

## Some Fixed Point Results for Multivalued Nonexpansive Mappings in Banach Spaces

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**Abstract:** In this paper, we deal with the approximation of fixed point for multivalued nonexpansive mappings via a new three-step algorithm which is independent and faster than the iterative process discussed by Agarwal et al. [3] and establish some strong convergence theorems and a weak convergence theorem in the setting of Banach spaces. Our results extend and generalize the previous works from the existing literature.

**Key Words:** Multivalued nonexpansive mapping, three-step iteration scheme, fixed point, weak and strong convergence, Banach space.

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### §1. Introduction

Throughout this paper, let  $\mathcal{X}$  be a real Banach space with the norm  $\|\cdot\|$ . Let  $\mathbb{N}$  denotes the set of all positive integers and let  $F(\mathcal{T})$  denotes the set of all fixed points of the mapping  $\mathcal{T}$ .

Let  $\mathcal{K}$  be a subset of  $\mathcal{X}$ . A subset  $\mathcal{K}$  is called proximal if for each  $x \in \mathcal{X}$ , there exists an element  $k \in \mathcal{K}$  such that  $d(x, k) = \inf\{\|x - y\| : y \in \mathcal{K}\} = d(x, \mathcal{K})$ . It is well known that a weakly compact convex subset of a Banach space and closed convex subsets of a uniformly convex Banach space are Proximal.

We shall denote  $CB(\mathcal{K})$ ,  $C(\mathcal{K})$  and  $P(\mathcal{K})$  by the families of all nonempty closed and bounded subsets, nonempty compact subsets and nonempty proximal subsets of  $\mathcal{K}$ , respectively. Let  $H$  denote the Hausdorff metric induced by the metric  $d$  of  $\mathcal{X}$ , that is,

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

for every  $A, B \in CB(\mathcal{X})$ , where  $d(x, B) = \inf\{\|x - y\| : y \in B\}$ .

A multivalued mapping  $\mathcal{T} : \mathcal{K} \rightarrow CB(\mathcal{K})$  is said to be a *contraction* if there exists a constant  $b \in [0, 1)$  such that for any  $x, y \in \mathcal{K}$ ,

$$H(\mathcal{T}x, \mathcal{T}y) \leq b \|x - y\|,$$

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and  $\mathcal{T}$  is said to be *nonexpansive* if

$$H(\mathcal{T}x, \mathcal{T}y) \leq \|x - y\|,$$

for all  $x, y \in \mathcal{K}$ . A point  $x \in \mathcal{K}$  is called a fixed point of  $\mathcal{T}$  if  $x \in \mathcal{T}x$ .

Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics (see [5] and references cited therein). Moreover, the existence of fixed points for multivalued nonexpansive mappings in uniformly convex Banach spaces was proved by Lim [7]. Many authors have studied the fixed point for multivalued mappings (e.g., see [4, 6, 8, 10, 15, 16, 19]).

In 2005, Sastry and Babu [11] obtained the convergence results from single valued mappings to multivalued mappings by defining Ishikawa and Mann iterates for multivalued mappings with a fixed point. They considered the following:

Let  $\mathcal{K}$  be a nonempty convex subset of  $\mathcal{X}$ ,  $\mathcal{T}: \mathcal{K} \rightarrow P(\mathcal{K})$  is a multivalued mapping with  $p \in F(\mathcal{T})$ .

(i) The *Mann iteration* is defined by

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n s_n, \quad n \in \mathbb{N}, \end{cases} \quad (1.1)$$

where  $\{\alpha_n\}$  is a real sequence in  $(0, 1)$  and  $s_n \in \mathcal{T}x_n$  such that  $\|s_n - p\| = d(p, \mathcal{T}x_n)$ .

(ii) The *Ishikawa iteration* is defined by

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n r_n, \\ y_n = (1 - \beta_n)x_n + \beta_n s_n, \quad n \in \mathbb{N}, \end{cases} \quad (1.2)$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are real sequences in  $(0, 1)$ ,  $\|s_n - r_n\| = d(\mathcal{T}x_n, \mathcal{T}y_n)$  and  $\|r_n - p\| = d(\mathcal{T}y_n, \mathcal{T}p)$  for  $s_n \in \mathcal{T}x_n$  and  $r_n \in \mathcal{T}y_n$ . They established some strong and weak convergence results of the above iterates for multivalued nonexpansive mappings  $\mathcal{T}$  under some appropriate conditions.

In 2007, Panyanak [10] extended the results of Sastry and Babu [11] to a uniformly convex Banach space and also modified the above Ishikawa iterative scheme as follows:

Let  $\mathcal{T}: \mathcal{K} \rightarrow P(\mathcal{K})$  be a multi-valued mapping.

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ y_n = (1 - \beta_n)x_n + \beta_n z_n, \quad n \in \mathbb{N}, \end{cases} \quad (1.3)$$

where  $\{\beta_n\}$  is a real sequence in  $[0, 1]$ ,  $z_n \in \mathcal{T}x_n$  and  $u_n \in \mathcal{T}x_n$  are such that  $\|z_n - u_n\| =$

$d(u_n, \mathcal{T}x_n)$  and  $\|x_n - u_n\| = d(x_n, F(\mathcal{T}))$ , respectively, and

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z'_n, \quad n \in \mathbb{N}, \end{cases} \quad (1.4)$$

where  $\{\alpha_n\}$  is a real sequence in  $[0, 1]$ ,  $z'_n \in \mathcal{T}x_n$  and  $v_n \in \mathcal{T}x_n$  are such that  $\|z'_n - v_n\| = d(v_n, \mathcal{T}x_n)$  and  $\|y_n - v_n\| = d(y_n, F(\mathcal{T}))$ , respectively and proved a convergence theorem of Mann iterates for a mapping defined on a noncompact domain. Later in 2008, Song and Wang [14] proved strong convergence theorems of Mann and Ishikawa iterates for multivalued nonexpansive mappings under some appropriate control conditions. Furthermore, they also gave an affirmative answer to Panyanak's open question in [10].

Recently, Abbas and Nazir [1] introduced and studied the following iteration scheme: let  $\mathcal{K}$  be a nonempty subset of a Banach space  $\mathcal{X}$  and  $\mathcal{T}$  be a nonlinear mapping of  $\mathcal{K}$  into itself. Then the sequence  $\{x_n\}$  in  $\mathcal{K}$  is defined by

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1 - \alpha_n)\mathcal{T}y_n + \alpha_n\mathcal{T}z_n, \\ y_n = (1 - \beta_n)\mathcal{T}x_n + \beta_n\mathcal{T}z_n, \\ z_n = (1 - \gamma_n)x_n + \gamma_n\mathcal{T}x_n, \quad n \in \mathbb{N}, \end{cases} \quad (1.5)$$

where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are real sequences in  $(0, 1)$ . They showed that this process converges faster than both Picard and the Agarwal et al. ([3]) and in support gave analytic proof by a numerical example (for more details, see [1]).

Motivated by Sastry and Babu [11], Panyanak [10] and Song and Wang [14], we first give a multivalued version of the iteration scheme (1.5) of Abbas and Nazir [1] and then study its convergence analysis in the setting of Banach spaces. We define our iteration scheme as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1 - \alpha_n)v_n + \alpha_n w_n, \\ y_n = (1 - \beta_n)u_n + \beta_n w_n, \\ z_n = (1 - \gamma_n)x_n + \gamma_n u_n, \quad n \in \mathbb{N}, \end{cases} \quad (1.6)$$

where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are real sequences in  $(0, 1)$ ,  $u_n \in \mathcal{T}x_n$ ,  $v_n \in \mathcal{T}y_n$  and  $w_n \in \mathcal{T}z_n$  such that  $\|w_n - u_n\| = d(\mathcal{T}z_n, \mathcal{T}x_n)$ ,  $\|v_n - w_n\| = d(\mathcal{T}y_n, \mathcal{T}z_n)$ ,  $\|v_n - u_n\| = d(\mathcal{T}y_n, \mathcal{T}x_n)$ ,  $\|u_{n+1} - v_n\| = d(\mathcal{T}x_{n+1}, \mathcal{T}y_n)$  and  $\|u_{n+1} - w_n\| = d(\mathcal{T}x_{n+1}, \mathcal{T}z_n)$ , respectively.

Now, we recall the following definitions.

**Definition 1.1** A Banach space  $X$  is said to satisfy Opial condition [9] if for any sequence  $\{x_n\}$  in  $X$ ,  $x_n$  converges to  $x$  weakly it follows that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|,$$

for all  $y \in X$  with  $y \neq x$ .

Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces  $l^p$  ( $1 < p < \infty$ ). On the other hand,  $L^p[0, 2\pi]$  with  $1 < p \neq 2$  fail to satisfy Opial condition.

**Definition 1.2** A multivalued mapping  $T: \mathcal{K} \rightarrow P(\mathcal{X})$  is called demiclosed at  $y \in \mathcal{K}$  if for any sequence  $\{x_n\}$  in  $\mathcal{K}$  weakly convergent to an element  $x$  and  $y_n \in Tx_n$  strongly convergent to  $y$ , we have  $y \in Tx$ .

The following is the multivalued version of condition (I) of Senter and Dotson [13].

**Definition 1.3** A multivalued nonexpansive mapping  $T: \mathcal{K} \rightarrow CB(\mathcal{K})$  where  $\mathcal{K}$  a subset of  $\mathcal{X}$  is said to satisfy condition (I) if there exists a nondecreasing function  $f: [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0$ ,  $f(t) > 0$  for all  $t \in (0, \infty)$  such that  $d(x, Tx) \geq f(d(x, F(T)))$  for all  $x \in \mathcal{K}$ , where  $F(T) \neq \emptyset$  is the fixed point set of the multivalued mapping  $T$ .

We need the following Lemmas to prove our main results.

**Lemma 1.4** ([18]) Let  $\{p_n\}$ ,  $\{q_n\}$ ,  $\{r_n\}$  be three sequences of nonnegative real numbers satisfying the following conditions:

$$p_{n+1} \leq (1 + q_n)p_n + r_n, \quad n \geq 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty.$$

Then,

- (1)  $\lim_{n \rightarrow \infty} p_n$  exists;
- (2) In addition, if  $\liminf_{n \rightarrow \infty} p_n = 0$ , then  $\lim_{n \rightarrow \infty} p_n = 0$ .

**Lemma 1.5** ([12]) Let  $E$  be a uniformly convex Banach space and  $0 < \alpha \leq t_n \leq \beta < 1$  for all  $n \in \mathbb{N}$ . Suppose further that  $\{x_n\}$  and  $\{y_n\}$  are sequences of  $E$  such that  $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$ ,  $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$  and  $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = a$  hold for some  $a \geq 0$ . Then  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .

**Lemma 1.6** ([16]) Let  $T: \mathcal{K} \rightarrow P(\mathcal{K})$  be a multivalued mapping and  $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$ . Then the following are equivalent:

- (1)  $x \in F(T)$ ;
- (2)  $P_T(x) = \{x\}$ ;
- (3)  $x \in F(P_T)$ .

Moreover,  $F(T) = F(P_T)$ .

## §2. Main Results

In this section we prove some strong and a weak convergence theorems using iteration scheme (1.6). First, we need the following lemmas to prove main results.

**Lemma 2.1** Let  $\mathcal{X}$  be a real Banach space and  $\mathcal{K}$  be a nonempty closed and convex subset of

$\mathcal{X}$ . Let  $\mathcal{T}: \mathcal{K} \rightarrow P(\mathcal{K})$  be a multivalued mapping such that  $F(\mathcal{T}) \neq \emptyset$  and  $P_{\mathcal{T}}$  is a nonexpansive mapping. Let  $\{x_n\}$  be the sequence defined by (1.6), where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are real sequences in  $(0, 1)$ . Then  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F(\mathcal{T})$ .

*Proof* Let  $p \in F(\mathcal{T})$ . Then  $p \in P_{\mathcal{T}}(p) = \{p\}$  by Lemma 1.6. It follows from (1.6) that

$$\begin{aligned} \|z_n - p\| &\leq (1 - \gamma_n)\|x_n - p\| + \gamma_n\|u_n - p\| \\ &\leq (1 - \gamma_n)\|x_n - p\| + \gamma_n H(P_{\mathcal{T}}(x_n), P_{\mathcal{T}}(p)) \\ &\leq (1 - \gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{2.1}$$

Again using (1.6) and (2.1), we obtain

$$\begin{aligned} \|y_n - p\| &\leq (1 - \beta_n)\|u_n - p\| + \beta_n\|w_n - p\| \\ &\leq (1 - \beta_n)H(P_{\mathcal{T}}(x_n), P_{\mathcal{T}}(p)) + \beta_n H(P_{\mathcal{T}}(z_n), P_{\mathcal{T}}(p)) \\ &\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\ &\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{2.2}$$

Now using (1.6), (2.1) and (2.2), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \alpha_n)\|v_n - p\| + \alpha_n\|w_n - p\| \\ &\leq (1 - \alpha_n)H(P_{\mathcal{T}}(y_n), P_{\mathcal{T}}(p)) + \alpha_n H(P_{\mathcal{T}}(z_n), P_{\mathcal{T}}(p)) \\ &\leq (1 - \alpha_n)\|y_n - p\| + \alpha_n\|z_n - p\| \\ &\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{2.3}$$

It follows from Lemma 1.4 that  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for each  $p \in F(\mathcal{T})$ . This completes the proof.  $\square$

**Lemma 2.2** *Let  $\mathcal{X}$  be a uniformly convex Banach space and  $\mathcal{K}$  be a nonempty closed and convex subset of  $\mathcal{X}$ . Let  $\mathcal{T}: \mathcal{K} \rightarrow P(\mathcal{K})$  be a multivalued mapping such that  $F(\mathcal{T}) \neq \emptyset$  and  $P_{\mathcal{T}}$  is a nonexpansive mapping. Let  $\{x_n\}$  be the sequence defined by (1.6), where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are real sequences in  $(0, 1)$ . Then  $\lim_{n \rightarrow \infty} d(x_n, \mathcal{T}x_n) = 0$ ,  $\lim_{n \rightarrow \infty} d(x_n, \mathcal{T}y_n) = 0$  and  $\lim_{n \rightarrow \infty} d(x_n, \mathcal{T}z_n) = 0$ .*

*Proof* From Lemma 2.1,  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for each  $p \in F(\mathcal{T})$ . We suppose that  $\lim_{n \rightarrow \infty} \|x_n - p\| = l$  for some  $l \geq 0$ .

Since  $\limsup_{n \rightarrow \infty} \|u_n - p\| \leq \limsup_{n \rightarrow \infty} H(P_{\mathcal{T}}(x_n), P_{\mathcal{T}}(p)) \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = l$ , so

$$\limsup_{n \rightarrow \infty} \|u_n - p\| \leq l. \tag{2.4}$$

Again, since  $\limsup_{n \rightarrow \infty} \|v_n - p\| \leq \limsup_{n \rightarrow \infty} H(P_{\mathcal{T}}(y_n), P_{\mathcal{T}}(p)) \leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = l$ , so

$$\limsup_{n \rightarrow \infty} \|v_n - p\| \leq l. \quad (2.5)$$

Similarly,

$$\limsup_{n \rightarrow \infty} \|w_n - p\| \leq l. \quad (2.6)$$

Applying Lemma 1.5, we get

$$\lim_{n \rightarrow \infty} \|u_n - v_n\| = 0 \quad (2.7)$$

$$\lim_{n \rightarrow \infty} \|w_n - u_n\| = 0 \quad (2.8)$$

and

$$\lim_{n \rightarrow \infty} \|v_n - w_n\| = 0. \quad (2.9)$$

Taking  $\limsup$  on both sides of (2.1) and (2.2), we obtain

$$\limsup_{n \rightarrow \infty} \|z_n - p\| \leq l \quad (2.10)$$

and

$$\limsup_{n \rightarrow \infty} \|y_n - p\| \leq l. \quad (2.11)$$

Also,

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)v_n + \alpha_n w_n - p\| \\ &= \|(v_n - p) + \alpha_n(w_n - v_n)\| \\ &\leq \|v_n - p\| + \alpha_n \|w_n - v_n\| \\ &\leq \|v_n - p\| + \|w_n - v_n\| \end{aligned}$$

implies that

$$l \leq \liminf_{n \rightarrow \infty} \|v_n - p\|. \quad (2.12)$$

Combining (2.5) and (2.12), we obtain

$$\lim_{n \rightarrow \infty} \|v_n - p\| = l. \quad (2.13)$$

Thus,

$$\begin{aligned} \|v_n - p\| &\leq \|v_n - w_n\| + \|w_n - p\| \\ &\leq \|v_n - w_n\| + H(P_{\mathcal{T}}(z_n), P_{\mathcal{T}}(p)) \\ &\leq \|v_n - w_n\| + \|z_n - p\| \end{aligned}$$

gives

$$l \leq \liminf_{n \rightarrow \infty} \|z_n - p\| \quad (2.14)$$

and by virtue of (2.10), we obtain

$$\lim_{n \rightarrow \infty} \|z_n - p\| = l. \quad (2.15)$$

Similarly,

$$\begin{aligned} \|w_n - p\| &\leq \|w_n - v_n\| + \|v_n - p\| \\ &\leq \|w_n - v_n\| + H(P_{\mathcal{T}}(y_n), P_{\mathcal{T}}(p)) \\ &\leq \|w_n - v_n\| + \|y_n - p\| \end{aligned}$$

gives

$$l \leq \liminf_{n \rightarrow \infty} \|y_n - p\| \quad (2.16)$$

and by virtue of (2.11), we obtain

$$\lim_{n \rightarrow \infty} \|y_n - p\| = l. \quad (2.17)$$

Applying Lemma 1.5 once again, we obtain

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (2.18)$$

Notice that

$$\|x_n - v_n\| \leq \|x_n - u_n\| + \|u_n - v_n\|.$$

Using (2.7) and (2.18), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - v_n\| = 0 \quad (2.19)$$

and

$$\|x_n - w_n\| \leq \|x_n - u_n\| + \|u_n - w_n\|.$$

Using (2.8) and (2.19), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - w_n\| = 0. \quad (2.20)$$

Since  $d(x_n, \mathcal{T}x_n) \leq \|x_n - u_n\|$ , we have

$$\lim_{n \rightarrow \infty} d(x_n, \mathcal{T}x_n) = 0. \quad (2.21)$$

Again since  $d(x_n, \mathcal{T}y_n) \leq \|x_n - v_n\|$ , we have

$$\lim_{n \rightarrow \infty} d(x_n, \mathcal{T}y_n) = 0. \quad (2.22)$$

Similarly, since  $d(x_n, \mathcal{T}z_n) \leq \|x_n - w_n\|$ , we have

$$\lim_{n \rightarrow \infty} d(x_n, \mathcal{T}z_n) = 0. \quad (2.23)$$

This completes the proof.  $\square$

Now we shall prove some strong convergence theorems using iteration scheme (1.6) in real Banach spaces.

**Theorem 2.3** *Let  $\mathcal{X}$  be a real Banach space and  $\mathcal{K}$  be a nonempty closed and convex subset of  $\mathcal{X}$ . Let  $\mathcal{T}: \mathcal{K} \rightarrow P(\mathcal{K})$  be a multivalued mapping such that  $F(\mathcal{T}) \neq \emptyset$  and  $P_{\mathcal{T}}$  is a nonexpansive mapping. Let  $\{x_n\}$  be the sequence defined by (??), where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are real sequences in  $(0, 1)$ . Then  $\{x_n\}$  converges strongly to a fixed point of  $\mathcal{T}$  if and only if  $\liminf_{n \rightarrow \infty} d(x_n, F(\mathcal{T})) = 0$ .*

*Proof* The necessity is obvious. Conversely, suppose that  $\liminf_{n \rightarrow \infty} d(x_n, F(\mathcal{T})) = 0$ . As proved in Lemma 1.6, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\|$$

which gives

$$d(x_{n+1}, F(\mathcal{T})) \leq d(x_n, F(\mathcal{T})).$$

This implies that  $\lim_{n \rightarrow \infty} d(x_n, F(\mathcal{T}))$  exists by Lemma 1.4 and so by hypothesis,  $\liminf_{n \rightarrow \infty} d(x_n, F(\mathcal{T})) = 0$ . Therefore, we must have  $\lim_{n \rightarrow \infty} d(x_n, F(\mathcal{T})) = 0$ .

Next, we have to show that  $\{x_n\}$  is a Cauchy sequence in  $\mathcal{K}$ . Let  $\varepsilon > 0$  be arbitrary chosen. Since  $\lim_{n \rightarrow \infty} d(x_n, F(\mathcal{T})) = 0$ , there exists a constant  $n_1$  such that for all  $n \geq n_1$  we have

$$d(x_n, F(\mathcal{T})) < \frac{\varepsilon}{4}.$$

In particular,  $\inf\{\|x_{n_1} - p\| : p \in F(\mathcal{T})\} < \frac{\varepsilon}{4}$ . There must exist a  $q \in F(\mathcal{T})$  such that

$$\|x_{n_1} - q\| < \frac{\varepsilon}{2}.$$

Now for  $m, n \geq n_1$ , we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\ &\leq 2\|x_{n_1} - q\| \\ &< 2\left(\frac{\varepsilon}{2}\right) = \varepsilon. \end{aligned}$$

Hence  $\{x_n\}$  is a Cauchy sequence in a closed subset  $\mathcal{K}$  of a Banach space  $\mathcal{X}$ , and so it must

converge in  $\mathcal{K}$ . Let  $\lim_{n \rightarrow \infty} x_n = q_1$ . Now

$$\begin{aligned} d(q_1, P_{\mathcal{T}}(q_1)) &\leq \|x_n - q_1\| + d(x_n, P_{\mathcal{T}}(x_n)) + H(P_{\mathcal{T}}(x_n), P_{\mathcal{T}}(q_1)) \\ &\leq 2\|x_n - q_1\| + \|x_n - u_n\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

which gives that  $d(q_1, \mathcal{T}q_1) = 0$ . But  $P_{\mathcal{T}}$  is a nonexpansive mapping and so  $F(\mathcal{T})$  is closed. Therefore,  $q_1 \in F(P_{\mathcal{T}}) = F(\mathcal{T})$ . This shows that  $\{x_n\}$  converges strongly to a point of  $F(\mathcal{T})$ . This completes the proof.  $\square$

**Theorem 2.4** *Let  $\mathcal{X}$  be a real Banach space and  $\mathcal{K}$  be a nonempty compact convex subset of  $\mathcal{X}$ . Let  $\mathcal{T}: \mathcal{K} \rightarrow P(\mathcal{K})$  be a multivalued mapping such that  $F(\mathcal{T}) \neq \emptyset$  and  $P_{\mathcal{T}}$  is a nonexpansive mapping. Let  $\{x_n\}$  be the sequence defined by (??), where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are real sequences in  $(0, 1)$ . Then  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  converges strongly to a fixed point of  $\mathcal{T}$ .*

*Proof* By Lemma 2.2, we have  $\lim_{n \rightarrow \infty} d(x_n, \mathcal{T}x_n) = 0$ . Since by hypothesis  $\mathcal{K}$  be a nonempty compact convex subset of  $\mathcal{X}$ , so there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\lim_{k \rightarrow \infty} \|x_{n_k} - u\| = 0$  for some  $u \in \mathcal{K}$ . Thus

$$\begin{aligned} d(u, P_{\mathcal{T}}(u)) &\leq \|x_{n_k} - u\| + d(x_{n_k}, P_{\mathcal{T}}(x_{n_k})) + H(P_{\mathcal{T}}(x_{n_k}), P_{\mathcal{T}}(u)) \\ &\leq 2\|x_{n_k} - u\| + \|x_{n_k} - u_{n_k}\| \\ &\rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

This shows that  $u$  is a fixed of  $\mathcal{T}$ . From Lemma 2.1, we get that  $\lim_{n \rightarrow \infty} \|x_n - u\| = 0$ . Again from Lemma 2.2, we get that

$$\begin{aligned} \|y_n - x_n\| &= \|(1 - \beta_n)u_n + \beta_n w_n - x_n\| \\ &\leq \|u_n - x_n\| + \beta_n \|w_n - u_n\| \\ &\leq \|u_n - x_n\| + \|w_n - u_n\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

and

$$\begin{aligned} \|z_n - x_n\| &= \|(1 - \gamma_n)x_n + \gamma_n u_n - x_n\| \\ &\leq \gamma_n \|u_n - x_n\| \\ &\leq \|u_n - x_n\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

It follows that  $\lim_{n \rightarrow \infty} \|y_n - u\| = 0$  and  $\lim_{n \rightarrow \infty} \|z_n - u\| = 0$ . Thus the conclusion follows. This completes the proof.  $\square$

Now, we apply Theorem 2.3 to obtain another strong convergence theorem in uniformly

convex Banach spaces satisfies condition (I) of Senter and Dotson [13].

**Theorem 2.5** *Let  $\mathcal{X}$  be a uniformly convex Banach space and  $\mathcal{K}$  be a nonempty closed and convex subset of  $\mathcal{X}$ . Let  $\mathcal{T}: \mathcal{K} \rightarrow P(\mathcal{K})$  be a multivalued mapping satisfying condition (I) such that  $F(\mathcal{T}) \neq \emptyset$  and  $P_{\mathcal{T}}$  is a nonexpansive mapping. Let  $\{x_n\}$  be the sequence defined by (1.6), where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are real sequences in  $(0, 1)$ . Then  $\{x_n\}$  converges strongly to a fixed point of  $\mathcal{T}$ .*

*Proof* By Lemma 2.1,  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F(\mathcal{T})$  and so the sequence  $\{x_n\}$  is bounded. Let  $\lim_{n \rightarrow \infty} \|x_n - p\| = r$  for some  $r > 0$ .

Now from Lemma 2.1, we know that

$$\|x_{n+1} - p\| \leq \|x_n - p\|$$

which implies that

$$\inf_{p \in F(\mathcal{T})} \|x_{n+1} - p\| \leq \inf_{p \in F(\mathcal{T})} \|x_n - p\|,$$

and also  $d(x_{n+1}, F(\mathcal{T})) \leq d(x_n, F(\mathcal{T}))$ . And so,  $\lim_{n \rightarrow \infty} d(x_n, F(\mathcal{T}))$  exists. By using condition (I) and Lemma 2.2, we have

$$\lim_{n \rightarrow \infty} f(d(x_n, F(\mathcal{T}))) \leq \lim_{n \rightarrow \infty} d(x_n, \mathcal{T}x_n) = 0.$$

That is,

$$\lim_{n \rightarrow \infty} f(d(x_n, F(\mathcal{T}))) = 0.$$

Since  $f$  is a nondecreasing function and  $f(0) = 0$ , it follows that  $\lim_{n \rightarrow \infty} d(x_n, F(\mathcal{T})) = 0$ . The conclusion follows from Theorem 2.3. This completes the proof.  $\square$

Now, we prove the weak convergence theorem of the sequence  $\{x_n\}$  defined by (1.6).

**Theorem 2.6** *Let  $\mathcal{X}$  be a uniformly convex Banach space satisfying Opial's condition and  $\mathcal{K}$  be a nonempty closed and convex subset of  $\mathcal{X}$ . Let  $\mathcal{T}: \mathcal{K} \rightarrow P(\mathcal{K})$  be a multivalued mapping such that  $F(\mathcal{T}) \neq \emptyset$  and  $P_{\mathcal{T}}$  is a nonexpansive mapping. Let  $\{x_n\}$  be the sequence defined by (??), where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are real sequences in  $(0, 1)$ . Let  $I - P_{\mathcal{T}}$  be demiclosed with respect to zero. Then  $\{x_n\}$  converges weakly to a fixed point of  $\mathcal{T}$ .*

*Proof* Let  $z \in F(\mathcal{T})$ . From Lemma 2.1, we know that  $\lim_{n \rightarrow \infty} \|x_n - z\|$  exists. Now we prove that  $\{x_n\}$  has a unique weak subsequential limit in  $F(\mathcal{T})$ . To prove this, let  $p_1$  and  $p_2$  be weak limits of the subsequences  $\{x_{n_i}\}$  and  $\{x_{n_j}\}$  of  $\{x_n\}$ , respectively. By (2.18), there exists  $u_n \in \mathcal{T}x_n$  such that  $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$ . Since  $I - P_{\mathcal{T}}$  is demiclosed with respect to zero, therefore we obtain  $p_1 \in F(\mathcal{T})$ . In the same way, we can prove that  $p_2 \in F(\mathcal{T})$ .

Next, we prove uniqueness. For this, suppose that  $p_1 \neq p_2$ . Then by Opial's condition, we

have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \|x_n - p_1\| &= \lim_{n_i \rightarrow \infty} \|x_{n_i} - p_1\| \\
&< \lim_{n_i \rightarrow \infty} \|x_{n_i} - p_2\| \\
&= \lim_{n \rightarrow \infty} \|x_n - p_2\| \\
&= \lim_{n_j \rightarrow \infty} \|x_{n_j} - p_2\| \\
&< \lim_{n_j \rightarrow \infty} \|x_{n_j} - p_1\| \\
&= \lim_{n \rightarrow \infty} \|x_n - p_1\|,
\end{aligned}$$

which is a contradiction. Hence  $\{x_n\}$  converges weakly to a fixed point of  $\mathcal{T}$ . This completes the proof.  $\square$

**Remark 2.7** Our results extend, generalize and improve several corresponding results from the existing literature and iterative schemes discussed by Panyanak [10], Sastry and Babu [11], Song and Wang [14] and Song and Cho [16] in the sense of faster iterative scheme.

Suzuki [17] introduced a condition on mappings called condition (C) which is weaker than nonexpansiveness.

Recently, Abkar and Eslamian [2] introduced the definition of condition (C) for multi-valued mapping. The definition is as follows.

**Definition 2.8**([2]) *A multivalued mapping  $\mathcal{T}: \mathcal{X} \rightarrow CB(\mathcal{X})$  is said to satisfy condition (C) provided that*

$$\frac{1}{2}d(x, \mathcal{T}x) \leq \|x - y\| \Rightarrow H(\mathcal{T}x, \mathcal{T}y) \leq \|x - y\|, \quad x, y \in \mathcal{X}.$$

The following result can be found in [2].

**Lemma 2.9**([2]) *Let  $\mathcal{T}: \mathcal{X} \rightarrow CB(\mathcal{X})$  be a multi-valued mapping. If  $\mathcal{T}$  is nonexpansive, then  $\mathcal{T}$  satisfies the condition (C).*

We mention that there exist single-valued and multi-valued mappings satisfying the condition (C) which are not nonexpansive.

**Example 2.10**([17]) Define a mapping  $\mathcal{T}$  on  $[0,3]$  by

$$\mathcal{T}(x) = \begin{cases} 0, & \text{if } x \neq 3, \\ 1, & \text{if } x = 3. \end{cases}$$

Then  $\mathcal{T}$  is a single-valued mapping satisfying condition (C), but  $\mathcal{T}$  is not nonexpansive.

**Example 2.11**([2]) Define a mapping  $\mathcal{T}: [0, 5] \rightarrow [0, 5]$  by

$$T(x) = \begin{cases} [0, \frac{x}{5}], & \text{if } x \neq 5, \\ \{1\}, & \text{if } x = 5. \end{cases}$$

Then it is easy to show that  $\mathcal{T}$  is a multi-valued mapping satisfying condition (C), but  $\mathcal{T}$  is not nonexpansive.

Now, we obtain some strong convergence results using iteration scheme (1.6) and condition (C).

**Theorem 2.12** *Let  $\mathcal{X}$  be a real Banach space and  $\mathcal{K}$  be a nonempty closed and convex subset of  $\mathcal{X}$ . Let  $\mathcal{T}: \mathcal{K} \rightarrow P(\mathcal{K})$  be a multivalued mapping such that  $F(\mathcal{T}) \neq \emptyset$  and  $P_{\mathcal{T}}$  satisfies condition (C). Let  $\{x_n\}$  be the sequence defined by (1.6), where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are real sequences in  $(0, 1)$ . Then  $\{x_n\}$  converges strongly to a fixed point of  $\mathcal{T}$  if and only if  $\liminf_{n \rightarrow \infty} d(x_n, F(\mathcal{T})) = 0$ .*

*Proof* The proof of Theorem 2.12 immediately follows from Lemma 2.1 and Theorem 2.3. This completes the proof.  $\square$

**Theorem 2.13** *Let  $\mathcal{X}$  be a real Banach space and  $\mathcal{K}$  be a nonempty closed and convex subset of  $\mathcal{X}$ . Let  $\mathcal{T}: \mathcal{K} \rightarrow P(\mathcal{K})$  be a multivalued mapping such that  $F(\mathcal{T}) \neq \emptyset$  and  $P_{\mathcal{T}}$  satisfies condition (C). Let  $\{x_n\}$  be the sequence defined by (1.6), where  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  are real sequences in  $(0, 1)$ . If the following condition is satisfied:*

(c<sub>1</sub>) *there exists an increasing function  $\psi: [0, \infty) \rightarrow [0, \infty)$  with  $\psi(r) > 0, \forall r > 0$  such that*

$$d(x_n, \mathcal{T}x_n) \geq \psi(d(x_n, F(\mathcal{T}))), \quad (2.24)$$

*then  $\{x_n\}$  converges strongly to a fixed point of  $\mathcal{T}$ .*

*Proof* As in the proof of Lemma 2.2, we know that  $\lim_{n \rightarrow \infty} d(x_n, \mathcal{T}x_n) = 0$ . Hence from (2.21) we obtain  $\lim_{n \rightarrow \infty} d(x_n, F(\mathcal{T})) = 0$ . The conclusion of Theorem 2.13 can be obtained from Theorem 2.12 immediately. This completes the proof.  $\square$

### §3. Concluding Remarks

In this paper, we study a new three-step iteration scheme for multivalued nonexpansive mappings in Banach spaces and establish some strong convergence theorems and a weak convergence theorem under some appropriate conditions applying on the space. Our results extend and generalize several results from the current existing literature.

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