International J.Math. Combin. Vol.1(2026), 132-143

PMC-Labeling of Subdivision of Path and Cycle Related Graphs

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Abstract: The graph $G = (V, E)$ consists of p vertices and q edges. Let

$$\rho = \begin{cases} \frac{p}{2}, & p \text{ is even} \\ \frac{p-1}{2}, & p \text{ is odd,} \end{cases}$$

and $\Gamma = \{\pm 1, \pm 2, \ldots, \pm \rho\}$. Consider a function $\lambda : V \to \Gamma$ that allocates unique labels from Γ to the various vertices of V when p is even and allocates a unique labels in Γ to $p-1$ vertices of V, repeating a label for the remaining one vertex when p is odd. Then the labeling as mentioned above is called a pair mean cordial labeling (PMC-labeling) if for every edge uv of G, there is a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ are denoted the number of edges labelled with 1 and the number of edges not labelled with 1, respectively. A graph G that has a pair mean cordial labeling is called a pair mean cordial graph (PMC-Graph). In this paper we prove that the subdivision of path, cycle, wheel, crown, helm, fan graph, friendship graph, coconut tree, double comb graph, jellyfish graph, flower graph, sunflower graph, gear graph and jewel graph are PMC-labeling.

Key Words: PMC-labeling, Smarandachely PMC-graph, path, cycle, wheel, helm and crown.

AMS(2010): 05C38, 05C78.

§1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. The cardinality of the vertex set is called the order of G, denoted by p. The cardinality of the edge set is called the size of G, denoted by q. A graph with p vertices and q edges is called a (p, q) graph. Terms not defined here are used in the sense of Harary [8]. The first half of the 18th century saw the introduction of graph Theory, following the solution of the Konigsberg Bridge problem in 1736. Since then, graph theory has emerged as a powerful tool in the field of mathematical research

[1]Received August 26, 2025, Accepted February 28, 2026

for its ability to represent any physical problem involving the arrangement of objects. A graph is a collection of vertices and edges between them. Ever since it broke into the mainstream of mathematical research, graph theory has found application in diverse fields ranging from biochemistry, architecture, psychology, economics, linguistics, sociology, electrical engineering and computer science to operations research. Graph labeling is a strong relation between number theory and graph structures. It is the potential area of research in graph theory. If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling. The concept of graph labeling is a frontier between number theory and structure of graphs. A systematic study of various applications of graph labeling is carried out in Bloom and Golomb [1]. It is interesting to note that the labelled graphs serve as useful models for a minor road range of applications. Labeled graphs are used in numerous areas like coding theory, X-ray crystallography, the design of good radar type codes, astronomy, circuit design, communication network addressing, data base management. A detailed study on graph labelling is reported in [7]. The concept of cordial labeling was introduced by Cahit [6]. The study of Zumkeller numbers [10] is a part of number theory which is one of the important branches of mathematics. The concept of k-Zumkeller labeling of graphs has been introduced and investigated in the literature [2,3,4,5,9]. The idea of PMC-labeling of some simple graphs has been previously reported in [11,12,13,14]. In this paper we investigate the PMC-labeling of the subdivision of path, cycle, wheel, crown, helm, fan graph, friendship graph, coconut tree, double comb graph, jellyfish graph, flower graph, sunflower graph, gear graph and jewel graph.

§2. Preliminaries

In this section, we present a few fundamental definitions that are essential for the upcoming section.

Definition 2.1 *A wheel graph W_n is the graph $K_1 + C_n$.*

Definition 2.2 *A crown $C_n \odot K_1$ is obtained by joining a pendent edge to each vertex of the cycle C_n.*

Definition 2.3 *A helm graph H_n is a graph obtained from a wheel by attaching a pendant vertex at each vertex of the cycle C_n.*

Definition 2.4 *A fan f_n, $n \geq 2$ is obtained by joining all vertices of P_n to a further vertex called the center.*

Definition 2.5 *A friendship graph F_n is a graph which consists of n triangles with a common vertex.*

Definition 2.6 *A coconut tree $CT_{m,n}$ is a graph obtained by connecting the center vertex of $K_{1,n}$ with a pendant vertex of the path P_m.*

Definition 2.7 *A double comb graph $P_n \odot 2K_1$ is obtained by joining the two pendant edge to* *each vertices of the path P_n.*

Definition 2.8 *A jelly fish graphs $J(m,n)$ obtained from a cycle $C_4 : u_1u_2u_3u_4u_1$ by joining u_1 and u_3 with an edge and appending m pendent edges to u_2 and n pendent edges to u_4.*

Definition 2.9 *A flower graph Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm.*

Definition 2.10 *A sunflower graph S_n is obtained by taking a wheel with central vertex u and the cycle $C_n : u_1u_2 \ldots u_nu_1$ and new vertices $v_1v_2 \ldots v_n$ where v_i is joined by vertices $u_i, u_{i+1\,(mod\,n)}$.*

Definition 2.11 *A gear graph G_n is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the cycle C_n.*

Definition 2.12 *A jewel graph J_n is the graph with vertex set $V(J_n) = \{u, v, x, y, u_i \mid 1 \le i \le n\}$ and edge set $E(J_n) = \{ux, uy, xy, xv, yv, uu_i, vu_i \mid 1 \le i \le n\}$.*

Definition 2.13 *A subdivision graph $S(G)$ of a graph G is obtained from G by inserting a new vertex of degree 2 on each edge of G.*

Definition 2.14 *Let $G = (V, E)$ be a graph consisting of p vertices, q edges,*

$$\rho = \begin{cases} \frac{p}{2}, & p \text{ is even} \\ \frac{p-1}{2}, & p \text{ is odd,} \end{cases}$$

and $\Gamma = \{\pm 1, \pm 2, \cdots, \pm\rho\}$. Consider a function $\lambda : V \to \Gamma$ that allocates unique labels from Γ to the various vertices of V when p is even and allocates a unique labels in Γ to $p-1$ vertices of V, repeating a label for the remaining one vertex when p is odd. Then the labeling as mentioned above is called a pair mean cordial labeling (PMC-labeling) if for every edge uv of G, there is a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \le 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ are denoted the number of edges labelled with 1 and the number of edges not labelled with 1, respectively. A graph G that has a pair mean cordial labeling is called a pair mean cordial graph (PMC-graph).

Otherwise, if there are $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \le 1$ for a graph G, it is called a Smarandachely PMC-graph.

An example of PMC-graph is shown in Figure 1.

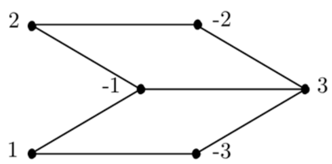

Figure 1. An example of PMC-graph

§3. Main Results

In this section, we investigate the PMC-labeling behavior of the subdivision of path, cycle, wheel, crown, helm, fan graph, friendship graph, coconut tree, double comb graph, jellyfish graph, flower graph, sunflower graph, gear graph and jewel graph.

Theorem 3.1 *The subdivision of path P_n, $S(P_n)$ is a PMC-graph for all $n \geq 1$.*

Proof We have $S(P_n) \simeq P_{2n-1}$ and P_{2n-1} is a PMC-graph [11]. □

Theorem 3.2 *The subdivision of cycle C_n, $S(C_n)$ is a PMC-graph for all $n \geq 3$.*

Proof We obtain $S(C_n) \simeq C_{2n}$ and C_{2n} is a PMC-graph [11]. □

Theorem 3.3 *The subdivision of wheel graph W_n, $S(W_n)$ is a PMC-graph for all $n \geq 3$.*

Proof Consider the subdivision of wheel graph $S(W_n)$, $n \geq 3$. Denote by $V(S(W_n)) = \{v_0, v_i, u_i, w_i \mid 1 \leq i \leq n\}$ and $E(S(W_n)) = \{v_0u_i, u_iv_i, v_iw_i \mid 1 \leq i \leq n\} \cup \{w_iv_{i+1}, w_nv_1 \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(W_n)$ respectively. Then the order and size of $S(W_n)$ respectively are $3n+1$ and $4n$. The theorem is established by discussing two cases.

Case 1. n is odd.

Define the injective function $\lambda : V(S(W_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n+1}{2}\}$. Take $\lambda(v_0) = n+2$. Then assign the labels $-1, -2, \cdots, -n$ and $2, 3, \cdots, n+1$ respectively to the vertices $w_1, w_2, \cdots, w_n$ and $v_1, v_2, \cdots, v_n$. Further assign the labels $-n-1, -n-2, \cdots, \frac{-3n-1}{2}$ and $n+3, n+4, \cdots, \frac{3n+1}{2}$ to the vertices $u_1, u_2, \cdots, u_{\frac{n+1}{2}}$ and $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \cdots, u_{n-1}$ respectively. Finally assign the label 1 to the vertex u_n.

Case 2. n is even.

Define the injective function $\lambda : V(S(W_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n}{2}\}$. Apply the Case 1 labeling to the vertices $v_0, v_i, w_i, 1 \leq i \leq n$. Next assign the labels $-n-1, -n-2, \cdots, \frac{-3n}{2}$ and $n+3, n+4, \cdots, \frac{3n}{2}$ to the vertices $u_1, u_2, \cdots, u_{\frac{n}{2}}$ and $u_{\frac{n+2}{2}}, u_{\frac{n+4}{2}}, \cdots, u_{n-2}$ respectively. Finally assign the labels $1, 1$ to the vertices u_{n-1}, u_n respectively. Hence the labeling in both cases results in $2n$ edges assigned the label 1 and $2n$ edges without the label 1. Therefore vertex labeling λ is a PMC-labeling of the subdivision of wheel graph $S(W_n)$. □

Example 3.4 A PMC-labeling of the subdivision of wheel graph $S(W_4)$ is shown in Figure 2.

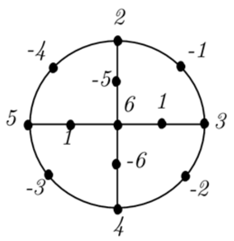

Figure 2. A PMC-labeling of the subdivision of wheel graph $S(W_4)$

Theorem 3.5 *The subdivision of crown graph $C_n \odot K_1$, $S(C_n \odot K_1)$ is a PMC-graph for all $n \geq 3$.*

Proof Let us consider the subdivision of crown graph $S(C_n \odot K_1)$, $n \geq 3$. Denote by $V(S(C_n \odot K_1)) = \{u_i, v_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(S(C_n \odot K_1)) = \{u_iv_i, u_iy_i, y_iw_i \mid 1 \leq i \leq n\} \cup \{v_iu_{i+1}, v_nu_1 \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(C_n \odot K_1)$ respectively. Then the order and size of $S(C_n \odot K_1)$ respectively are $4n$ and $4n$. Define the injective function $\lambda : V(S(C_n \odot K_1)) \rightarrow \{\pm 1, \pm 2, \cdots, \pm 2n\}$. First assign the labels $-1, -2, \cdots, -n$ and $2, 3, \cdots, n+1$ respectively to the vertices $y_1, y_2, \cdots, y_n$ and $x_1, x_2, \cdots, x_n$. Further assign the labels $-n-1, -n-2$ and $n+4, n+5, \cdots, 2n$ to the vertices u_1, u_2 and $u_3, u_4, \cdots, u_{n-1}$ respectively. Fix the label $n+2$ to the vertex u_n. Next assign the labels $n+3$ and $-n-3, -n-4, \cdots, -2n$ to the vertices v_1 and $v_2, v_3, \cdots, v_{n-1}$ respectively. Finally assign the label 1 to the vertex v_n. From the above labeling technique, we obtain $2n$ edges labelled with 1 and $2n$ edges not labelled with 1. Therefore, the vertex labeling λ is a PMC-labeling of the subdivision of crown graph $S(C_n \odot K_1)$. □

Theorem 3.6 *The subdivision of helm graph H_n, $S(H_n)$ is a PMC-graph for all $n \geq 3$.*

Proof Consider the subdivision of helm graph $S(H_n)$, $n \geq 3$. Denote by $V(S(H_n)) = \{v_0, v_i, u_i, w_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(S(H_n)) = \{v_0u_i, u_iv_i, v_iy_i, y_ix_i, v_iw_i \mid 1 \leq i \leq n\} \cup \{w_iv_{i+1}, w_nv_1 \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(H_n)$ respectively. Then the order and size of $S(H_n)$ respectively are $5n+1$ and $6n$. This theorem is analyzed by considering two cases.

Case 1. n is odd.

Define the injective function $\lambda : V(S(H_n)) \rightarrow \{\pm 1, \pm 2, \cdots, \pm \frac{5n+1}{2}\}$. Take $\lambda(v_0) = 2n+2$. Then assign the labels $-1, -2, \cdots, -n$ and $2, 3, \ldots, n+1$ respectively to the vertices $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$. Moreover assign the labels $-n-1, -n-2, \cdots, -2n$ and $n+2, n+3, \ldots, 2n+1$ to the vertices $w_1, w_2, \cdots, w_n$ and $v_1, v_2, \cdots, v_n$ respectively. Thereafter assign the labels $-2n-1, -2n-2, \cdots, \frac{-5n-1}{2}$ and $2n+3, 2n+4, \cdots, \frac{5n+1}{2}$ respectively to the vertices $u_1, u_2, \cdots, u_{\frac{n+1}{2}}$ and $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \cdots, u_{n-1}$. Finally, fix the label 1 to vertex u_n.

Case 2. n is even.

Define the injective function $\lambda : V(S(H_n)) \rightarrow \{\pm 1, \pm 2, \cdots, \pm \frac{5n}{2}\}$. Apply the Case 1 labeling to the vertices $v_0, v_i, w_i, x_i, y_i 1 \leq i \leq n$. Then assign the labels $-2n-1, -2n-2, \cdots, \frac{-5n}{2}$ and $2n+3, 2n+4, \cdots, \frac{5n}{2}$ respectively to the vertices $u_1, u_2, \cdots, u_{\frac{n}{2}}$ and $u_{\frac{n+2}{2}}, u_{\frac{n+4}{2}}, \cdots, u_{n-2}$. Finally assign the labels 1, 1 to the vertices u_{n-1}, u_n respectively. Hence the labeling in both cases results in $3n$ edges assigned the label 1 and $3n$ edges without the label 1. Therefore, the vertex labeling λ is a PMC-labeling of the subdivision of helm graph $S(H_n)$. □

Theorem 3.7 *The subdivision of fan graph f_n, $S(f_n)$ is a PMC-graph for all $n \geq 3$.*

Proof Let us consider the subdivision of fan graph $S(f_n)$, $n \geq 3$. Denote by $V(S(f_n)) = \{v_0, v_i, u_i, w_j \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n-1\}$ and $E(S(f_n)) = \{v_0u_i, u_iv_i \mid 1 \leq i \leq n\} \cup$

$\{v_iw_i, w_iv_{i+1} \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(f_n)$ respectively. Then the order and size of $S(f_n)$ respectively are $3n$ and $4n-2$. This theorem is analyzed by considering two cases.

Case 1. n is odd.

Define the injective function $\lambda : V(S(f_n)) \to \{\pm1, \pm2, \cdots, \pm\frac{3n-1}{2}\}$. Let $\lambda(v_0) = 1$ and $\lambda(u_1) = 2$. Then assign the labels $-1, -2, \cdots, -n$ and $3, 4, \cdots, n+1$ respectively to the vertices $v_1, v_2, \cdots, v_n$ and $w_1, w_2, \cdots, w_{n-1}$. Moreover assign the labels $-n-1, -n-2, \cdots, \frac{-3n+1}{2}$ and $n+2, n+3, \cdots, \frac{3n-1}{2}$ to the vertices $u_2, u_3, \cdots, u_{\frac{n+1}{2}}$ and $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \cdots, u_{n-1}$ respectively. Finally, fix the label $\frac{-3n+1}{2}$ to the vertex u_n.

Case 2. n is even.

Define the injective function $\lambda : V(S(f_n)) \to \{\pm1, \pm2, \cdots, \pm\frac{3n}{2}\}$. Apply the Case 1 labeling to the vertices $v_0, v_i, w_i, u_1, 1 \leq i \leq n$. Then assign the labels $-n-1, -n-2, \cdots, \frac{-3n}{2}$ and $n+2, n+3, \cdots, \frac{3n}{2}$ respectively to the vertices $u_1, u_2, \cdots, u_{\frac{n+2}{2}}$ and $u_{\frac{n+4}{2}}, u_{\frac{n+6}{2}}, \cdots, u_n$. Hence the labeling in both cases results in $2n-1$ edges assigned the label 1 and $2n-1$ edges without the label 1. Therefore, the vertex labeling λ is a PMC-labeling of the subdivision of fan graph $S(f_n)$. □

Theorem 3.8 *The subdivision of friendship graph F_n, $S(F_n)$ is a PMC-graph for all $n \geq 1$.*

Proof Let us consider the subdivision of friendship graph $S(F_n)$, $n \geq 1$. Denote by $V(S(F_n)) = \{v_0, v_i, u_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(S(F_n)) = \{v_0x_i, v_0y_i, x_iv_i, y_iu_i, v_iz_i, z_iu_i \mid 1 \leq i \leq n\}$ the vertex set and edge set of $S(F_n)$ respectively. Then the order and size of $S(F_n)$ respectively are $5n+1$ and $6n$. This theorem is analyzed by considering two cases.

Case 1. n is odd.

Define the injective function

$$\lambda : V(S(F_n)) \to \{\pm1, \pm2, \ldots, \pm\frac{5n+1}{2}\}.$$

Let $\lambda(v_0) = 1$. Then assign the labels $-1, -3, \cdots, -2n+1$ and $-2, -4, \cdots, -2n$ respectively to the vertices $v_1, v_2, \cdots, v_n$ and $u_1, u_2, \cdots, u_n$. Moreover assign the labels $3, 5, \cdots, 2n+1$ and $2, 4, \cdots, 2n$ to the vertices $z_1, z_2, \cdots, z_n$ and $x_1, x_2, \cdots, x_n$ respectively. Finally, assign the labels $-2n-1, -2n-2, \cdots, \frac{-5n-1}{2}$ and $2n+2, 2n+3, \cdots, \frac{5n+1}{2}$ respectively to the vertices $y_1, y_3, \cdots, y_n$ and $y_2, y_4, \cdots, y_{n-1}$.

Case 2. n is even.

Define the injective function

$$\lambda : V(S(F_n)) \to \{\pm1, \pm2, \cdots, \pm\frac{5n}{2}\}.$$

Apply the Case 1 labeling to the vertices $v_i, u_i, z_i, x_i 1 \leq i \leq n$. Then assign the labels $-2n-1, -2n-2, \cdots, \frac{-5n}{2}$ and $2n+2, 2n+3, \cdots, \frac{5n}{2}$ respectively to the vertices $y_1, y_3, \cdots, y_{n-1}$

and $y_2, y_4, \cdots, y_{n-2}$. Hence the labeling in both cases results in $3n$ edges assigned the label 1 and $3n$ edges without the label 1. Therefore, the vertex labeling λ is a PMC-labeling of the subdivision of friendship graph $S(F_n)$. □

Example 3.9 A PMC-labeling of the subdivision of friendship graph $S(F_4)$ is shown in Figure 3.

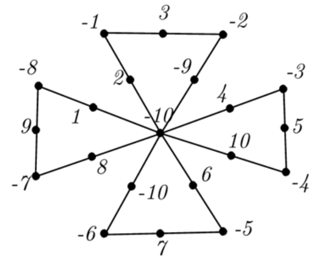

Figure 3. A PMC-labeling of the subdivision of friendship graph $S(F_4)$

Theorem 3.10 *The subdivision of coconut tree $CT_{n,m}$, $S(CT_{n,m})$ is a PMC-graph for all $n, m \geq 1$.*

Proof Let us consider the subdivision of coconut tree $S(CT_{n,m})$, $n, m \geq 1$. Denote by $V(S(CT_{n,m})) = \{u_i, v_j, y_j \mid 1 \leq i \leq n \,\text{and}\, 1 \leq j \leq m\} \cup \{x_i \mid 1 \leq i \leq n-1\}$ and $E(S(CT_{n,m})) = \{u_n y_j, y_j v_j \mid 1 \leq i \leq m\} \cup \{u_i x_i, x_i u_{i+1} \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(CT_n)$ respectively. Then the order and size of $S(CT_n)$ respectively are $2n+2m-1$ and $2n+2m-2$. Define the injective function $\lambda : V(S(CT_{n,m})) \rightarrow \{\pm 1, \pm 2, \cdots, \pm(n+m-1)\}$. First assign the labels $-1, -2, \cdots, -m$ and $2, 3, \cdots, m+1$ respectively to the vertices $y_1, y_2, \cdots, y_m$ and $v_1, v_2, \cdots, v_m$. Then assign the labels $-m-1, -m-2, \cdots, -m-n+1$ and 1 to the vertices $u_1, u_2, \cdots, u_{n-1}$ and u_n respectively. Finally assign the labels $m+2, m+3, \cdots, m+n-1$ and 1 to the vertices $x_1, x_2, \cdots, x_{n-2}$ and x_{n-1} respectively. Hence the number of edges labeled with 1 is $n+m-1$ while the number of the edges not labeled with 1 is $n+m-1$. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of coconut tree $S(CT_{n,m})$. □

Theorem 3.11 *The subdivision of double comb graph DC_n, $S(DC_n)$ is a PMC-graph for all $n \geq 2$.*

Proof Consider the subdivision of double comb graph $S(DC_n)$, $n \geq 2$. Denote by $V(S(DC_n)) = \{u_i, v_i, w_i, x_j, y_i, z_i \mid 1 \leq i \leq n \,\text{and}\, 1 \leq j \leq n-1\}$ and $E(S(DC_n)) = \{u_i y_i, y_i v_i, u_i z_i, z_i w_i, \mid 1 \leq i \leq n\} \cup \{u_i x_i, x_i u_{i+1} \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(DC_n)$ respectively. Then the order and size of $S(DC_n)$ respectively are $6n-1$ and $6n-2$. Define the injective function $\lambda : V(S(DC_n)) \rightarrow \{\pm 1, \pm 2, \cdots, \pm(3n-1)\}$. First assign the labels $-1, -2, \cdots, -n$ and $2, 3, \cdots, n+1$ respectively to the vertices $y_1, y_2, \cdots, y_n$ and $v_1, v_2, \cdots, v_n$. Then assign the labels $-n-1, -n-2, \cdots, -2n$ and $n+2, n+3, \cdots, 2n+1$ to the vertices $z_1, z_2, \cdots, z_n$ and $w_1, w_2, \cdots, w_n$ respectively. Moreover assign the labels $-2n-1, -2n-2, \cdots, -3n+1$ and 1 respectively to the vertices $u_1, u_2, \cdots, u_{n-1}$ and u_n. Finally assign the labels $2n+2, 2n+3, \cdots, 3n-1$ and 1 to the vertices $x_1, x_2, \cdots, x_{n-2}$ and x_{n-1}

respectively. Hence the number of edges labeled with 1 is $3n-1$ while the number of the edges not labeled with 1 is $3n-1$. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of double comb graph $S(DC_n)$. $\Box$

Theorem 3.12 *The subdivision of jelly fish graph $J_{n,m}$, $S(J_{n,m})$ is a PMC-graph for all $n, m \geq 1$.*

Proof Let us consider the subdivision of jelly fish graph $S(J_{n,m})$, $n, m \geq 1$. Denote by $V(S(J_{n,m})) = \{a, b, c, d, w, a', b', c', d', u_i, x_i, v_j, y_j \mid 1 \leq i \leq n \,\text{and}\, 1 \leq j \leq m\}$ and $E(S(J_{n,m})) = \{aa', a'b, bb', b'c, cc', c'd, dd', d'a, aw, wc, dx_i, x_iu_i, by_j, y_jv_j, \mid 1 \leq i \leq n \,\text{and}\, 1 \leq j \leq m\}$ the vertex set and edge set of $S(J_{n,m})$ respectively. Then the order and size of $S(J_{n,m})$ respectively are $2n+2m+9$ and $2n+2m+10$. Define the injective function $\lambda : V(S(J_{n,m})) \to \{\pm 1, \pm 2, \ldots, \pm(n+m+4)\}$. First assign the labels $-3, -4, -2, -1, 4$ and $6, 5, 2, 3$ respectively to the vertices a, b, c, d, w and a', b', c', d'. Then assign the labels $-5, -6, \cdots, -n-4$ and $6, 7, \cdots, n+5$ to the vertices $x_1, x_2, \cdots, x_n$ and $u_1, u_2, \cdots, u_n$ respectively. Moreover assign the labels $-n-5, -n-6, \cdots, -n-m-4$ and $n+6, n+7, \cdots, n+m+4$ respectively to the vertices $y_1, y_2, \cdots, y_m$ and $v_1, v_2, \cdots, v_{m-1}$. Finally assign the label 1 to the vertex v_m respectively. Hence the number of edges labeled with 1 is $n+m+5$ while the number of the edges not labeled with 1 is $n+m+5$. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of jelly fish graph $S(J_{n,m})$. $\Box$

Example 3.13 A PMC-labeling of the subdivision of jelly fish graph $S(J_{3,4})$ is shown in Figure 4.

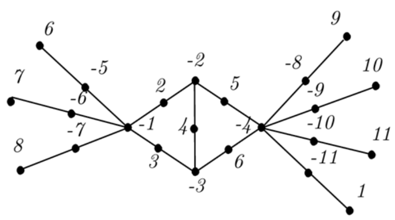

Figure 4. A PMC-labeling of the subdivision of jelly fish graph $S(J_{3,4})$

Theorem 3.14 *The subdivision of flower graph fl_n, $S(fl_n)$ is a PMC-graph for all $n \geq 3$.*

Proof Consider the subdivision of flower graph $S(fl_n)$, $n \geq 3$. Denote by $V(S(fl_n)) = \{v_0, v_i, u_i, x_i, y_i, z_i, w_i \mid 1 \leq i \leq n\}$ and $E(S(fl_n)) = \{v_0x_i, x_iv_i, v_iy_i, y_iu_i, v_0z_i, z_iu_i, v_iw_i \mid 1 \leq i \leq n\} \cup \{w_iv_{i+1}, w_nv_1 \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(fl_n)$ respectively. Then the order and size of $S(fl_n)$ respectively are $6n+1$ and $8n$. This theorem is analyzed by considering two cases.

Case 1. n is even.

Define the injective function $\lambda : V(S(fl_n)) \to \{\pm 1, \pm 2, \cdots, \pm 3n\}$. Let $\lambda(v_0) = 1$. First assign the labels $-1, -4, \cdots, \frac{-3n+4}{2}$ and $4, 7, \cdots, \frac{3n+2}{2}$ respectively to the vertices $u_1, u_3, \cdots, u_{n-1}$ and $u_2, u_4, \cdots, u_n$. Moreover assign the labels $2, 5, \cdots, \frac{3n-2}{2}$ and $-2, -5, \cdots, \frac{-3n+2}{2}$ to the vertices $y_1, y_3, \cdots, y_{n-1}$ and $y_2, y_4, \cdots, y_n$ respectively. Now assign the labels $3, 6, \cdots, \frac{3n}{2}$

and $-3, -6, \cdots, \frac{-3n}{2}$ respectively to the vertices $z_1, z_3, \cdots, z_{n-1}$ and $z_2, z_4, \cdots, z_n$. Assign the labels $\frac{3n+4}{2}, \frac{3n+6}{2}, \cdots, \frac{5n+2}{2}$ and $\frac{-3n-2}{2}, \frac{-3n-4}{2}, \cdots, \frac{-5n}{2}$ to the vertices $v_1, v_2, \cdots, v_n$ and $w_1, w_2, \cdots, w_n$ respectively. Further assign the labels $\frac{5n+4}{2}, \frac{5n+6}{2}, \cdots, 3n$ and $\frac{-5n-2}{2}, \frac{-5n-4}{2}, \cdots, -3n$ to the vertices $x_1, x_3, \cdots, x_{n-3}$ and $x_2, x_4, \cdots, x_n$ respectively. Finally fix the label 1 to the vertex x_{n-1}.

Case 2. n is odd.

Let us define the injective function $\lambda : V(S(fl_n)) \to \{\pm 1, \pm 2, \cdots, \pm 3n\}$. Let $\lambda(v_0) = 1$. First assign the labels $-1, -4, \cdots, \frac{-3n+1}{2}$ and $4, 7, \cdots, \frac{3n-1}{2}$ respectively to the vertices $u_1, u_3, \cdots, u_n$ and $u_2, u_4, \cdots, u_{n-1}$. Moreover, assign the labels $2, 5, \cdots, \frac{3n+1}{2}$ and $-2, -5, \cdots, \frac{-3n+5}{2}$ to the vertices $y_1, y_3, \cdots, y_n$ and $y_2, y_4, \cdots, y_{n-1}$, respectively. Assign the labels $3, 6, \cdots, \frac{3n+3}{2}$ and $-3, -6, \cdots, \frac{-3n+3}{2}$ respectively to the vertices $z_1, z_3, \cdots, z_n$ and $z_2, z_4, \cdots, z_{n-1}$. Assign the labels $\frac{3n+5}{2}, \frac{3n+7}{2}, \cdots, \frac{5n+3}{2}$ and $\frac{-3n-1}{2}, \frac{-3n-3}{2}, \cdots, \frac{-5n+1}{2}$ to the vertices $w_1, w_2, \cdots, w_n$ and $v_1, v_2, \cdots, v_n$ respectively. Further assign the labels $\frac{5n+5}{2}, \frac{5n+7}{2}, \cdots, 3n$ and $\frac{-5n-1}{2}, \frac{-5n-3}{2}, \cdots, -3n$ to the vertices $x_2, x_4, \cdots, x_{n-3}$ and $x_1, x_3, \cdots, x_n$ respectively. Finally fix the label 1 to the vertex x_{n-1}. Hence the labeling in both cases results in $4n$ edges assigned the label 1 and $4n$ edges without the label 1. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of flower graph $S(fl_n)$. □

Example 3.15 A PMC-labeling of the subdivision of flower graph $S(fl_4)$ is shown in Figure 5.

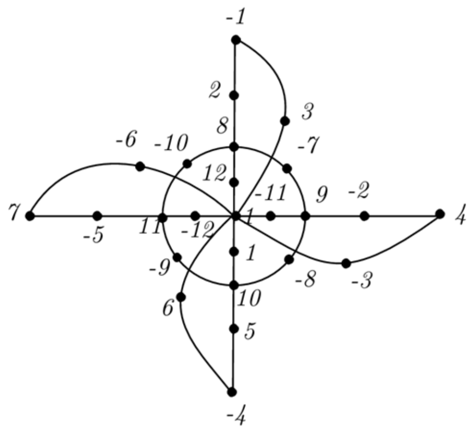

Figure 5. A PMC-labeling of the subdivision of flower graph $S(fl_4)$

Theorem 3.16 *The subdivision of sun flower graph SF_n, $S(SF_n)$ is a PMC-graph for all $n \geq 3$.*

Proof Consider the subdivision of sun flower graph $S(SF_n)$, $n \geq 3$. The vertex set and edge set of $S(SF_n)$ are denoted by $V(S(SF_n)) = \{v_0, v_i, u_i, x_i, y_i, z_i, w_i \mid 1 \leq i \leq n\}$ and $E(S(SF_n)) = \{v_0x_i, x_iv_i, v_iw_i, v_iy_i, y_iu_i, z_iu_i \mid 1 \leq i \leq n\} \cup \{w_iv_{i+1}, w_nv_1, z_iv_{i+1}, z_nv_1 \mid 1 \leq i \leq n-1\}$ respectively. Then the order and size of $S(SF_n)$ respectively are $6n + 1$ and $8n$. This theorem is analyzed by considering two cases.Let us define the injective function $\lambda : V(S(SF_n)) \to \{\pm 1, \pm 2, \cdots, \pm 3n\}$. Let $\lambda(v_0) = -3n$.

Case 1. n is even.

In this case, apply the Case 1 of Theorem 3.14 labeling to the vertices $u_i, y_i, z_i, v_i, w_i 1 \leq i \leq n$. After that assign the labels $\frac{-5n-2}{2}, \frac{-5n-4}{2}, \cdots, -3n$ and $\frac{5n+4}{2}, \frac{5n+6}{2}, \cdots, 3n$ to the vertices $x_1, x_3, \cdots, x_{n-1}$ and $x_2, x_4, \cdots, x_{n-2}$ respectively. Finally fix the label 1 to the vertex x_n.

Case 2. n is odd.

Apply the Case 2 of Theorem 3.14 labeling to the vertices $u_i, y_i, z_i, v_i, w_i 1 \leq i \leq n$. Thereafter assign the labels $\frac{-5n-1}{2}, \frac{-5n-3}{2}, \cdots, -3n$ and $\frac{5n+5}{2}, \frac{5n+7}{2}, \cdots, 3n$ to the vertices $x_1, x_3, \cdots, x_n$ and $x_2, x_4, \cdots, x_{n-3}$ respectively. Finally fix the label 1 to the vertex x_{n-1}. Hence the labeling in both cases results in $4n$ edges assigned the label 1 and $4n$ edges without the label 1. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of sun flower graph $S(SF_n)$. □

Example 3.17 A PMC-labeling of the subdivision of sun flower graph $S(SF_4)$ is shown in Figure 6.

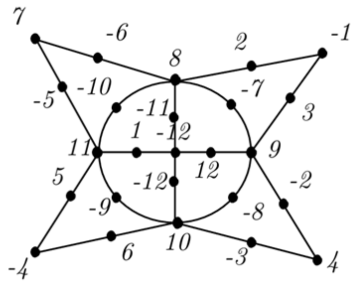

Figure 6. A PMC-labeling of the subdivision of sun flower graph $S(SF_4)$

Theorem 3.18 *The subdivision gear graph G_n, $S(G_n)$ is a PMC-graph for all $n \geq 3$.*

Proof Consider the subdivision of gear graph $S(G_n)$, $n \geq 3$. Denote by $V(S(G_n)) = \{v_0, v_i, u_i, x_i, y_i, z_i \mid 1 \leq i \leq n\}$ and $E(S(G_n)) = \{v_0x_i, x_iv_i, v_iy_i, y_iu_i, z_iu_i \mid 1 \leq i \leq n\} \cup \{z_iv_{i+1}, z_nv_1 \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(G_n)$ respectively. Then the order and size of $S(G_n)$ respectively are $5n+1$ and $6n$. This theorem is analyzed by considering two cases.

Case 1. n is odd.

Define the injective function $\lambda : V(S(G_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{5n+1}{2}\}$. Let $\lambda(v_0) = \frac{-5n-1}{2}$. In this case, apply the Case 2 of Theorem 3.14 labeling to the vertices $u_i, y_i, z_i, v_i 1 \leq i \leq n$. After that assign the labels $\frac{3n+5}{2}, \frac{3n+7}{2}, \cdots, \frac{5n+1}{2}$ to the vertices $x_1, x_2, \cdots, x_{n-1}$ respectively. Finally fix the label 1 to the vertex x_n.

Case 2. n is even.

Let us define the injective function $\lambda : V(S(G_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{5n+1}{2}\}$. Let $\lambda(v_0) = 1$. Apply the Case 1 of Theorem 3.14 labeling to the vertices $u_i, y_i, z_i, v_i 1 \leq i \leq n$. After that assign the labels $\frac{3n+4}{2}, \frac{3n+6}{2}, \cdots, \frac{5n}{2}$ to the vertices $x_1, x_2, \cdots, x_{n-1}$ respectively. Finally fix the label 1 to the vertex x_n. Hence the labeling in both cases results in $3n$ edges assigned the label 1 and $3n$ edges without the label 1. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of gear graph $S(G_n)$. □

Theorem 3.19 *The subdivision of jewel graph JL_n, $S(JL_n)$ is a PMC-graph for all $n \geq 1$.*

Proof Consider the subdivision of jewel graph $S(JL_n)$, $n \geq 1$. The vertex set and edge set of $S(JL_n)$ are denoted by $V(S(JL_n)) = \{a, b, c, d, x, a', b', c', d', u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(S(JL_n)) = \{aa', a'b, bb', b'c, cc', c'd, dd', d'a, ax, xc, dv_i, v_iu_i, bw_i, w_iu_i \mid 1 \leq i \leq n\}$ respectively. Then the order and size of $S(JL_n)$ respectively are $3n + 9$ and $4n + 10$. This theorem is analyzed by considering two cases.

Case 1. n is odd.

Define the injective function $\lambda : V(S(JL_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n+9}{2}\}$. First assign the labels $2, 4, 3, -4, -1$ and $-3, -2, 5, 6$ respectively to the vertices a, b, c, d, x and a', b', c', d'. Assign the labels $7, 10, \cdots, \frac{3n+5}{2}$ and $-7, -10, \cdots, \frac{-3n-5}{2}$ to the vertices $u_1, u_3, \cdots, u_{n-2}$ and $u_2, u_4, \cdots, u_{n-1}$ respectively. Assign the label 1 to the vertex u_n. Next assign the labels $-5, -8, \cdots, \frac{-3n-7}{2}$ and $8, 11, \cdots, \frac{3n+7}{2}$ respectively to the vertices $v_1, v_3, \cdots, v_n$ and $v_2, v_4, \cdots, v_{n-1}$. Also assign the labels $-6, -9, \cdots, \frac{-3n-9}{2}$ and $9, 12, \cdots, \frac{3n+9}{2}$ respectively to the vertices $w_1, w_3, \cdots, w_n$ and $w_2, w_4, \cdots, w_{n-1}$.

Case 2. n is even.

Let us define the injective function $\lambda : V(S(JL_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n+8}{2}\}$. First assign the labels to the vertices a, b, c, d, x and a', b', c', d' as in Case (1). Then assign the labels $7, 10, \cdots, \frac{3n+8}{2}$ and $-7, -10, \cdots, \frac{-3n-8}{2}$ to the vertices $u_1, u_3, \cdots, u_{n-1}$ and $u_2, u_4, \cdots, u_n$ respectively. Next assign the labels $-5, -8, \cdots, \frac{-3n-4}{2}$ and $8, 11, \cdots, \frac{3n+4}{2}$ respectively to the vertices $v_1, v_3, \cdots, v_{n-1}$ and $v_2, v_4, \cdots, v_{n-2}$. Then fix the label 1 to the vertex v_n. Also assign the labels $-6, -9, \cdots, \frac{-3n-6}{2}$ and $9, 12, \cdots, \frac{3n+6}{2}$ respectively to the vertices $w_1, w_3, \cdots, w_{n-1}$ and $w_2, w_4, \cdots, w_{n-2}$. Finally fix the label 1 to the vertex w_n. Hence the labeling in both cases results in $2n + 5$ edges assigned the label 1 and $2n + 5$ edges without the label 1. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of jewel graph $S(JL_n)$. □

§4. Conclusion

In this paper, we have examined the PMC-labeling behavior of the subdivision of path, cycle, wheel, crown, helm, fan graph, friendship graph, coconut tree, double comb graph, jellyfish graph, flower graph, sunflower graph, gear graph and jewel graph. The exploration of other classes of graphs and the investigation of more complex PMC-labeling properties remain open challenges.

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