International J.Math. Combin. Vol.2 (2011), 117-129

Some Results on
Generalized Multi Poly-Bernoulli and Euler Polynomials

Hassan Jolany, Hossein Mohebbi

(Department of Mathematics, statistics, and computer Science, University of Tehran, Iran)

R.Eizadi Alikelaye

(Islamic Azad University of QAZVIN, Iran)

E-mail: jolany@ut.ac.ir, re.eizadi@gmail.com

Abstract: The Arakawa-Kaneko zeta function has been introduced ten years ago by T. Arakawa and M. Kaneko in [22]. In [22], Arakawa and Kaneko have expressed the special values of this function at negative integers with the help of generalized Bernoulli numbers $B^{(k)}$ called *poly-Bernoulli numbers*. Kim-Kim [4] introduced Multi poly- Bernoulli numbers and proved that special values of certain zeta functions at non-positive integers can be described in terms of these numbers. The study of Multi poly-Bernoulli and Euler numbers and their combinatorial relations has received much attention [2,4,6,7,12,13,14,19,22,27]. In this paper we introduce the generalization of Multi poly-Bernoulli and Euler numbers and consider some combinatorial relationships of the Generalized Multi poly-Bernoulli and Euler numbers of higher order. The present paper deals with Generalization of Multi poly-Bernouli numbers and polynomials of higher order. In 2002, Q. M. Luo and et al (see [11, 23, 24]) defined the generalization of Bernoulli polynomials and Euler numbers. Some earlier results of Luo in terms of generalized Multi poly-Bernoulli and Euler numbers, can be deduced. Also we investigate some relationships between Multi poly-Bernoulli and Euler polynomials.

Key Words: Generalized Multi poly-Bernoulli polynomials, generalized Multi poly-Euler polynomials, stirling numbers, polylogarithm, Multi- polylogarithm.

AMS(2010): 05A10, 05A19

§1. Introduction

Bernoulli numbers are the signs of a very strong bond between elementary number theory, complex analytic number theory, homotopy theory(the J-homomorphism, and stable homotopy groups of spheres), differential topology(differential structures on spheres), the theory of modular forms(Eisenstein series) and p-adic analytic number theory(the p-adic L-function) of

[1]Received December 29, 2010. Accepted June 8, 2011.

Mathematics. For $n \in Z, n \geq 0$, Bernulli numbers B_n originally arise in the study of finite sums of a given power of consecutive integers. They are given by $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, ...$, with $B_{2n+1} = 0$ for $n > 1$, and

$$B_n = -\frac{1}{n+1}\sum_{m=0}^{n-1}\binom{n+1}{m}B_m, \quad n \geq 1 \tag{1}$$

The modern definition of Bernoulli numbers B_n can be defined by the contour integral

$$B_n = \frac{n!}{2\pi i}\oint \frac{z}{e^z - 1}\frac{dz}{z^{n+1}}, \tag{2}$$

where the contour encloses the origin, has radius less than 2π.

Also Bernoulli polynomials $B_n(x)$ are usualy defined(see[1], [4], [5])by the generating function

$$G(x,t) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi \tag{3}$$

and consequently, Bernoulli numbers $B_n(0) := B_n$ can be obtained by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}$$

Bernoulli polynomials, first studied by Euler (see[1]), are employed in the integral representation of differentiable periodic functions, and play an important role in the approximation of such functions by means of polynomials (see[14]-[18]).

Euler polynomials $E_n(x)$ are defined by the generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi \tag{4}$$

Euler numbers E_n can be obtained by the generating function

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!} \tag{5}$$

The first four such polynomials, are

$$B_0(x) = 1, B_1(x) = x - 1/2, B_2(x) = x^2 - x + 1/6$$
$$B_3(x) = x^3 - 3/2x^2 + 1/2x, ...$$

and

$$E_0(x) = 1, E_1(x) = x - 1/2, E_2(x) = x^2 - x,$$
$$E_3(x) = x^3 - 3/2x^2 + 1/4, ...$$

Euler polynomials are strictly connected with Bernoulli ones, and are used in the Taylor expansion in a neighborhood of the origin of trigonometric and hyperbolic secant functions.

In the sequel, we list some properties of Bernoulli and Euler numbers and polynomials as well as recurrence relations and identities.

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \tag{6}$$

$$E_n(x) = \frac{1}{n+1} \sum_{k=1}^{n+1} (2 - 2^{k+1}) \binom{n+1}{k} B_k x^{n+1-k}. \tag{7}$$

$$B_n(x+1) - B_n(x) = nx^{n-1}, \tag{8}$$

$$E_n(x+1) + E_n(x) = 2x^n. \tag{9}$$

Lemma 1.1(see[20],[21]) *For any integer $n \geq 0$, we have*

$$B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) \tag{10}$$

$$E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) \tag{11}$$

Consequently, from (8), (9) and lemma 1.1, we obtain,

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)x^n \tag{12}$$

$$\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) = 2x^n. \tag{13}$$

Lemma 1.3 *For any positive integer $n \geq 0$, we have*

$$B_n(px) = p^{n-1} \sum_{r=0}^{p-1} B_n(x + \frac{r}{p}) \ (p \textit{ is a positive integer}) \tag{14}$$

$$E_n(px) = p^n \sum_{r=0}^{p-1} (-1)^r E_n(x + \frac{r}{p}) \ (p \textit{ is an odd integer}) \tag{15}$$

Let us briefly recall $k-th$ polylogarithm. The polylogarithm is a special function $Li_k(z)$, that is defined by the sum

$$Li_k(z) := \sum_{s=1}^{\infty} \frac{z^s}{s^k} \tag{16}$$

For formal power series $Li_k(z)$ is the $k-th$ polylogarithm if $k \geq 1$, and a rational function if $k \leq 0$. The name of the function come from the fact that it may alternatively be defined as the repeated integral of itself, namely that

$$Li_{k+1}(z) = \int_0^z \frac{Li_k(t)}{t} dt \tag{17}$$

for integer values of k, we have the following explicit expressions

$$Li_1(z) = -\log(1-z),\ Li_0(z) = \frac{z}{1-z}\ Li_{-1}(z) = \frac{z}{(1-z)^2}$$
$$Li_{-2}(z) = \frac{z(1+z)}{(1-z)^3},\ Li_{-3}(z) = \frac{z(1+4z+z^2)}{(1-z)^4}, ...$$

The integral of the Bose-Einstein distribution is expressed in terms of a polylogarithm,

$$Li_{k+1}(z) = \frac{1}{\Gamma(k+1)} \int_0^{\infty} \frac{t^k}{\frac{e^t}{z} - 1} dt \tag{18}$$

Lemma 1.3(see[18]) *For $n \in N \cup \{0\}$, we have an explicit formula for $Li_{-n}(z)$ as follow*

$$Li_{-n}(z) = \sum_{k=1}^{n+1} \frac{(-1)^{n+k+1}(k-1)!S(n+1,k)}{(1-z)^k} \tag{19}$$
$$(n = 1, 2, ...)$$

where $s(n,k)$ are Stirling numbers of the second kind.

Now, we introduce the generalization of $Li_k(z)$. Let r be an integer with a value greater than one.

Definition 1.1 *Let $k_1, k_2, ...k_r$ be integers. The generalization of polylogarithm are defined by*

$$Li_{k_1,k_2,...,k_r}(z) = \sum_{\substack{m_1,m_2,...,m_r \in \mathcal{Z} \\ 0<m_1<m_2<...<m_r}} \frac{z^{m_r}}{m_1^{k_1}...m_r^{k_r}} \tag{20}$$

The rational numbers $B_n^{(k)}, (n = 0, 1, 2, ...)$ are said to be poly-Bernoulli numbers if they satisfy

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} \tag{21}$$

In addition, for any $n \geq 0$, $B_n^{(1)}$ is the classical Bernoulli number, B_n(see[7], [12]). Also , the rational numbers $H_n^{(k)}(u), (n = 0, 1, 2, ...)$are said to be poly-Euler numbers if they satisfy

$$\frac{Li_k(1-e^{(1-u)})}{u-e^t} = \sum_{n=0}^{\infty} H_n^{(k)}(u) \frac{t^n}{n!} \tag{22}$$

where u is an algebraic real number and $k \geq 1$.(see[13],[19])

Let us now introduce a generalization of poly-Bernoulli numbers, making use of $Li_{k_1,...,k_r}(z)$.

Definition 1.1(see[7]) *Multi poly-Bernoulli numbers $B_n^{(k_0,...,k_r)}, (n = 0, 1, 2, ...)$ are defined for each integer $k_1, k_2, ..., k_r$ by the generating series*

$$\frac{Li_{(k_1,k_2,...,k_r)}(1-e^{-t})}{(1-e^{-t})^r} = \sum_{n=0}^{\infty} B_n^{(k_1,...,k_r)} \frac{t^n}{n!} \tag{23}$$

By Definition 1.2, the left hand side of (23) is

$$\frac{1}{1^{k_1}2^{k_2}...r^{k_r}} + \sum_{\substack{0<m_1<...<m_r \\ m_r \neq r}} \frac{(1-e^{-t})^{m_r - r}}{m_1^{k_1}...m_r^{k_r}} \tag{24}$$

hence we have

$$B_0^{(k_1,..,k_r)} = \frac{1}{1^{k_1}2^{k_2}...r^{k_r}} \tag{25}$$

$$B_1^{(k_1,...k_r)} = \sum_{0<m_1<...<m_r} \frac{1}{m_1^{k_1}...m_{r-1}^{k_{r-1}}(r+1)^{k_r}} \tag{26}$$

Definition 1.3 *Multi poly-Euler numbers* $H_n^{(k_1,...,k_r)}, (n = 0, 1, ...)$ *are defined for each integer* $k_1, ..., k_r$ *by the generating series*

$$\frac{Li_{(k_1,...,k_r)}(1-e^{(1-u)})}{(u-e^t)^r} = \sum_{n=0}^{\infty} H_n^{(k_1,...,k_r)}(u)\frac{t^n}{n!} \tag{27}$$

Kaneko [6] presented the following recurrence formulae for poly-Bernoulli numbers which we state hear.

Theorem 1.1(Kaneko)([2,6,14,22]) *For any* $k \in Z$ *and* $n \geq 0$,*we have*

$$B_n^{(k)} = \frac{1}{n+1}\left\{B_n^{(k-1)} - \sum_{m=1}^{n-1}\binom{n}{m-1}B_m^{(k)}\right\} \tag{28}$$

$$B_n^{(k)} = (-1)^n \sum_{k=1}^{n+1} \frac{(-1)^{m-1}(m-1)!\left\{\begin{matrix} n \\ m-1 \end{matrix}\right\}}{m^k} \tag{29}$$

$$B_n^{(-k)} = \sum_{j=0}^{\min(n,k)} (j!)^2 \left\{\begin{matrix} n+1 \\ j+1 \end{matrix}\right\}\left\{\begin{matrix} k+1 \\ j+1 \end{matrix}\right\} \quad (n, k \geq 0) \tag{30}$$

$$B_n^{(-k)} = B_k^{(-n)} \quad (n, k \geq 0) \tag{31}$$

$$B_n^{(k)} = \sum_{m=0}^{n} (-1)^m \binom{n}{m} B_{n-m}^{(k-1)} \left\{\sum_{l=0}^{m} \frac{(-1)^l}{n-l+1}\binom{m}{l}B_l^{(1)}\right\} \tag{32}$$

where

$$\left\{\begin{matrix} n \\ m \end{matrix}\right\} = \frac{(-1)^m}{m!}\sum_{l=0}^{m}(-1)^l\binom{m}{l}l^n \quad n, m \geq 0 \tag{33}$$

called the second type stirling numbers.

Y.Hamahata and H.Masubuchi in [12], presented the following recurrence formulae for Multi poly-Bernoulli numbers.

Theorem 1.2(H.Masubuchi & Y.Hamahata) *For $n \geq 0$ and $(k_1, ..., k_r \in Z)$ we have*

$$B_n^{(k_1,...,k_r)} = \tag{34}$$
$$(-1)^n \sum_{m_r=r}^{n+r} \left(\sum_{0<m_1<...<m_r} \frac{(-1)^{m_r-r}(m_r-r)! \left\{ \begin{matrix} n \\ m_r - r \end{matrix} \right\}}{m_1^{k_1}...m_r^{k_r}} \right)$$

If $k_r \neq 1$and $n \geq 1$, then

$$B_n^{(k_1,...,k_r)} = \frac{1}{n+r} \left\{ B_n^{(k_1,...,k_{r-1},k_r-1)} - \sum_{m=1}^{n-1} \binom{n}{m-1} B_m^{(k_1,...,k_r)} \right\} \tag{35}$$

If $k_r = 1$ and $n \geq 1$,then

$$B_n^{(k_1,...,k_{r-1},1)} = \frac{1}{n+r} \tag{36}$$
$$\left\{ B_n^{(k_1,...,k_r-1)} - \sum_{m=0}^{n-1} (-1)^{n-m} \left\{ r\binom{n}{m} + \binom{n}{m-1} \right\} B_m^{(k_1,...,k_{r-1},1)} \right\}$$

Also, they proved (see[1]) if

$$B[r]_n^{(k)} = B_n^{(\overbrace{0,...,0}^{r-1},k)} \tag{37}$$

then for $n, k \geq 0$, we have

$$B[r]_n^{(-k)} = B[r]_k^{(-n)} \tag{38}$$

In [23], [24], Q.M.Luo, F.Oi and L.Debnath defined the generalization of Bernoulli and Euler polynomials $B_n(x, a, b, c)$ and $E_n(x, a, b, c)$ respectively, which are expressed as follows

$$\frac{t}{b^t - a^t} c^{xt} = \sum_{k=0}^{\infty} B_k(x, a, b, c) \frac{t^k}{k!} \tag{39}$$

$$\frac{2c^{xt}}{b^t + a^t} = \sum_{k=0}^{\infty} E_k(x, a, b, c) \frac{t^k}{k!} \tag{40}$$

In this paper, by the method of Q.M.Luo and et al [11], we give some properties on generalized Multi poly-Bernoulli and Euler polynomials

Definition 1.4 *Let $a, b > 0$ and $a \neq b$. The generalized Multi poly-Bernoulli numbers $B_n^{(k_1,\cdots,k_r)}(a, b)$, the generalized Multi poly-Bernoulli polynomials*

$$B_n^{(k_1,\cdots,k_r)}(x, a, b) \quad and \quad B_n^{(k_1,\cdots,k_r)}(x, a, b, c)$$

are defined by the following generating functions, respectively;

$$\frac{Li_{(k_1,\dots,k_r)}(1-(ab)^{-t})}{(b^t-a^{-t})^r} = \sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(a,b)\frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|\ln a + \ln b|} \tag{41}$$

$$\frac{Li_{(k_1,\dots,k_r)}(1-(ab)^{-t})}{(b^t-a^{-t})^r}e^{rxt} = \sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(x;a,b)\frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|\ln a + \ln b|} \tag{42}$$

$$\frac{Li_{(k_1,\dots,k_r)}(1-(ab)^{-t})}{(b^t-a^{-t})^r}c^{rxt} = \sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(x;a,b,c)\frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|\ln a + \ln b|} \tag{43}$$

Definition 1.5 *Let $a, b > 0$, and $a \neq b$, the generalized Multi poly-Euler numbers $H_n^{(k_1,\dots,k_r)}(u;a,b)$,the generalized multi poly-Euler polynomial $H_n^{k_1,\dots,k_r}(x;u,a,b)$ and $H_n^{k_1,\dots,k_r}(x;u,a,b,c)$ are defined by the following generating functions, respectively,*

$$\frac{Li_{(k_1,\dots,k_r)}(1-e^{(1-u)})}{(ua^{-t}-b^t)^r} = \sum_{n=0}^{\infty} H_n^{(k_1,\dots,k_r)}(u,a,b)\frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|\ln a + \ln b|} \tag{44}$$

$$\frac{Li_{(k_1,\dots,k_r)}(1-e^{(1-u)})}{(ua^{-t}-b^t)^r}e^{rxt} = \sum_{n=0}^{\infty} H_n^{(k_1,\dots,k_r)}(x;u,a,b)\frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|\ln a + \ln b|} \tag{45}$$

$$\frac{Li_{(k_1,\dots,k_r)}(1-e^{(1-u)})}{(ua^{-t}-b^t)^r}c^{rxt} = \sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_r)}(x;u,a,b,c)\frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|\ln a + \ln b|} \tag{46}$$

§2. Main Theorems

In this section, we introduce our main results. We give some theorems and corollaries which are related to generalized Multi poly-Bernoulli numbers and generalized Multi poly-Euler polynomials. We present some recurrence formulae for generalized Multi-poly-Bernoulli and Euler polynomials.

Theorem 2.1 *Let $a, b > 0$ and $a \neq b$, we have*

$$B_n^{(k_1,\dots,k_r)}(a,b) = B_n^{(k_1,\dots,k_r)}\left(\frac{-\ln b}{\ln a + \ln b}\right)(\ln a + \ln b)^n \tag{47}$$

proof By applying Definition 1.4, we have

$$\frac{Li_{(k_1,\dots,k_r)}(1-(ab)^{-x})}{(b^x-a^{-x})^r} = \sum_{n=0}^{\infty} \frac{B_n^{(k_1,\dots,k_r)}(a,b)}{n!}x^n$$

$$\frac{Li_{(k_1,\dots,k_r)}(1-(ab)^{-x})}{(b^x-a^{-x})^r} = \frac{1}{b^{xr}}\left(\frac{Li_{(k_1,\dots,k_r)}(1-e^{-x\ln ab})}{(1-e^{-x\ln ab})^r}\right)$$

$$= e^{-xr\ln b}\left(\frac{Li_{(k_1,\dots,k_r)}(1-e^{-x\ln ab})}{(1-e^{-x\ln ab})^r}\right)$$

So, we get

$$\frac{Li_{(k_1,\ldots,k_r)}(1-e^{-x\ln ab})}{(b^x-a^{-x})^r}=\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}\left(\frac{-\ln b}{\ln a+\ln b}\right)(\ln a+\ln b)^n\frac{x^n}{n!}$$

Therefore, by comparing the coefficients of $\frac{t^n}{n!}$ on both sides, proof will be complete

$$B_n^{(k_1,\ldots,k_r)}(a,b)=B_n^{(k_1,\ldots,k_r)}\left(\frac{-\ln b}{\ln a+\ln b}\right)(\ln a+\ln b)^n$$

□

The generalized Multi poly-Bernoulli and Euler numbers process a number of interesting properties which we state here

Theorem 2.2 *Let $a,b>0$ and $a\neq b$. For real algebraic u we have*

$$H_n^{(k_1,\ldots,k_r)}(u;a,b)=H_n^{(k_1,\ldots,k_r)}\left(u;\frac{\ln a}{\ln a+\ln b}\right)(\ln a+\ln b)^n. \tag{48}$$

Next, we investigate a strong relationships between $B_n^{(k_1,\ldots,k_r)}(a,b)$ and $B_n^{(k_1,\ldots,k_r)}$.

Theorem 2.3 *Let $a,b>0$,$a\neq b$and $a>b>0$, we have*

$$B_n^{(k_1,\ldots,k_r)}(a,b)=\sum_{i=0}^{j}(-r)^{j-i}(\ln a+\ln b)^i(\ln b)^{j-i}\binom{j}{i}B_i^{(k_1,\ldots,k_r)}. \tag{49}$$

□

By applying Definition 1.4, we have

$$\begin{aligned}\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-x})}{(b^x-a^{-x})^r}&=\frac{1}{b^{xr}}\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-x})}{(1-e^{-x\ln ab})^r}\\ &=\left(\sum_{k=0}^{\infty}\frac{(\ln b)^k}{k!}x^kr^k(-1)^k\right)\left(\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(\ln a+\ln b)^n\frac{x^n}{n!}\right)\\ &=\sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}(-r)^{j-i}\frac{B_j^{(k_1,\ldots,k_r)}(\ln a+\ln b)^i(\ln b)^{j-i}}{i!(j-i)!}x^j\right)\end{aligned}$$

By comparing the coefficient of $\frac{t^n}{n!}$ on both sides, we get.

$$B_j^{(k_1,\ldots,k_r)}(a,b)=\sum_{i=0}^{j}(-r)^{j-i}(\ln a+\ln b)^i(\ln b)^{j-i}\binom{j}{i}B_i^{(k_1,\ldots,k_r)}$$

□

By the same method proceeded in the proof of Theorem 2.3, we obtained similar relations for $H_n^{(k_1,\ldots,k_r)}(u;a,b)$ and $H_n^{(k_1,\ldots,k_r)}$.

Theorem 2.4 *Let $a, b > 0$, and $b > a > 0$. For algebraic real number u, we have*

$$H_n^{(k_1,\dots,k_r)}(u;a,b) = \sum_{i=0}^{n} r^i(\ln a + \ln b)^i(\ln a)^{n-i}\binom{n}{i}H_i^{(k_1,\dots,k_r)} \tag{50}$$

Theorem 2.5 *Let $x \in R$ and conditions of Theorem 2.3 holds true, then we get*

$$B_n^{(k_1,\dots,k_r)}(x;a,b,c) = \sum_{l=0}^{n}\binom{n}{l}r^{n-l}(\ln c)^{n-l}B_l^{(k_1,\dots,k_r)}(a,b)x^{n-l} \tag{51}$$

$$H_n^{(k_1,\dots,k_r)}(u;x,a,b,c) = \sum_{l=0}^{n}\binom{n}{l}r^{n-l}(\ln c)^{n-l}H_l^{(k_1,\dots,k_r)}(u;a,b)x^{n-l} \tag{52}$$

Proof By applying Definitions 1.4 and 1.5, proof will be complete. □

Theorem 2.6 *Let conditions of Theorem 2.5 holds true, we obtain*

$$H_n^{(k_1,\dots,k_r)}(u;x,a,b,c) = \sum_{k=0}^{n}\binom{n}{k}r^{n-k}(\ln c)^{n-k}H_k^{(k_1,\dots,k_r)}(u,\frac{\ln a}{\ln a + \ln b})(\ln a + \ln b)^k x^{n-k} \tag{53}$$

$$B_n^{(k_1,\dots,k_r)}(x;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}r^{n-k}(\ln c)^{n-k}B_k^{(k_1,\dots,k_r)}(\frac{-\ln b}{\ln a + \ln b})(\ln a + \ln b)^k x^{n-k}. \tag{54}$$

Proof By applying Theorems 2.1 and 2.5, we get (53), and Obviously, the result of (54) is similar with (53). □

Theorem 2.7 *Let conditions of Theorem 2.5 holds true, then we get*

$$B_n^{(k_1,\dots,k_r)}(x;a,b,c) = \tag{55}$$
$$= \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{n}{k}\binom{k}{j}r^{n-k}(\ln c)^{n-k}(\ln b)^{k-j}(\ln a + \ln b)^j B_j^{(k_1,\dots,k_r)}x^{n-k}$$

$$H_n^{(k_1,\dots,k_r)}(x;a,b,c) = \tag{56}$$
$$= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}r^{n-k}(\ln c)^{n-k}(\ln a)^{k-j}(\ln c + \ln b)^j H_j^{(k_1,\dots,k_r)}x^{n-k}$$

$$B_n^{(k_1,\dots,k_r)}(x+1;a,b,c) = B_n^{(k_1,\dots,k_r)}(x;ac,\frac{b}{c},c) \tag{57}$$

$$H_n^{(k_1,\dots,k_r)}(u,1-x,ac,b,c) = B_n^{(k_1,\dots,k_r)}(u,-x,ac,\frac{b}{c},c) \tag{58}$$

$$B_n^{(k_1,\dots,k_r)}(x+y;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}r^{n-k}(\ln c)^{n-k}B_n^{(k_1,\dots,k_r)}(x;a,b,c)y^{n-k} \tag{59}$$

$$H_n^{(k_1,\dots,k_r)}(u;x+y,a,b,c) = \sum_{k=0}^{n}\binom{n}{k}r^{n-k}(\ln c)^{n-k}H_n^{(k_1,\dots,k_r)}(x;a,b,c)y^{n-k} \tag{60}$$

Proof We only prove (59) and (55)-(60) can be derived by Definitions 1.4 and 1.5.

$$\begin{aligned}
&\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(b^t-a^{-t})^r}c^{(x+y)rt}=\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x+y,a,b,c)\frac{t^n}{n!}\\
&=\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(b^t-a^{-t})^r}c^{xrt}.c^{yrt}\\
&=\left(\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!}\right)\left(\sum_{i=0}^{n}\frac{y^i(\ln c)^i r^i}{i!}t^i\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}r^{n-k}y^{n-k}(\ln c)^{n-k}B_k^{(k_1,\ldots,k_r)}(x;a,b,c)\right)\frac{t^n}{n!}
\end{aligned}$$

So by comparing the coefficients of $\frac{t^n}{n!}$ in the two expressions, we obtain the desired result 2.13. □

Theorem 2.8 *By the same method proceeded in the proof of previous Theorems, we find similar relations for* $B_n^{(k_1,\ldots,k_r)}(t)$ *and* $H_n^{(k_1,\ldots,k_r)}(u,t)$.

$$B_n^{(k_1,\ldots,k_r)}(t)=B_n^{(k_1,\ldots,k_r)}(e^{1+t},e^{-t}) \tag{61}$$

$$H_n^{(k_1,\ldots,k_r)}(u,t)=H_n^{(k_1,\ldots,k_r)}(u;e^{t},e^{1-t}) \tag{62}$$

Now, we present formulae which show a deeper motivation of generalized poly-Bernoulli and Euler polynomials.

Theorem 2.9 *Let* $x,y\in R$ *and conditions of Theorem 2.5 holds true, we get*

$$B_n^{(k_1,\ldots,k_r)}(x,a,b,c)=(\ln a+\ln b)^n B_n^{(k_1,\ldots,k_r)}\left(\frac{-\ln b+x\ln c}{\ln a+\ln b}\right) \tag{63}$$

$$H_n^{(k_1,\ldots,k_r)}(u;x,a,b,c)=H_n^{(k_1,\ldots,k_r)}\left(u;\frac{\ln a+x\ln c}{\ln a+\ln b}\right) \tag{64}$$

Proof We can write

$$\begin{aligned}
\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!}&=\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(b^t-a^{-t})^r}c^{xrt}\\
&=\frac{1}{b^{rt}}\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(1-(ab)^{-t})^r}c^{xrt}\\
&=e^{r(-\ln b+x\ln c)t}\left(\frac{Li_{(k_1,\ldots,k_r)}(1-e^{-t\ln ab})}{(1-e^{-t\ln ab})^r}\right)
\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get

$$B_n^{(k_1,\ldots,k_r)}(x;a,b,c)=(\ln a+\ln b)^n B_n^{(k_1,\ldots,k_r)}\left(\frac{-\ln b+x\ln c}{\ln a+\ln b}\right).$$

□

GI-Sang Cheon and H.M.Srivastava in [8],[10] investigated the classical relationship between Bernoulli and Euler polynomials . Now we present a relationship between generalized Multi poly-Bernoulli and generalized Euler polynomials. The following relation (65) are given by Q.M.Luo, So by applying this recurrence formula, we obtain Theorem 2.10,

$$E_k(x+1,1,b,b)+E_k(x,1,b,b)=2x^k(\ln b)^k \tag{65}$$

Theorem 2.10 *Let $a,b>0$, we have*

$$B_n^{(k_1,\ldots,k_r)}(x+y;a,b)= \tag{66}$$
$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[B_k^{(k_1,\ldots,k_r)}(y,a,b)+B_k^{(k_1,\ldots,k_r)}(y+1,a,b)\right]r^{n-k}E_{n-k}(x,1,b,b)$$

Proof We know

$$B_n^{(k_1,\ldots,k_r)}(x+y;1,b,b)=\sum_{k=0}^{n}\binom{n}{k}r^{n-k}(\ln b)^{n-k}B_k^{(k_1,\ldots,k_r)}(y;1,b,b)x^{n-k},$$
$$E_k(x+y,1,b,b)+E_k(x,1,,b,b)=2x^k(\ln b)^k$$

So, we obtain

$$B_n^{(k_1,\ldots,k_r)}(x+y,1,b,b)=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}r^{n-k}(\ln b)^{n-k}B_k^{(k_1,\ldots,k_r)}(y;1,b,b)\times$$
$$\left[\frac{1}{(\ln b)^{n-k}}(E_{n-k}(x;1,b,b)+E_{n-k}(x+1,1,b,b))\right]$$
$$=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}r^{n-k}B_k^{(k_1,\ldots,k_r)}(y;1,b,b)\times$$
$$\left[(E_{n-k}(x;1,b,b)+\sum_{j=0}^{n-k}\binom{n-k}{j}E_j(x,1,b,b))\right]$$

$$=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}r^{n-k}B_k^{(k_1,\ldots,k_r)}(y;1,b,b)E_{n-k}(x;1,b,b)$$
$$+\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}r^{n-k}E_j(x;1,b,b)\sum_{k=0}^{n-j}\binom{n-j}{k}B_k^{(k_1,\ldots,k_r)}(y;1,b,b)$$
$$=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}r^{n-k}B_k^{(k_1,\ldots,k_r)}(y;1,b,b)E_{n-k}(x;1,b,b)$$
$$+\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}r^{n-k}B_{n-j}^{(k_1,\ldots,k_r)}(y+1;1,b,b)E_j(x;1,b,b)$$

So we have

$$
\begin{aligned}
&B_n^{(k_1,\ldots,k_r)}(x+y;a,b)\\
&=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[B_k^{(k_1,\ldots,k_r)}(y,a,b)+B_k^{(k_1,\ldots,k_r)}(y+1,a,b)\right]r^{n-k}E_{n-k}(x,1,b,b)
\end{aligned}
$$

Therefore we obtain the desired result (66). □

The following corollary is a straightforward consequence of Theorem 2.10.

Corollary 2.1(see [8],[10]) *In Theorem 2.10, if we set $r=1$, $k=1$ and $b=e$, we obtain*

$$
B_n(x)=\sum_{\substack{k=0\\k\neq 1}}^{n}\binom{n}{k}B_kE_{n-k}(x). \tag{67}
$$

Further work: In [25], Jang et al. gave new formulae on Genocchi numbers. They defined poly-Genocchi numbers to give the relation between Genocchi numbers, Euler numbers, and poly-Genocchi numbers. After Y. Simsek [26], gave a new generating functions which produce Genocchi zeta functions. So by applying a similar method of Kim-Kim [4], we can introduce generalized Genocchi Zeta functions and next define Multi poly-Genocchi numbers and obtain several properties in this area.

Acknowledgments: The authors wishes to express his sincere gratitude to the referee for his/her valuable suggestions and comments and Professor Mohammad Maleki for his cooperations and helps.

References

[1] M.Abramowits, I. A.Stegun, *Handbook of Mathematical functions with formulas, Graphs and Mathematical tables,*National Bureau of standards, Washington, DC, 1964.

[2] T.Arakawa and M.Kaneko, On poly-Bernoulli numbers, *Comment Math. univ. st. Pauli*, 48, (1999), 159-167.

[3] B. N.Gue and F.Qi, Generalization of Bernoulli polynomials, *I. J. Math. Ed. Sci. Tech*, 33, (2002), N0 3, 428-431

[4] M.-S.-Kim and T.Kim, An explicit formula on the generalized Bernoulli number with other n, *Indian. J. Pure and applied Math.*, 31, (2000), 1455-1466.

[5] Hassan jolany and M.R.Darafsheh, Some other remarks on the generalization of Bernoulli and Euler numbers, *Journal's name??*, Vol. 5 (2009), No. 3, 118-129.

[6] M.Kaneko, Poly-bernoulli numbers, *Journal de Theorides de bordeaux*, 9(1997), 221-228

[7] Y.Hamahata, H.Masubuch, Special multi-poly-Bernoulli numbers, *Journal of Integer sequences*, Vol 10, (2007).

[8] H. M.Srivastava and A.Pinter, Remarks on some relationships between the Bernoulli and Euler polynomials, *Applied. Math. Letter*, 17,(2004)375-380.

[9] Chad Berwbaker, A combinatorial interpretion of the poly-Bernoulli numbers and two fermat analogues, *Integers Journal*, 8, (2008).

[10] GI-Sang Cheon, A note on the Bernoulli and Euler polynomials, *Applied. Math. Letter*, 16, (2003), 365-368.

[11] Q. M.Lue, F.Oi and L.Debnath, Generalization of Euler numbers and polynomials, *Int. J. Math. Sci.*, (2003) 3893-3901.

[12] Y.Hamahata, H.Masubuchi, Recurrence formulae for multi-poly-Bernoulli numbers, *Integers Journal*, 7(2007).

[13] Jin-Woo Son and Min Soo kim, On Poly-Eulerian numbers, *Bull.Korean Math. Sco.*, 1999 36:47-61.

[14] M.Kaneko, N.Kurokawa, M.Wakayama, A variation of Euler's approach to the Riemann zeta function, *Kyushu J.Math.*, 57(2003) 175-192.

[15] M.Ward, A calculus of sequences, *Amer.J.Math.*, 58(1936) 255-266.

[16] H.Tsumura, A note on q-analogues of the Dirichlet series and q-Bernoulli numbers, *J.Number Theory*, 39(1991) 251-256.

[17] T.Kim, On a q-analogue of the p-adic log gamma functions and related integrals, *J.Number Theory*, 76(1999)320-329.

[18] H.M.Srivastava and J.Choi, *Series Associated with The zeta and Related functions*, Kluwer Academic Publisher, Dordrechet,Boston, London, 2001.

[19] Hassan Jolany and M.R.Darafsheh, Generalization on poly-Bernoulli numbers and polynomials, *International J.Math. Combin.*, Vol.2 (2010), 07-14.

[20] Q-M-Luo, H.M.Srivastava, Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials, *Comput.Math.Appl.*, 51(2006) 631-642.

[21] L.Carlitz,q-Bernoulli and Eulerian numbers, *Trans.Amer.math.Soc.*, 76 (1954) 332-350

[22] T.Arakawa, M.Kaneko, Multiple zeta values, Poly-Bernoulli numbers and related zeta functions, *Nagoya Math. J.*, 153 (1999), 189-209.

[23] Q.M.Luo, B. N.Guo, F.Qi, and L.Debnath,Generalization of Bernoulli numbers and polynomials, IJMMS, Vol. 2003, Issue 59, 2003, 3769-3776.

[24] Q.M.Luo, F.Qi, and L.Debnath, Generalizations of Euler numbers and polynomials, *IJMMS*, Vol. 2003, Issue 61, 2003, 3893-3901.

[25] L.C.Jang, T.Kim, D.H. Lee and D.W. Park, An application of polylogarithms in the analogs of Genocchi numbers, *Notes on Number Theory and Discrete Mathematics*, Vol.7, No. 3, pp. 65-69, 2001.

[26] Y.Simsek, q-Hardy-Berndt type sums associated with q-Genocchi type zeta and l-functions, *http://arxiv.org/abs/0710.5681v1.*

[27] Hassan Jolany and Hamed Faramarzi, Generalizations on poly-Eulerian numbers and polynomials , *Scientia Magna*, Vol. 6 (2010), No. 1, 9-18.