*International J.Math. Combin. Vol.1(2016), 27-33*

# Super Edge-Antimagic Labeling of Subdivided Star Trees

A.Raheem and A.Q.Baig

Department of Mathematics

COMSATS Institute of information Technology, Islamabad, Pakistan

E-mail: rahimciit7@gmail, makbiyik@yildiz.edu.tr, aqbaig1@gmail.com

**Abstract**: Let $G$ be a graph with $V(G)$ and $E(G)$ as the vertex set and the edge set respectively. An $(a,d)$-edge-antimagic total labeling of a graph $G$ is a bijection $\lambda$ from the set $V(G) \cup E(G) \rightarrow \{1,2,3,\cdots,|V(G|+|E(G)|\}$ such that the set of edge-weights $\{\lambda(x)+\lambda(xy)+\lambda(y) : xy \in E(G)\}$ is equal to $\{a, a+d, a+2d, \cdots, a+(|E(G)|-1)d\}$ where the integers $a>0$ and $d \geq 0$. An $(a,d)$-edge-antimagic total labeling of a graph $G$ is called super $(a,d)$-EAT labeling if the smallest possible labels are assigned to the vertices of the graph $G$.

**Key Words**: Labeling, super $(a,d)$-EAT labeling, subdivision of star trees.

**AMS(2010)**: 05C78.

## §1. Introduction

All graphs in this paper are finite, undirected and simple. For a graph $G$ we denote the vertex-set and edge-set by $V(G)$ and $E(G)$, respectively. A $(v,e)$-graph $G$ is a graph such that $v=|V(G)|$ and $e=|E(G)|$. A general reference for graph-theoretic ideas can be seen in [24]. In the present paper the domain will be the set of all the elements of a graph $G$ and such a labeling is called a total labeling. The more details on antimagic total labeling can be seen in [14, 9]. The subject of edge-magic total labeling of graphs has its origin in the works of Kotzig and Rosa [17, 18] on what they called magic valuations of graphs. The definition of $(a,d)$-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller in [21] as a natural extension of edge-magic labeling defined by Kotzig and Rosa.

**Conjecture** 1.1([11]) Every tree admits a super edge-magic total labeling.

In the support of this conjecture, many authors have considered super edge-magic total labeling for many particular classes of trees for example[23, 1, 20, 2, 22, 310, 15, 16, 12, 13, 21]. Lee and Shah [19] verified this conjecture by a computer search for trees with at most 17 vertices. However, this conjecture is still as an open problem.

A star is a particular type of tree graph and many authors have proved the magicness for subdivided stars. Ngurah et. al. [20] proved that $T(m,n,k)$ is also super edge-magic if

[1]Received April 17, 2015, Accepted December 7, 2015.

$k = n+3$ or $n+4$. In [23], Salman et. al. found the super edge-magic total labeling of a subdivision of a star $S_n^m$ for $m = 1, 2$. Javaid et. al. [16] proved super edge-magic total labeling on subdivided star $K_{1,4}$ and w-trees.

However, super $(a, d)$-edge-antimagic total labeling of $G \cong T(n_1, n_2, n_3, \cdots, n_r)$ for different $\{n_i : 1 \leq i \leq r\}$ is still open.

**Definition** 1.1 *A graph $G$ is called $(a, d)$-edge-antimagic total ($(a, d) - EAT$) if there exist integers $a > 0$, $d \geq 0$ and a bijection*

$$\lambda : V(G) \cup E(G) \rightarrow \{1, 2, 3, \cdots, v+e\}$$

*such that $W = \{w(xy) : xy \in E(G)\}$ forms an arithmetic sequence starting from $a$ with the common difference $d$, where $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$ for every $xy \in E(G)$. $W$ is called the set of edge-weights of the graph $G$.*

**Definition** 1.2 *A $(a, d)$-edge-antimagic total labeling $\lambda$ is called super $(a, d)$-edge-antimagic total labeling if $\lambda(V(G)) = \{1, 2, 3, \cdots, v\}$.*

**Definition** 1.3 *For $n_i \geq 1$ and $r \geq 3$, let $G \cong T(n_1, n_2, n_3, \cdots, n_r)$ be a graph obtained by inserting $n_i - 1$ vertices to each of the $i$-th edge of the star $K_{1,r}$ where $1 \leq i \leq r$.*

The notion of a dual labeling has been introduced by Kotzig and Rosa [17]. According to him, if $f$ is an $(a, 0)$-EAT labeling with magic constant $a$ then $f_1$ is also an $(a, 0)$-EAT labeling with magic constant $a_1 = 3(v+e+1) - a$. The following is defined as $f_1(x) = v + e + 1 - f(x)$ for all $x \in V(G) \cup E(G)$.

**Lemma** 1.1[12] *If $f$ is a super edge-magic total labeling of $G$ with the magic constant $c$, then the function $f_1 : V(G) \cup E(G) \rightarrow \{1, 2, 3, \cdots, v+e\}$ defined by*

$$f_1(x) = \begin{cases} v + 1 - f(x), & \text{for } x \in V(G), \\ 2v + e + 1 - f(x), & \text{for } x \in E(G). \end{cases}$$

*is also a super edge-magic total labeling of $G$ with the magic constant $c_1 = 4v + e + 3 - c$.*

We consider the following proposition which we will use frequently in the main results.

**Proposition** 1.1([8]) *If a $(v, e)$-graph $G$ has a $(s, d)$-EAV labeling then*

(1) *$G$ has a super $(s + v + 1, d + 1)$-EAT labeling;*
(2) *$G$ has a super $(s + v + e, d - 1)$-EAT labeling.*

## §2. Super $(a, d)$-EAT Labeling of Subdivided Stars

In this section we deal with the main results related to the super $(a, d)$-EAT labelings. on generalized families of subdivided stars for all possible values of $d$.

**Theorem** 2.1 *For $n \geq 1$ and $r \geq 4$, $G \cong T(n+1, n+2, 2n+4, n_4, \cdots, n_r)$ admits a super $(a, 0)$-EAT labeling with $a = 2v + s - 1$ and a super $(a, 2)$-EAT labeling with $a = v + s + 1$, where $v = |V(G)|$ and $s = (n+5) + \sum_{m=4}^{r} [2^{m-4}(n+2)]$ and $n_m = 2^{m-2}(n+2)$ for $4 \leq m \leq r$.*

*Proof* The vertices and the edges of the graph $G$ are $v = (2n+4) + \sum_{m=4}^{r} [2^{m-3}(n+2)]$ and $e = v - 1$. Define the vertex labeling $\lambda : V(G) \to \{1, 2, \cdots, v\}$ as follows:

Let $\lambda(c) = 1$. For even $1 \leq l_i \leq n_i$, where $i = 1, 2, 3$ and $4 \leq i \leq r$:

$$\lambda(u) = \begin{cases} 1 + \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ (n+3) - \frac{l_2}{2}, & \text{for } u = x_2^{l_2} \\ (2n+5) - \frac{l_3}{2}., & \text{for } u = x_2^{l_3}. \end{cases}$$

$$\lambda(x_i^{l_i}) = (2n+5) + \sum_{m=4}^{i} [2^{m-3}(n+2)] - \frac{l_i}{2} \text{ respectively.}$$

For odd $1 \leq l_i \leq n_i$ and $\alpha = (2n+5) + \sum_{m=4}^{r} [2^{m-3}(n+2)]$, where $i = 1, 2, 3$ and $4 \leq i \leq r$:

$$\lambda(u) = \begin{cases} \alpha + \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ (\alpha + n + 3) - \frac{l_2+1}{2}, & \text{for } u = x_2^{l_2}, \\ (\alpha + 2n + 5) - \frac{l_3+1}{2}, & \text{for } u = x_2^{l_2}. \end{cases}$$

and $\lambda(x_i^{l_i}) = (\alpha + 2n + 5) + \sum_{m=4}^{i} [2^{m-3}(n+2)] - \frac{l_i+1}{2}$ respectively.

The set of all edge-sums $\{\lambda(x) + \lambda(y) : xy \in E(G)\}$ generated by the above formulas forms an integer sequence $(\alpha+1)+1, (\alpha+1)+2, \cdots, (\alpha+1)+e$, where $s = \alpha + 2$. Therefore, by Proposition 1.1, $\lambda$ can be extended to a super $(a, 0)$-EAT labeling with $a = 2v - 1 + s = 2v + (n+3) + \sum_{m=4}^{r} [2^{m-3}(n+2)]$ and to a super $(a, 2)$-EAT labeling with $a = v + 1 + s = v + (n+4) + \sum_{m=4}^{r} [2^{m-3}(n+2)]$. □

**Theorem** 2.2 *For $n \geq 1$ and $r \geq 3$, $G \cong T(n+1, n+2, 2n+4, n_4, \cdots, n_r)$ admits a super $(a, 1)$-EAT labeling with $a = 2v + s - 1$ and a super $(a, 3)$-EAT labeling with $a = v + s + 1$, where $v = |V(G)|$ and $s = 3$ and $n_m = 2^{m-2}(n+2)$ for $4 \leq m \leq r$.*

*Proof* Let us consider the vertices and edges are defined as in Theorem 2.1. Now, define $\lambda : V(G) \to \{1, 2, \cdots, v\}$ as follows:

$\lambda(c) = 1$. For $1 \leq l_i \leq n_i$, where $i = 1, 2, 3$ and $4 \leq i \leq r$:

$$\lambda(u) = \begin{cases} l_1 + 1, & \text{for} \quad u = x_1^{l_1}, \\ (2n+5) - l_2, & \text{for} \quad u = x_2^{l_2}, \\ (4n+9) - l_2, & \text{for} \quad u = x_3^{l_3}, \end{cases}$$

and $\lambda(x_i^{l_i}) = (4n+9) + \sum\limits_{m=4}^{i} [2^{m-2}(n+2)] - l_i$ respectively.

The set of all edge-sums $\{\lambda(x) + \lambda(y) : xy \in E(G)\}$ generated by the above formulas forms an integer sequence $3, 3+2, \cdots, 3+2(e-1)$, where $s = 3$. Therefore, by Proposition 1.1, $\lambda$ can be extended to a super $(a,1)$-EAT labeling with $a = 2v - 1 + s = 2v + 2$ and to a super $(a,3)$-EAT labeling with $a = v + 1 + s = v + 4$. □

As a consequence of Lemma 1.1. and the Theorem 2.1., we have the following corollaries:

**Corollary** 2.3 *For $n \geq 1$ and $r \geq 4$, $G \cong T(n+1, n+2, 2n+4, n_4, \cdots, n_r)$ admits a super $(a,0)$-EAT labeling with magic constant $a = (3v - n - 1) - \sum\limits_{m=4}^{r} [2^{m-3}(n+2)]$, where $n_m = 2^{m-2}(n+2)$ for $4 \leq m \leq r$.*

**Corollary** 2.4 *For $n \geq 1$ and $r \geq 4$, $G \cong T(n+1, n+2, 2n+4, n_4, \cdots, n_r)$ admits a super $(a,2)$-EAT labeling with minimum edge weight is $a = (2v - n + 1) - \sum\limits_{m=4}^{r} [2^{m-3}(n+2)]$, where $n_m = 2^{m-2}(n+2)$ for $4 \leq m \leq r$.*

We construct relation between the Super $(a,d)$-EAT labelings and the $(a,d)$-EAT labelings deduce from Theorem 2.2. and according to the concept of Kotzig and Rosa related to a dual labeling, we have the following corollary.

**Corollary** 2.5 *For $n \geq 1$ and $r \geq 4$, $G \cong T(n+1, n+2, 2n+4, n_4, \cdots, n_r)$ admits a $(a,1)$-EAT labeling with minimum edge weight is $a = 3v$ and $(a,3)$-EAT labeling with minimum edge weight $a = 2v + 2$, where $n_m = 2^{m-2}(n+2)$ for $4 \leq m \leq r$.*

**Theorem** 2.6 *For $n \geq 1$ and $r \geq 4$, $G \cong T(n+1, n+1, n+2, n_4, \cdots, n_r)$ admits a super $(a,0)$-EAT labeling with $a = 2v + s - 1$ and a super $(a,2)$-EAT labeling with $a = v + s + 1$, where $v = |V(G)|$ and*

$$s = 1 + \lceil \frac{3(n+2)}{2} \rceil + \sum_{m=4}^{r} [2^{m-4}(n+2)]$$

*and $n_m = 2^{m-3}(n+2)$ for $4 \leq m \leq r$.*

*Proof* The vertices and edges of the graph $G$ are $v = (3n+4) + \sum\limits_{m=4}^{r} [2^{m-3}(n+2)]$ and $e = v - 1$. Define the vertex labeling $\lambda : V(G) \rightarrow \{1, 2, \cdots, v\}$ as follows:

$\lambda(c) = \lceil \frac{n+2}{2} \rceil$. For even $1 \le l_i \le n_i$, where $i = 1, 2, 3$ and $4 \le i \le r$:

$$\lambda(u) = \begin{cases} \frac{n+2}{2} - \frac{l_1}{2}, & \text{for } u = x_1^{l_1}, \\ \\ \frac{n+2}{2} + \frac{l_2}{2}, & \text{for } u = x_2^{l_2}, \\ \\ \lceil \frac{3(n+2)}{2} \rceil - \frac{l_3}{2}, & \text{for } u = x_3^{l_3}. \end{cases}$$

$$\lambda(x_i^{l_i}) = \lceil \frac{3(n+2)}{2} \rceil + \sum_{m=4}^{i} [2^{m-4}(n+2)] - \frac{l_i}{2} \text{ respectively.}$$

For odd $1 \le l_i \le n_i$ and $\alpha = \lceil \frac{3(n+2)}{2} \rceil + \sum\limits_{m=4}^{r} [2^{m-4}(n+2)]$, where $i = 1, 2, 3$ and $4 \le i \le r$:

$$\lambda(u) = \begin{cases} \alpha + \lceil \frac{n+3}{2} \rceil - \frac{l_1+1}{2}, & \text{for } u = x_1^{l_1}, \\ \alpha + \lceil \frac{n+1}{2} \rceil + \frac{l_2+1}{2}, & \text{for } u = x_2^{l_2}, \\ \alpha + 1 + \lfloor \frac{3(n+2)}{2} \rfloor - \frac{l_3+1}{2}, & \text{for } u = x_3^{l_3}. \end{cases}$$

and

$$\lambda(x_i^{l_i}) = \alpha + 1 + \lfloor \frac{3(n+1)}{2} \rfloor + \sum_{m=4}^{i} [2^{m-4}(n+2)] - \frac{l_i+1}{2} \text{ respectively.}$$

The set of all edge-sums $\{\lambda(x) + \lambda(y) : xy \in E(G)\}$ generated by the above formulas forms a consecutive integer sequence $(\alpha+1)+1, (\alpha+1)+2, \cdots, (\alpha+1)+e$, where $s = \alpha+2$. Therefore, by Proposition 2.1, $\lambda$ can be extended to a super $(a, 0)$-EAT labeling with

$$a = 2v + s - 1 = 2v + \lceil \frac{3(n+1)}{2} \rceil + \sum_{m=4}^{r} [2^{m-4}(n+2)]$$

and to a super $(a, 2)$-EAT labeling with

$$a = v + 1 + s = v + 2 + \lceil 3n + 72 \rceil + \sum_{m=4}^{r} [2^{m-4}(n+2)]. \qquad \square$$

**Theorem** 2.7 *For $n \ge 1$ and $r \ge 4$, $G \cong T(n+1, n+1, n+2, n_4, \cdots, n_r)$ admits a super $(a, 1)$-EAT labeling with $a = 2v + s - 1$ and a super $(a, 3)$-EAT labeling with $a = v + s + 1$, where $v = |V(G)|$ and $s = 3$ and $n_m = 2^{m-3}(n+2)$ for $4 \le m \le r$.*

*Proof* Let us consider the vertices and edges are defined as in Theorem 2.3. Now, we define $\lambda : V(G) \to \{1, 2, \cdots, v\}$ as follows:

$\lambda(c) = n + 2$. For $1 \le l_i \le n_i$, where $i = 1, 2, 3$ and $4 \le i \le r$:

$$\lambda(u) = \begin{cases} (n+2) - l_1, & \text{for } u = x_1^{l_1}, \\ (n+2) + l_2, & \text{for } u = x_2^{l_2}, \\ 3(n+2) - l_3, & \text{for } u = x_3^{l_3}, \end{cases}$$

and

$$\lambda(x_i^{l_i}) = 3(n+2) + \sum_{m=4}^{i} [2^{m-3}(n+2)] - l_i \text{ respectively.}$$

The set of all edge-sums $\{\lambda(x)+\lambda(y) : xy \in E(G)\}$ generated by the above formulas forms an integer sequence $3, 3+2, \cdots, 3+2(e-1)$, where $s = 3$. Therefore, by Proposition 2.1, $\lambda$ can be extended to a super $(a,1)$-EAT labeling with $a = 2v - 1 + s = 2v + 2$ and to a super $(a,3)$-EAT labeling with $a = v + 1 + s = v + 4$. □

## §3. Conclusion

In this paper, we have proved the super edge anti-magicness of subdivided stars for all possible values of $d$, However the problem of the anti-magicness is still open for different values of magic constant.

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