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Reverse Hyper-Zagreb Indices of the Cartesian Product of Graphs

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Abstract: In this paper, found some exact expressions for the first and second reverse hyper Zagreb indices of Cartesian product of two simple connected graphs are determined. And repeated same for the forgotten topological index F and F_1 reverse index of a molecular graphs.

Key Words: Reverse Zagreb index, reverse hyper-Zagreb index, Cartesian product of graph.

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§1. Introduction

Let G be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $\Delta(G)$ be the maximum degree of vertex among the set $V(G)$. The degree of the vertex v is denoted by d_v. In Chemical Science, a molecular graph is a graph in which atoms are taken as vertices and bonds as edges.Topological index is used to characterize some property of molecular graphs and those are used in theoretical Chemistry [1].

Forty years ago Gutman and Trinajestic [2], in which they defined first and second Zagreb indices as,

$$M_1 = M_1(G) = \sum_{v\in V(G)} d_v^2 \tag{1}$$

and

$$M_2 = M_2(G) = \sum_{uv\in E(G)} d_u d_v \tag{2}$$

respectively, where uv denotes an edge connecting the vetices u and v. Before we move to main results, first look to the know definitions which is in [3].

Definition 1.1 *Let G be a simple connected graph and v be a vertex of G. Then, the reverse vertex degree C_v of the vertex v is defined as $C_v = \Delta(G) - d_v + 1$.*

Definition 1.2 *Let G be a simple connected graph and v be a vertex of G.Then, the total reverse vertex degree $TR(G)$ of the graph G is the sum of the reverse vertex degree of the vertices of G.*

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That is,

$$TR(G) = \sum_{u \in V(G)} C_u$$

Definition 1.3 *Let G be a simple connected graph and v be a vertex of G. Then, the first reverse Zagreb alpha index of G defined as*

$$CM_1^{\alpha}(G) = \sum_{v \in V(G)} C_v^2$$

Definition 1.4 *Let G be a simple connected graph and v be a vertex of G. Then, the first reverse Zagreb beta index of G defined as*

$$CM_1^{\beta}(G) = \sum_{uv \in E(G)} (C_u + C_v)$$

Definition 1.5 *Let G be a simple connected graph and v be a vertex of G. Then, the second reverse Zagreb index of G defined as*

$$CM_2(G) = \sum_{uv \in E(G)} (C_u C_v)$$

Definition 1.6([6]) *The first and second reverse hyper-Zagreb indices of a simple graph G is defined as*

$$HCM_1(G) = \sum_{uv \in E(G)} (C_u + C_v)^2 \quad and \quad HCM_2(G) = \sum_{uv \in E(G)} (C_u C_v)^2$$

Definition 1.7[7] *The F-reverse index $FC(G)$ and $F_1C(G)$ index of a graph G is defined as,*

$$FC(G) = \sum_{uv \in E(G)} (c_u^2 + c_v^2) \quad and \quad F_1C(G) = \sum_{u \in V(G)} c_u^3.$$

Definition 1.8 *The Cartesian product $G \times H$ of graph G and H is a graph such that,*

(1) *the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$;*

(2) *$(a,b)(c,d)$ is an edge of $G \times H$ if $a = c$ and $bd \in E(G)$, or $b = d$ and $ac \in E(G)$.*

In [5] first and second Zagreb indices of the Cartesian product of graphs are computed and other topological indices of the product of graphs are found in [8], [9] and [10]. Further, in [3] S. Ediz found expressions for the reverse Zagreb indices of cartesian product of two simple connected graphs. By the motivation of [3] in this paper, Reverse hyper-Zagreb indices and forgotten reverse index of cartesian product of two simple connected graphs are determined.

§2. Preliminarily

The following are the preliminarily that have helped in our results and those are found in [3].

Lemma 2.1 *Let G and H be two simple connected graphs and $(a,b) \in E(G \times H)$. Then*

(a) $|V(G \times H)| = |V(G)||V(H)|$;
(b) $|E(G \times H)| = |V(G)||E(H)| + |E(G)||V(H)|$;
(c) $d_{G\times H}((a,b)) = d_a + d_b$.

Corollary 2.2 *Let G and H be two simple connected graphs. Then $\Delta_{G\times H} = \Delta_G + \Delta_H$.*

Proof Let $d_a = \Delta_G$ and $d_b = \Delta_H$. Then $V(G \times H)$. Clearly from above lemma, we have the result. □

Proposition 2.3 *Let $(a,b) \in V(G \times H)$. Then $C_{(a,b)} = C_a + C_b - 1$.*

§3. Reverse Hyper-Zagreb Indices of Cartesian Product of Graphs

In this section, we found reverse hyper-Zagreb indices of Cartesian product of two simply connected graph.

Theorem 3.1 *If G and H be two connected simple graphs then,*

$$\begin{aligned} HCM_1(G \times H) =& 4|E(H)|CM_1^{\alpha}(G) - 8|E(H)|TR(G) + 4|E(H)||V(G)| + |V(G)|HCM_1(H) \\ &+ 4[\ TR(G) - |V(G)|\ ]CM_1^{\beta}(H) + 4|E(G)|CM_1^{\alpha}(H) - 8|E(G)|TR(H) \\ &+ 4|E(G)||V(H)| + |V(H)|HCM_1(G) + 4[\ TR(H) - |V(H)|\ ]CM_1^{\beta}(G). \end{aligned}$$

Proof By Proposition 2.3, $C_{(a,b)} = C_a + C_b - 1$. So

$$\begin{aligned} HCM_1(G \times H) =& \sum_{(a,b)(c,d)\in E(G\times H)} [C_{(a,b)} + C_{(c,d)}]^2 \\ =& \sum_{u\in V(G)} \sum_{bd\in E(H)} [(C_u + C_b - 1) + (C_u + C_d - 1)]^2 \\ &+ \sum_{v\in V(H)} \sum_{ac\in E(G)} [(C_v + C_a - 1) + (C_v + C_c - 1)]^2 \\ =& \sum_{u\in V(G)} \sum_{bd\in E(H)} [2(C_u - 1) + (C_b + C_d)]^2 \\ &+ \sum_{v\in V(H)} \sum_{ac\in E(G)} [2(C_v - 1) + (C_a + C_c)]^2 \\ =& 4 \sum_{u\in V(G)} \sum_{bd\in E(H)} (C_u - 1)^2 + \sum_{u\in V(G)} \sum_{bd\in E(H)} C_b^2 + \sum_{u\in V(G)} \sum_{bd\in E(H)} C_d^2 \end{aligned}$$

$$
\begin{aligned}
&+4\sum_{u\in V(G)}\sum_{bd\in E(H)}(C_u-1)C_b+4\sum_{u\in V(G)}\sum_{bd\in E(H)}(C_u-1)C_d\\
&+\sum_{u\in V(G)}\sum_{bd\in E(H)}C_bC_d+\sum_{v\in V(H)}\sum_{ac\in E(G)}(C_v-1)^2\\
&+\sum_{v\in V(H)}\sum_{ac\in E(G)}C_a^2+\sum_{v\in V(H)}\sum_{ac\in E(G)}C_c^2+4\sum_{v\in V(H)}\sum_{ac\in E(G)}(C_v-1)C_a\\
&+4\sum_{v\in V(H)}\sum_{ac\in E(G)}(C_v-1)C_c+\sum_{v\in V(H)}\sum_{ac\in E(G)}C_aC_c\\
=&4\sum_{u\in V(G)}\sum_{bd\in E(H)}[C_u^2-2C_u+1]+\sum_{u\in V(G)}\sum_{bd\in E(H)}[C_b^2+C_d^2+2C_bC_d]\\
&+4\sum_{u\in V(G)}\sum_{bd\in E(H)}(C_u-1)(C_b+C_d)+4\sum_{v\in V(H)}\sum_{ac\in E(G)}[C_v^2-2C_v+1]\\
&+\sum_{v\in V(H)}\sum_{ac\in E(G)}[C_a^2+C_c^2+2C_aC_c]+4\sum_{v\in V(H)}\sum_{ac\in E(G)}(C_v-1)(C_a+C_c)\\
=&4|E(H)|CM_1^{\alpha}(G)-8|E(H)|TR(G)+4|E(H)||V(G)|+|V(G)|HCM_1(H)\\
&+4[\ TR(G)-|V(G)|\ ]CM_1^{\beta}(H)+4|E(G)|CM_1^{\alpha}(H)-8|E(G)|TR(H)\\
&+4|E(G)||V(H)|+|V(H)|HCM_1(G)+4[\ TR(H)-|V(H)|\ ]CM_1^{\beta}(G)
\end{aligned}
$$

This completes the proof. □

Theorem 3.2 *If G and H be two connected simple graphs then,*

$$
\begin{aligned}
HCM_2(G\times H)=&|E(H)|[CM_1^2(G)]^2-4|E(H)|TR(G)CM_1^2(G)+6|E(H)|CM_1^2(G)\\
&-4TR(G)|E(H)|+|E(H)||V(G)|+\big(CM_1^2(G)-2TR(G)\\
&+|V(G)|\big)\left(HCM_1(H)+2CM_2(H)\right)+2CM_2(G)CM_1^{\beta}(H)\left(TR(G)-|V(G)|\right)\\
&-6CM_1^{\alpha}(G)CM_1^{\alpha}(H)+6TR(G)CM_1^{\beta}(G)-2|V(G)|CM_1^{\beta}(H)\\
&+HCM_2(G)|V(H)|+|E(G)|[CM_1^2(H)]^2-4|E(G)|TR(H)CM_1^2(H)\\
&+6|E(G)|CM_1^2(H)-4TR(H)|E(G)|+|E(G)||V(H)|+\big(CM_1^2(H)-2TR(H)\\
&+|V(H)|\big)\left(HCM_1(G)+2CM_2(G)\right)+2CM_2(H)CM_1^{\beta}(G)\left(TR(H)-|V(H)|\right)\\
&-6CM_1^{\alpha}(H)CM_1^{\alpha}(G)+6TR(H)CM_1^{\beta}(H)-2|V(H)|CM_1^{\beta}(G)+HCM_2(H)|V(G)|
\end{aligned}
$$

Proof By Proposition 2.3, $C_(a,b)=C_a+C_b-1$. So

$$
\begin{aligned}
HCM_2(G\times H)=&\sum_{(a,b)(c,d)\in E(G\times H)}(C_{(a,b)}C_{(c,d)})^2\\
=&\sum_{u\in V(G)}\sum_{bd\in E(H)}[(C_u+C_b-1)(C_u+C_d-1)]^2\\
&+\sum_{v\in V(H)}\sum_{ac\in E(G)}[(C_v+C_a-1)(C_v+C_c-1)]^2\\
=&\sum_{u\in V(G)}\sum_{bd\in E(H)}[(C_u-1)^2+(C_b+C_d)(C_u-1)+C_bC_d]^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{v\in V(H)}\sum_{ac\in E(G)}[(C_v-1)^2+(C_a+C_c)(C_u-1)+C_aC_c]^2\\
=&\sum_{u\in V(G)}\sum_{bd\in E(H)}[C_u^4-4C_u^3+6C_u^2-4C_u+1]\\
&+[(C_b+C_d)^2+2C_bC_d](C_u^2-2C_u+1)\\
&+2C_bC_d(C_b+C_d)(C_u-1)-6C_u^2(C_b+C_d)+6C_u(C_b+C_d)\\
&-2(C_b+C_d)+C_b^2C_d^2+\sum_{v\in V(H)}\sum_{ac\in E(G)}[C_v^4-4C_v^3+6C_v^2-4C_v+1]\\
&+[(C_+C_c)^2+2C_aC_c](C_v^2-2C_v+1)+2C_aC_c(C_a+C_c)(C_v-1)\\
&-6C_v^2(C_a+C_c)+6C_v(C_a+C_c)-2(C_a+C_c)+C_a^2C_c^2\\
=&\sum_{u\in V(G)}\sum_{bd\in E(H)}[C_u^4-4C_u^3+6C_u^2-4C_u+1]\\
&+\sum_{u\in V(G)}\sum_{bd\in E(H)}[(C_b+C_d)^2+2C_bC_d](C_u^2-2C_u+1)\\
&+2\sum_{u\in V(G)}\sum_{bd\in E(H)}C_bC_d(C_b+C_d)(C_u-1)-6\sum_{u\in V(G)}\sum_{bd\in E(H)}C_u^2(C_b+C_d)\\
&+6\sum_{u\in V(G)}\sum_{bd\in E(H)}C_u(C_b+C_d)-2\sum_{u\in V(G)}\sum_{bd\in E(H)}(C_b+C_d)\\
&+\sum_{u\in V(G)}\sum_{bd\in E(H)}C_b^2C_d^2+\sum_{v\in V(H)}\sum_{ac\in E(G)}[C_v^4-4C_v^3+6C_v^2-4C_v+1]\\
&+\sum_{v\in V(H)}\sum_{ac\in E(G)}[(C_+C_c)^2+2C_aC_c](C_v^2-2C_v+1)\\
&+2\sum_{v\in V(H)}\sum_{ac\in E(G)}C_aC_c(C_a+C_c)(C_v-1)-6\sum_{v\in V(H)}\sum_{ac\in E(G)}C_v^2(C_a+C_c)\\
&+6\sum_{v\in V(H)}\sum_{ac\in E(G)}C_v(C_a+C_c)-2\sum_{v\in V(H)}\sum_{ac\in E(G)}(C_a+C_c)\\
&+\sum_{v\in V(H)}\sum_{ac\in E(G)}C_a^2C_c^2\\
=&|E(H)|[CM_1^2(G)]^2-4|E(H)|TR(G)CM_1^2(G)+6|E(H)|CM_1^2(G)-4TR(G)|E(H)|\\
&+|E(H)||V(G)|+\left(CM_1^2(G)-2TR(G)+|V(G)|\right)(HCM_1(H)+2CM_2(H))\\
&+2CM_2(G)CM_1^\beta(H)\left(TR(G)-|V(G)|\right)-6CM_1^\alpha(G)CM_1^\alpha(H)\\
&+6TR(G)CM_1^\beta(G)-2|V(G)|CM_1^\beta(H)+HCM_2(G)|V(H)|+|E(G)|[CM_1^2(H)]^2\\
&-4|E(G)|TR(H)CM_1^2(H)+6|E(G)|CM_1^2(H)-4TR(H)|E(G)|+|E(G)||V(H)|\\
&+\left(CM_1^2(H)-2TR(H)+|V(H)|\right)(HCM_1(G)+2CM_2(G))\\
&+2CM_2(H)CM_1^\beta(G)\left(TR(H)-|V(H)|\right)-6CM_1^\alpha(H)CM_1^\alpha(G)+6TR(H)CM_1^\beta(H)\\
&-2|V(H)|CM_1^\beta(G)+HCM_2(H)|V(G)|.
\end{aligned}
$$

This completes the proof. $\square$

Theorem 3.3 *If G and H be two connected molecular graphs then,*

$$\begin{aligned}FC(G\times H)=&2|E(H)|CM_1^{\alpha}(G)+|V(G)|FC(H)+2TR(G)CM_1^{\beta}(H)-4|E(H)|TR(G)\\&-2|V(G)|CM_1^{\beta}(H)+2|V(G)||E(H)|+2|E(G)|CM_1^{\alpha}(H)+|V(H)|FC(G)\\&+2TR(H)CM_1^{\beta}(G)-4|E(G)|TR(H)-2|V(H)|CM_1^{\beta}(G)+2|V(H)||E(G)|\end{aligned}$$

Proof By Proposition 2.3, $C_(a,b)=C_a+C_b-1$. So

$$\begin{aligned}FC(G\times H)&=\sum_{(a,b)(c,d)\in E(G\times H)}(C^2_{(a,b)}+C^2_{(c,d)})\\&=\sum_{u\in V(G)}\sum_{bd\in E(H)}[(C_u+C_b-1)^2+(C_u+C_d-1)^2]\\&\quad+\sum_{v\in V(H)}\sum_{ac\in E(G)}[(C_v+C_a-1)^2+(C_v+C_c-1)^2]\\&=\sum_{u\in V(G)}\sum_{bd\in E(H)}2[C_u^2]+\sum_{u\in V(G)}\sum_{bd\in E(H)}[C_b^2+C_d^2]\\&\quad+\sum_{u\in V(G)}\sum_{bd\in E(H)}2C_u(C_b+C_d)-\sum_{u\in V(G)}\sum_{bd\in E(H)}4C_u\\&\quad-\sum_{u\in V(G)}\sum_{bd\in E(H)}2(C_b+C_d)+\sum_{u\in V(G)}\sum_{bd\in E(H)}2\\&\quad+\sum_{v\in V(H)}\sum_{ac\in E(G)}2[C_v^2]+\sum_{v\in V(H)}\sum_{ac\in E(G)}[C_a^2+C_c^2]\\&\quad+\sum_{v\in V(H)}\sum_{ac\in E(G)}2C_v(C_a+C_c)-\sum_{v\in V(H)}\sum_{ac\in E(G)}4C_v\\&\quad-\sum_{v\in V(H)}\sum_{ac\in E(G)}2(C_a+C_c)+\sum_{v\in V(H)}\sum_{ac\in E(G)}2\\&=2|E(H)|CM_1^{\alpha}(G)+|V(G)|FC(H)+2TR(G)CM_1^{\beta}(H)\\&\quad-4|E(H)|TR(G)-2|V(G)|CM_1^{\beta}(H)\\&\quad+2|V(G)||E(H)|+2|E(G)|CM_1^{\alpha}(H)\\&\quad+|V(H)|FC(G)+2TR(H)CM_1^{\beta}(G)\\&\quad-4|E(G)|TR(H)-2|V(H)|CM_1^{\beta}(G)+2|V(H)||E(G)|\end{aligned}$$

This completes the proof. □

Theorem 3.4 *If G and H be two connected molecular graphs then,*

$$\begin{aligned}F_1C(G\times H)=&|V(H)|F_1C(G)+|V(G)|F_1C(H)-3TR(G)TR(H)[TR(G)+TR(H)]\\&-3|V(H)|CM_1^{\alpha}(G)-3|V(H)|CM_1^{\alpha}(G)-3|V(G)|CM_1^{\alpha}(H)-6TR(G)TR(H)\\&+3|V(H)|TR(G)+3|V(G)|TR(H)-|V(G)||V(H)|.\end{aligned}$$

Proof By Proposition 2.3, $C_(a,b) = C_a + C_b - 1$. So

$$\begin{aligned}
F_1C(G\times H) =& \sum_{a\in V(G)}\sum_{b\in V(H)} (C_a + C_b - 1)^3\\
=& \sum_{a\in V(G)}\sum_{b\in V(H)} [(C_a + C_b)^3 - 3(C_a + C_b)^2 + 3(C_a + C_b) - 1]\\
=& \sum_{a\in V(G)}\sum_{b\in V(H)} C_a^3 + \sum_{a\in V(G)}\sum_{b\in V(H)} C_b^3 - \sum_{a\in V(G)}\sum_{b\in V(H)} 3C_aC_b(C_a + C_b)\\
&- 3\sum_{a\in V(G)}\sum_{b\in V(H)} (C_a^2 + C_b^2) - \sum_{a\in V(G)}\sum_{b\in V(H)} 6C_aC_b\\
&+ \sum_{a\in V(G)}\sum_{b\in V(H)} (C_a + C_b) - \sum_{a\in V(G)}\sum_{b\in V(H)} 1\\
=& |V(H)|F_1C(G) + |V(G)|F_1C(H) - 3TR(G)TR(H)[TR(G) + TR(H)]\\
&- 3|V(H)|CM_1^{\alpha}(G) - 3|V(G)|CM_1^{\alpha}(H) - 6TR(G)TR(H)\\
&+ 3|V(H)|TR(G) + 3|V(G)|TR(H) - |V(G)||V(H)|
\end{aligned}$$

This completes the proof. □

§4. Conclusions

In this paper, we have obtained the exact values of the first and second reverse hyper Zagreb indices of Cartesian product of two simple connected graphs. And repeated same for the F and F_1 reverse index of a molecular graphs. It would be useful to study mathematical properties and formulas for some important graphs families. Same can be done with other operations (such as the composition, join, disjunction and symmetric difference of graphs, bridge graphs and Kronecker product of graphs) can be derived similarly.

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