

Reversible DNA Codes Over a Family of the Finite Rings

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Abstract: In this paper, the results introduced in [4] are extended. We define a non trivial automorphism θ_a over $R_a = F_4[u_1, \dots, u_a]/\langle u_i^2 = u_i, u_i u_j = u_j u_i \rangle$, where $i, j = 1, 2, \dots, a$ and a generalized Gray map over R_a which preserves DNA reversibility. The reversibility problem for DNA codes over a family of the finite rings R_a is solved, by using the skew cyclic codes over R_a . 2^a -mers are matched with the elements of the finite ring R_a .

Key Words: Reversibility, DNA codes, finite ring, Gray map.

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§1. Introduction

The reversibility problem is very important in DNA computing. Let $(\alpha_1, \alpha_2) \in R_1^2$ be a code-word corresponding to *ATGC*. The reverse of (α_1, α_2) is (α_2, α_1) . The vector (α_2, α_1) corresponding to *GCAT*. It is not reverse of *ATGC*. The reverse of *ATGC* is *CGTA*.

Some authors use the different approachers in order to solve this problem [1-11].

In [4], by defining a nontrivial automorphism, the skew cyclic codes over the finite ring R_2 were introduced DNA 4-bases were matched with the elements 256 of the finite R_2 . The reversible DNA codes were obtained.

In this paper, motivated by the previous work [4], we study the reversibility problem for DNA 2^a -bases, by using the skew cyclic codes over the finite ring R_a .

§2. Preliminaries

A family of the finite rings $R_a = F_4[u_1, \dots, u_a]/\langle u_i^2 = u_i, u_i u_j = u_j u_i \rangle$, where $i, j = 1, 2, \dots, a$ contains the commutative the finite rings with characteristic 2 and cardinality 4^{2^a} . The finite rings of the family are written as recursively

$$R_j = R_{j-1} + u_j R_{j-1}$$

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where $j = 1, 2, \dots, a$ and $R_1 = F_4 + u_1F_4, u_1^2 = u_1$, where $R_0 = F_4 = \{0, 1, w, w^2 = w + 1\}$.

We defined the Gray map as follows,

$$\begin{aligned} \phi_i & : R_i \longrightarrow R_{i-1}^2 \\ x_{i-1} + u_i y_{i-1} & \longmapsto (x_{i-1} + y_{i-1}, x_{i-1}) \end{aligned}$$

where $i = 1, 2, \dots, a$ and

$$\begin{aligned} \phi_1 & : R_1 \longrightarrow R_0^2 \\ x_0 + u_1 y_0 & \longmapsto (x_0 + y_0, x_0) \end{aligned}$$

where $R_0 = F_4$.

In [4], by using the matching the elements of R_0 and $S_{D_4} = \{A, T, C, G\}$ which is given as $\xi_0(0) = A, \xi_0(1) = T, \xi_0(w) = C, \xi_0(w^2) = G$ by using the Gray map from $R_1 = F_4 + u_1F_4$ to F_4^2 , it is defined a ξ_1 correspondence between the elements of the finite ring $R_1 = F_4 + u_1F_4$ and DNA double pairs as follows

elements α	DNA double pairs $\xi_1(\alpha)$
0	AA
1	TT
w	CC
1 + w	GG
u_1	TA
1 + u_1	AT
$u_1 + w$	GC
1 + $u_1 + w$	CG
$u_1 w$	CA
1 + $u_1 w$	GT
$w + u_1 w$	AC
1 + w + $u_1 w$	TG
$u_1 + u_1 w$	GA
1 + $u_1 + u_1 w$	CT
$w + u_1 + u_1 w$	TC
1 + w + $u_1 + u_1 w$	AG

§3. Skew Cyclic Codes over R_a

Definition 3.1 Let B be a finite ring and θ be a non trivial automorphism on B . A subset C

of B^n is called a skew cyclic code of length n if C satisfies the following conditions:

- (1) C is a submodule of B^n ;
- (2) if $c = (c_0, c_1, \dots, c_{n-1}) \in C$, then $\sigma_\theta(c) = (\theta(c_{n-1}), \theta(c_0), \dots, \theta(c_{n-2})) \in C$, where σ_θ is the skew cyclic shift operator.

By defining a non trivial automorphism on R_a as follows, we can define the skew cyclic codes over R_a .

$$\begin{aligned} \theta_i & : R_i \longrightarrow R_i \\ x_{i-1} + u_i y_{i-1} & \longmapsto \theta_{i-1}(x_{i-1} + y_{i-1}) + u_i \theta_{i-1}(y_{i-1}) \end{aligned}$$

and

$$\begin{aligned} \theta_1 & : R_1 \longrightarrow R_1 \\ x_0 + u_1 y_0 & \longmapsto (x_0 + y_0) + u_1 y_0 \end{aligned}$$

where $i = 2, 3, \dots, a$. The order of θ_i is 2, where $i = 1, 2, \dots, a$.

The rings

$$R_i[x, \theta_i] = \{b_0^i + b_1^i x + \dots + b_{n-1}^i x^{n-1} : b_j^i \in R_i, n \in N, i = 1, \dots, a, j = 0, \dots, n-1\}$$

are called skew polynomial rings with the usual polynomial addition and the multiplication as follows

$$(\varrho x^s)(\eta x^t) = \varrho \theta_i^s(\eta) x^{s+t}$$

where $i = 1, \dots, a$. They are non commutative rings.

In polynomial representation, a skew cyclic code of length n over R_i is defined as a left ideal of the quotient ring $R_{\theta_i, n} = R_i[x, \theta_i] / \langle x^n - 1 \rangle$, if the order of θ_i divides n , that is n is even. If the order of θ_i does not divides n , a skew cyclic code of length n over R_i is defined as a left $R_i[x, \theta_i]$ -submodule of $R_{\theta_i, n}$, since the set $R_{\theta_i, n} = R_i[x, \theta_i] / \langle x^n - 1 \rangle = \{f_i(x) + \langle x^n - 1 \rangle : f_i(x) \in R_i[x, \theta_i]\}$ is a left $R_i[x, \theta_i]$ -module with the multiplication from left defined by

$$r_i(x)(f_i(x) + \langle x^n - 1 \rangle) = r_i(x)f_i(x) + \langle x^n - 1 \rangle$$

where for any $r_i(x) \in R_i[x, \theta_i]$, for $i = 1, \dots, a$.

In both case, the following is hold.

Theorem 3.2 *Let C_i be a skew cyclic code over R_i and let $f_i(x)$ be a polynomial in C_i of minimal degree, $i = 1, \dots, a$. If the leading coefficient of $f_i(x)$ is a unit in R_i , then $C_i = \langle f_i(x) \rangle$, where $f_i(x)$ is a right divisor of $x^n - 1$.*

Definition 3.3 *For $\mathbf{x} = (x_0^i, x_1^i, \dots, x_{n-1}^i) \in R_i^n$, the vector $(x_{n-1}^i, x_{n-2}^i, \dots, x_1^i, x_0^i)$ is called the reverse of \mathbf{x} and is denoted by \mathbf{x}^r . A linear code C_i of length n over R_i is called reversible if $\mathbf{x}^r \in C_i$ for every $\mathbf{x} \in C_i$, where $i = 1, \dots, a$.*

We can express the matching the elements R_1 and $S_{D_{16}} = \{AA, TT, \dots, GG\}$ by means of the automorphism θ_1 as follows.

Each element $\alpha_1 = x_0 + u_1y_0 \in R_1$ and $\theta_1(\alpha_1)$ are mapped to DNA 2-mers which are reverse of each other. Let ξ_1 be a correspondence the elements of the finite ring R_1 and DNA 2-mers. For example

$$\xi_1(u_1) = TA, \text{ while } \xi_1(\theta_1(u_1)) = AT$$

This can be extended to a map γ_i from R_{i-1}^2 to 2^i -mers as follows,

$$\gamma_i(s_{i-1}, t_{i-1}) = (\xi_{i-1}(s_{i-1}), \xi_{i-1}(t_{i-1}))$$

where $s_{i-1}, t_{i-1} \in R_{i-1}$, for $i = 1, \dots, a$.

By using a map $\Psi_i = \gamma_i \circ \phi_i$, we can explain a relationship between skew cyclic codes and DNA codes. $\Psi_i(r_i)$ and $\Psi_i(\theta_i(r_i))$ are DNA reverse of each other, where $r_i = a_{i-1} + u_i b_{i-1}$, $a_{i-1}, b_{i-1} \in R_{i-1}$, where $i = 1, \dots, a$.

For $r_i = a_{i-1} + u_i b_{i-1} \in R_i$, we have

$$\begin{aligned} \Psi_i(r_i) &= \gamma_i(\phi_i(a_{i-1} + u_i b_{i-1})) = \gamma_i(a_{i-1} + b_{i-1}, a_{i-1}) \\ &= (\xi_{i-1}(a_{i-1} + b_{i-1}), \xi_{i-1}(a_{i-1})) \end{aligned}$$

On the other hand,

$$\begin{aligned} \Psi_i(\theta_i(r_i)) &= \Psi_i(\theta_{i-1}(a_{i-1} + b_{i-1}) + u_i \theta_{i-1}(b_{i-1})) \\ &= \gamma_i(\phi_i(\theta_{i-1}(a_{i-1} + b_{i-1}) + u_i \theta_{i-1}(b_{i-1}))) \\ &= \gamma_i(\theta_{i-1}(a_{i-1}), \theta_{i-1}(a_{i-1} + b_{i-1})) \\ &= (\xi_{i-1}(\theta_{i-1}(a_{i-1})), \xi_{i-1}(\theta_{i-1}(a_{i-1} + b_{i-1}))) \end{aligned}$$

where $i = 1, \dots, a$. This map can be extended as follows.

For any $r_i = (r_0^i, \dots, r_{n-1}^i) \in R_i^n$, where $i = 1, 2, \dots, a$.

$$(\Psi_i(r_0^i), \Psi_i(r_1^i), \dots, \Psi_i(r_{n-1}^i))^r = (\Psi_i(\theta_i(r_{n-1}^i)), \dots, \Psi_i(\theta_i(r_1^i)), \Psi_i(\theta_i(r_0^i)))$$

Example 3.4 If $r_3 = u_3((w + u_1) + u_2(1 + u_1w)) \in R_3$, then we have

$$\begin{aligned} \Psi_3(r_3) &= \gamma_3(\phi_3(r_3)) = \gamma_3(w + u_1 + u_2(1 + u_1w), 0) \\ &= (\xi_2(w + u_1 + u_2(1 + u_1w)), \xi_2(0)) = (AGGC, AAAA) \end{aligned}$$

On the other hand,

$$\begin{aligned}
\Psi_3(\theta_3(r_3)) &= \Psi_3(\theta_2(w + u_1 + u_2(1 + u_1w)) + u_3\theta_2(w + u_1 + u_2(1 + u_1w))) \\
&= \gamma_3(\theta_2(0), \theta_2(w + u_1 + u_2(1 + u_1w))) \\
&= (\xi_2(\theta_2(0)), \xi_2(\theta_2(w + u_1 + u_2(1 + u_1w)))) \\
&= (AAAA, CGGA)
\end{aligned}$$

Definition 3.5 Let C_i be a code of length n over R_i , for $i = 1, \dots, a$. If $\Psi_i(\mathbf{c})^r \in \Psi_i(C_i)$ for all $\mathbf{c} \in C_i$, then C_i or equivalently $\Psi_i(C_i)$ is called a reversible DNA code.

Definition 3.6 Let $g_i(x) = b_0^i + b_1^i x + b_2^i x^2 + \dots + b_s^i x^s$ be a polynomial of degree s over R_i , for $i = 1, \dots, a$. $g_i(x)$ is called a palindromic polynomial if $b_j^i = b_{s-j}^i$ for all $j \in \{0, 1, \dots, s\}$. $g_i(x)$ is called a θ_i -palindromic polynomial if $b_j^i = \theta_i(b_{s-j}^i)$ for all $j \in \{0, 1, \dots, s\}$, for $i = 1, \dots, a$.

As the order of θ_i is 2, a skew cyclic code of odd length n over R_a with respect to θ_i is an ordinary cyclic code. So we will take the length n to be even.

Theorem 3.7 Let $C_i = \langle f_i(x) \rangle$ be a skew cyclic code of length n over R_i , for $i = 1, \dots, a$, where $f_i(x)$ is a right divisor of $x^n - 1$ and $\deg(f_i(x))$ is odd. If $f_i(x)$ is a θ_i -palindromic polynomial then $\Psi_i(C_i)$ is a reversible DNA code.

Proof Let $f_i(x)$ be a θ_i -palindromic polynomial and $f_i(x) = a_0^i + a_1^i x + \dots + a_{2s-1}^i x^{2s-1}$. So $a_d^i = \theta_i(a_{2s-1-d}^i)$, for all $d = 0, 1, \dots, s-1$. Let $h_i(x) = h_0^i + h_1^i x + \dots + h_{2k-1}^i x^{2k-1}$. Let b_l^i be the coefficient of x^l in $h_i(x)f_i(x)$ where $l = 1, \dots, n-1$. For any $t < n/2$, the coefficient of x^t in $h_i(x)f_i(x)$ is

$$b_t^i = \sum_{j=0}^t h_j^i \theta_i^j(a_{t-j}^i)$$

and the coefficient of x^{n-t} is $b_{n-t}^i = \sum_{j=0}^t h_{2k-1-j}^i \theta_i^{2k-1-j}(a_{2s-1-(t-j)}^i)$.

The polynomial $h_i(x)f_i(x) = \sum_{p=0}^{2k-1} h_p^i x^p f_i(x)$ corresponds a vector $b = (b_0^i, b_1^i, \dots, b_{n-1}^i) \in C_i$ for $i = 1, \dots, a$.

The vector $\Psi_i(b)^r = ((\Psi_i(b_0^i), \Psi_i(b_1^i), \dots, \Psi_i(b_{n-1}^i)))^r$ is equal to the vector $\Psi_i(z)$, where the vector z corresponds the polynomial $\sum_{p=0}^{2k-1} \theta_i(h_p^i) x^{2k-1-p} f_i(x)$, for $i = 1, \dots, a$.

Since $z = (z_1^i, \dots, z_n^i) \in C_i$, then $\Psi_i(C_i)$ is a reversible DNA code, for $i = 1, \dots, a$. \square

Theorem 3.8 Let $C_i = \langle f_i(x) \rangle$ be a skew cyclic code of length n over R_i , for $i = 1, \dots, a$, where $f_i(x)$ is a right divisor of $x^n - 1$ and $\deg(f_i(x))$ is even. If $f_i(x)$ is a palindromic polynomial then $\Psi_i(C_i)$ is a reversible DNA code.

Proof Let $f_i(x)$ be a palindromic polynomial with even degree. $f_i(x) = a_0^i + a_1^i x + \dots + a_{2s}^i x^{2s}$ and $a_d^i = a_{2s-d}^i$, for all $d = 0, 1, \dots, s$. Let $h_i(x) = h_0^i + h_1^i x + \dots + h_{2k}^i x^{2k}$. Let b_l^i be the coefficient of x^l in $h_i(x)f_i(x)$ where $l = 1, \dots, n-1$. For any $t < n/2$, the coefficient of x^t

in $h_i(x)f_i(x)$ is

$$b_t^i = \sum_{j=0}^t h_j^i \theta_j^i (a_{t-j}^i)$$

and the coefficient of x^{n-t} is $b_{n-t}^i = \sum_{j=0}^t h_{2k-j}^i \theta_j^{2k-j} (a_{2s-(t-j)}^i)$.

The polynomial $h_i(x)f_i(x) = \sum_{p=0}^{2k} h_p^i x^p f_i(x)$ corresponds a vector $b = (b_0^i, b_1^i, \dots, b_{n-1}^i) \in C_i$ for $i = 1, \dots, a$.

The vector $\Psi_i(b)^r = ((\Psi_i(b_0^i), \Psi_i(b_1^i), \dots, \Psi_i(b_{n-1}^i)))^r$ is equal to the vector $\Psi_i(z)$, where the vector z corresponds the polynomial $\sum_{p=0}^{2k} \theta_i(h_p^i) x^{2k-p} f_i(x)$, for $i = 1, \dots, a$.

Since $z = (z_1^i, \dots, z_n^i) \in C_i$, then $\Psi_i(C_i)$ is a reversible DNA code, for $i = 1, \dots, a$. \square

§4. Conclusion

We have shown that skew cyclic codes over the ring R_a can be used to construct the reversible DNA codes.

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