

Pair Difference Cordial Number of Some Degree Splitting Graph

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Abstract: In this paper we determine the pair difference cordial number of degree splitting graph of bistar, complete bipartite graph, ladder, wheel.

Key Words: Pair difference cordial labeling, Smarandachely pair difference cordial labeling, Smarandachely pair difference cordial graph, bipartite graph, bistar, degree splitting graph, ladder, wheel.

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§1. Introduction

We consider only finite, undirected and simple graphs. The notion of pair difference cordial labeling of a graph was introduced in [4]. Also we have investigated pair difference cordial labeling behavior of several graphs like path, cycle, star, wheel, some snake and butterfly graphs, graphs derived from ladder graph, degree splitting graph of some graphs have been investigated in [4,5,6,7,8,9,10]. Recently pair difference cordial number of a graph was introduced in [14]. In this paper we determine the pair difference cordial number of degree splitting graph of bistar, complete bipartite, ladder, wheel.

§2. Preliminaries

Definition 2.1([4]) Let $G = (V, E)$ be a (p, q) graph. Define

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \dots, \pm \rho\}$ called the set of labels. Consider a mapping $f : V \rightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $|\Delta_{f_1} - \Delta_{f_1^c}| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1. Otherwise, if

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$|\Delta_{f_1} - \Delta_{f_2}| \geq 2$, such a labeling f is said to be a Smarandachely pair difference cordial labeling and a graph G for which there exists a pair difference cordial labeling or a Smarandache pair difference cordial labeling is called a pair difference cordial labeling graph or Smarandachely pair difference cordial graph.

Definition 2.2([5]) Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup \dots \cup S_t \cup T$ where each S_i is a set of vertices having at least two vertices and having the same degree and $T = V - \bigcup_{i=1}^t S_i$. The degree splitting graph of G denoted by $DS(G)$ is obtained from G by adding vertices w_1, w_2, \dots, w_t and joining w_i to each vertex of S_i ($1 \leq i \leq t$).

Theorem 2.3([13]) $DS(K_{n,n})$ is pair difference cordial if and only if $n = 2$, where $K_{n,n}$ is complete bipartite graph.

Theorem 2.4([13]) $DS(L_n)$ is pair difference cordial if and only if $n \leq 3$, where L_n is the ladder.

Theorem 2.5([13]) $DS(B_{n,n})$ is pair difference cordial if and only if $n \leq 2$, where $B_{n,n}$ is the bistar.

Theorem 2.6([13]) $DS(B_{1,n})$ is pair difference cordial if and only if $n \leq 4$.

Theorem 2.7([13]) $DS(W_n)$ is not pair difference cordial for all $n \geq 3$, where W_n is the wheel.

Theorem 2.8([4]) The cycle C_n is pair difference cordial if and only if $n > 3$.

§3. Pair Difference Cordial Number of a Graph

Definition 3.1 Let G be a (p, q) graph. Pair difference cordial number of a graph G is the least positive integer m such that $G \cup K_2$ is pair difference cordial. It is denoted by $PDC_\eta(G)$.

Remark 3.2 If G is pair difference cordial graph then $PDC_\eta(G) = 0$.

Theorem 3.3 For any integer $n \geq 1$,

$$PDC_\eta(B_{1,n}) = \begin{cases} 0 & \text{if } n \leq 4, \\ 2n - 6 & \text{if } n \geq 5. \end{cases}$$

Proof Let $V(B_{1,n} \cup mK_2) = \{x, v, w, u, u_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(B_{1,n} \cup mK_2) = \{vw, wu, xu_n\} \cup \{uu_i : 1 \leq i \leq n\} \cup \{v_iw_i : 1 \leq i \leq m\}$. Clearly $B_{1,n} \cup mK_2$ has $n + 2m + 4$ vertices and $2n + m + 3$ edges. There are two cases arises.

Case 1. $n \leq 4$.

The proof follows from Theorem 2.6.

Case 2. $n \geq 5$.

Subcase 2.1 n is odd.

Take $m = 2n - 6$. Define $f : V(B_{1,n} \cup (2n - 6)K_2) \rightarrow \{\pm 1, \pm 2, \dots, \pm \lfloor \frac{n+2m+3}{2} \rfloor\}$ as follows:

Assign the labels 1, 2, 3, 4, 4 to the vertices v, w, u, u_1, u_2 and assign the labels $-1, -2, -3, -4$ to the vertices x, u_n, u_3, u_4 . Next, assign the labels 5, 7, 9, $\dots, (n+2)$ to vertices $v_1, v_2, v_3, \dots, v_{\frac{m}{2}}$ and assign the labels 6, 8, 10, $\dots, (n+3)$ to the vertices $w_1, w_2, w_3, \dots, w_{\frac{m}{2}}$. Next, assign the label $-5, -7, -9, \dots, -(n+2)$ to the vertices $v_{\frac{m}{2}+1}, v_{\frac{m}{2}+1}, v_{\frac{m}{2}+1}, \dots, v_m$ and assign the label $-6, -8, -10, \dots, -(n+3)$ to the vertices $w_{\frac{m}{2}+1}, w_{\frac{m}{2}+1}, w_{\frac{m}{2}+1}, \dots, w_m$. Finally assign the distinct labels to the vertices $u_5, u_6, u_7, \dots, u_{n-1}$ from $\pm(n+1), \pm(n-1), \dots, \pm(\frac{n+2m+3}{2})$.

Therefore, f is pair difference cordial labeling of $B_{1,n} \cup (2n - 6)K_2$. The maximum number of edges with labels 1 from the $DS(B_{1,n})$ is, $\Delta_{f_1} = 5$, if n is odd. But the size of the $DS(B_{1,n})$ is $2n + 3$. Hence $2n - 6$ is the least integer such that $B_{1,n} \cup (2n - 6)K_2$ is pair difference cordial.

Subcase 2.2 n is even.

Take $m = 2n - 6$. Define $f : V(B_{1,n} \cup (2n - 6)K_2) \rightarrow \{\pm 1, \pm 2, \dots, \pm \lfloor \frac{n+2m+4}{2} \rfloor\}$ as follows:

Assign the labels 1, 2, 3, 4 to the vertices v, w, u, u_1 and assign the labels $-1, -2, -3, -4$ to the vertices x, u_n, u_2, u_3 . Next assign the labels 5, 7, 9, $\dots, (n+2)$ to the vertices $v_1, v_2, v_3, \dots, v_{\frac{m}{2}}$ and assign the labels 6, 8, 10, $\dots, (n+3)$ to the vertices $w_1, w_2, w_3, \dots, w_{\frac{m}{2}}$. Next assign the label $-5, -7, -9, \dots, -(n+2)$ to the vertices $v_{\frac{m}{2}+1}, v_{\frac{m}{2}+1}, v_{\frac{m}{2}+1}, \dots, v_m$ and assign the label $-6, -8, -10, \dots, -(n+3)$ to the vertices $w_{\frac{m}{2}+1}, w_{\frac{m}{2}+1}, w_{\frac{m}{2}+1}, \dots, w_m$. Finally assign the distinct labels to the vertices $u_4, u_5, u_6, \dots, u_{n-1}$ from $\pm(n+1), \pm(n-1), \dots, \pm(\frac{n+2m+3}{2})$.

Therefore, f is pair difference cordial labeling of $B_{1,n} \cup (2n - 6)K_2$. The maximum number of edges with labels 1 from the $DS(B_{1,n})$ is, $\Delta_{f_1} = 4$, if n is even. But the size of the $DS(B_{1,n})$ is $2n + 3$. Hence $2n - 6$ is the least integer such that $B_{1,n} \cup (2n - 6)K_2$ is pair difference cordial. \square

Theorem 3.4 For any integer $n \geq 1$,

$$PDC_\eta(DS(L_n)) = \begin{cases} 0 & \text{if } n \leq 3 \\ n - 1 & \text{if } n \geq 5 \text{ and } n \text{ is odd,} \\ n - 2 & \text{if } n \geq 4 \text{ and } n \text{ is even.} \end{cases}$$

Proof Let $V(DS(L_n) \cup mK_2) = \{x, x_i, y, y_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(DS(L_n) \cup mK_2) = \{xx_1, xx_n, xy_1, xy_n\} \cup \{yy_i, yx_i : 2 \leq i \leq n - 1\} \cup \{v_i w_i : 1 \leq i \leq m\}$. Clearly $DS(L_n) \cup mK_2$ has $2n + 2m + 2$ vertices and $5n + m - 2$ edges. There are three cases arises.

Case 1. $n \leq 3$.

The proof follows from Theorem 2.4.

Case 2. $n \geq 5$ and n is odd.

Take $m = n - 1$. Define $f : V(DS(L_n) \cup (n - 1)K_2) \rightarrow \{\pm 1, \pm 2, \dots, \pm n + m + 1\}$ as follows:

Assign the labels 1, 2, 3, $-1, -2, -3$ to the vertices x_1, x, x_n, y_1, y_2, y and assign the la-

bels $4, 5, 6, \dots, (n + 1)$ to the vertices $x_{n-1}, x_{n-2}, x_{n-3}, \dots, u_3, u_2$. Next assign the labels $-4, -5, -6, \dots, -(n + 1)$ to the vertices $y_3, y_4, y_5, \dots, y_n$. Next assign the labels $(n + 2), (n + 4), (n + 6), \dots, (n + m)$ to the vertices $v_1, v_2, v_3, \dots, v_{\frac{n-1}{2}}$ and assign the labels $(n + 3), (n + 5), (n + 7), \dots, (n + m + 1)$ to the vertices $w_1, w_2, w_3, \dots, w_{\frac{n-1}{2}}$. Now we assign the labels $-(n + 2), -(n + 4), -(n + 6), \dots, -(n + m)$ to the vertices $v_{\frac{n-1}{2}+1}, v_{\frac{n-1}{2}+2}, v_{\frac{n-1}{2}+3}, \dots, v_{n-1}$ and assign the labels $-(n + 3), -(n + 5), -(n + 7), \dots, -(n + m + 1)$ to the vertices $w_{\frac{n-1}{2}+1}, w_{\frac{n-1}{2}+2}, w_{\frac{n-1}{2}+3}, \dots, w_{n-1}$.

Therefore, f is pair difference cordial labeling of $DS(L_n) \cup (n - 1)K_2$. The maximum number of edges with labels 1 from the $DS(L_n)$ is, $\Delta_{f_1} = 2n$. But the size of the $DS(L_n)$ is $5n - 2$. Hence $n - 1$ is the least integer such that $DS(L_n) \cup (n - 1)K_2$ is pair difference cordial.

Case 3. $n \geq 4$ and n is even.

Take $m = n - 2$. Define $f : V(L_n \cup (n - 2)K_2) \rightarrow \{\pm 1, \pm 2, \dots, \pm n + m + 1\}$ as follows:

Assign the labels $1, 2, 3, -1, -2, -3$ to the vertices x_1, x, x_n, y_1, y_2, y and assign the labels $4, 5, 6, \dots, (n + 1)$ to the vertices $x_{n-1}, x_{n-2}, x_{n-3}, \dots, u_3, u_2$. Next assign the labels $-4, -5, -6, \dots, -(n + 1)$ to the vertices $y_3, y_4, y_5, \dots, y_n$. Next assign the labels $(n + 2), (n + 4), (n + 6), \dots, (n + m)$ to the vertices $v_1, v_2, v_3, \dots, v_{\frac{n-2}{2}}$ and assign the labels $(n + 3), (n + 5), (n + 7), \dots, (n + m + 1)$ to the vertices $w_1, w_2, w_3, \dots, w_{\frac{n-2}{2}}$. Now we assign the labels $-(n + 2), -(n + 4), -(n + 6), \dots, -(n + m)$ to the vertices $v_{\frac{n-2}{2}+1}, v_{\frac{n-2}{2}+2}, v_{\frac{n-2}{2}+3}, \dots, v_{n-2}$ and assign the labels $-(n + 3), -(n + 5), -(n + 7), \dots, -(n + m + 1)$ to the vertices $w_{\frac{n-2}{2}+1}, w_{\frac{n-2}{2}+2}, w_{\frac{n-2}{2}+3}, \dots, w_{n-2}$.

Therefore, f is pair difference cordial labeling of $DS(L_n) \cup (n - 2)K_2$ and the maximum number of edges with labels 1 from the $DS(L_n)$ is $\Delta_{f_1} = 2n$. But the size of the $DS(L_n)$ is $5n - 2$. Hence $n - 2$ is the least integer such that $DS(L_n) \cup (n - 2)K_2$ is a pair difference cordial graph. □

A pair difference cordial labeling on $DS(L_6) \cup 4K_2$ is shown in Figure 1.

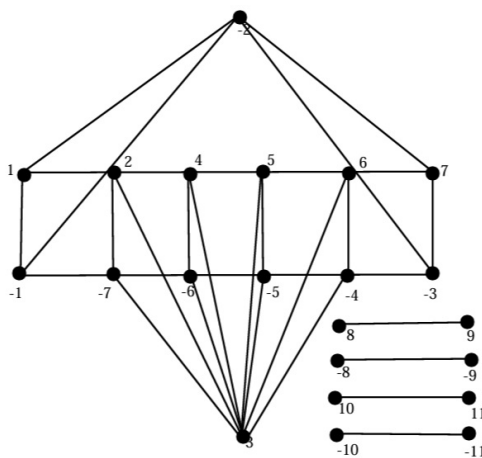


Figure 1

Theorem 3.5 *If $n \geq 3$, then*

$$PDC_{\eta}(DS(W_n)) = \begin{cases} n - 1 & \text{if } n \text{ is odd,} \\ n - 2 & \text{if } n \text{ is even.} \end{cases}$$

Proof Let $V(DS(W_n) \cup mK_2) = \{x, x_i, y : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(DS(W_n) \cup mK_2) = \{x_1x_n, x_ix_{i+1} : 1 \leq i \leq n-1\} \cup \{xx_i, yx_i : 1 \leq i \leq n\} \cup \{v_iw_i : 1 \leq i \leq m\}$.

Clearly, $DS(W_n) \cup mK_2$ has $n + 2m + 2$ vertices and $3n + m$ edges. There are two cases arises.

Case 1. n is odd.

Take $m = n - 1$. Define

$$f : V(DS(W_n) \cup (n - 1)K_2) \rightarrow \{\pm 1, \pm 2, \dots, \pm \frac{n + 2m + 2}{2}\}$$

as follows:

The maximum possible number of 1 occurs only when we assign the labels $3, 4, 5, \dots, n + 1$ to the vertices $x_3, x_4, x_5 \dots, x_n$ and assign the labels $1, 2, n + 2$ to the vertices x_1, x_2, x . Next assign the labels $-1, -3, -5, \dots, -n$ to the vertices $v_1, v_2, v_3, \dots, v_{\frac{n+1}{2}}$ and assign the labels $-2, -4, -6, \dots, -(n + 1)$ to the vertices $w_1, w_2, w_3, \dots, w_{\frac{n+1}{2}}$ and assign the labels $-(n + 2), -(n + 1)$ to the vertices $v_{\frac{n+3}{2}}, w_{\frac{n+3}{2}}$. Finally assign the labels $(n + 3), (n + 4), -(n + 3), -(n + 4)$ to the vertices $v_{\frac{n+5}{2}}, w_{\frac{n+5}{2}}, v_{\frac{n+7}{2}}, w_{\frac{n+7}{2}}$ and assign the labels $(n + 5), (n + 6), -(n + 5), -(n + 6)$ to the vertices $v_{\frac{n+9}{2}}, w_{\frac{n+9}{2}}, v_{\frac{n+11}{2}}, w_{\frac{n+11}{2}}$. Proceeding like this until we reach v_{n-1}, w_{n-1} .

Therefore, f is pair difference cordial labeling of $DS(W_n) \cup (n - 1)K_2$. The maximum number of edges with labels 1 from the $DS(W_n)$ is, $\Delta_{f_1} = n + 1$. But the size of the $DS(W_n)$ is $3n$. Hence $n - 1$ is the least integer such that $DS(W_n) \cup (n - 1)K_2$ is pair difference cordial.

Case 2. n is even.

Take $m = n - 2$. Define $f : V(DS(W_n) \cup (n - 2)K_2) \rightarrow \{\pm 1, \pm 2, \dots, \pm \frac{n + 2m + 2}{2}\}$ as follows:

The maximum possible number of 1 occurs only when we assign the labels $3, 4, 5, \dots, n + 1$ to the vertices $x_3, x_4, x_5 \dots, x_n$ and assign the labels $1, 2, n + 2$ to the vertices x_1, x_2, x . Next assign the labels $-1, -3, -5, \dots, -(n + 1)$ to the vertices $v_1, v_2, v_3, \dots, v_{\frac{n+2}{2}}$ and assign the labels $-2, -4, -6, \dots, -(n + 2)$ to the vertices $w_1, w_2, w_3, \dots, w_{\frac{n+2}{2}}$. Finally assign the labels $(n + 3), (n + 4), -(n + 3), -(n + 4)$ to the vertices $v_{\frac{n+4}{2}}, w_{\frac{n+4}{2}}, v_{\frac{n+6}{2}}, w_{\frac{n+6}{2}}$ and assign the labels $(n + 5), (n + 6), -(n + 5), -(n + 6)$ to the vertices $v_{\frac{n+8}{2}}, w_{\frac{n+8}{2}}, v_{\frac{n+10}{2}}, w_{\frac{n+10}{2}}$. Proceeding like this until we reach v_{n-2}, w_{n-2} .

Therefore, f is pair difference cordial labeling of $DS(W_n) \cup (n - 2)K_2$. The maximum number of edges with labels 1 from the $DS(W_n)$ is $\Delta_{f_1} = n + 1$. But the size of the $DS(W_n)$ is $3n$. Hence $n - 2$ is the least integer such that $DS(W_n) \cup (n - 2)K_2$ is pair difference cordial. This completes the proof. \square

Theorem 3.6 For any integer $n \geq 1$,

$$PDC_{\eta}(DS(K_{n,n})) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n = 2, \\ 5 & \text{if } n = 3, \\ n^2 - 2n - 2 & \text{if } n \geq 4. \end{cases}$$

Proof Let $V(DS(K_{n,n}) \cup mK_2) = \{x_i, y_i, x : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(DS(K_{n,n}) \cup mK_2) = E(K_{n,n}) \cup \{xx_i, xy_i : 1 \leq i \leq n\} \cup \{v_iw_i : 1 \leq i \leq m\}$. Clearly, $DS(K_{n,n}) \cup mK_2$ has $2n + 2m + 1$ vertices and $n^2 + 2n + m$ edges.

Case 1. $n = 1$.

In this case, $K_{n,n} \cong C_3$. Assign the labels 1, 1, 2, -1, -2 to the vertices x, x_1, y_1, v_1, w_1 .

Case 2. $n = 2$.

The proof follows from Theorem 2.3.

Case 3. $n = 3$.

In this case, A pair difference cordial labeling on $DS(K_{3,3}) \cup 5K_2$ is shown in Figure 2.

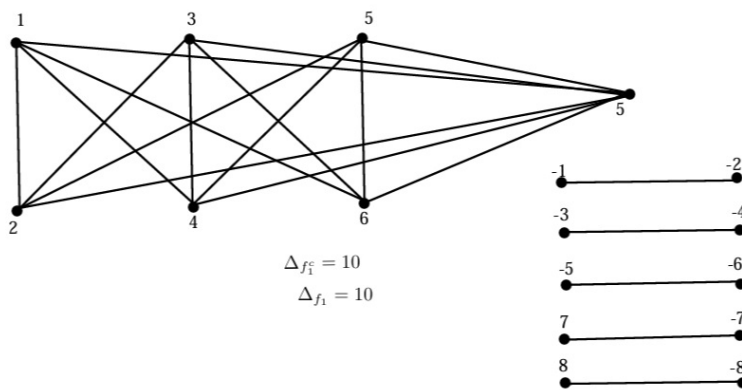


Figure 2

Case 4. $n \geq 4$.

Take $m = n^2 - 2n - 2$. Define $f : V(DS(K_{n,n}) \cup (n^2 - 2n - 2)K_2) \rightarrow \{\pm 1, \pm 2, \dots, \pm \lfloor \frac{n+2m+4}{2} \rfloor\}$ as follows:

Assign the labels 1, 3, 5, \dots , $2n - 1$ to the vertices $x_1, x_2, x_3, \dots, x_n$ and assign the labels 2, 4, 6, \dots , $2n$ to the vertices $y_1, y_2, y_3, \dots, y_n$. Next assign the labels -1, -3, -5, \dots , $-(2n - 1)$ to the vertices $v_1, v_2, v_3, \dots, v_n$ and assign the labels -2, -4, -6, \dots , $-2n$ to the vertices $w_1, w_2, w_3, \dots, w_m$. Next assign the labels $(2n + 1), (2n + 2), -(2n + 1), -(2n + 2)$ to the vertices $v_{n+1}, w_{n+1}, v_{n+2}, w_{n+2}$ and assign the labels $(2n + 3), (2n + 4), -(2n + 3), -(2n + 4)$ to the vertices $v_{n+3}, w_{n+3}, v_{n+4}, w_{n+4}$. Proceeding like this until we reach $v_{n^2 - 2n - 2}, w_{n^2 - 2n - 2}$.

Therefore, f is pair difference cordial labeling of $K_{n,n} \cup (n^2 - 2n - 2)K_2$. The maximum number of edges with labels 1 from the $DS(K_{n,n})$ is, $\Delta_{f_1} = 2n + 1$. But the size of the $DS(K_{n,n})$ is $n^2 + 2n$. Hence $n^2 - 2n - 2$ is the least integer such that $K_{n,n} \cup (n^2 - 2n - 2)K_2$ is pair difference cordial. \square

Theorem 3.7 For any integer $n \geq 1$,

$$PDC_\eta(DS(K_{n,n+1})) = \begin{cases} 0 & \text{if } n = 1, 2, \\ n^2 - n - 2 & \text{if } n \geq 3. \end{cases}$$

Proof Let $V(DS(K_{n,n+1}) \cup mK_2) = \{x_i, y_i, x, y : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(DS(K_{n,n+1}) \cup mK_2) = E(K_{n,n+1}) \cup \{xx_i, yy_i : 1 \leq i \leq n\} \cup \{v_iw_i : 1 \leq i \leq m\}$. Clearly $DS(K_{n,n+1}) \cup mK_2$ has $2n + 3$ vertices and $n^2 + 3n + 1$ edges.

Case 1. $n = 1$.

In this case, $K_{n,n} \cong C_4$. The proof follows from Theorem 2.8.

Case 2. $n = 2$.

In this case, a pair difference cordial labeling on $DS(K_{2,3})$ is shown in Figure 3.

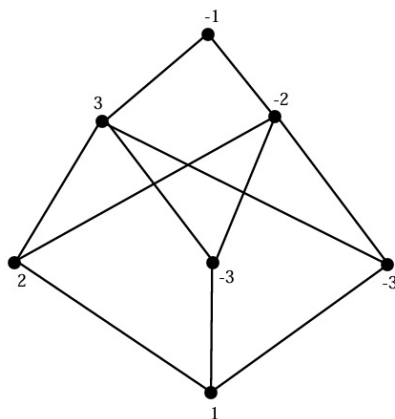


Figure 3

Case 3. $n \geq 3$.

Subcase 3.1 n is even.

Take $m = n^2 - n - 2$. Define $f : V(DS(K_{n,n+1}) \cup (n^2 - n - 2)K_2) \rightarrow \{\pm 1, \pm 2, \dots, \pm \lfloor \frac{2n+3}{2} \rfloor\}$ as follows:

Assign the labels $3, 5, 7, \dots, n + 1$ to the vertices $y_1, y_2, y_3, \dots, y_{\frac{n}{2}}$ and assign the labels $2, 4, 6, \dots, n$ to the vertices $x_1, x_2, x_3, \dots, x_{\frac{n}{2}}$. Next assign the labels $-3, -5, -7, \dots, -(n + 1)$ to the vertices $x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, x_{\frac{n}{2}+3}, \dots, x_n$ and assign the labels $-2, -4, -6, \dots, -n$ to the vertices $y_{\frac{n}{2}+1}, y_{\frac{n}{2}+2}, y_{\frac{n}{2}+3}, \dots, y_n$. Now assign the labels $1, -1, -(n - 1)$ to the vertices x, y, x_{n+1} . Next assign the labels $(n + 2), (n + 3), -(n + 2), -(n + 3)$ to the vertices v_1, w_1, v_2, w_2 and assign

the labels $(n+4), (n+5), -(n+4), -(n+5)$ to the vertices v_{n+3}, w_3, v_4, w_4 . Proceeding like this until we reach v_{n^2-n-2}, w_{n^2-n-2} .

Therefore, f is pair difference cordial labeling of $K_{n,n+1} \cup (n^2 - 2n - 2)K_2$. The maximum number of edges with labels 1 from the $DS(K_{n,n+1})$ is, $\Delta_{f_1} = 2n + 1$. But the size of the $DS(K_{n,n+1})$ is $n^2 + 3n + 1$. Hence $n^2 - n - 2$ is the least integer such that $DS(K_{n,n+1}) \cup (n^2 - n - 2)K_2$ is pair difference cordial.

Subcase 3.2 n is odd.

Take $m = n^2 - n - 2$. Define $f : V(DS(K_{n,n+1}) \cup (n^2 - n - 2)K_2) \rightarrow \{\pm 1, \pm 2, \dots, \pm \lfloor \frac{2n+3}{2} \rfloor\}$ as follows:

Assign the labels $3, 5, 7, \dots, n$ to the vertices $y_1, y_2, y_3, \dots, y_{\frac{n-1}{2}}$ and assign the labels $2, 4, 6, \dots, n+1$ to the vertices $x_1, x_2, x_3, \dots, x_{\frac{n+1}{2}}$. Next assign the labels $-3, -5, -7, \dots, -n$ to the vertices $x_{\frac{n+1}{2}+1}, x_{\frac{n+1}{2}+2}, x_{\frac{n+1}{2}+3}, \dots, x_n$ and assign the labels $-2, -4, -6, \dots, -(n+1)$ to the vertices $y_{\frac{n-1}{2}+1}, y_{\frac{n-1}{2}+2}, y_{\frac{n-1}{2}+3}, \dots, y_n$. Now assign the labels $1, -1, -n$ to the vertices x, y, x_{n+1} . Next assign the labels $(n+2), (n+3), -(n+2), -(n+3)$ to the vertices v_1, w_1, v_2, w_2 and assign the labels $(n+4), (n+5), -(n+4), -(n+5)$ to the vertices v_{n+3}, w_3, v_4, w_4 . Proceeding like this until we reach v_{n^2-n-2}, w_{n^2-n-2} .

Therefore, f is pair difference cordial labeling of $K_{n,n+1} \cup (n^2 - 2n - 2)K_2$. The maximum number of edges with labels 1 from the $DS(K_{n,n+1})$ is, $\Delta_{f_1} = 2n + 1$. But the size of the $DS(K_{n,n+1})$ is $n^2 + 3n + 1$. Hence $n^2 - n - 2$ is the least integer such that $DS(K_{n,n+1}) \cup (n^2 - n - 2)K_2$ is pair difference cordial. \square

References

- [1] Cahit I., Cordial graphs, a weaker version of graceful and harmonious graphs, *Ars Combin.*, 23(1987), 201–207.
- [2] Gallian J. A., A Dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 19(2016).
- [3] Harary F., *Graph Theory*, Addison wesley, New Delhi, 1969.
- [4] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordial labeling of graphs, *J.Math. Comp.Sci.*, Vol.11(3), (2021), 2551–2567.
- [5] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordiality of some snake and butterfly graphs, *Journal of Algorithms and Computation*, Vol.53(1), (2021), 149–163.
- [6] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordial graphs obtained from the wheels and the paths, *J. Appl. and Pure Math.*, Vol.3 No. 3-4, (2021), pp. 97–114.
- [7] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordiality of some graphs derived from ladder graph, *J.Math. Comp.Sci.*, Vol.11 No 5, (2021), 6105–6124.
- [8] Ponraj R., Gayathri A. and Soma Sundaram S., Some pair difference cordial graphs, *Ikonion Journal of Mathematics*, Vol.3(2), (2021), 17–26.
- [9] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordial labeling of planar grid and mangolian tent, *Journal of Algorithms and Computation*, Vol.53(2), December (2021), 47–56.

- [10] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordiality of some special graphs, *J. Appl. and Pure Math.*, Vol.3 No. 5-6, (2021), pp. 263–274.
- [11] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordial labeling of some star related graphs, *Journal of Mehani Mathematical Research*, Vol.12(2), (2023), 255–266.
- [12] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordial labeling of m – copies of some graphs, *J. Appl. and Pure Math.*, Vol.4 No. 5–6, (2022), pp. 317–329.
- [13] Ponraj R., Gayathri A. and Somasundaram S., Pair difference cordiality of degree splitting graph of some graphs (Submitted to the journal).
- [14] Ponraj R. and Gayathri A., Pair difference cordial number of graphs, *J. Appl. & Pure Math.*, No.3-4, 6(2024), 127-139.
- [15] Ponraj R. and S. Somasundaram, On the degree splitting of graph of a graph, *National Academy Science Letter*, 27(2004), 275-278.
- [16] Seoud M. A. and Salman M. S., On difference cordial graphs, *Mathematica Aeterna*, Vol.5(2015), 189–199.
- [17] Seoud M. A. and Salman M. S., Some results and examples on difference cordial graphs, *Turkish Journal of Mathematics*, 40(2016), 417–427.