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Some Results on α-graceful Graphs

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Abstract: In this paper we have proved that the graph obtained by merging t consecutive vertices of two cycles C_{4r} and C_{4s} is α-graceful graph. We also proved that G_1 be an α-graceful graph and G_2 be a graceful graph joining by path P_n is graceful in addition we proved G_1 and G_2 be α-graceful graphs joining by path P_n is an α-graceful graph too.

Key Words: Cycle, path, graceful labeling, α-labeling.

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§1. Introduction

A.Rosa [1] defined α-labeling or (α-valuation) as a graceful labeling with the additional property that there is an integer k $(0 \leq k < E(G))$ such that for every $e = (x, y) \in E(G)$, either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. If follows that such a k must be the smaller of the two vertex labels that yield the edge labeled 1.

In [4] Kaneria et al. proved that the α-labeling of double path union of some α-graceful graph like cycle, complete bipartite graph and path. We will consider a simple finite and undirected graph $G = (V, E)$ on $|V| = p$ vertices and $|E| = q$ edges. For a comprehensive bibliography of papers on graph labeling we have refereed Gallian [3]. Here we recall some definitions which are used in this paper.

Definition 1.1 *A function f is called graceful labeling of a graph $G = (V, E)$ if $f : V \longrightarrow \{0, 1, \cdots, q\}$ is injective and the induced function $f^\star : E \longrightarrow \{1, 2, \cdots, q\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E$. A graph G is called graceful graph if it admits a graceful labeling.*

Definition 1.2 *A function f is called α-labeling of a graph $G = (V, E)$ if f is a graceful labeling for G and there exist an integer k $(0 \leq k < q - 1)$ such that for every $e = (x, y) \in E(G)$, either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. A graph G with an α-labeling is necessarily bipartite graph. A graph which admits α-labeling, we call here α-graceful graph.*

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§2. Main Results

Theorem 2.1 *The graph obtained by merging t consecutive vertices of two cycle C_{4r} and C_{4s} is α-graceful, where $t \leq 2[min\{r, s\}]$ and $r, s, t \in N$.*

Proof Let $u_i (i \leq i \leq 4r)$ be consecutive vertices of the cycle C_{4r} and $v_1 = u_{4r-t+1}$, $v_2 = u_{4r-t+2}$, $\cdots$, $v_t = u_{4r}$, v_{t+1}, v_{t+2}, $\cdots$, v_{4s} be consecutive vertices of the second cycle C_{4s}, among the consecutive vertices v_1, v_2, $\cdots$, v_t of the second cycle are merged with the consecutive vertices u_{4r-t+1}, u_{4r-t+2}, $\cdots$, u_{4r} of the first cycle C_{4r} respectively.

Let G be the graph obtained by merging t consecutive vertices stated above of the cycle C_{4r} and C_{4s}, where $t \leq 2[\min\{r, s\}]$ and $r, s, t \in N$.

The vertex labeling function $f : V(C_{4r}) \longrightarrow \{0, 1, \cdots, 4r\}$ is defined by

$$f(u_i) = \begin{cases} 4r - \frac{i-1}{2}, \forall i = 1, 3, \cdots, 4r-1, \\ \frac{i-2}{2}, \forall i = 2, 4, \cdots, 2r, \\ \frac{i}{2}, \forall i = 2r+2, 2r+4, \cdots, 4r \end{cases}$$

is an α-graceful labeling for the cycle C_{4r}, where $k_1 = 2r$.

Also the vertex labeling function $g_i : V(C_{4s}) \longrightarrow \{0, 1, \cdots, 4s\}$ $(i = 1, 2)$ is defined by

$$g_1(v_j) = \begin{cases} 4s - \frac{j-1}{2}, \forall j = 1, 3, \cdots, 4s-1, \\ \frac{j-2}{2}, \forall j = 2, 4, \cdots, 2s, \\ \frac{j}{2}, \forall j = 2s+2, 2s+4, \cdots, 4s. \end{cases}$$

$$g_2(v_j) = \begin{cases} \frac{j-1}{2}, \forall j = 1, 3, \cdots, 2s-1, \\ \frac{j+1}{2}, \forall j = 2s+1, 2s+3, \cdots, 4s-1, \\ 4s - \frac{j-2}{2}, \forall j = 2, 4, \cdots, 4s. \end{cases}$$

are α-graceful labeling for the cycle C_{4s}, where $k_2 = 2s$.

To define a vertex labeling function $h : V(G) \longrightarrow \{0, 1, \cdots, q\}$, where $q = 4(r+s) - t + 1$, we take following two cases.

Case 1. t is odd.

Define $h(u) = f(u)$ if $f(u) \leq 2r$ and $f(u) + 4s - t + 1$if $f(u) > 2r$ for $\forall u \in V(C_{4r})$, $h(v) = g_2(v) + 2r - \frac{t-1}{2}$ for $\forall\ v \in V(C_{4s}) - V(C_{4r})$.

Case 2. t is even.

Define $h(u) = f(u)$ if $f(u) \leq 2r$ and $f(u) + 4s - t + 1$ if $f(u) > 2r$ for $\forall\ u \in V(C_{4r})$, $h(v) = g_1(v) + 2r - \frac{t-2}{2}$ for $\forall\ v \in V(C_{4s}) - V(C_{4r})$.

Above defined labeling pattern give the vertex labels $0, 1, \cdots, r-1, r+1, r+2, \cdots, 2r$, q, $q-1, \cdots, q+1-2r$ to the vertices of C_{4r} and the vertex labels $2r+1, 2r+2, \cdots$, $2r + \lfloor\frac{2s-t}{2}\rfloor$, $2r + \lfloor\frac{2s+4-t}{2}\rfloor$, $2r + \lfloor\frac{2s+6-t}{2}\rfloor$, $\cdots, q-2r$ to the vertices of C_{4s} which are not common vertices of C_{4r} and C_{4s}.

So f is an injective map. Moreover it produces the edge labels $q, q-1, \cdots, q-4r+1$ to the edges of C_{4r} and the edge labels $1, 2, \cdots, q-4r$ to the edges of C_{4s}, which are not common edges of C_{4r} and C_{4s}. Thus $f^\star : E(G) \longrightarrow \{0, 1, \cdots, q\}$ defined by absolute difference of end vertices labels is a bijective map.

It is observed that $h(v_{4s}) = 2(r+s)+1-\frac{t-1}{2}$, $h(v_{4s-1}) = 2(r+s)-\frac{t-1}{2}$ in Case 1 and $h(v_{4s}) = 2(r+s)-\frac{t-2}{2}$, $h(v_{4s-1}) = 2(r+s)+1-\frac{t-2}{2}$ in the Case 2. These produce $h^\star(e = (v_{4s-1}, v_{4s})) = 1$. Take $k = \lceil \frac{q}{2} \rceil$. Now,

$$\begin{aligned}
\left\lceil \frac{q}{2} \right\rceil &= \left\lceil \frac{4(r+s)-t+1}{2} \right\rceil \\
&= 2(r+s) - \left\lfloor \frac{t-1}{2} \right\rfloor \\
&= \begin{cases} 2(r+s) - \frac{t-1}{2} \text{ when } t \text{ is odd} \\ 2(r+s) - \frac{t-2}{2} \text{ when } t \text{ is even} \end{cases} \\
&= \min\{h(v_{4s}), h(v_{4s-1})\}.
\end{aligned}$$

Therefore, $k = \lceil \frac{q}{2} \rceil$ is non-negative integer $(0 \leq k < q)$, which satisfied for every $e = (x, y) \in E(G)$, $\min\{h(x), h(y)\} \leq k < \max\{h(x), h(y)\}$. Hence, G admits an α-graceful labeling h and so, it is an α-graceful graph. □

Illustration 2.2 Graph obtained by merging 7 consecutive vertices of C_{16}, C_{20} and its α-graceful labeling are shown in Figure 1.

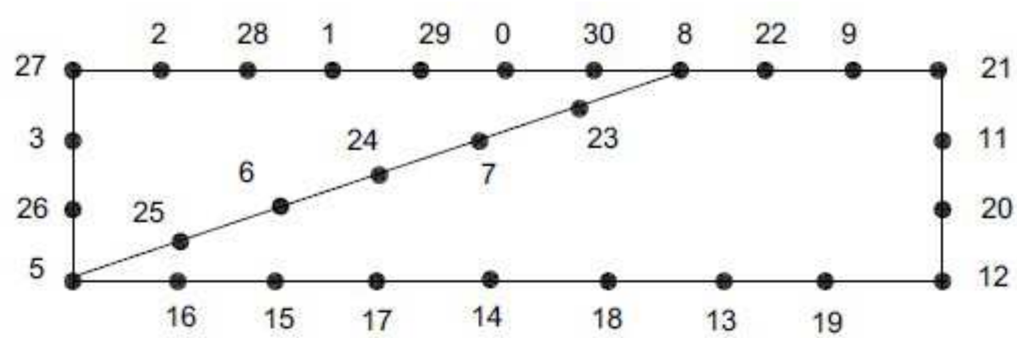

Figure 1 $C_{16} \bigcup C_{20}$ with seven consecutive common vertices and its α-labeling, here $k = 15$.

Theorem 2.3 *Let G_1 be an α-graceful graph, G_2 be a graceful graph and $q_1 = |E(G_1)|$, $q_2 = |E(G_2)|$. Let f_1 be an α-graceful labeling for G_1 and f_2 be a graceful labeling for G_2. Let k $(0 \leq k < q_1)$ be a non-negative integer such that for every $e_1 = (x, y) \in E(G_1)$, $min\{f_1(x), f_1(y)\} \leq k < max\{f_1(x), f_1(y)\}$. Let $v \in V(G_1)$ with $f_1(v) = k$ and $w = V(G_2)$ with $f_2(w) = 0$. Then the graph obtained by joining v with w by a path P_n is graceful.*

Proof Take $V_1 = \{u \in V(G_1)/f_1(u) \leq k\}$ and $V_2 = V(G_1) - V_1$. Let $u_1 = v$, $u_2, u_3, \cdots, u_n = w$ be the vertices of path P_n. Let G be the graph obtained by joining v with w by a path P_n.

The vertex labeling function $g : V(P_n) \longrightarrow \{0, 1, \cdots, n-1\}$ defined by

$$g(u_i) = \begin{cases} \frac{i-1}{2} \text{ if } i \text{ is odd,} \\ q - \frac{i-2}{2} \text{ if } i \text{ is even}, \forall i = 1, 2, \cdots, n \end{cases}$$

is an α-graceful for the path P_n, where $k_1 = \lfloor \frac{n-1}{2} \rfloor$. Let $V_3 = \{u \in V(P_n)/f(u) \leq k_1\}$ and $V_4 = V(P_n) - V_3$. To define a vertex labeling function $h : V(G) \longrightarrow \{0, 1, \ldots, q\}$, where $q = q_1 + q_2 + n - 1$, we take following two cases.

Case 1. n is odd.

Let $h|_{V_1} = f_1|_{V_1}$, $h|_{V_2} = f_1|_{V_2} + q_2 + n - 1$, $h|_{V_3} = g|_{V_3} + k$, $h|_{V_4} = g|_{V_4} + k + q_2$ and $h|_{V(G_2)} = f_2|_{V(G_2)} + k + \frac{n-1}{2}$.

Case 2. n is even.

Let $h|_{V_1} = f_1|_{V_1}$, $h|_{V_2} = f_1|_{V_2} + q_2 + n - 1$, $h|_{V_3} = g|_{V_3} + k$, $h|_{V_4} = g|_{V_4} + k + q_2$ and $h|_{V(G_2)} = q_2 + k + \frac{n}{2} - f_2|_{V(G_2)}$.

It is observe that $h(v) = f_1(v) = g(u_1) + k = k$, which is common vertex of G_1 and P_n. Also

$$\begin{aligned} h(w) &= \begin{cases} g(w) + k = k + \lfloor \frac{n}{2} \rfloor, & \text{where } n \text{ is odd as } w \in V_3 \\ g(w) + k + q_2 = k + q_2 + \lfloor \frac{n}{2} \rfloor, & \text{where } n \text{ is even as } w \in V_4 \end{cases} \\ &= \begin{cases} k + \lfloor \frac{n}{2} \rfloor + f_2(w), & \text{where } n \text{ is odd as } f_2(w) = 0 \\ k + q_2 + \lfloor \frac{n}{2} \rfloor - f_2(w), & \text{where } n \text{ is even as } f_2(w) = 0 \end{cases}, \end{aligned}$$

which is common vertex of P_n and G_2. Above defined labeling pattern give the vertex labels from $\{0, 1, \cdots, k, q_2 + k + n, \cdots, q\}$ to the vertices of G_1, the vertex labels $k+1, k+2, \cdots, k + \lfloor \frac{n-2}{2} \rfloor, k + q_2 + \lfloor \frac{n+2}{2} \rfloor, \ldots, q_2 + k + n - 1$ to the vertices of P_n except terminal vertices v, w and the vertex labels from $\{k + \lfloor \frac{n}{2} \rfloor, k + \lfloor \frac{n}{2} \rfloor + 1, \cdots, q_2 + k + \lfloor \frac{n}{2} \rfloor\}$ to the vertices of G_2.

So, h is an injective map. Moreover it produced the edge labels $q, q-1, \cdots, q_2 + n$ to the graph G_1, the edge labels $q_2 + 1, q_2 + 2, \cdots, q_2 + n - 1$ to the graph P_n and the edge labels $1, 2, \cdots, q_2$ to the graph G_2. Thus $f^\star : E(G) \longrightarrow \{0, 1, \cdots, q\}$ defined by absolute difference of end vertices labels is a bijective map. Therefore G admits a graceful labeling h and so, it is a graceful graph. □

Illustration 2.4 The graphs obtained by joining a vertex of an α-graceful graph $K_{4,3}$ and a vertex of a graceful graph C_7 by a path P_4 with their require labeling are shown in Figures 2 and 3.

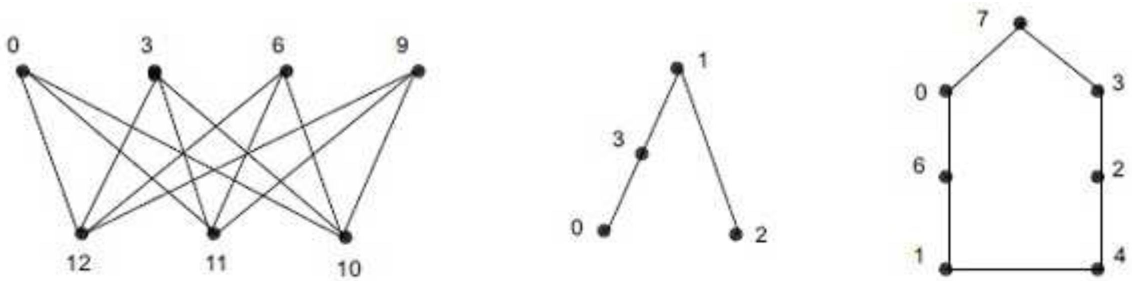

Figure 2 α-graceful labeling for $K_{4,3}$ and P_4,where $k = 9, 1$ respectively and graceful labeling for C_7.

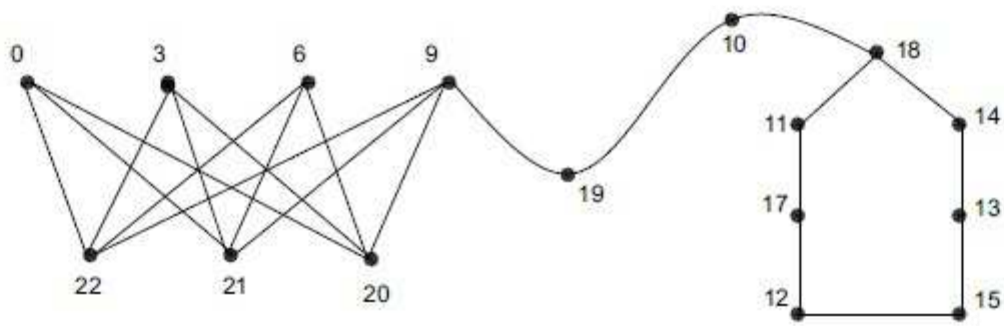

Figure 3 A graceful labeling for the graph obtained by joining $K_{4,3}$, C_7 by a path P_4.

Theorem 2.5 *Let G_1, G_2 be α-graceful graphs and $q_1 = |E(G_1)|$, $q_2 = |E(G_2)|$. Let f_1, f_2 be α-graceful labeling for G_1 and G_2 respectively. Let k_1 $(0 \leq k_1 < q_1)$, k_2 $(0 \leq k_2 < q_2)$ be two non-negative integers such that for every $e_i = (x_i, y_i) \in E(G_i)$, $\min\{f_i(x_i), f_i(y_i)\} \leq k_i < max\{f_i(x_i), f_i(y_i)\}$, $\forall\ i = 1, 2$. Let $v \in V(G_1)$ with $f_1(v) = k_1$ and $w \in V(G_2)$ with $f_2(w) = 0$. Then the graph obtained by joining v with w by a path P_n is α-graceful graph.*

Proof Take $V_1 = \{u \in V(G_1)/f_1(u) \leq k_1\}$ and $V_2 = V(G_1) - V_1$. Let $u_1 = v, u_2, u_3, \cdots, u_n = w$ be the vertices of path P_n. Let G be the graph obtained by joining v with w by a path P_n. The vertex labeling function g on P_n defined in Theorem 2.3 is an α-graceful labeling for the path P_n, where $k_3 = \lfloor \frac{n-1}{2} \rfloor$. Let $V_3 = \{u \in V(P_n)/f(u) \leq k_3\}$ and $V_4 = V(P_n) - V_3$.

We define a vertex labeling function $h : V(G) \longrightarrow \{0, 1, \cdots, q\}$, where $q = q_1 + q_2 + n - 1$ as it defined in Theorem 2.3.

It is observed that $h(v) = f_1(v) = g(u_1) + k = k$, which is common vertex of G_1 and P_n.

Also

$$h(w) = \begin{cases} g(w)+k, & \text{where } n \text{ is odd as } w \in V_3 \\ g(w)+k+q_2, & \text{where } n \text{ is even as } w \in V_4 \end{cases}$$
$$= \begin{cases} k+\lfloor \frac{n}{2} \rfloor + f_2(w), & \text{where } n \text{ is odd} \\ k+q_2+\lfloor \frac{n}{2} \rfloor - f_2(w), & \text{where } n \text{ is even} \end{cases},$$

which is common vertex of P_n and G_2. Above defined labeling pattern give rise a graceful labeling h to the graph G, as discussed in Theorem 2.3.

Since f_2 is an α-graceful labeling for G_2, Therefor $e=(u,v) \in E(G)$ such that $f_2^\star(e) = |f_2(u)-f_2(v)| = 1$. Science $\min\{f_2(u), f_2(v)\} \le k_2 < \max\{f_2(u), f_2(v)\}$ and $|f_2(u)-f_2(v)| = 1$, we must say $\min\{f_2(u), f_2(v)\} = k_2$ and $\max\{f_2(u), f_2(v)\} = k_2+1$.

Now

$$h^\star(e) = |h(u)-h(v)| = |f_2(u)-f_2(v)|$$
$$= f^\star(e) = 1.$$

Take

$$k = \min\{h(u), h(v)\}$$
$$= \begin{cases} \min\{f_2(u)+k_1+\lfloor \frac{n}{2} \rfloor, f_2(v)+k_1+\lfloor \frac{n}{2} \rfloor\}, & \text{where } n \text{ is odd} \\ \min\{q_2+k_1+\lfloor \frac{n}{2} \rfloor - f_2(u), q_2+k_1+\lfloor \frac{n}{2} \rfloor - f_2(v)\}, & \text{where } n \text{ is even} \end{cases}$$
$$= \begin{cases} k_1+\lfloor \frac{n}{2} \rfloor + \min\{f_2(u), f_2(v)\}, & \text{where } n \text{ is odd} \\ q_2+k_1+\lfloor \frac{n}{2} \rfloor - \max\{f_2(u), f_2(v)\}, & \text{where } n \text{ is even} \end{cases}$$
$$= \begin{cases} k_1+k_2+\lfloor \frac{n}{2} \rfloor, & \text{where } n \text{ is odd} \\ q_2+k_1-(k_2+1)+\lfloor \frac{n}{2} \rfloor, & \text{where } n \text{ is even} \end{cases}$$
$$= \begin{cases} k_1+k_2+\frac{n-1}{2}, & \text{where } n \text{ is odd} \\ q_2-k_2+k_1+\frac{n-2}{2}, & \text{where } n \text{ is even} \end{cases}.$$

Then it is observed that for any $e=(x,y) \in E(G)$, $\min\{f(x), f(y)\} \le k < max\{f(x), f(y)\}$ and so, h is an α-graceful labeling for G. Hence G is an α-graceful graph. □

Illustration 2.6 The graphs obtained by joining a vertex of an α-graceful graph $P_3 \times P_4$ and a vertex of another α-graceful graph C_8 by a path P_5 and their related labelings are shown in Figures 4 and 5.

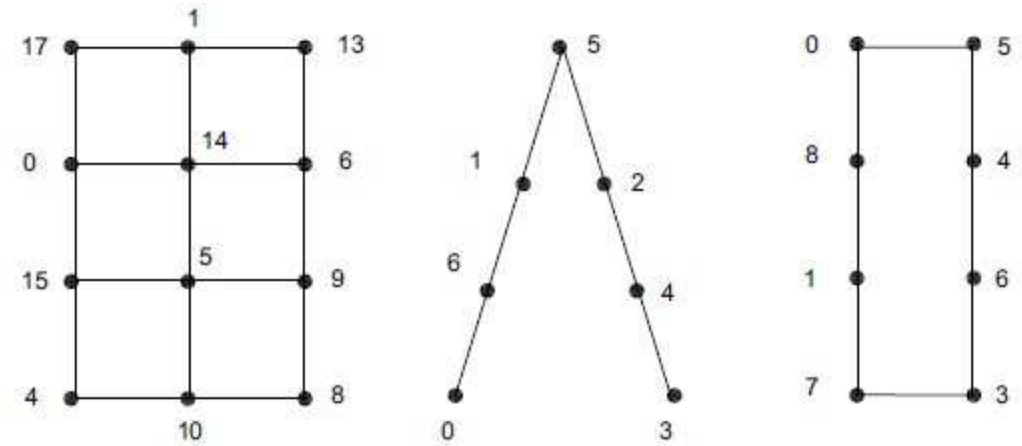

Figure 4 α-graceful labeling for $P_3 \times P_4$,P_7 and C_8, where $k = 8, 3, 4$ respectively.

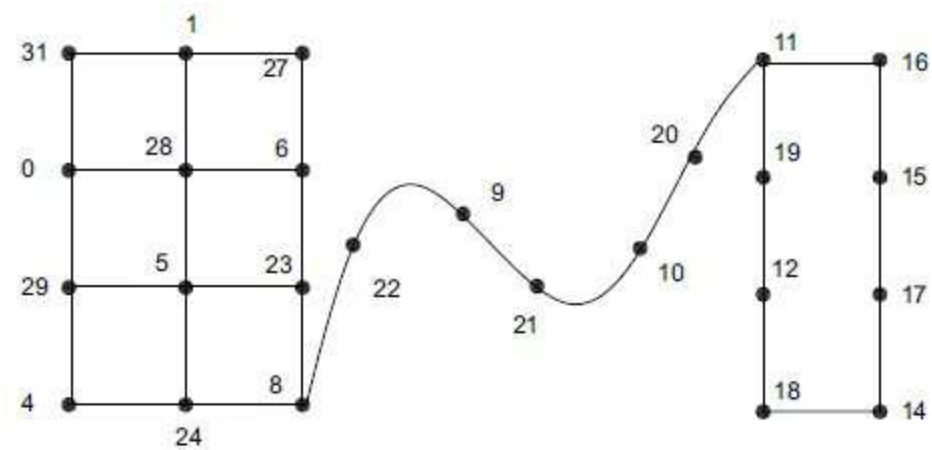

Figure 5 α-graceful labeling for a graph obtained by joining $P_3 \times P_4$ and C_8 by a path P_7.

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