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Some Subclasses of Meromorphic with p-Valent q-Spirallike Functions

Read.S.A.Qahtan, Hamid Shamsan and S.Latha

(Department of Mathematics, Yuvaraja's College, University of Mysore, Mysore 570005, India)

E-mail: readsaleh2015@gmail.com, hmas19771@gmail.com, drlatha@gmail.com

Abstract: In this paper, we introduce and investigate two new subclasses of the function class $\Sigma\mathcal{MS}(p,\lambda,\beta,q)$ and $\Sigma\mathcal{MC}(p,\lambda,\beta,q)$ of q-spirallike meromorphic functions defined in the punctured open unit disc.

Key Words: Univalent functions, meromorphic functions, meromorphic q-spirallike functions.

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§1. Introduction

Let Σ_p be the class of functions f of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=1-p}^{\infty} a_n\, z^n,\ (p \in N = 1, 2, 3, \cdots), \tag{1}$$

which are analytic in the open disc $E^* = \{z : z \in \mathbb{C}\ and\ 0 < |z| < 1\}$. Let $\mathcal{S}$ be the subclass of functions in Σ_p which are univalent in E. Let $\mathcal{P}$ be the class of functions p given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n\, z^n\ (z \in E), \tag{2}$$

which are analytic in the open disc E and satisfy the condition:

$$\Re\{p(z)\} > 0\ (z \in E). \tag{3}$$

If $f \in \Sigma_p$ and satisfies

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\ (z \in E, 0 \le \beta < p), \tag{4}$$

then we say that f is meromorphic p-valent starlike of order β $(0 \le \beta < p)$ and we denote this class by $\Sigma\mathcal{MS}(p,\beta)$.

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If $f \in \Sigma_p$ and satisfies

$$-\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta \ (z \in E, 0 \leq \beta < p), \tag{5}$$

then we say that f is meromorphic p-valent convex of order β and we denote this class by $f \in \Sigma\mathcal{MC}(p,\beta)$.

A function $f \in \Sigma_p$ is said to be λ-spirallike of order β in the unit disk E if

$$-\Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > \beta,\ (z \in E,\ 0 \leq \beta < p,\ |\lambda| < \frac{\pi}{2}).$$

In [8] Jackson introduced and studied the concept of the q-derivative operator ∂_q as follows

$$\partial_q f(z) = \frac{f(z)-f(qz)}{z\ (1-q)}, \quad (z \neq 0, \quad 0<q<1, \quad \partial_q f(0) = f'(0)). \tag{6}$$

Equivalently (6) may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} \ [n]_q \ \ a_n \ \ z^{n-1}, \quad z \neq 0, \tag{7}$$

where $[n]_q = \frac{1-\ q^n}{1-\ q}$. Note that as $q \to 1^-$, $[n]_q \to n$.

Definition 1.1 *A function $f \in \Sigma_p$ is said to be meromorphic p-valent λ-q-spirallike functions of order β if it satisfies the following:*

$$-\Re\left\{e^{i\lambda}\frac{z\partial_q f(z)}{f(z)}\right\} > \beta \ (z \in E,\ |\lambda| < \frac{\pi}{2},\ 0 \leq \beta > p\cos\lambda,\ 0 < q \leq 1), \tag{8}$$

we denote this class by $\Sigma\mathcal{MS}(p,\lambda,\beta,q)$.

Definition 1.2 *A function $f \in \Sigma_p$ is said to be meromorphic p-valent convex λ-q-spirallike functions of order β if it satisfies the following*

$$-\Re\left\{e^{i\lambda}\frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)}\right\} > \beta \ (z \in E, 0 \leq \beta < 1), \tag{9}$$

we denote this class by $\Sigma\mathcal{MC}(p,\lambda,\beta,q)$.

Remark 1.1 *$f \in \mathcal{MS}(p,\lambda,\beta,q)$ iff*

$$-e^{i\lambda}\frac{z\partial_q f(z)}{f(z)} \prec \frac{pe^{i\lambda}-(2\beta-pe^{-i\lambda})z}{1-z}, \tag{10}$$

and $f \in \mathcal{MC}(p,\lambda,\beta,q)$ iff

$$-e^{i\lambda}\left(\frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)}\right) \prec \frac{pe^{i\lambda}-(2\beta-pe^{-i\lambda})z}{1-z}. \tag{11}$$

§2. Main Results

Theorem 2.1 *If the sequence* $\{A_{p+m}\}_0^\infty$ *defined by*

$$\begin{cases} A_p = \frac{2(\beta - p\cos\lambda)}{p+[p]_q}, & m=0, \\ A_{p+m} = \frac{2(\beta - p\cos\lambda)}{p+[p+m]_q}\left(1+\sum_{k=0}^{m-1}|a_{p+k}|\right), & m \in N, \end{cases} \tag{12}$$

and $p \in N$. *Then*

$$A_{p+m} = \frac{2(\beta - p\cos\lambda)}{2\beta + [m+p]_q + p - 2p\cos\lambda}\prod_{k=0}^{m}\frac{2\beta + [k+p]_q + p - 2p\cos\lambda}{p+[p+k]_q}, \tag{13}$$

where $m \in N_0 = N \setminus \{0\}$.

Proof By virtue of (12), we have

$$p + [p+m+1]_q A_{p+m+1} = 2(\beta - p\cos\lambda)\left(1 + \sum_{k=0}^{m} A_{p+k}\right), \tag{14}$$

and

$$p + [p+m]_q A_{p+m} = 2(\beta - p\cos\lambda)\left(1 + \sum_{k=0}^{m-1} A_{p+k}\right). \tag{15}$$

From (14) and (15), we have

$$\frac{A_{p+m+1}}{A_{p+m}} = \frac{2\beta + [m+p]_q + p - 2p\cos\lambda}{p + [p+m+1]_q} \quad m \in N_0. \tag{16}$$

$$\begin{aligned} A_{p+m} &= \frac{A_{p+m}}{A_{p+m-1}}.\frac{A_{p+m-1}}{A_{p+m-2}}\cdots\frac{A_{p+1}}{A_p}.A_p \\ &= \frac{2\beta + [m+p-1]_q + p - 2p\cos\lambda}{p+[p+m]_q}\cdots\frac{2\beta + [p]_q + p - 2p\cos\lambda}{p+[p+1]_q}.\frac{2\beta - 2p\cos\lambda}{p+[p]_q} \\ &= \frac{2(\beta - p\cos\lambda)}{2\beta + [m+p]_q + p - 2p\cos\lambda}\prod_{k=0}^{m}\frac{2\beta + [k+p]_q + p - 2p\cos\lambda}{p+[p+k]_q} \ (m \in N). \end{aligned} \tag{17}$$

The proof of Theorem 2.1 is completed. □

As $q \to 1^-$, we get the following result proved by Shi.et al. [13].

Corollary 2.1 *If* $\{A_{p+m}\}_0^\infty$ *defined by*

$$\begin{cases} A_p = \frac{\beta - p\cos\lambda}{p}, & m=0, \\ A_{p+m} = \frac{2(\beta - p\cos\lambda)}{2p+m}\left(1+\sum_{k=0}^{m-1}|a_{p+k}|\right), & m \in N, \end{cases} \tag{18}$$

and $p \in N$. *Then*

$$A_{p+m} \leq \frac{2(\beta - p\cos\lambda)}{2\beta + m + 2p - 2p\cos\lambda} \prod_{k=0}^{m} \frac{2\beta + k + 2p - 2p\cos\lambda}{2p+k}, \tag{19}$$

where, $m \in N_0 = N \setminus \{0\}$.

Theorem 2.2 *Let* $f(z) = \frac{1}{z^p} + \sum_{m=0}^{\infty} a_{p+m} z^{p+m} \in \mathcal{MS}(p,\lambda,\beta,q)$. *Then*

$$|a_{p+m}| \leq \frac{2(\beta - p\cos\lambda)}{2\beta + [m+p]_q + p - 2p\cos\lambda} \prod_{k=0}^{m} \frac{2\beta + [k+p]_q + p - 2p\cos\lambda}{p + [p+k]_q} \ (m \in N_0). \tag{20}$$

Proof Let

$$L(z) = \frac{\beta + e^{i\lambda}\frac{z\partial_q f(z)}{f(z)} + ip\sin\lambda}{\beta - p\cos\lambda} \ (z \in E, \ f \in \mathcal{MS}(p,\lambda,\beta,q)). \tag{21}$$

We know that $L \in \mathcal{P}$. It follows that

$$e^{i\lambda} z\partial_q f(z) = (\beta - p\cos\lambda) f(z) L(z) - (\beta - ip\sin\lambda) f(z). \tag{22}$$

Let

$$L(z) = 1 + l_1 z + l_2 z^2 + \cdots. \tag{23}$$

Then

$$\begin{aligned} & e^{i\beta}\left(\frac{-[p]_q}{z^p} + [p]_q a_p z^p + [p+1]_q a_{p+1} z^{p+1} + \cdots + [p+m]_q a_{p+m} z^{p+m} + \cdots\right) \\ & = (\beta - p\cos\lambda)\left(\frac{1}{z^p} + a_p z^p + a_{p+1} z^{p+1} + \cdots\right) \times (1 + l_1 z + l_2 z^2 + \cdots) \\ & \quad - (\beta - ip\sin\lambda)\left(\frac{1}{z^p} + a_p z^p + a_{p+1} z^{p+1} + \cdots + a_{p+m} z^{p+m} + \cdots\right). \end{aligned} \tag{24}$$

We have from sides (24)

$$\begin{aligned} e^{i\lambda}[p+m]_q a_{p+m} &= (\beta - p\cos\lambda)\left(l_{2p+m} + a_p l_m + a_{p+m} l_{m-1}\right. \\ & \quad \left. + \cdots + a_{p+m}\right) - (\beta + ip\sin\lambda) a_{p+m}. \end{aligned} \tag{25}$$

Moreover, we know that

$$|l_n| \leq 2 \ (n \in N). \tag{26}$$

From (25) and(26) we have

$$|a_p| \leq \frac{2(\beta - p\cos\lambda)}{p + [p]_q} \tag{27}$$

and

$$|a_{p+m}| \leq \frac{2(\beta - p\cos\lambda)}{p + [p+m]_q}\left(1 + \sum_{k=0}^{m-1} |a_{p+k}|\right) \tag{28}$$

with supposing $p \in N$. We define $\{A_{p+m}\}_{m=0}^{\infty}$ by

$$\begin{cases} A_p = \frac{2(\beta - p\cos\lambda)}{p+[p]_q}, & m = 0; \\ A_{p+m} = \frac{2(\beta - p\cos\lambda)}{p+[p+m]_q}\left(1 + \sum_{k=0}^{m-1} |a_{p+k}|\right), & m \geq 1. \end{cases} \tag{29}$$

Now by the mathematical induction principle we will prove that

$$|a_{p+m}| \leq A_{p+m} (m \in N_0). \tag{30}$$

We can easily verify that

$$|a_p| \leq A_p = \frac{2(\beta - p\cos\lambda)}{p + [p]_q}. \tag{31}$$

Thus, assuming that

$$|a_{p+j}| \leq A_{p+j} (j = 0, 1, \cdots, m.m \in N_0), \tag{32}$$

from (28)) and (32) we have

$$\begin{aligned} |a_{p+m+1}| \leq & \frac{2(\beta - p\cos\lambda)}{p + [p+m+1]_q}\left(1 + \sum_{k=0}^{m} |a_{p+k}|\right) \\ \leq & \frac{2(\beta - p\cos\lambda)}{p + [p+m+1]_q}\left(1 + \sum_{k=0}^{m} |A_{p+k}|\right) \\ = & A_{p+m+1} \ (m \in N_0). \end{aligned} \tag{33}$$

Therefore, by the principle of mathematical induction, we have

$$|a_{p+m}| \leq A_{p+m} \ (m \leq N_0). \tag{34}$$

By means of Theorem 2.1 and (29), we know that

$$A_{p+m} = \frac{2(\beta - p\cos\lambda)}{2\beta + [m+p]_q + p - 2p\cos\lambda} \prod_{k=0}^{m} \frac{2\beta + [k+p]_q + p - 2p\cos\lambda}{p + [p+k]_q} (m \in n_0). \tag{35}$$

So, from (34) and (35) we get proof of the Theorem 2.2. □

As $q \to 1^-$ we get the following result proved by Shi.et al. [13].

Corollary 2.2 *Let* $f(z) = \frac{1}{z^p} + \sum_{m=0}^{\infty} a_{p+m} z^{p+m} \in \mathcal{MS}(p, \lambda, \beta)$. *Then*

$$A_{p+m} = \frac{2(\beta - p\cos\lambda)}{2\beta + m + 2p - 2p\cos\lambda} \prod_{k=0}^{m} \frac{2\beta + k + 2p - 2p\cos\lambda}{2p + k} (m \in n_0). \tag{36}$$

From Theorem 2.2 we get the following result.

Corollary 2.3 *Let* $f(z) = \frac{1}{z^p} + \sum_{m=0}^{\infty} a_{p+m} z^{p+m} \in \mathcal{MC}(p, \lambda, \beta, q)$. *Then*

$$A_{p+m} = \frac{2p(\beta - p\cos\lambda)}{[p+m](2\beta + [m+p]_q + p - 2p\cos\lambda)} \prod_{k=0}^{m} \frac{2\beta + [k+p]_q + p - 2p\cos\lambda}{p + [p+k]_q} (m \in n_0). \quad (37)$$

Theorem 2.3 *Let* $f(z) = \frac{1}{z^p} + \sum_{m=0}^{\infty} a_{p+m} z^{p+m} \in \mathcal{MS}(p, \lambda, \beta, q)$. *Then*

$$\frac{p\cos\lambda - 2(\beta - p\cos\lambda)r}{1-r} \leq \Re\left(-e^{i\vartheta} \frac{z\partial_q f(z)}{f(z)}\right) \leq \frac{p\cos\lambda + 2(\beta - p\cos\lambda)r}{1+r} \quad (38)$$

for $|z| = r < 1$.

Proof Suppose the function ϕ defined by

$$\phi(z) = \frac{pe^{i\lambda} - (2\beta - pe^{-i\lambda} z)}{1-z} \ (z \in E). \quad (39)$$

Let $z = re^{i\lambda}$ $(0 < r < 1)$. We have

$$\Re\{\phi(z)\} = p\cos\lambda - \frac{2(\beta - p\cos\lambda)r(\cos\vartheta - r)}{1 + r^2 - 2r\cos\vartheta}. \quad (40)$$

Let

$$\varphi(\tau) = p\cos\lambda - \frac{2(\beta - p\cos\lambda)r(\tau - r)}{1 + r^2 - 2\tau r} \ (\tau = \cos\vartheta). \quad (41)$$

Then

$$\partial_q \varphi(\tau) = \frac{-2r(\beta - p\cos\lambda)[(1 + r^2 - 2\tau r) - r[2r]_q(\tau - r)]}{(1 + r^2 - 2qr\tau)(1 + r^2 - 2r\tau)}. \quad (42)$$

This means that

$$p\cos\lambda - \frac{2(\beta - p\cos\lambda)r}{1-r} \leq \Re(\phi(z)) \leq p\cos\lambda + \frac{2(\beta - p\cos\lambda)r}{1+r}, \quad (43)$$

which is equivalent to

$$\frac{p\cos\lambda - 2(\beta - p\cos\lambda)r}{1-r} \leq \Re\{\phi(z)\} \leq \frac{p\cos\lambda + 2(\beta - p\cos\lambda)r}{1+r}. \quad (44)$$

We know that

$$-e^{i\lambda} \frac{z\partial_q f(z)}{f(z)} \prec \phi(z)$$

and $\phi(z)$ is univalent in E, this is prove the inequality (38). □

As $q \to 1^-$ we get the following result proved by Shi.et al. [13].

Corollary 2.4 *Let* $f(z) = \frac{1}{z^p} + \sum_{m=0}^{\infty} a_{p+m} z^{p+m} \in \mathcal{MS}(p, \lambda, \beta)$. *Then*

$$\frac{p\cos\lambda - 2(\beta - p\cos\lambda)r}{1-r} \leq \Re\left(-e^{i\lambda}\frac{zf'(z)}{f(z)}\right) \leq \frac{p\cos\lambda + 2(\beta - p\cos\lambda)r}{1+r}, \tag{45}$$

for $|z| = r < 1$.

Corollary 2.5 *Let* $f(z) = \frac{1}{z^p} + \sum_{m=0}^{\infty} a_{p+m} z^{p+m} \in \mathcal{MC}(p, \lambda, \beta, q)$. *Then*

$$\frac{p\cos\lambda - 2(\beta - p\cos\lambda)r}{1-r} \leq \Re\left(-e^{i\lambda}\frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)}\right) \leq \frac{p\cos\lambda + 2(\beta - p\cos\lambda)r}{1+r} \tag{46}$$

for $|z| = r < 1$.

Theorem 2.4 *If* $f \in \Sigma_p$ *satisfies*

$$q\sum_{n=1-p}^{\infty}\left(|[n]_q e^{i\lambda} + \gamma| + |[n]_q e^{i\lambda} + 2\beta - \gamma|\right)|a_n| \leq |[p]_q e^{i\lambda} - 2q\beta + q\gamma| - |[p]_q e^{i\lambda} - q\gamma| \tag{47}$$

for some real λ, β *and* γ $(0 \leq \gamma \leq p\cos\lambda)$, *then* $f \in \mathcal{MS}(p, \lambda, \beta, q)$

Proof To prove $f \in \mathcal{MS}(p, \lambda, \beta, q)$, it suffices to show that

$$\left|\frac{e^{i\lambda}\frac{z\partial_q f(z)}{f(z)} + \gamma}{e^{i\lambda}\frac{z\partial_q f(z)}{f(z)} + (2\beta - \gamma)}\right| < 1 \ (z \in E, 0 \leq \gamma \leq p\cos\lambda). \tag{48}$$

From (47), we know that

$$\begin{aligned} \left|[p]_q e^{i\lambda} - 2q\beta + q\gamma\right| + q\sum_{n=1-p}^{\infty}\left|[n]_q e^{i\lambda} + 2\beta - \gamma\right||a_n| \quad &\geq \quad \left|[p]_q e^{i\lambda} - q\gamma\right| \\ &+q\sum_{n=1-p}^{\infty}|[n]_q e^{i\lambda} + \gamma||a_n| > 0. \end{aligned} \tag{49}$$

Now, by the maximum modulus principle, we deduce from (1) and (49) that

$$\begin{aligned} \left|\frac{e^{i\lambda}\frac{z\partial_q f(z)}{f(z)} + \gamma}{e^{i\lambda}\frac{z\partial_q f(z)}{f(z)} + (2\beta - \gamma)}\right| &= \left|\frac{e^{i\lambda}\left(\frac{-[p]_q}{qz^p} + \sum_{n=1-p}^{\infty}[n]_q a_n z^n\right) + \frac{\gamma}{z^p} + \gamma\sum_{n=1-p}^{\infty} a_n z^n}{e^{i\lambda}\left(\frac{-[p]_q}{qz^p} + \sum_{n=1-p}^{\infty}[n]_q a_n z^n\right) + (2\beta - \gamma)\left(\frac{1}{z^p} + \sum_{n=1-p}^{\infty} a_n z^n\right)}\right| \\ &= \left|\frac{(-[p]_q e^{i\lambda} + q\gamma) + q\sum_{n=1-p}^{\infty}([n]_q e^{i\lambda} + \gamma)a_n z^n}{(-[p]_q e^{i\lambda} + 2q\beta - q\gamma) + q\sum_{n=1-p}^{\infty}([n]_q e^{i\lambda} + 2\beta - \gamma)a_n z^n}\right| \\ &< \frac{|[p]_q e^{i\lambda} - q\gamma| + q\sum_{n=1-p}^{\infty}|[n]_q e^{i\lambda} + \gamma||a_n|}{|[p]_q e^{i\lambda} - 2q\beta + q\gamma| - q\sum_{n=1-p}^{\infty}|[n]_q e^{i\lambda} + 2\beta - \gamma||a_n|} \\ &\leq 1. \end{aligned} \tag{50}$$

This completes the proof. □

As $q \to 1^-$ we get the following result proved by Shi.et al.[13].

Corollary 2.6 *If $f \in \Sigma_p$ satisfies the*

$$\sum_{n=1-p}^{\infty} \left(|ne^{i\lambda} + \gamma| + |ne^{i\lambda} + 2\beta - \gamma|\right) |a_n| \leq |pe^{i\lambda} - 2\beta + \gamma| - |pe^{i\lambda} - \gamma| \tag{51}$$

for some real λ, β and γ $(0 \leq \gamma \leq p\cos\lambda)$, then $f \in \mathcal{MC}(p, \lambda, \beta)$.

Corollary 2.7 *If $f \in \sum @_p$ satisfies the*

$$\begin{aligned} q\sum_{n=1-p}^{\infty} |[n]_q| \left(|[n]_q e^{i\lambda} + \gamma| + |[n]_q e^{i\lambda} + 2\beta - \gamma|\right) |a_n| \quad &\leq \quad [p]_q \left(|[p]_q e^{i\lambda} - 2q\beta + q\gamma|\right. \\ &\left. - |[p]_q e^{i\lambda} - q\gamma|\right) \end{aligned} \tag{52}$$

for some real λ, β and γ $(0 \leq \gamma \leq p\cos\lambda)$, then $f \in \mathcal{MC}(p, \lambda, \beta, q)$.

Lemma 2.1([7]) *If $|\phi|$ attains its maximum value on the circle $|z| = r < 1$ at z_0 and ϕ is a nonconstant regular function in E then*

$$z_0\phi'(z_0) = k\phi(z_0), \ k \geq 1, \ k \in R.$$

Theorem 2.5 *If $f \in \Sigma_p$ satisfies*

$$\left|\frac{f(z)}{f(qz)} + \frac{zf(z)\partial_q^2 f(z)}{f(qz)\partial_q f(z)} - \frac{z\partial_q f(z)}{f(qz)}\right| < \frac{\beta - p}{2\beta} \tag{53}$$

for some real $\beta > p$, than $f \in \mathcal{MS}(p, 0, \beta, q)$.

Proof Define the function φ by

$$\varphi(z) = \frac{\frac{z\partial_q f(z)}{f(z)} + p}{\frac{z\partial_q f(z)}{f(z)} + (2\beta - p)} \ (z \in E). \tag{54}$$

Note that φ is analytic in E and $\varphi(0) = 0$. From (54), we have

$$\frac{z\partial_q f(z)}{f(z)} = \frac{-p + (2\beta - p)\varphi(z)}{1 - \varphi(z)}. \tag{55}$$

Taking q-differentiating of (55) logarithmically, we get

$$\frac{f(z)}{f(qz)} + \frac{zf(z)\partial_q^2 f(z)}{f(qz)\partial_q f(z)} - \frac{z\partial_q f(z)}{f(qz)} = \frac{z(1 - \varphi(z))(2\beta - p)\partial_q\varphi(z)}{(-p + (2\beta - p)\varphi(z))(1 - \varphi(qz))} + \frac{z\partial_q\varphi(z)}{(1 - \varphi(qz))}. \tag{56}$$

From (53) and (56), we get that

$$\left|\frac{f(z)}{f(qz)}+\frac{zf(z)\partial_q^2 f(z)}{f(qz)\partial_q f(z)}-\frac{z\partial_q f(z)}{f(qz)}\right|=\left|\frac{2(\beta-p)\partial_q\varphi(z)}{[-p+(2\beta-p)\varphi(z)](1-\varphi(qz))}\right|<\frac{\beta-p}{2\beta}. \tag{57}$$

Consider $z_0 \in E$ such that

$$\max_{|z|\leq|z_0|}|\varphi(z)|=|\varphi(z_0)|=1.$$

By Lemma 2.1, let $\varphi(z_0)=e^{i\vartheta}$ and $z_0\partial_q\varphi(z_0)=Le^{i\vartheta}$ $(L\geq 1)$. For such a point z_0, we have that

$$\begin{aligned}&\left|\frac{f(z_0)}{f(qz_0)}+\frac{z_0 f(z_0)\partial_q^2 f(z_0)}{f(qz_0)\partial_q f(z_0)}-\frac{z_0\partial_q f(z_0)}{f(qz_0)}\right|\\ &=\left|\frac{2(\beta-p)Le^{i\vartheta}}{[-p+(2\beta-p)e^{i\vartheta}](1-e^{i\vartheta})}\right|\\ &=\frac{2(\beta-p)L}{\sqrt{p^2+(2\beta-p)^2-2p(2\beta-p)\cos\vartheta}\sqrt{2(1-\cos\vartheta)}}\\ &\geq\frac{\beta-p}{2\beta}.\end{aligned} \tag{58}$$

This contradicts our condition (53). Therefore ,there is no $z_0 \in E$ such that $|\varphi(z_0)|=1$. This implies that $|\varphi(z)|<1$ $(z\in E^*)$, that is,

$$\left|\frac{\frac{z\partial_q f(z)}{f(z)}+p}{\frac{z\partial_q f(z)}{f(z)}+(2\beta-p)}\right|<1,\ (z\in E)$$

thus, we conclude that $f\in\mathcal{MS}(p,0,\beta,q)$. □

As $q\to 1^-$ we get the following result proved by Shi.et al. [13].

Corollary 2.8 *If $f\in\Sigma_p$ satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|<\frac{\beta-p}{2\beta} \tag{59}$$

for some real $\beta>p$, than $f\in\mathcal{MS}(p,0,\beta)$.

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