

Some New Families of Odd Graceful Graphs

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Abstract: A labeling or numbering of a graph G is an assignment f of labels to the vertices of G that induces for each edge uv a labeling depending on the vertex labels $f(u)$ and $f(v)$. In this paper we study some new families of odd graceful graphs.

Key Words: Labeling, odd-even graceful graph, tree.

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§1. Introduction

Unless mentioned otherwise, a graph in this paper shall mean a simple finite graph without isolated vertices. For all terminology and notations in Graph Theory, we follow [1], and all terminology regarding to labeling, we follow [2] and [3].

Gnanajothi [3] introduced the concept of odd graceful graphs as an extension of graceful graphs. A graph $G = (V, E)$ with p vertices and q edges is said to admit an *odd graceful labeling* if $f : V(G) \rightarrow \{0, 1, 2, \dots, 2q - 1\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1, 3, 5, \dots, 2q - 1\}$ defined as $f^*(uv) = |f(u) - f(v)|$ is bijective. The graph which admits odd graceful labeling is called an odd graceful graph. In the present paper, we investigate some new families of odd graceful graphs generated from various graph operations on the given graph.

§2. Main Results

Definition 2.1 Let G_n^* be a graph with vertex set $V = \{a_i, b_i/i = 1, 2, \dots, n\}$ and $E = \{a_i a_{i+1}, b_i b_{i+1}, a_i b_{i+1}, b_i a_{i+1}/i = 1, 2, \dots, n - 1\}$.

Definition 2.2 D_n^* be a graph with $V = \{a_{ij}/i = 1, 2, \dots, n; j = 1, 2, 3, 4\}$ and $E = \{a_{i,1} a_{i+1,1}/i = 1, 2, \dots, n - 1\} \cup \{a_{i,3} a_{i+1,3}/i = 1, 2, \dots, n - 1\} \cup \{a_{i,1} a_{i,2}; a_{i,2} a_{i,3}; a_{i,3} a_{i,4}; a_{i,4} a_{i,1}/i = 1, 2, \dots, n\}$.

Theorem 2.1 Let G_n^* be a graph with vertex $V = \{a_i, b_i/i = 1, 2, \dots, n\}$ and $E = \{a_i a_{i+1}, b_i b_{i+1}, a_i b_{i+1}, b_i a_{i+1}/i = 1, 2, \dots, n - 1\}$. Then G is odd graceful.

Proof Let G_n^* be a graph with vertex set $V = \{a_i, b_i/i = 1, 2, \dots, n\}$ and $E = \{a_i a_{i+1}, b_i b_{i+1}, a_i b_{i+1}, b_i a_{i+1}/i = 1, 2, \dots, n - 1\}$. Note that, it has $2n$ vertices and $4n - 4$ edges.

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Define a function $f : V(G_n^*) \rightarrow \{0, 1, 2, \dots, 8n - 9 = 2(4n - 4) - 1\}$ such that

$$\begin{aligned} f(a_1) &= 0 \\ f(a_{2i-1}) &= 2(4i - 3); 2 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even, or } 2 \leq i \leq \frac{n+1}{2} \text{ if } n \text{ is odd} \\ f(a_{2i}) &= 8n - 5 - 8i; 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even, or } 1 \leq i \leq \frac{n-1}{2} \text{ if } n \text{ is odd} \\ f(b_1) &= 2 \\ f(b_{2i-1}) &= 8(i - 1); 2 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even, or } 2 \leq i \leq \frac{n+1}{2} \text{ if } n \text{ is odd} \\ f(b_{2i}) &= 8n - 1 - 8i; 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even, or } 1 \leq i \leq \frac{n-1}{2} \text{ if } n \text{ is odd} \end{aligned}$$

It is easy to show that f is injective. Also $\max_{v \in V(G_n^*)} f(v) = 8n - 9$. Thus, $f(v) \in \{0, 1, 2, \dots, 8n - 9\}$, for all $v \in V(G_n^*)$.

Now, it can be easily verified that all the edge values are in the interval $[1, 8n - 9]$. Thus, f is an odd graceful numbering. Hence, the graph G_n^* is odd graceful. \square

An odd graceful labelling of the graph G_8^* is displayed in Figure 2.1.

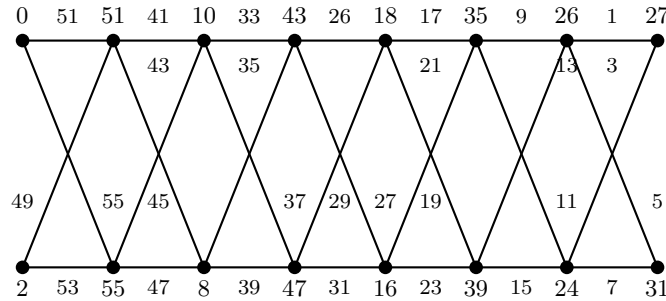


Figure 2.1

Theorem 2.2 Let D_n^* be a graph with $V = \{a_{ij}/i = 1, 2, \dots, n, j = 1, 2, 3, 4\}$ and $E = \{a_{i,1}a_{i+1,1}/i = 1, 2, \dots, n-1\} \cup \{a_{i,3}a_{i+1,3}/i = 1, 2, \dots, n-1\} \cup \{a_{i,1}a_{i,2}; a_{i,2}a_{i,3}; a_{i,3}a_{i,4}; a_{i,4}a_{i,1}, i = 1, 2, \dots, n\}$. Then D_n^* is odd graceful for any n .

Proof Let $\{a_{ij}/i = 1, 2, \dots, n; j = 1, 2, 3, 4\}$ be the set of vertices of D_n^* . Note that, D_n^* has $4n$ vertices and $6n - 2$ edges.

Define a function $f : V(D_n^*) \rightarrow \{0, 1, 2, \dots, 12n - 5 = 2(6n - 2) - 1\}$ such that

$$\begin{aligned} f(a_{i,1}) &= \begin{cases} 12i - 4 & \text{if } i \text{ is even} \\ 12(n - i) + 7 & \text{if } n \text{ is odd} \end{cases} \\ f(a_{i,2}) &= \begin{cases} 12(n - i) - 3 & \text{if } i \text{ is even} \\ 12(i - 1) & \text{if } i \text{ is odd} \end{cases} \\ f(a_{i,3}) &= \begin{cases} 12i - 6 & \text{if } i \text{ is even} \\ 12(n - i) + 3 & \text{if } i \text{ is odd} \end{cases} \end{aligned}$$

$$f(a_{i,4}) = \begin{cases} 12(n-i) + 1 & \text{if } i \text{ is even} \\ 12i - 10 & \text{if } i \text{ is odd} \end{cases}$$

for $i = 1, 2, \dots, n$. Let f^* be the edge labelling induced by f such that $f^*(uv) = |f(u) - f(v)|$. Now we split the edge set of D_n^* into three disjoint sum.

Let $P = \{f^*(a_{ij}, a_{i,j+1})/i = 1, 2, \dots, n; j = 1, 2, 3\} \cup \{f^*(a_{i,4}a_{i1})/i = 1, 2, \dots, n\}$. Then the values of edges under P is $\{1, 3, 5, 7; 13, 15, 17, 19; 25, 27, 29, 31; \dots; 12n - 11, 12n - 9, 12n - 7, 12n - 5\}$.

Let $Q = \{f^*(a_{i1}, a_{i+1,1})/i = 1, 2, \dots, n-1\}$. Then the corresponding edge values are $\{11, 23, 35, \dots, 12n - 13\}$.

Let $R = \{f^*(a_{i,3}a_{i+1,3})/i = 1, 2, \dots, n-1\}$. Then the corresponding edge values are equal to $\{9, 21, 31, \dots, 12n - 15\}$.

Next, consider $P \cup Q \cup R$.

$$P \cup Q \cup R = \{1, 3, 5, 7, \dots, 12n - 5 = 2q - 1\}$$

That is, the edge values of D_n^* are $\{1, 3, 5, 7, \dots, 12n - 5 = 2q - 1\}$. It can be easily shown that f is injective on $V(D_n^*)$ and it is obvious that f is an odd graceful numbering. Hence, the graph D_n^* is odd graceful. \square

Figure 2.2 gives an odd graceful labelling of the graph D_6^* .

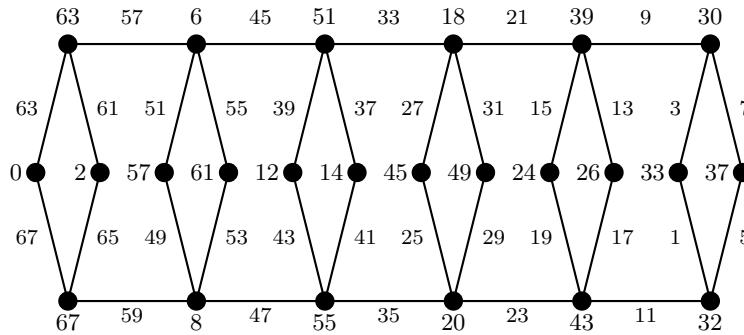


Figure 2.2

Observation 2.1 Let K_2 be a complete graph on two vertices. Take $2n$ copies of K_2 . Keep n copies of K_2 in one set and n copies in the second set. Let $u_{i,1}, u_{i,2}$ for $i = 1, 2, \dots, n$ to the vertices of one set and $u'_{i,1}, u'_{i,2}$ for $i = 1, 2, \dots, n$ be the vertices in the second set. Now join $u'_{i,2}$ to $u'_{i+1,1}$ and $u'_{i,1}$ to $u_{i+1,1}$. The resultant graph is denoted as $G(K_2)$ and it has $4n$ vertices and $6n - 2$ edges.

The graph $G(K_2)$ obtained by 3 copies of K_2 is shown in Figure 2.3.

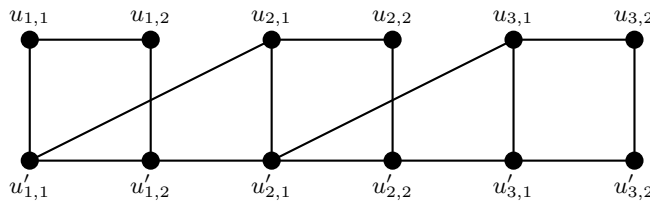


Figure 2.3

Theorem 2.3 *The graph $G(K_2)$ is odd graceful.*

Proof Let $u_{i,1}, u_{i,2}, u'_{i,1}, u'_{i,2}$ for $i = 1, 2, \dots, n$ be the set of vertices of $G(K_2)$. Define a function $f : V(G) \rightarrow \{0, 1, 2, \dots, 12n - 5 = [2(6n - 2)] - 1\}$ such that

$$\begin{aligned} f(u_{1,1}) &= 0 \\ f(u_{i+1,1}) &= 8i - 4, \text{ for } i = 1, 2, \dots, n - 1 \\ f(u_{1,2}) &= 12n - 5 \\ f(u_{2i-1,2}) &= 12n - 17 - 8(i - 2); 2 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even} \\ & \qquad \qquad \qquad 2 \leq i \leq \frac{n+1}{2} \text{ if } n \text{ is odd} \\ f(u_{2i,2}) &= 12n - 15 - 8(i - 1), 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even} \\ & \qquad \qquad \qquad 1 \leq i \leq \frac{n-1}{2} \text{ if } n \text{ is odd} \\ f(u'_{1,1}) &= 12n - 9 \\ f(u'_{2i-1,1}) &= 12n - 19 - 8(i - 2); 2 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even} \\ & \qquad \qquad \qquad 2 \leq i \leq \frac{n+1}{2} \text{ if } n \text{ is odd} \\ f(u'_{2i,1}) &= 12n - 13 - 8(i - 1), 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even} \\ & \qquad \qquad \qquad 1 \leq i \leq \frac{n-1}{2} \text{ if } n \text{ is odd} \\ f(u'_{1,2}) &= 2 \\ f(u'_{i+1,2}) &= 8i; 1 \leq i \leq n - 1 \end{aligned}$$

Easily, it can be verified that f is injective. Also, $\max f(v) = 12n - 5, v \in V(K_2)$. Thus, $f(v) \in \{0, 1, 2, \dots, 12n - 5\}$. Finally, it can be easily proved that, the values of the edges are in the interval $[1, 12n - 5]$. Thus, f is an odd graceful labeling. Hence, the graph $G(K_2)$ is odd graceful. \square

Figure 2.4 gives an odd graceful labeling on graph $G(K_2)$ obtained by 6 copies of K_2 .

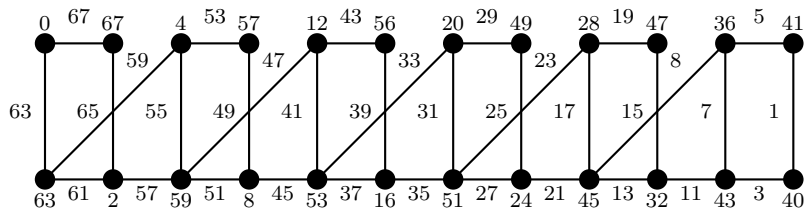


Figure 2.4

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