

Some Results on Generalized Sasakian Space Forms Admitting Almost Quasi-Yamabe Solitons

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Abstract: In this paper we have studied generalized Sasakian space form admitting almost quasi-Yamabe soliton and gradient almost quasi-Yamabe soliton. It is shown that if a generalized Sasakian space form admits a closed almost quasi-Yamabe soliton, then either soliton vector field is pointwise collinear with ζ or the structure functions are connected by a relation and the manifold becomes Ricci semi-symmetric generalized Sasakian space form. Next, it is proven that, if the metric of a generalized Sasakian space form is a gradient almost quasi-Yamabe soliton, then either the gradient of ψ is pointwise collinear with ζ or the structure functions are connected by a relation.

Key Words: Generalized Sasakian space form, almost quasi-Yamabe soliton, gradient almost quasi-Yamabe soliton

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§1. Introduction

The notion of Yamabe flow was proposed by R. S. Hamilton [14] as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on a Riemannian manifold of dimension greater than or equal to three. The Yamabe soliton as a self-similar solutions to the Yamabe flow

$$\frac{\partial}{\partial t}(g(t)) = -\sigma(g(t)),$$

where $g(t)$ is an one parameter family of metrics on a some smooth manifold and σ stands for scalar curvature of the manifold (see [6, 10, 14]). It is well-known that a Riemannian metric g defined on a n -dimensional smooth manifold M is said to be an Yamabe soliton if, for some real constant ω , there exists a smooth vector field \bar{V} on smooth manifold M satisfying the following equation

$$\frac{1}{2}\mathcal{L}_{\bar{V}}g = (\sigma - \omega)g, \tag{1.1}$$

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where $\mathcal{L}_{\bar{V}}$ indicates the Lie-derivative operator along \bar{V} on M and σ being the scalar curvature of M . The Yamabe soliton is called shrinking, steady and expanding according as ω is positive, zero and negative respectively. Moreover, if \bar{V} is Killing, the Yamabe soliton is said to be trivial, otherwise it is non-trivial. Yamabe solitons have been investigated under different types of manifolds by several geometers (see [8, 9, 15, 21, 23, 26, 28]) and many others.

In 2013, E. Barbosa et al.[3] introduced the concept of almost Yamabe soliton, which is a generalization of the classical Yamabe soliton by setting ω to be a real valued smooth function on M . We denote an almost Yamabe soliton by (g, \bar{V}, ω) . Furthermore, T. Seko et al. [25] completely classified almost Yamabe solitons in the context of hypersurfaces in Euclidean spaces. The Yamabe soliton becomes the so called gradient soliton if $\bar{V} = \text{grad}(\psi)$, for some smooth function $\psi : M \rightarrow \mathbb{R}$, where \mathbb{R} is the set of real numbers. In this case, the soliton Eq. (1.1) becomes

$$\nabla^2\psi = (\sigma - \omega)g, \quad (1.2)$$

where ∇^2 is the Hessian operator. The function ψ is called potential function of the soliton.

In 2014, G. Huang et al. [16] introduced the concept of quasi-Yamabe gradient soliton, which is a generalization of gradient Yamabe soliton (1.2). The quasi-Yamabe gradient soliton equation is given by

$$\nabla^2\psi = \frac{1}{m}d\psi \otimes d\psi + (\sigma - \omega)g, \quad (1.3)$$

where m is a positive constant and $\omega \in \mathbb{R}$. If $m \rightarrow \infty$, the Eq. (1.3) recovers gradient Yamabe soliton. For more details about the quasi-Yamabe gradient solitons, we refer to [16, 19, 27].

In 2017, V. Pirhadi et al. [22] defined gradient almost quasi-Yamabe soliton by setting ω to be a smooth real valued function on M . In their paper, the authors proved that a necessary and sufficient condition under which an arbitrary compact almost Yamabe soliton is necessarily gradient [22].

Very recently in 2020, X. Chen first initiated almost quasi-Yamabe solitons in [7] as follows.

Definition 1.1([7]) *A smooth manifold M of dimension n equipped with a Riemannian metric g is said to be almost quasi-Yamabe soliton if there exist a smooth vector field \bar{V} and a real valued smooth function ω on M satisfying the equation*

$$\frac{1}{2}\mathcal{L}_{\bar{V}}g = \frac{1}{m}\bar{V}^b \otimes \bar{V}^b + (\sigma - \omega)g, \quad (1.4)$$

where \bar{V}^b is the 1-form associated to \bar{V} and σ is the scalar curvature of M .

An almost quasi-Yamabe metric is closed if 1-form \bar{V}^b is closed and is trivial if \bar{V} is identically zero. We denote an almost quasi-Yamabe soliton by (g, \bar{V}, m, ω) . From definition, if $m \rightarrow \infty$, almost quasi-Yamabe solitons are Yamabe solitons. Furthermore, the almost quasi-Yamabe soliton is said to be gradient if $\bar{V} = \text{grad}(\psi) = D\psi$, for some real valued smooth function ψ on M , denoted by (g, ψ, m, ω) . After that several authors have studied almost quasi-Yamabe solitons on various geometric contexts like Kenmotsu manifolds [13] and within the frame-work of paracontact geometry [11] and obtained some interesting results.

Motivated by the above studies, the present authors to consider generalized Sasakian s-

pace form whose metric as an almost quasi-Yamabe soliton and gradient almost quasi-Yamabe soliton.

The present paper is organized as follows: After the brief introduction, we discuss some fundamental definitions related to generalized Sasakian space forms and curvature formulas, which are contained in Section 2. Section 3 is devoted to study of almost quasi-Yamabe soliton on generalized Sasakian space form. It is proved that if the metric of a generalized Sasakian space form admits a closed almost quasi-Yamabe soliton, then either soliton vector field \bar{V} is pointwise collinear with ζ or the structure functions are connected by relation and the manifold becomes a Ricci semi-symmetric generalized Sasakian space form. Next, it is proven that, if the metric of a generalized Sasakian space form is a gradient almost quasi-Yamabe soliton, then either the gradient of ψ is pointwise collinear with ζ or the structure functions are connected by a relation, which is contained in Section 4.

§2. Preliminaries

A smooth manifold M of dimension n equipped with a Riemannian metric g is said to be an almost contact metric manifold [5], if it admits a $(1, 1)$ -tensor field φ , a characteristic vector field ζ , a 1-form η on M such that

$$\varphi^2(\bar{X}) = -\bar{X} + \eta(\bar{X})\zeta, \quad \eta(\zeta) = 1, \quad (2.1)$$

$$g(\bar{X}, \zeta) = \eta(\bar{X}), \quad \varphi(\zeta) = 0, \quad \eta(\varphi\bar{X}) = 0, \quad (2.2)$$

$$g(\varphi\bar{X}, \varphi\bar{Y}) = g(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y}), \quad (2.3)$$

$$g(\bar{X}, \varphi\bar{Y}) + g(\varphi\bar{X}, \bar{Y}) = 0 \quad (2.4)$$

for any smooth vector fields \bar{X}, \bar{Y} on M . An almost contact metric manifold is denoted by $(M, \varphi, \zeta, \eta, g)$.

A Sasakian space form is a Sasakian manifold with constant φ -sectional curvature c . Similarly, a Kenmotsu manifold with constant φ -sectional curvature c is called Kenmotsu space form. In 2004, P. Alegre et al. [1] initiated the concept of generalized Sasakian space form as a generalization of these spaces. After that several authors ([2, 4, 12, 18]) investigated generalized Sasakian space forms and Sasakian space forms and many beautiful results have been obtained on these spaces. An almost contact metric manifold $(M, \varphi, \zeta, \eta, g)$ is called a generalized Sasakian space form if its curvature tensor R satisfies

$$\begin{aligned} R(\bar{X}, \bar{Y})\bar{Z} = & f_1\{g(\bar{Y}, \bar{Z})\bar{X} - g(\bar{X}, \bar{Z})\bar{Y}\} + f_2\{g(\bar{X}, \varphi\bar{Z})\varphi\bar{Y} - g(\bar{Y}, \varphi\bar{Z})\varphi\bar{X} \\ & + 2g(\bar{X}, \varphi\bar{Y})\varphi\bar{Z}\} + f_3\{g(\bar{X}, \bar{Z})\eta(\bar{Y})\zeta - g(\bar{Y}, \bar{Z})\eta(\bar{X})\zeta \\ & + \eta(\bar{X})\eta(\bar{Z})\bar{Y} - \eta(\bar{Y})\eta(\bar{Z})\bar{X}\}, \end{aligned} \quad (2.5)$$

where f_1, f_2, f_3 are real valued smooth functions on M . In such a case, we also denote a $(2n+1)$ -dimensional generalized Sasakian space form by $M(f_1, f_2, f_3)$. In particular, if $f_1 = \frac{c+3}{4}, f_2 =$

$f_3 = \frac{c-1}{4}$, then generalized Sasakian space form turns into Sasakian space form. Similarly, it turns into Kenmotsu space form if $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ (see [17]). Also, if $f_1 = f_2 = f_3 = \frac{c}{4}$, it reduces to cosymplectic space form (see [20]).

In a $(2n + 1)$ -dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$, the following relations are satisfied [1]:

$$\nabla_{\bar{X}}\zeta = (f_3 - f_1)\varphi(\bar{X}), \quad (2.6)$$

$$(\nabla_{\bar{X}}\eta)\bar{Y} = (f_3 - f_1)g(\varphi\bar{X}, \bar{Y}), \quad (2.7)$$

$$(\nabla_{\bar{X}}\varphi)\bar{Y} = (f_3 - f_1)\{\eta(\bar{Y})\bar{X} - g(\bar{X}, \bar{Y})\zeta\}, \quad (2.8)$$

$$R(\bar{X}, \bar{Y})\zeta = (f_1 - f_3)\{\eta(\bar{Y})\bar{X} - \eta(\bar{X})\bar{Y}\}, \quad (2.9)$$

$$R(\zeta, \bar{X})\bar{Y} = (f_3 - f_1)\{\eta(\bar{Y})\bar{X} - g(\bar{X}, \bar{Y})\zeta\}, \quad (2.10)$$

$$S(\bar{X}, \bar{Y}) = (2nf_1 + 3f_2 - f_3)g(\bar{X}, \bar{Y}) - \{3f_2 + (2n - 1)f_3\}\eta(\bar{X})\eta(\bar{Y}), \quad (2.11)$$

$$Q\bar{X} = (2nf_1 + 3f_2 - f_3)\bar{X} - \{3f_2 + (2n - 1)f_3\}\eta(\bar{X})\zeta, \quad (2.12)$$

$$S(\bar{X}, \zeta) = 2n(f_1 - f_3)\eta(\bar{X}), \quad (2.13)$$

$$Q\zeta = 2n(f_1 - f_3)\zeta \quad (2.14)$$

for any smooth vector fields \bar{X}, \bar{Y} on M , where S and Q denotes the Ricci tensor and Ricci operator respectively.

For a Riemannian manifold (M, g) admitting an almost quasi-Yamabe gradient soliton, we have the following lemma.

Lemma 2.1([7]) *For an almost quasi-Yamabe gradient soliton $(M, g, \sigma, m, \omega)$, the curvature tensor R is given by*

$$R(\bar{X}, \bar{Y})D\psi = \bar{X}(\sigma - \omega)\bar{Y} - \bar{Y}(\sigma - \omega)\bar{X} - \frac{\sigma - \omega}{m}\{\bar{X}(\psi)\bar{Y} - \bar{Y}(\psi)\bar{X}\} \quad (2.15)$$

for any smooth vector fields \bar{X}, \bar{Y} on M .

§3. Almost Quasi-Yamabe Soliton on a Generalized Sasakian Space Form $M(f_1, f_2, f_3)$

In this section, we study on the generalized Sasakian space form admitting almost quasi-Yamabe solitons. First we consider the 1-form \bar{V}^b is closed.

Theorem 3.1 *If a $(2n + 1)$ -dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a closed almost quasi-Yamabe soliton whose soliton vector field \bar{V} , then either \bar{V} is pointwise collinear with ζ or the real valued smooth functions f_2 and f_3 are connected by*

$$3f_2 + (2n - 1)f_3 = 0.$$

Proof Since V^b is closed, Eq. (1.4) is identical to

$$\nabla_{\bar{Y}}\bar{V} = (\sigma - \omega)\bar{Y} + \frac{1}{m}g(\bar{V}, \bar{Y})\bar{V} \quad (3.1)$$

for all smooth vector field \bar{Y} on M . Covariantly, differentiating the Eq. (3.1) along \bar{X} on M , we obtain

$$\begin{aligned} \nabla_{\bar{X}}\nabla_{\bar{Y}}\bar{V} &= (\bar{X}(\sigma - \omega))\bar{Y} + (\sigma - \omega)\nabla_{\bar{X}}\bar{Y} + \frac{1}{m}\{g(\nabla_{\bar{X}}\bar{V}, \bar{Y}) \\ &+ g(\bar{V}, \nabla_{\bar{X}}\bar{Y})\}\bar{V} + \frac{1}{m}g(\bar{V}, \bar{Y})\nabla_{\bar{X}}\bar{V}. \end{aligned} \quad (3.2)$$

Exchanging \bar{X} and \bar{Y} in Eq. (3.2) gives

$$\begin{aligned} \nabla_{\bar{Y}}\nabla_{\bar{X}}\bar{V} &= (\bar{Y}(\sigma - \omega))\bar{X} + (\sigma - \omega)\nabla_{\bar{Y}}\bar{X} + \frac{1}{m}\{g(\nabla_{\bar{Y}}\bar{V}, \bar{X}) \\ &+ g(\bar{V}, \nabla_{\bar{Y}}\bar{X})\}\bar{V} + \frac{1}{m}g(\bar{V}, \bar{X})\nabla_{\bar{Y}}\bar{V}. \end{aligned} \quad (3.3)$$

Also from Eq. (3.1) it can be written that

$$\nabla_{[\bar{X}, \bar{Y}]}\bar{V} = (\sigma - \omega)\{\nabla_{\bar{X}}\bar{Y} - \nabla_{\bar{Y}}\bar{X}\} + \frac{1}{m}\{g(\bar{V}, \nabla_{\bar{X}}\bar{Y}) - g(\bar{V}, \nabla_{\bar{Y}}\bar{X})\}\bar{V}. \quad (3.4)$$

By virtue of the well-known Riemannian curvature formula

$$R(\bar{X}, \bar{Y})\bar{V} = \nabla_{\bar{X}}\nabla_{\bar{Y}}\bar{V} - \nabla_{\bar{Y}}\nabla_{\bar{X}}\bar{V} - \nabla_{[\bar{X}, \bar{Y}]}\bar{V},$$

and Eqs. (3.1), (3.3), (3.4), we have

$$R(\bar{X}, \bar{Y})\bar{V} = (\bar{X}(\sigma - \omega))\bar{Y} - (\bar{Y}(\sigma - \omega))\bar{X} + \frac{\sigma - \omega}{m}\{g(\bar{V}, \bar{Y})\bar{X} - g(\bar{V}, \bar{X})\bar{Y}\}. \quad (3.5)$$

Now, taking inner product of Eq. (3.5) with ζ and using (2.2) we obtain

$$\begin{aligned} g(R(\bar{X}, \bar{Y})\bar{V}, \zeta) &= \{(\bar{X}(\sigma - \omega)) - \frac{\sigma - \omega}{m}g(\bar{V}, \bar{X})\}\eta(\bar{Y}) \\ &- \{(\bar{Y}(\sigma - \omega)) - \frac{\sigma - \omega}{m}g(\bar{V}, \bar{Y})\}\eta(\bar{X}). \end{aligned} \quad (3.6)$$

Again, in view of the Eq. (2.9) we obtain

$$g(R(\bar{X}, \bar{Y})\bar{V}, \zeta) = (f_1 - f_3)\{g(\bar{Y}, \bar{V})\eta(\bar{X}) - g(\bar{X}, \bar{V})\eta(\bar{Y})\}. \quad (3.7)$$

Equating the Eqs. (3.6) and (3.7), we get

$$\begin{aligned} \{(\bar{X}(\sigma - \omega)) - \frac{\sigma - \omega}{m}g(\bar{V}, \bar{X})\}\eta(\bar{Y}) - \{(\bar{Y}(\sigma - \omega)) - \frac{\sigma - \omega}{m}g(\bar{V}, \bar{Y})\}\eta(\bar{X}) \\ = (f_1 - f_3)\{g(\bar{Y}, \bar{V})\eta(\bar{X}) - g(\bar{X}, \bar{V})\eta(\bar{Y})\}. \end{aligned} \quad (3.8)$$

Replacing \bar{X} by $\varphi\bar{X}$ and \bar{Y} by ζ respectively in Eq. (3.8) and using Eq. (2.2), we have

$$\varphi D(\sigma - \omega) = \left(\frac{\sigma - \omega}{m} - f_1 + f_3\right)\varphi V. \quad (3.9)$$

Now consider an orthonormal basis $\{e_1, e_2, e_3, \dots, e_{2n+1}\}$ of the tangent space at each point of the generalized Sasakian space form $M(f_1, f_2, f_3)$. Then contracting Eq. (3.5) over \bar{Y} and using Eq. (2.11) entails that

$$(2nf_1 + 3f_2 - f_3 - \frac{2n(\sigma - \omega)}{m})\bar{V} - (2nf_3 + 3f_2 - f_3)\eta(\bar{V})\zeta = -2nD(\sigma - \omega). \quad (3.10)$$

Now applying φ on both sides of Eq. (3.10) and making use of Eq. (2.2), we infer that

$$(2nf_1 + 3f_2 - f_3 - \frac{2n(\sigma - \omega)}{m})\varphi\bar{V} = -2n\varphi D(\sigma - \omega). \quad (3.11)$$

In view of Eq. (3.9), the Eq. (3.11) becomes

$$\{3f_2 + (2n - 1)f_3\}\varphi V = 0. \quad (3.12)$$

If $\varphi V = 0$, then it follows from Eq. (2.1) that $\bar{V} = \eta(\bar{V})\zeta$, and hence \bar{V} is pointwise collinear with ζ . Suppose if $\varphi V \neq 0$, then we have from Eq. (3.12) that $3f_2 + (2n - 1)f_3 = 0$, and this proves the theorem. \square

From [24], it is known that a $(2n+1)$ -dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ is Ricci semi-symmetric if and only if $3f_2 + (2n - 1)f_3 = 0$. Thus we can state the following.

Theorem 3.2 *If a $(2n + 1)$ -dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a closed almost quasi-Yamabe soliton whose soliton vector field \bar{V} , then either \bar{V} is pointwise collinear with ζ or the manifold M is Ricci semisymmetric.*

Furthermore, if we put $3f_2 + (2n - 1)f_3 = 0$ in Eq. (2.11), then we get

$$S(\bar{X}, \bar{Y}) = 2n(f_1 - f_3)g(\bar{X}, \bar{Y}) \quad (3.13)$$

for all smooth vector fields \bar{X}, \bar{Y} on M . Thus, we are in a position to state the following.

Corollary 3.3 *If a $(2n + 1)$ -dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a closed almost quasi-Yamabe soliton whose soliton vector field \bar{V} , then either \bar{V} is pointwise collinear with ζ or the manifold M is an Einstein manifold of scalar curvature $\sigma = 2n(2n + 1)(f_1 - f_3)$.*

Again on this generalized Sasakian space form $M(f_1, f_2, f_3)$, considering $f_1 = \frac{c+3}{4}$ and $f_3 = \frac{c-1}{4}$ in Eq. (3.13), we compute that

$$S(\bar{X}, \bar{Y}) = 2ng(\bar{X}, \bar{Y})$$

for all smooth vector fields \bar{X}, \bar{Y} on M . Thus, we are in a position to state the following.

Corollary 3.4 *Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ -dimensional Sasakian space form admitting a closed almost quasi-Yamabe soliton whose soliton vector field \bar{V} . Then either \bar{V} is pointwise collinear with ζ or the manifold M is an Einstein manifold of constant scalar curvature $\sigma =$*

$2n(2n + 1)$.

Similarly in a Kenmotsu space form $f_1 = \frac{c-3}{4}$ and $f_3 = \frac{c+1}{4}$. Using these values of f_1 and f_3 in Eq. (3.13) infers that

$$S(\bar{X}, \bar{Y}) = -2ng(\bar{X}, \bar{Y})$$

for all smooth vector fields \bar{X}, \bar{Y} on M .

Thus, we are in a position to state the following.

Corollary 3.5 *Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ -dimensional Kenmotsu space form admitting a closed almost quasi-Yamabe soliton with soliton vector field \bar{V} . Then either \bar{V} is pointwise collinear with ζ or the manifold M is an Einstein manifold of constant scalar curvature $\sigma = -2n(2n + 1)$.*

Again if $m \rightarrow \infty$, then the Eq. (3.9) and Eq. (3.11) reduces to

$$\varphi D(\sigma - \omega) = (f_3 - f_1)\varphi V. \quad (3.14)$$

and

$$(2nf_1 + 3f_2 - f_3)\varphi \bar{V} = -2n\varphi D(\sigma - \omega), \quad (3.15)$$

respectively. Combining Eq. (3.14) and Eq. (3.15) we obtain

$$\{3f_2 + (2n - 1)f_3\}\varphi V = 0. \quad (3.16)$$

Thus, we are in a position to state the following.

Corollary 3.6 *If a $(2n + 1)$ -dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a closed almost Yamabe soliton whose soliton vector field \bar{V} , then either \bar{V} is pointwise collinear with ζ or the manifold M is an Einstein manifold of scalar curvature $\sigma = 2n(2n + 1)(f_1 - f_3)$.*

Corollary 3.7 *Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ -dimensional Sasakian space form admitting a closed almost Yamabe soliton whose soliton vector field \bar{V} . Then either \bar{V} is pointwise collinear with ζ or the manifold M is an Einstein manifold of constant scalar curvature $\sigma = 2n(2n + 1)$.*

Corollary 3.8 *Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ -dimensional Kenmotsu space form admitting a closed almost Yamabe soliton whose soliton vector field \bar{V} . Then either \bar{V} is pointwise collinear with ζ or the manifold M is an Einstein manifold of constant scalar curvature $\sigma = -2n(2n + 1)$.*

§4. Gradient Almost Quasi-Yamabe Soliton on a Generalized Sasakian Space

Form $M(f_1, f_2, f_3)$

In this section, we investigate some properties on a generalized Sasakian space form admitting gradient almost quasi-Yamabe solitons whose soliton vector field \bar{V} is gradient of some real

valued smooth function ψ on M . In this regard our next result is

Theorem 4.1 *Let a $(2n+1)$ -dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admit gradient almost quasi-Yamabe soliton, whose potential function ψ . Then either the gradient of ψ is pointwise collinear with ζ or the real valued smooth functions f_2 and f_3 are related by $3f_2 + (2n-1)f_3 = 0$.*

Proof Let us consider a $(2n+1)$ -dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ and assume that it admits a gradient almost quasi-Yamabe soliton, whose potential function ψ . Taking inner product both sides of Eq. (2.15) in the direction of ζ and then with the help of Eq. (2.2), we get

$$g(R(\bar{X}, \bar{Y})D\psi, \zeta) = \left\{ \bar{X}(\sigma - \omega) - \frac{\sigma - \omega}{m} \bar{X}(\psi) \right\} \eta(\bar{Y}) - \left\{ \bar{Y}(\sigma - \omega) - \frac{\sigma - \omega}{m} \bar{Y}(\psi) \right\} \eta(\bar{X}). \quad (4.1)$$

On the other hand, in view of Eq. (2.11), we obtain

$$g(R(\bar{X}, \bar{Y})D\psi, \zeta) = (f_1 - f_3) \{ \bar{Y}(\psi) \eta(\bar{X}) - \bar{X}(\psi) \eta(\bar{Y}) \}. \quad (4.2)$$

Comparing the Eqs. (4.1) and (4.2), we get

$$(f_1 - f_3) \{ \bar{Y}(\psi) \eta(\bar{X}) - \bar{X}(\psi) \eta(\bar{Y}) \} = \left\{ \bar{X}(\sigma - \omega) - \frac{\sigma - \omega}{m} \bar{X}(\psi) \right\} \eta(\bar{Y}) - \left\{ \bar{Y}(\sigma - \omega) - \frac{\sigma - \omega}{m} \bar{Y}(\psi) \right\} \eta(\bar{X}). \quad (4.3)$$

Now replacing \bar{X} and \bar{Y} by $\varphi\bar{X}$ and ζ respectively and using Eq. (2.2), we reveal that

$$(f_3 - f_1 + \frac{\sigma - \omega}{m})(\varphi\bar{X})\psi = (\varphi\bar{X})(\sigma - \omega). \quad (4.4)$$

Now, contracting the Eq. (2.15) over \bar{Y} , and then recalling the Eq. (2.11), infers that

$$\left\{ 2n \left(\frac{\sigma - \omega}{m} - f_1 \right) - 3f_2 + f_3 \right\} \bar{X}(\psi) + \{ 3f_2 + (2n-1)f_3 \} \eta(\bar{X}) \zeta(\psi) = 2n\bar{X}(\sigma - \omega). \quad (4.5)$$

Replacing \bar{X} by $\varphi\bar{X}$ in Eq. (4.5) and using Eq. (2.2), we get

$$\left\{ 2n \left(\frac{\sigma - \omega}{m} - f_1 \right) - 3f_2 + f_3 \right\} (\varphi\bar{X})\psi = 2n(\varphi\bar{X})(\sigma - \omega). \quad (4.6)$$

Then, with the help of Eq. (4.4), the Eq. (4.6) becomes

$$\{ 3f_2 + (2n-1)f_3 \} (\varphi\bar{X})\psi = 0. \quad (4.7)$$

If $(\varphi\bar{X})\psi = 0$, then it follows from Eq. (2.1) that $\bar{X} = \eta(\bar{X})\zeta(\psi)$, from which we have $grad(\psi) = \zeta(\psi)\zeta$ and hence $grad(\psi)$ is pointwise collinear with ζ . Suppose if $(\varphi\bar{X})\psi \neq 0$, then we have from Eq. (4.7) that $3f_2 + (2n-1)f_3 = 0$, and this proves the theorem. \square

The following corollaries can be achieved as the similar manner as the Section 3.

Corollary 4.2 *If a $(2n + 1)$ -dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a gradient almost quasi-Yamabe soliton, whose soliton vector field \bar{V} , then either \bar{V} is pointwise collinear with ζ or the manifold M is Ricci semisymmetric.*

Corollary 4.3 *If a $(2n + 1)$ -dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a gradient almost quasi-Yamabe soliton, whose potential function ψ , then either the gradient of ψ is pointwise collinear with ζ or the manifold M is an Einstein manifold of scalar curvature $\sigma = 2n(2n + 1)(f_1 - f_3)$.*

Corollary 4.4 *Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ -dimensional Sasakian space form admitting a gradient almost quasi-Yamabe soliton, whose potential function ψ . Then either the gradient of ψ is pointwise collinear with ζ or the manifold M is an Einstein manifold of constant scalar curvature $\sigma = 2n(2n + 1)$.*

Corollary 4.5 *Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ -dimensional Kenmotsu space form admitting a gradient almost quasi-Yamabe soliton, whose potential function ψ . Then either the gradient of ψ is pointwise collinear with ζ or the manifold M is an Einstein manifold of constant scalar curvature $\sigma = -2n(2n + 1)$.*

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