International J.Math. Combin. Vol.4(2022), 21-31

Some Hermite-Hadamard Type Inequalities For Trigonometrically ρ-Convex Functions via by an Identity

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Abstract: In this paper, using an identity with integral inequalities Hölder, power-mean, Hölder-İşcan and improved power-mean integral inequalities, we get some refinements of the Hermite-Hadamard type integral inequalites for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically ρ-convex.

Key Words: Convex function, trigonometrically convex function, trigonometrically ρ-convex function, Hermite-Hadamard integral inequality.

AMS(2010): 26A51, 26D10, 26D15.

§1. Preliminaries and Fundamentals

Throughout the paper I is a non-empty interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. One of the most famous inequality for the class of convex functions is so called Hermite-Hadamard inequality, which states that: For a convex mapping $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

[1]Received November 4, 2022, Accepted December 4, 2022.

is known as the Hermite-Hadamard integral inequality (for more information, see [5]). Since then, some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [2, 3, 4, 11].

Definition 1.1([10]) *Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0, 1)$ we have*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$.

Definition 1.2([7]) *A non-negative function $f : I \to \mathbb{R}$ is called trigonometrically convex function on interval $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0, 1]$,*

$$f(tx + (1-t)y) \leq \left(\sin\frac{\pi t}{2}\right) f(x) + \left(\cos\frac{\pi t}{2}\right) f(y).$$

We will denote by $TC(I)$ the class of all trigonometrically convex functions on interval I. For $h(t) = \sin\frac{\pi t}{2}$, every trigonometrically convex function is also h-convex function.

Theorem 1.1(Hölder Inequality for Integrals, [9]) *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|^q$, $|g|^q$ are integrable functions on interval $[a, b]$, $q > 1$ then*

$$\int_a^b |f(x)g(x)|\, dx \leq \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, dx\right)^{\frac{1}{q}}$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$, almost everywhere, where A and B are constants.

Theorem 1.2(Power-mean Integral Inequality) *Let $q \geq 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|$, $|f||g|^q$ are integrable functions on interval $[a, b]$, then*

$$\int_a^b |f(x)g(x)|\, dx \leq \left(\int_a^b |f(x)|\, dx\right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)|\,|g(x)|^q\, dx\right)^{\frac{1}{q}}.$$

In [6], İşcan achieved the following integral inequality which gives better approach than the classical Hölder integral inequality:

Theorem 1.3(Hölder-İşcan Inequality for Integrals) *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and $|f|^q$ and $|g|^q$ are integrable functions on interval*

$[a, b]$, *then*

$$\int_a^b |f(x)g(x)|\,dx \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}.$$

In [8], a different representation of the Hölder-İşcan inequality is given as follows:

Theorem 1.4(Improved power-Mean Integral Inequality) *Let* $p \ge 1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *If* f *and* g *are real functions defined on interval* $[a,b]$ *and* $|f|$ *and* $|f|\,|g|^q$ *are integrable functions on interval* $[a,b]$, *then*

$$\int_a^b |f(x)g(x)|\,dx \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}.$$

§2. Some Integral Inequalities for Trigonometrically ρ-Convexity

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard type integral inequalities for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically ρ-convex. Alomari and Darus [1] used the following lemma.

Lemma 2.1([1]) *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I° *where* $a, b \in I$ *with* $a < b$. *If* $f' \in L[a,b]$, *then the following equality holds:*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2}+(1-t)a\right)dt + \int_0^1 (t-1)f'\left(tb+(1-t)\frac{a+b}{2}\right)dt\right].$$

Note that we will use the followings in this section:

$$\int_0^1 |t|\sin\frac{\pi t}{2}dt = \int_0^1 |t-1|\cos\frac{\pi t}{2}dt = \frac{4}{\pi^2},$$

$$\int_0^1 |t-1|\sin\frac{\pi t}{2}dt = \int_0^1 |t|\cos\frac{\pi t}{2}dt = \frac{2\pi-4}{\pi^2},$$

$$\int_0^1 |t|^p\,dt = \int_0^1 |t-1|^p\,dt = \frac{1}{p+1},$$

$$\begin{aligned}\int_0^1 \sin\frac{\pi t}{2}dt &= \int_0^1 \cos\frac{\pi t}{2}dt = \frac{2}{\pi}\\ \int_0^1 |t|\,dt &= \int_0^1 |t-1|\,dt = \frac{1}{2}\\ A &= A(u,v) = \frac{u+v}{2} \text{ arithmetic mean}\end{aligned}$$

Theorem 2.1 *Let $f : I \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I and assume that $f' \in L[a,b]$. If $|f'|$ is trigonometrically ρ-convex function on interval $[a,b]$, then the following inequality*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le \frac{b-a}{2}\left(\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right| + \left(\frac{2\pi-4}{\pi^2}\right)A\left(\left|f'(a)\right|, \left|f'(b)\right|\right)\right)\end{aligned}$$

holds for $t \in [0,1]$.

Proof Using Lemma 2.1 and the inequalities

$$\left|f'(t\frac{a+b}{2} + (1-t)a\right| \le \sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right| + \cos\frac{\pi t}{2}\left|f'(a)\right|$$

and

$$\left|f'(tb + (1-t)\left(\frac{a+b}{2}\right)\right| \le \sin\frac{\pi t}{2}\left|f'(b)\right| + \cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|$$

we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le \left|\frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2} + (1-t)a\right)dt + \int_0^1 (t-1)f'\left(tb + (1-t)\frac{a+b}{2}\right)dt\right]\right|\\ &\le \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|dt + \int_0^1 |t-1|\left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|dt\right]\\ &\le \frac{b-a}{4}\int_0^1 |t|\left[\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right| + \cos\frac{\pi t}{2}\left|f'(a)\right|\right]dt\\ &\quad + \frac{b-a}{4}\int_0^1 |t-1|\left[\sin\frac{\pi t}{2}\left|f'(b)\right| + \cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|\right]dt\\ &= \frac{b-a}{4}\left[\left|f'\left(\frac{a+b}{2}\right)\right|\int_0^1 |t|\sin\frac{\pi t}{2}dt + \left|f'(a)\right|\int_0^1 |t|\cos\frac{\pi t}{2}dt\right]\\ &\quad + \frac{b-a}{4}\left[\left|f'(b)\right|\int_0^1 |t-1|\sin\frac{\pi t}{2}dt + \left|f'\left(\frac{a+b}{2}\right)\right|\int_0^1 |t-1|\cos\frac{\pi t}{2}dt\right]\\ &= \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|\frac{4}{\pi^2} + \left|f'(a)\right|\left(\frac{2\pi-4}{\pi^2}\right) + \left|f'(b)\right|\left(\frac{2\pi-4}{\pi^2}\right) + \left|f'\left(\frac{a+b}{2}\right)\right|\frac{4}{\pi^2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{b-a}{2}\left(\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right| + \left(\frac{2\pi-4}{\pi^2}\right)\frac{|f'(a)|+|f'(b)|}{2}\right) \\
&= \frac{b-a}{2}\left[\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right| + \left(\frac{2\pi-4}{\pi^2}\right)A\left(|f'(a)|,|f'(b)|\right)\right].
\end{aligned}
$$

This completes the proof of theorem. □

Theorem 2.2 *Let $f : I \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I assume that $q > 1$. If the mapping $|f'|^q$ is trigonometrically ρ-convex function on interval $[a, b]$, then the following inequality*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq 2^{\frac{1}{q}}\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}\left[\left(A\left|f'\left(\frac{a+b}{2}\right)\right|^q, |f'(a)|^q\right)\right]^{\frac{1}{q}} \\
&\quad + 2^{\frac{1}{q}}\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}\left[A\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{\frac{1}{q}},
\end{aligned}
$$

holds for $t \in [0, 1]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Using Lemma 2.1, Hölder's integral inequality and inequalities

$$
\left|f'(t\frac{a+b}{2} + (1-t)a\right|^q \leq \sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \cos\frac{\pi t}{2}|f'(a)|^q
$$

and

$$
\left|f'(tb + (1-t)\left(\frac{a+b}{2}\right)\right|^q \leq \sin\frac{\pi t}{2}|f'(b)|^q + \cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q
$$

which is the trigonometrically ρ-convexity of $|f'|^q$, we get

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \left|\frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2} + (1-t)a\right)dt + \int_0^1 (t-1)f'\left(tb + (1-t)\frac{a+b}{2}\right)dt\right]\right| \\
&\leq \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|dt + \int_0^1 |t-1|\left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|dt\right] \\
&\leq \frac{b-a}{4}\left(\int_0^1 |t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{b-a}{4}\left(\int_0^1 |t-1|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&= \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \cos\frac{\pi t}{2}|f'(a)|^q\right)dt\right)^{\frac{1}{q}} \\
&\quad + \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left(\sin\frac{\pi t}{2}|f'(b)|^q + \cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\int_0^1 \sin\frac{\pi t}{2}dt + |f'(a)|^{q}\int_0^1 \cos\frac{\pi t}{2}dt\right]^{\frac{1}{q}} \\
&\quad + \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[|f'(b)|^{q}\int_0^1 \sin\frac{\pi t}{2}dt + \left|f'\left(\frac{a+b}{2}\right)\right|^{q}\int_0^1 \cos\frac{\pi t}{2}dt\right]^{\frac{1}{q}} \\
&= 2^{\frac{1}{q}}\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}\left[\left(A\left|f'\left(\frac{a+b}{2}\right)\right|^{q}, |f'(a)|^{q}\right)\right]^{\frac{1}{q}} \\
&\quad + 2^{\frac{1}{q}}\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}\left[A\left(|f'(b)|^{q}, \left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof of theorem. □

Theorem 2.3 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I such that $0 < \rho(b-a) < \pi$ and assume that $q \geq 1$. If the mapping $|f'|^q$ is trigonometrically ρ-convex function on interval $[a, b]$, then the following inequality*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\frac{4}{\pi^2} + |f'(a)|^{q}\frac{2\pi-4}{\pi^2}\right)^{\frac{1}{q}} \\
&\quad + \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(|f'(b)|^{q}\frac{2\pi-4}{\pi^2} + \left|f'\left(\frac{a+b}{2}\right)\right|^{q}\frac{4}{\pi^2}\right)^{\frac{1}{q}},
\end{aligned}
$$

holds for $t \in [0, 1]$.

Proof Firstly, let $q > 1$. From Lemma 2.1, well known Power-mean integral inequality and the following inequalities

$$
\begin{aligned}
\left|f'(t\frac{a+b}{2} + (1-t)a\right|^{q} &\leq \sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^{q} + \cos\frac{\pi t}{2}|f'(a)|^{q} \\
\left|f'(tb + (1-t)\left(\frac{a+b}{2}\right)\right|^{q} &\leq \sin\frac{\pi t}{2}|f'(b)|^{q} + \cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^{q},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|dt + \int_0^1 |t-1|\left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|dt\right] \\
&\leq \frac{b-a}{4}\left(\int_0^1 |t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t|\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}} \\
&\quad + \frac{b-a}{4}\left(\int_0^1 |t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t-1|\left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|^{q}dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{b-a}{4}\left(\int_0^1 |t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t|\left(\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \cos\frac{\pi t}{2}\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&+ \frac{b-a}{4}\left(\int_0^1 |t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t-1|\left(\sin\frac{\pi t}{2}\left|f'(b)\right|^q + \cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&= \frac{b-a}{4}\left(\int_0^1 |t|\,dt\right)^{1-\frac{1}{q}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q\int_0^1 |t|\sin\frac{\pi t}{2}dt + \left|f'(a)\right|^q\int_0^1 |t|\cos\frac{\pi t}{2}dt\right)^{\frac{1}{q}} \\
&+ \frac{b-a}{4}\left(\int_0^1 |t-1|\,dt\right)^{1-\frac{1}{q}}\left(\left|f'(b)\right|^q\int_0^1 |t-1|\sin\frac{\pi t}{2}dt + \left|f'\left(\frac{a+b}{2}\right)\right|^q\int_0^1 |t-1|\cos\frac{\pi t}{2}dt\right)^{\frac{1}{q}} \\
&= \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q\frac{4}{\pi^2} + \left|f'(a)\right|^q\frac{2\pi-4}{\pi^2}\right)^{\frac{1}{q}} \\
&+ \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'(b)\right|^q\frac{2\pi-4}{\pi^2} + \left|f'\left(\frac{a+b}{2}\right)\right|^q\frac{4}{\pi^2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

For $q = 1$ we use the estimates from the proof of Theorem 2.1, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 2.1 *Under the assumption of Theorem* 2.3 *with* $q = 1$*, we have the following inequality:*

$$
\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left(\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right| + \frac{2\pi-4}{\pi^2}A\left(\left|f'(a)\right|, \left|f'(ab|\right)\right)
$$

This inequality coincides with inequality in Theorem 2.1.

Theorem 2.4 *Let* $f : I \to \mathbb{R}$ *be a continuously differentiable function, let* $a < b$ *in* I *assume that* $q > 1$*. If the mapping* $|f'|^q$ *is trigonometrically* ρ*-convex function on interval* $[a, b]$*, then the following inequality*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \frac{b-a}{4}\left\{\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2\pi-4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{4}{\pi^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right. \\
&+ \left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{2\pi-4}{\pi^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\
&+ \left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{2\pi-4}{\pi^2}\left|f'(b)\right|^q + \frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} \\
&\left.+ \left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{4}{\pi^2}\left|f'(b)\right|^q + \frac{2\pi-4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}
$$

holds for $t \in [0, 1]$*, where* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Using Lemma 2.1, Hölder-İşcan integral inequality and inequalities

$$\begin{aligned} \left|f'(t\frac{a+b}{2}+(1-t)a\right|^q &\leq \sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\cos\frac{\pi t}{2}\left|f'(a)\right|^q \\ \left|f'(tb+(1-t)\left(\frac{a+b}{2}\right)\right|^q &\leq \sin\frac{\pi t}{2}\left|f'(b)\right|^q+\cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q \end{aligned}$$

which is the trigonometrically ρ-convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt+\int_0^1 |t-1|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right] \\
&\leq \frac{b-a}{4}\left\{\left(\int_0^1 (1-t)\left|t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad+\left(\int_0^1 t\left|t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1 (1-t)\left|t-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left|f'(tb+(1-t)\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\left.+\left(\int_0^1 t\left|t-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'(tb+(1-t)\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
&\leq \frac{b-a}{4}\left\{\left(\int_0^1 (1-t)\left|t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left(\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\cos\frac{\pi t}{2}\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&\quad+\left(\int_0^1 t\left|t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left(\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\cos\frac{\pi t}{2}\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&\quad+\left(\int_0^1 (1-t)\left|t-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left(\sin\frac{\pi t}{2}\left|f'(b)\right|^q+\cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&\quad\left.+\left(\int_0^1 t\left|t-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left(\sin\frac{\pi t}{2}\left|f'(b)\right|^q+\cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right\} \\
&= \frac{b-a}{4}\left\{\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2\pi-4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{4}{\pi^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{2\pi-4}{\pi^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{2\pi-4}{\pi^2}\left|f'(b)\right|^q+\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} \\
&\quad\left.+\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{4}{\pi^2}\left|f'(b)\right|^q+\frac{2\pi-4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 (1-t)\,|t|^p\,dt &= \int_0^1 t\,|t-1|^p\,dt = \frac{1}{(p+1)(p+2)},\\
\int_0^1 t\,|t|^p\,dt &= \int_0^1 (1-t)\,|t-1|^p\,dt = \frac{1}{p+2},\\
\int_0^1 (1-t)\sin\frac{\pi t}{2}dt &= \int_0^1 t\cos\frac{\pi t}{2}dt = \frac{2\pi-4}{\pi^2},\\
\int_0^1 (1-t)\cos\frac{\pi t}{2}dt &= \int_0^1 t\sin\frac{\pi t}{2}dt = \frac{4}{\pi^2}.
\end{aligned}$$

This completes the proof. □

Theorem 2.5 *Let $f : I \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I assume that $q \geq 1$. If the mapping $|f'|^q$ is trigonometrically ρ-convex function on interval $[a,b]$, then the following inequality*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\leq \frac{b-a}{4}\left\{\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{16-4\pi}{\pi^3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{16-4\pi}{\pi^3}|f'(a)|^q\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{8\pi-16}{\pi^3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{2\pi^2-16}{\pi^3}|f'(a)|^q\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{2\pi^2-16}{\pi^3}|f'(b)|^q+\frac{8\pi-16}{\pi^3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{16-4\pi}{\pi^3}|f'(b)|^q+\frac{16-4\pi}{\pi^3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

holds for $t \in [0,1]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof Using Lemma 2.1, Improved power-mean integral inequality and inequalities

$$\begin{aligned}
\left|f'(t\frac{a+b}{2}+(1-t)a\right|^q &\leq \sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\cos\frac{\pi t}{2}|f'(a)|^q\\
\left|f'(tb+(1-t)\left(\frac{a+b}{2}\right)\right|^q &\leq \sin\frac{\pi t}{2}|f'(b)|^q+\cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q
\end{aligned}$$

which is the trigonometrically ρ-convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\leq \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt+\int_0^1 |t-1|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{b-a}{4}\left\{\left(\int_0^1 (1-t)\,|t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,|t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1 t\,|t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 (1-t)\,|t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,|t-1|\left|f'(tb+(1-t)\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1 t\,|t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|t-1|\left|f'(tb+(1-t)\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
&\leq \frac{b-a}{4}\left\{\left(\int_0^1 (1-t)\,|t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,|t|\left(\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\cos\frac{\pi t}{2}\,|f'(a)|^q\right)dt\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1 t\,|t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|t|\left(\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\cos\frac{\pi t}{2}\,|f'(a)|^q\right)dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 (1-t)\,|t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,|t-1|\left(\sin\frac{\pi t}{2}\,|f'(b)|^q+\cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1 t\,|t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|t-1|\left(\sin\frac{\pi t}{2}\,|f'(b)|^q+\cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right\}\\
&= \frac{b-a}{4}\left\{\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{16-4\pi}{\pi^3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{16-4\pi}{\pi^3}\,|f'(a)|^q\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{8\pi-16}{\pi^3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{2\pi^2-16}{\pi^3}\,|f'(a)|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{2\pi^2-16}{\pi^3}\,|f'(b)|^q+\frac{8\pi-16}{\pi^3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\\
&\left.+\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{16-4\pi}{\pi^3}\,|f'(b)|^q+\frac{16-4\pi}{\pi^3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^1 (1-t)\,|t|\,dt &= \int_0^1 t\,|t-1|\,dt=\frac{1}{6}\\
\int_0^1 t\,|t|\,dt &= \int_0^1 (1-t)\,|t-1|\,dt=\frac{1}{3}\\
\int_0^1 (1-t)\,|t|\sin\frac{\pi t}{2}dt &= \int_0^1 (1-t)\,|t|\cos\frac{\pi t}{2}dt=\frac{16-4\pi}{\pi^3}\\
\int_0^1 t\,|t-1|\sin\frac{\pi t}{2}dt &= \int_0^1 t\,|t-1|\cos\frac{\pi t}{2}dt=\frac{16-4\pi}{\pi^3}\\
\int_0^1 t\,|t|\sin\frac{\pi t}{2}dt &= \int_0^1 (1-t)\,|t-1|\cos\frac{\pi t}{2}dt=\frac{8\pi-16}{\pi^3},\\
\int_0^1 t\,|t|\cos\frac{\pi t}{2}dt &= \int_0^1 (1-t)\,|t-1|\sin\frac{\pi t}{2}dt=\frac{2\pi^2-16}{\pi^3}.
\end{aligned}
$$

This completes the proof. □

Corollary 2.2 *If we choose $q = 1$ in Theorem* 2.5*, we get the following inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}\left(\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|+\left(\frac{2\pi-4}{\pi^2}\right)A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\right).$$

This inequality coincides with the inequality in Theorem 2.1*.*

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