+2}{2}}, v_{\frac{m+4}{2}}, \cdots, v_{m-2}$. Finally label the vertex v_{m-1} by 2. Hence $\bar{\mathbb{S}}_{\Lambda_1} = m-1$ and $\bar{\mathbb{S}}_{\Lambda_1^c} = m-1$.

Subcase 2.2 $m - n = -1$.

We have to show that $CT(m, m+1)$ is a PMC-graph. If m is odd, assign the labels to the vertices u_i, $1 \leq i \leq m$ as in Subcase 2.1 of Case 2. Next, we assign the labels $\frac{-m-1}{2}, \frac{-m-3}{2}, \cdots, -m$ and $\frac{m+3}{2}, \frac{m+5}{2}, \cdots, m$ to the vertices $v_1, v_2, \cdots, v_{\frac{m+1}{2}}$ and $v_{\frac{m+3}{2}}, v_{\frac{m+5}{2}}, \cdots, v_m$ respectively. Then, label the vertex v_{m+1} by $\frac{m+1}{2}$. If m is even, then assign the labels to the vertices u_i, $1 \leq i \leq m$ as in Subcase 2.1 of Case 2. Also we assign the labels $\frac{-m}{2}, \frac{-m-2}{2}, \cdots, -m$ and $\frac{m+4}{2}, \frac{m+6}{2}, \cdots, m$ to the vertices $v_1, v_2, \cdots, v_{\frac{m+2}{2}}$ and $v_{\frac{m+4}{2}}, v_{\frac{m+6}{2}}, \cdots, v_m$ correspondingly. Finally, label the vertex v_{m+1} by $\frac{-m}{2}$. Consequently, $\bar{\mathbb{S}}_{\Lambda_1} = m = \bar{\mathbb{S}}_{\Lambda_1^c}$.

Case 3. $|m - n| = 2$.

Then, $m - n = 2$ or $m - n = -2$.

Subcase 3.1 $m - n = 2$.

Now we have to prove that $CT(m, m-2)$ is a PMC-graph. Assign the labels to the vertices u_i, v_j, $1 \leq i \leq m, 1 \leq j \leq m-2$ as in subcase (i) of case (ii). Hence $\bar{\mathbb{S}}_{\Lambda_1} = m-1$ and $\bar{\mathbb{S}}_{\Lambda_1^c} = m-2$.

Subcase 3.2 $m - n = -2$.

We have to show that $CT(m, m+2)$ is a PMC-graph. If m is odd, assign the labels $2, 3, \cdots, \frac{m+3}{2}$ and $-1, -2, \cdots, \frac{-m+1}{2}$ according to the vertices $u_1, u_3, \cdots, u_m$ and $u_2, u_4, \cdots, u_{m-1}$. Next, assign the labels $\frac{-m-1}{2}, \frac{-m-3}{2}, \cdots, -m-1$ and $\frac{m+5}{2}, \frac{m+7}{2}, \cdots, m+1$ to the vertices $v_1, v_2, \cdots, v_{\frac{m+3}{2}}$ and $v_{\frac{m+5}{2}}, v_{\frac{m+7}{2}}, \ldots, v_{m+1}$ respectively. Then label the vertex v_{m+2} by 1. If m is even, we assign the labels $2, 3, \cdots, \frac{m+2}{2}$ and $-1, -2, \cdots, \frac{-m}{2}$ corresponding to vertices $u_1, u_3, \cdots, u_{m-1}$ and $u_2, u_4, \cdots, u_m$. Moreover, we assign the labels $\frac{-m-2}{2}, \frac{-m-4}{2}, \cdots, -m-1$ and $\frac{m+4}{2}, \frac{m+6}{2}, \cdots, m+1$ according to the vertices $v_1, v_2, \cdots, v_{\frac{m+2}{2}}$ and $v_{\frac{m+4}{2}}, v_{\frac{m+6}{2}}, \cdots, v_{m+1}$. Finally label the vertex v_{m+2} by 1. Therefore, $\bar{\mathbb{S}}_{\Lambda_1} = m$ and $\bar{\mathbb{S}}_{\Lambda_1^c} = m+1$.

Case 4. $|m - n| = 3$.

Then, $m - n = 3$ or $m - n = -3$.

Subcase 4.1 $m - n = 3$.

We have to show that $CT(m, m+3)$ is a PMC-graph. Define $\Lambda(u_1) = -m+2$ and $\Lambda(u_2) = 1$. If m is odd, we assign the labels $2, 3, \ldots, \frac{m+1}{2}$ and $-1, -2, \cdots, \frac{-m+3}{2}$ according to the vertices $u_3, u_5, \cdots, u_m$ and $u_4, u_6, \cdots, u_{m-1}$. Next, we assign the labels $\frac{-m+1}{2}, \frac{-m-1}{2}, \cdots, -m+3$

and $v_{\frac{m-1}{2}}, v_{\frac{m+1}{2}}, \cdots, v_{m-4}$ to the vertices $v_1, v_2, \ldots, v_{\frac{m-3}{2}}$ and $\frac{m+3}{2}, \frac{m+5}{2}, \cdots, m-2$ respectively. Then label the vertex v_{m-3} by 1. If m is even, then assign the labels $2, 3, \ldots, \frac{m}{2}$ and $-1, -2, \cdots, \frac{-m+2}{2}$ corresponding to the vertices $u_3, u_5, \cdots, u_{m-1}$ and $u_4, u_6, \cdots, u_m$. Further, we assign the labels $\frac{-m}{2}, \frac{-m-2}{2}, \cdots, -m+3$ and $\frac{m+2}{2}, \frac{m+4}{2}, \cdots, m-3$ according to the vertices $v_1, v_2, \cdots, v_{\frac{m-4}{2}}$ and $v_{\frac{m-2}{2}}, v_{\frac{m}{2}}, \cdots, v_{m-4}$. Finally, label the vertex v_{m-3} by 1. Subsequently, $\bar{\mathbb{S}}_{\Lambda_1} = \bar{\mathbb{S}}_{\Lambda_1^c} = m-2$.

Subcase 4.2 $m-n=-3$.

We have to show that $CT(m, m+3)$ is a PMC-graph. Now assign the labels to the vertices u_i, v_j, $1 \leq i \leq m$, $1 \leq j \leq m+2$ as in Subcase 3.1 of Case 3. If m is odd, then label the vertex v_{m+3} by $\frac{-m-1}{2}$. If m is even, label the vertex v_{m+3} by $\frac{m+2}{2}$. Hence $\bar{\mathbb{S}}_{\Lambda_1} = \bar{\mathbb{S}}_{\Lambda_1^c} = m+1$. □

Theorem 3.9 *The coconut tree $CT(m,n)$ is not PMC-graph for every m, n with $n-m \geq 4$.*

Proof If possible, let $CT(m,n)$ is a PMC-graph. If the edge uv receives the label 1, the possible results are either $\Lambda(u)+\Lambda(v)=1$ or $\Lambda(u)+\Lambda(v)=2$.

Case 1. $n-m$ is odd.

Then, the maximum possible number of edges designated with a label 1 is $m+1$. Subsequently, the minimum number of edges that are not designated with a label 1 is $q-(m+1) = n-2$. Therefore, $\bar{\mathbb{S}}_{\Lambda_1^c} - \bar{\mathbb{S}}_{\Lambda_1} \geq n-2-(m+1) = n-m-3 \geq 2 > 1$, we get a contradiction.

Case 2. $n-m$ is even.

Then, the maximum possible number of edges designated with a label 1 is m. Consequently, the minimum number of edges that are not designated with a label 1 is $q-m=n-1$. Thus, $\bar{\mathbb{S}}_{\Lambda_1^c} - \bar{\mathbb{S}}_{\Lambda_1} \geq n-1-m = n-m-1 \geq 3 > 1$, a contradiction arises. □

Theorem 3.10 *The coconut tree $CT(m,n)$ is a PMC-graph for every m, n with $m-n \geq 4$.*

Proof Clearly, the coconut tree $CT(m,n)$ has $m+n-1$ edges and $m+n$ vertices.

Case 1. $m \equiv 0 \pmod 4$.

Subcase 1.1 $n \equiv 0 \pmod 4$.

Let us assign the labels $2, 3, \cdots, \frac{m+n+4}{4}$ and $-1, -2, \cdots, \frac{-m-n}{4}$ according to the vertices $u_1, u_3, \cdots, u_{\frac{m+n-2}{2}}$ and $u_2, u_4, \cdots, u_{\frac{m+n}{2}}$. Label the vertex $u_{\frac{m+n+2}{2}}$ by $\frac{-m-n-4}{4}$. Also we assign the labels $\frac{-m-n-8}{4}, \frac{m+n+8}{4}$ and $\frac{-m-n-12}{4}, \frac{m+n+12}{4}$ to the vertices $u_{\frac{m+n+4}{2}}, u_{\frac{m+n+6}{2}}$ and $u_{\frac{m+n+8}{2}}, u_{\frac{m+n+10}{2}}$ respectively. This process should be repeated until the label 1 is assigned to u_m. Subsequently, assign the labels $\frac{-m-2}{2}, \frac{m+2}{2}$ and $\frac{-m-4}{2}, \frac{m+4}{2}$ corresponding to the vertices v_1, v_2 and v_3, v_4. This process should be repeated until the labels $\frac{-m-n}{2}, \frac{m+n}{2}$ are assigned to v_{n-1}, v_n.

Subcase 1.2 $n \equiv 1 \pmod 4$.

Assign the labels $2, 3, \cdots, \frac{m+n+7}{4}$ and $-1, -2, \cdots, \frac{-m-n+1}{4}$ to vertices $u_1, u_3, \cdots, u_{\frac{m+n+1}{2}}$ and $u_2, u_4, \cdots, u_{\frac{m+n-1}{2}}$ respectively. So assign the labels $\frac{-m-n-7}{4}, \frac{-m-n-3}{4}$ according to the vertices $u_{\frac{m+n+3}{2}}, u_{\frac{m+n+5}{2}}$. Assign the labels $\frac{-m-n-11}{4}, \frac{m+n+11}{4}$ and $\frac{-m-n-15}{4}, \frac{m+n+15}{4}$ to the vertices $u_{\frac{m+n+7}{2}}, u_{\frac{m+n+9}{2}}$ and $u_{\frac{m+n+11}{2}}, u_{\frac{m+n+13}{2}}$ respectively. This process should be repeated until the label 1 is assigned to u_m. Subsequently, assign the labels $\frac{-m-2}{2}, \frac{m+2}{2}$ and $\frac{-m-4}{2}, \frac{m+4}{2}$ according to the vertices v_1, v_2 and v_3, v_4 respectively. This process should be repeated until the labels $\frac{-m-n+1}{2}, \frac{m+n-1}{2}$ are assigned to v_{n-2}, v_{n-1}. Finally, label the vertex v_n by $\frac{-m-n+1}{2}$.

Subcase 1.3 $n \equiv 2 \pmod 4$.

Assign the labels $2, 3, \cdots, \frac{m+n+6}{4}$ and $-1, -2, \cdots, \frac{-m-n+2}{4}$ to the vertices $u_1, u_3, \cdots, u_{\frac{m+n}{2}}$ and $u_2, u_4, \cdots, u_{\frac{m+n-2}{2}}$ respectively. Then, assign the labels $\frac{-m-n-6}{4}, \frac{-m-n-2}{4}$ according to the vertices $u_{\frac{m+n+2}{2}}, u_{\frac{m+n+4}{2}}$. Next, assign the labels $\frac{-m-n-10}{4}, \frac{m+n+10}{4}$ and $\frac{-m-n-14}{4}, \frac{m+n+14}{4}$ to the vertices $u_{\frac{m+n+6}{2}}, u_{\frac{m+n+8}{2}}$ and $u_{\frac{m+n+10}{2}}, u_{\frac{m+n+12}{2}}$ respectively. This process should be repeated until the label 1 is assigned to u_m. Subsequently, assign the labels to the vertices v_j, $1 \leq j \leq n$ as in Subcase 1.1 of Case 1.

Subcase 1.4 $n \equiv 3 \pmod 4$.

Also assign the labels $2, 3, \cdots, \frac{m+n+5}{4}$ and $-1, -2, \cdots, \frac{-m-n-1}{4}$ according to the vertices $u_1, u_3, \cdots, u_{\frac{m+n-1}{2}}$ and $u_2, u_4, \cdots, u_{\frac{m+n+1}{2}}$. Label the vertex $u_{\frac{m+n+3}{2}}$ by $\frac{-m-n-5}{4}$. Next, assign the labels $\frac{-m-n-9}{4}, \frac{m+n+9}{4}$ and $\frac{-m-n-11}{4}, \frac{m+n+11}{4}$ according to the vertices $u_{\frac{m+n+5}{2}}, u_{\frac{m+n+7}{2}}$ and $u_{\frac{m+n+9}{2}}, u_{\frac{m+n+11}{2}}$. This process should be repeated until the label 1 is assigned to u_m. Consequently, assign the labels to the vertices v_j, $1 \leq j \leq n$ as in Subcase 1.2 of Case 1.

Case 2 $m \equiv 1 \pmod 4$.

Subcase 2.1 $n \equiv 0 \pmod 4$.

Now, assign the labels to the vertices u_i, $1 \leq i \leq m-2$ as in Subcase 1.2 of Case 1. Then, assign the labels $\frac{-m-1}{2}, 1$ according to the vertices u_{m-1}, u_m. Label the vertex v_1 by $\frac{m+1}{2}$. Assign the labels $\frac{-m-3}{2}, \frac{m+3}{2}$ and $\frac{-m-5}{2}, \frac{m+5}{2}$ to the vertices v_2, v_3 and v_4, v_5 respectively. This process should be repeated until the labels $\frac{-m-n+1}{2}, \frac{m+n-1}{2}$ are assigned to v_{n-2}, v_{n-1}. Finally, label the vertex v_n by $\frac{-m-n+1}{2}$.

Subcase 2.2 $n \equiv 1 \pmod 4$.

In this case, assign the labels to the vertices u_i, $1 \leq i \leq m-2$ as in Subcase 1.3 of Case 1. Then, assign the labels $\frac{-m-1}{2}, 1$ according to the vertices u_{m-1}, u_m. Label the vertex v_1 by $\frac{m+1}{2}$. Also, assign the labels $\frac{-m-3}{2}, \frac{m+3}{2}$ and $\frac{-m-5}{2}, \frac{m+5}{2}$ corresponding to the vertices v_2, v_3 and v_4, v_5. This process should be repeated until the labels $\frac{-m-n}{2}, \frac{m+n}{2}$ are assigned to v_{n-1}, v_n.

Subcase 2.3 $n \equiv 2 \pmod 4$.

Assign the labels to the vertices u_i, $1 \leq i \leq m-2$ as in Subcase 1.4 of Case 1. Next, assign the labels $\frac{-m-1}{2}, 1$ according to the vertices u_{m-1}, u_m. Consequently, assign the labels to the vertices v_j, $1 \leq j \leq n$ as in Subcase 2.1 of Case 2.

Subcase 2.4 $n \equiv 3 \pmod 4$.

Subsequently, assign the labels to the vertices u_i, $1 \leq i \leq m-2$ as in Subcase 1.1 of Case

1. Also, assign the labels $\frac{-m-1}{2}, 1$ corresponding to the vertices u_{m-1}, u_m. Assign the labels to the vertices v_j, $1 \leq j \leq n$ as in Subcase 2.2 of Case 2.

Case 3. $m \equiv 2 \pmod 4$.

Subcase 3.1 $n \equiv 0 \pmod 4$.

This proof is consistent with Subcase 1.3 of Case 1.

Subcase 3.2 $n \equiv 1 \pmod 4$.

This proof is consistent with Subcase 1.4 of Case 1.

Subcase 3.3 $n \equiv 2 \pmod 4$.

This proof is consistent with Subcase 1.1 of Case 1.

Subcase 3.4 $n \equiv 3 \pmod 4$.

This proof is consistent with Subcase 1.2 of Case 1.

Case 4. $m \equiv 3 \pmod 4$.

Subcase 4.1 $n \equiv 0 \pmod 4$.

This proof is consistent with Subcase 2.3 of Case 2.

Subcase 4.2 $n \equiv 1 \pmod 4$.

This proof is consistent with Subcase 2.4 of Case 2.

Subcase 4.3 : $n \equiv 2 \pmod 4$.

This proof is consistent with Subcase 2.1 of Case 2.

Subcase 4.4 $n \equiv 3 \pmod 4$.

This proof is consistent with Subcase 2.2 of Case 2. □

Example 3.11 A PMC-labeling of the coconut tree $CT(8, 4)$ is given in Figure 4.

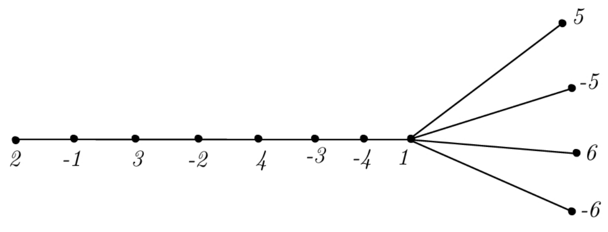

Figure 4

§4. Conclusion

The PMC-labeling behavior of some tree related graphs like the X-tree, Y-tree, prism of wheel graph, subdivision of bistar graph and coconut tree have been investigated in this paper. It is still available to future work to establish the PMC-labeling for more graph families.

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