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Radio Number for Special Family of Graphs with Diameter 2, 3 and 4

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Abstract: A radio labeling of a graph G is an injective function [4] $f : V(G) \to \mathbb{N} \cup \{0\}$ such that for every $u, v \in V(G)$,

$$|f(u) - f(v)| \geqslant \text{diam}(G) - d(u, v) + 1.$$

The span of f is the difference of the largest and the smallest channels used, that is,

$$\max_{u,v \in V(G)} \{f(u) - f(v)\}.$$

The radio number of G is defined as the minimum span of a radio labeling of G and denoted as $rn(G)$. In this paper, we present algorithms to get the radio labeling of special family of graphs like double cones, books and nC_4 with a common vertex whose diameters are 2,3 and 4 respectively.

Key Words: Radio labeling, radio number, channel assignment, distance two labeling.

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§1. Introduction

The channel assignment problem is a telecommunication problem in which our aim is to assign a channel (non-negative integer) to each TV or Radio station so that we do not have any interference in the communication. The level of interference between the TV or Radio stations correlates with the geographic locations of these stations. Earlier designers of TV networks considered close locations and very close locations so that the transmitters at the close locations receive different channels and the transmitters at very close locations are at least two apart for clear communication.

This can be modeled by graph models in which vertices represent the stations and two vertices are joined by an edge if they are very close and they are joined by a path of length 2 if they are close.

The mathematical abstraction of the above concept is distance two labeling or $L(2,1)$-labeling. A $L(2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x) - f(y)| \geqslant 2$ if x and y are adjacent and $|f(x) - f(y)| \geqslant 1$ if x and y are at distance 2, for all x and y in $V(G)$.

Practically speaking the interference among channels may go beyond two levels. So we have to

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extend the interference level from two to the largest possible - the diameter of the corresponding graph. So the concept of $L(2,1)$ labeling was generalized to radio labeling.

Radio labeling was originally introduced by G.Chartrand et.al., [1] in 2001. A radio labeling of a graph G is an injective function $f : V(G) \to \mathbb{N} \cup \{0\}$ such that for every $u, v \in V(G)$,

$$|f(u) - f(v)| \geqslant \text{diam}(G) - d(u,v) + 1.$$

The span of f is the difference of the largest and the smallest channels used, that is,

$$\max_{u,v \in V(G)} \{f(u) - f(v)\}.$$

The radio number of G is defined as the minimum span of a radio labeling of G and denoted as $rn(G)$.

§2. Some Existing Results

(1) D.D.F. Liu and X. Zhu [2] have discussed the radio number for paths and cycles.

(2) D.D.F. Liu [3] has obtained lower bounds for the radio number of trees and characterized trees achieving this bound.

(3) Mustapha et al. [4] have discussed radio k-labeling for cartesian products of graphs.

(4) The radio labeling of cube and fourth power of cycles have been discussed by B. Sooryanarayana and P. Raghunath [5, 6].

(5) The radio labeling of k^{th} power of a path is discussed by P. Devadas Rao et al. [7]

(6) The radio number for the split graph and the middle graph of cycle C_n is discussed by S.K. Vaidya et al. [8].

In the survey of literature available on radio labeling, we found that only two types of problems are considered in this area till this date.

(1) To investigate bounds for the radio number of a graph;

(2) To completely determine the radio number of a graph.

§3. Results

In this section, we develop and justify algorithms to get Radio labeling of the family of graphs - double cones whose diameter is 2, books whose diameter is 3 and nC_4 with a common vertex whose diameter is 4.

We note that the concept of radio labeling and $L(2,1)$ labeling coincides when the diameter of the graph is 2. In general, to prove a labeling is a radio labeling, we have to prove that every $u, v \in V(G)$, $|f(u) - f(v)| \geqslant \text{diam}(G) - d(u,v) + 1$.

Double Cones are graphs obtained by joining two isolated vertices to every vertex of C_n. A Double Cone on $n+2$ vertices is denoted by $CO_{n+2} = C_n + 2K_1$. We note that CO_{n+2} has two vertices with maximum degree n.

Algorithm 1:

Input : A double cone $CO_{n+2} = C_n + 2K_1$, $n \geqslant 5$.

Output : Radio labeling of CO_{n+2}.

Step 1: Arrange the vertices of C_n as $v_1, v_2, \ldots, v_n$ and let u_1, u_2 denotes $2K_1$.

Step 2: First label v_is of odd i and then label v_is of even i starting from the label 3 and ending with $n+2$ consecutively.

Step 3 : Label u_1 with 0 and u_2 with 1.

The justification of the desired output is proved in Theorem 3.1.

Theorem 3.1 *For the double cone CO_{n+2}, $n \geqslant 5$, $rn(CO_{n+2}) = n+2$.*

Proof Consider the double cone CO_{n+2}, $n \geqslant 5$. Let $V(Cn) = \{v_1, v_2, \ldots, v_n\}$ and u_1, u_2 denotes $2K_1$. Define $f : V(CO_{n+2}) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} f(u_1) &= 0 \\ f(u_2) &= 1 \\ f(v_1) &= 3 \\ f(v_i) &= f(v_1) + \frac{i-1}{2} \text{ if } i \text{ is odd and } i > 1. \\ f(v_2) &= f(v_1) + \left\lceil \frac{n}{2} \right\rceil \\ f(v_i) &= f(v_1) + \left\lceil \frac{n}{2} \right\rceil + \frac{i-2}{2} \text{ if } i \text{ is even and } i > 2. \end{aligned}$$

When $d(v_i, v_j) = 1$, we have $|f(v_i) - f(v_j)| = \lceil \frac{n}{2} \rceil$ or $\lceil \frac{n}{2} \rceil - 1$, $n-1$. That is, when $d(v_i, v_j) = 1$, we have $|f(v_i) - f(v_j)| \geqslant 2$. When $d(v_i, v_j) = 2$, we have $|f(v_i) - f(v_j)| \geqslant 1$ since $f(v_i)$s are distinct. Since $f(v_i) \geqslant 3$ and $f(u_1) = 0$, we have $|f(v_i) - f(u_1)| \geqslant 3$. Since $f(v_i) \geqslant 3$ and $f(u_2) = 1$, we have $|f(v_i) - f(u_2)| \geqslant 2$. Hence f is a radio labeling and $rn(CO_{n+1}) \leqslant n+2$.

Since CO_{n+2} has two vertices u_1 and u_2 with maximum degree n and that too, they are adjacent with the same vertices $v_1, v_2, \ldots, v_n$, the consecutive $n+2$ labels $0, 1, 2, \ldots, n+1$ are not sufficient to produce a radio-labeling. Therefore, $rn(CO_{n+2}) \geqslant n+2$ and hence $rn(CO_{n+2}) = n+2$. □

A book B_n is the product of the star $K_{1,n}$ with K_2. A book B_n has $p = 2n+2$ vertices, $q = 3n+1$ edges, n pages, maximum degree $n+1$ and diameter 3. Next we present an algorithm to get a radio labeling of a book B_n.

Algorithm 2:

Input : A book B_n, $n \geqslant 4$, with p vertices.

Output : Radio labeling of B_n.

Step 1: Let the vertices of the n^{th} page of B_n be v, v_n, w_n and w where v and w lie on the common edge vw.

Step 2: Label v with the label $p+1$ and w with the label 0.

Step 3: Label v_i with $2i$, $i = 1, 2, \cdots, n$.

Step 4: Label w_i with $f(v_i) + 5$, $i = 1, 2, \ldots, n-2$ and w_{n-1} with the label 3 and w_n with the label 5, where $f(v_i)$ is the label of v_i.

The justification of the desired output is proved in Theorem 3.2.

Theorem 3.2 *For a Book B_n, $n \geqslant 4$, $rn(B_n) = p+1$.*

Proof Consider a book B_n. A book B_n has $p = 2n + 2$ vertices, $q = 3n + 1$ edges, n pages, maximum degree $n + 1$ and diameter 3. Let the vertices of the n^{th} page of B_n be v, v_n, w_n and w where v and w lie on the common edge vw.

Define $f : V(B_n) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned}
f(v) &= p+1\\
f(w) &= 0\\
f(v_i) &= 2i, \quad i = 1, 2, \cdots, n\\
f(w_i) &= f(v_i) + 5, \quad i = 1, 2, \cdots, n-2\\
f(w_{n-1}) &= 3\\
f(w_n) &= 5
\end{aligned}$$

We note that $d(v, v_i) = 1$, $d(w, w_i) = 1$, $d(v_i, w_i) = 1$, $i = 1, 2, \ldots, n$ and $d(v, w) = 1$. Now,

$$\begin{aligned}
|f(v) - f(v_i)| &= p + 1 - 2i = 2n + 2 + 1 - 2i = 2n + 3 - 2i \geqslant 3, \quad i = 1, 2, \cdots, n.\\
|f(w) - f(w_i)| &= f(v_i) + 5 = 2i + 5 \geqslant 3, \quad i = 1, 2, \cdots, n-2.\\
|f(w) - f(w_{n-1})| &= 3.\\
|f(w) - f(w_n)| &= 5.
\end{aligned}$$

Consider $f(v_i) - f(w_i) = 2i + 5 - 2i = 5, \quad i = 1, 2, \cdots, n-2.$

$$\begin{aligned}
|f(v_{n-1}) - f(w_{n-1})| &= 2n - 2 - 3 = 2n - 5 \geqslant 3.\\
|f(v_n) - f(w_n)| &= 2n - 5 \geqslant 3.\\
\text{Also} \quad |f(v) - f(w)| &= p + 1 = 2n + 3.
\end{aligned}$$

Thus, $|f(s_i) - f(t_i)| \geqslant 3$ when $d(s_i, t_i) = 1$, $s_i, t_i \in V(B_n)$. We note that $d(v, w_i) = 2$, $d(w, v_i) = 2$, $i = 1, 2, \cdots, n$, $d(v_i, v_j) = 2$, $i \neq j$, $i, j = 1, 2, \cdots, n$ and $d(w_i, w_j) = 2$, $i \neq j$, $i, j = 1, 2, \cdots, n$.

When $d(v, w_i) = 2$, $i = 1, 2, \cdots, n$,

$$\begin{aligned}
|f(v) - f(w_i)| &= p + 1 - 2i - 5 = 2n + 2 + 1 - 2i - 5 = 2n - 2 - 2i \geqslant 2,\\
&\qquad\qquad\qquad\qquad i = 1, 2, \cdots, n-2.\\
|f(v) - f(w_{n-1})| &= p + 1 - 3 = p - 2 \geqslant 2\\
|f(v) - f(w_n)| &= p + 1 - 5 = p - 4 \geqslant 2
\end{aligned}$$

When $d(w, v_i) = 2$, $i = 1, 2, \cdots, n$, $|f(w) - f(v_i)| = 2i \geqslant 2$, $i = 1, 2, \cdots, n$. When $d(v_i, v_j) = 2$, $i \neq j$, $i, j = 1, 2, \cdots, n$, consider for $i > j$, $|f(v_i) - f(v_j)| = 2i - 2j \geqslant 2$. When $d(w_i, w_j) = 2$, $i \neq j$, $i, j = 1, 2, \cdots, n-2$, consider for $i > j$, $|f(w_i) - f(w_j)| = |f(v_i) - f(v_j)| = 2i - 2j \geqslant 2$. Since $f(w_{n-1}) = 3$ and $f(w_n) = 5$, $|f(w_i) - f(w_j)| \geqslant 2$, for all $i, j = 1, 2, \cdots, n$. Thus, $|f(s_i) - f(t_i)| \geqslant 2$ when $d(s_i, t_i) = 2$, $s_i, t_i \in V(B_n)$.

Since all the vertex labels are distinct, we have,

$$|f(s_i) - f(t_i)| \geqslant 1 \text{ when } d(s_i, t_i) = 3, \ s_i, t_i \in V(B_n).$$

Hence f is a radio labeling and $rn(B_n) \leqslant p + 1$.

Since there are p vertices, there should be p distinct labels. The p labels $0, 1, 2, \cdots, p-1$ are not sufficient to produce a radio labeling of the Book B_n.

Suppose they produce a radio labeling, we note that B_n has 2 adjacent vertices v and w such that each is adjacent with a different set of $\frac{p-2}{2}$ vertices, say, A and B respectively such that

(i) each vertex in A should have label difference at least 3 with v and among themselves they should have label difference at least 2;

(ii) each vertex in B should have label difference at least 3 with w and among themselves they should have label difference at least 2 and

(iii) the label difference of v and w should be 3.

This is not possible with p consecutive non-negative integers $0, 1, 2, \cdots, p-1$. Therefore B_n should have more than these p labels $0, 1, 2, \cdots, p-1$. Since we have 2 partitions of the vertex set of B_n, say, $A \cup \{v\}$ and $B \cup \{w\}$ with the above three properties, B_n should have at least 2 more labels to have a radio labeling. Therefore, $rn(B_n) \geqslant p+1$. Hence $rn(B_n) = p+1$. □

Now we consider the graph, the collection of n copies of C_4s, all of which have a common vertex, that is, nC_4 with a common vertex.

Algorithm 3:

Input : nC_4 with a common vertex, $n \geqslant 3$.

Output : Radio labeling of the graph, nC_4 with a common vertex.

Step 1: Let the common vertex be u and the vertices of the i^{th} cycle be u, v_{i_1}, v_{i_2} and v_{i_3}, $i = 1, 2, \cdots, n$.

Step 2: Label the common vertex u with $7n+2$.

Step 3: Label v_{i_2} with label $i-1$, $i = 1, 2, \cdots, n$.

Step 4: Arrange the remaining vertices as $v_{11}, v_{13}, v_{21}, v_{23}, \cdots, v_{n_1}, v_{n_3}$. Label these vertices starting from $n+1, n+4, n+7, \ldots, 7n-2$.

The justification of the desired output is proved in Theorem 3.3.

Theorem 3.3 *Let G denotes nC_4 with a common vertex. Then $rn(G) = 7n+2$, $n \geqslant 3$.*

Proof Consider the graph, nC_4 with a common vertex. Let the common vertex be u and the vertices of the i^{th} cycle be u, v_{i_1}, v_{i_2} and v_{i_3}, $i = 1, 2, \cdots, n$.

Define $f : V(G) \to \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} f(u) &= 7n+2 \\ f(v_{i_2}) &= i-1, \quad i = 1, 2, \cdots, n \\ f(v_{i_1}) &= n+1+(i-1)6, \quad i = 1, 2, \cdots, n \\ f(v_{i_3}) &= n+4+(i-1)6, \quad i = 1, 2, \cdots, n \end{aligned}$$

We note that $d(u, v_{ij}) = 1$, $i = 1, 2, \cdots, n$ and $j = 1, 3$. Also $d(v_{i_1}, v_{i_2}) = 1$ and $d(v_{i_3}, v_{i_2}) = 1$, $i = 1, 2, \cdots, n$. Now, $|f(u) - f(v_{i_1}) = 7n+2-n-1-(i-1)6 = 6n+1-(i-1)6 = 6n+7-6i \geqslant 4, i = 1, 2, \cdots, n$ and $|f(u) - f(v_{i_3}) = 7n+2-n-4-(i-1)6 = 6n-2-(i-1)6 = 6n+4-6i \geqslant 4, i = 1, 2, \cdots, n$.

Consider

$$|f(v_{i_1}) - f(v_{i_2})| = n+1+(i-1)6-(i-1) = n+1+(i-1)5 \geqslant 4, \ i = 1, 2, \cdots, n.$$

Consider

$$|f(v_{i_3})-f(v_{i_2})|=n+4+(i-1)6-(i-1)=n+4+(i-1)5\geqslant 4,\ \ i=1,2,\cdots,n.$$

We note that, for $i=1,2,\ldots,n$ and $j=2$, $d(u,v_{ij})=2$ and for $i,\ \alpha=1,2,\cdots,n$ and $j,k=1,3$, $d(v_{ij},v_{\alpha k})=2$ $(i,\alpha,j,k$ all are not equal).

Since $f(u)=7n+2$ and for $i=1,2,\cdots,n$ and $j=2$, $f(v_{ij})=i-1$, we have,

$$|f(u)-f(v_{ij})|=7n+2-i+1=7n+3-i\geqslant 3,\quad i=1,2,\cdots,n \text{ and } j=2.$$

Now, since, for $i=1,2,\cdots,n$, $f(v_{i_1})=n+1+(i-1)6$ and $f(v_{i_3})=n+4+(i-1)6$, we have, for $i,\alpha=1,2,\cdots,n$ and $j,k=1,3$, $(i,\alpha,j,k$ all are not equal), $|f(v_{ij})-f(v_{\alpha k})\geqslant 3$.
Next, we note that, for $i=1,2,\cdots,n$, $d(v_{i_2},v_{j_k})=3$ where $j=1,2,\cdots,n$ & $j\neq i$ and $k=1,3$. In this case, $|f(v_{j_1})-f(v_{i_2})=n+1+(j-1)6-(i-1)=n+6j-i-4\geqslant 2$. Also, $|f(v_{j_3})-f(v_{i_2})|=n+4+(j-1)6-(i-1)=n+6j-i-1\geqslant 2$. Finally, for $i\neq j$, $d(v_{i_2},v_{j_2})=4$. Since all the labels of v_{i_2}, $i=1,2,\cdots,n$ are distinct, $|f(v_{i_2})-f(v_{j_2})|\geqslant 1$. Hence f is a radio labeling and $rn(G)\leqslant 7n+2$.

Now we find the optimal value of $rn(G)$. The minimum possible label of u is 0. Since the distance between u and v_{11} is 1, the minimum label we can use at v_{11} is at least 4. Since the vertices $v_{13},v_{21},v_{23},v_{31},v_{33},\cdots,v_{n_1},v_{n_3}$ are at a distance 2 from v_{11}, each of this should have a label difference 3 with v_{11} and among themselves. And so the minimum range of the labels for these vertices is $\{7,10,\cdots,6n+1\}$. So, without loss of generality,we assume that the minimum label v_{n_3} should receive is $6n+1$. There are n more vertices, namely, $v_{12},v_{22},\cdots,v_{n_2}$. We cannot use the label $6n+2$ to any of these vertices because these vertices are at a distance 3 from v_{n_3} and hence the label difference at these vertices should be at least 2. So the minimum label we can use to any one of these vertices is $6n+3$ and there are n such vertices. Therefore the minimum label we required to label all these vertices is $6n+1+(n+1)=7n+2$.

Therefore, the minimum label which can give a radio labeling for the graph G is $7n+2$. That is, $rn(G)\geqslant 7n+2$. Hence, $rn(G)=7n+2$. □

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