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# Supereulerian Locally Semicomplete Multipartite Digraphs

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**Abstract**: A digraph $D$ is *eulerian* if $D$ is strongly connected and for every vertex $v \in V(D)$, $d^+(v) = d^-(v)$. Bang-Jensen and Thomassé gave the conjecture for a digraph $D$ that if $\lambda(D) \geq \alpha(D)$, then $D$ is supereulerian. Bang-Jensen and Maddaloni [Journal of Graph Theory, 79(2015)8-20] proved that this conjecture holds for every semicomplete multipartite digraph. In this paper, we generalized the above known results and show that this conjecture holds for every strong locally semicomplete multipartite digraph.

**Key Words**: Digraph, supereulerian digraph, semicomplete digraph, locally semicomplete multipartite digraph.

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## §1. Introduction

A graph $G$ is *eulerian* if $G$ is connected without vertices of odd degree, and $G$ is supereulerian if $G$ has a spanning eulerian subgraph. The purpose of this paper is to investigate the digraph version of the supereulerian problem. Given a digraph $D$, its underling graph, denoted by $\overline{D}$, is gotten from $D$ by overlooking the directions. Let $A(D)$ and $V(D)$ be the set of arcs and vertices in $D$, respectively. A *walk* in $D$ is an alternating sequence $W = x_1e_1x_2e_2 \ldots x_{k-1}e_{k-1}x_k$ of vertices $x_i$ and arcs $e_j$ from $D$ such that $e_i = (x_i, x_{i+1}) \in A(D)$ for $i = 1, \ldots, k-1$. A walk is *closed* if $x_1 = x_k$. If all the arcs of a closed walk are distinct, we call it a *closed trail* (also called a *cycle*). If all the vertices of a walk $W$ are distinct, we call it an $(x_1, x_k)$-*dipath*. A digraph $D = (V(D), A(D))$ is *strongly connected* if there is a $(u, v)$-dipath for any two vertices $u$ and $v$. A digraph $D$ is *weakly connected* if $\overline{D}$ is connected. Furthermore, $D$ is said to be *eulerian* if $D$ is strongly connected and for every vertex $v \in V(D)$, $d^+(v) = d^-(v)$. Equivalently, $D$ is eulerian if and only if $D$ itself is a spanning closed trail. A digraph $D$ is *supereulerian* if $D$ has a spanning eulerian subdigraph. If $X$ and $Y$ are disjoint subsets of $V(D)$, let $A(X, Y) := \{(x, y) \in A(D) : x \in X, y \in Y\}$ (sometime, $A(X, Y)$ is simply written as $(X, Y)$) and $d(X, Y) := |(X, Y)|$. *A circuit* $C$ of a digraph $D$ is a connected subdigraph of $D$ such that

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${d_C}^+(v) = {d_C}^-(v) = 1$ for each vertex $v \in C$. Let $[n] = \{1, 2, \ldots, n\}$.

A *semicomplete digraph* is a digraph without nonadjacent vertices. A *locally semicomplete digraph* [3] is a digraph D that satisfies the following conditions: for every vertex $x$ of $D$, the set $N^+(x)$ of vertices dominated by $x$ (respectively, the set $N^-(x)$ of vertices that dominate $x$) induces a semicomplete digraph. Hence a semicomplete digraph is also locally semicomplete. If each vertex in a semicomplete digraph is blown up, then we get a semicomplete multipartite digraph. A digraph $D = (V, A)$ is *semicomplete multipartite* if there is a partition $V_1, V_2, \ldots, V_c$ of $V$ into independent sets so that every vertex in $V_i$ shares an arc with every vertex in $V_j$ for $1 \leq i < j \leq c$.

Similarly, a *locally semicomplete multipartite digraph* $D$ is obtained from a locally semicomplete digraph $F$ with $V(F) = \{v_1, v_2, \ldots, v_c\}$ by blowing up each vertex $v_i \in V(F)$ into one independent set $V_i$ in $D$, such that $N_D^{\lambda}(x) = V_{i_1} \cup \ldots \cup V_{i_p}$ for any $x \in V_i$ if and only if $N_F^{\lambda}(v_i) = \{v_{i_1}, \ldots, v_{i_p}\}$, where $\lambda \in \{+, -\}$ and $\{v_{i_1}, \ldots, v_{i_p}\} \subseteq V(F)$. If a digraph $D = (V, A)$ is locally semicomplete multipartite, then there is a partition $V_1, V_2, \ldots, V_c$ of $V$ satisfying the following two conditions:

(1) each $V_i$, called a partite set, for $i \in [c]$ is an independent set;

(2) the set of vertices dominated by $x$ (respectively, the set of vertices that dominate $x$) induces a semicomplete multipartite digraph for any vertex $x \in V$.

It can be seen that a semicomplete multipartite digraph is locally semicomplete multipartite but the inverse is not right.

Let $\kappa(D)$ and $\lambda(D)$ be the vertex-strong connectivity and the arc-strong connectivity of a digraph $D$, respectively. While $\alpha(D)$ and $\alpha'(D)$ denote the independence number and the matching number of $D$ respectively, which equal to the independence number and the matching number of a graph $\overline{D}$, respectively.

An eulerian factor of $D$ is a collection of arc-disjoint cycles spanning $V(D)$. There are much results about the graph to be supereulerian, for examples to see the surveys [7],[8],[12]. Contrary to the supereulerian properties of graphs, not much work has been done yet for supereulerian digraphs. Gutin [8] discussed supereulerian digraphs by using the connected $(g, f)$-factors; Hong et al. [10] and J. Bang-Jensen et al. [4] gave the degree sum conditions for a graph being supereulerian; Bang-Jensen and Thomassé gave the following conjecture in 2011, Bang-Jensen and Maddaloni [4] said the conjecture is unpublished.

**Conjecture** 1.1 *Let $D$ be a digraph. If $\lambda(D) \geq \alpha(D)$, then $D$ is supereulerian.*

Then by using the Hoffman's circulation theorem, the first author and Maddaloni [4] proved that for a digraph $D$ if $\lambda(D) \geq \alpha(D)$, then $D$ has an eulerian factor and the Conjecture 1.1 hold for all semicomplete multipartite digraphs. Algefari and Lai [2] proved that if $\lambda(D) \geq \alpha'(D) > 0$, then $D$ has a spanning eulerian subdigraph. These two authors, Alsatami and Liu [1] gave the sufficient condition of symmetric digraphs to be supereulerian, and got the result that partially symmetric digraphs are supereulerian.

In this paper, we prove the Conjecture 1.1 for strong locally semicomplete multipartite digraphs.

## §2. Locally Semicomplete Multipartite Digraphs

**Lemma** 2.1([4]) *Let $D$ be a digraph. If $\lambda(D) \geq \alpha(D)$, then $D$ has an Eulerian factor.*

From the definition of a locally semicomplete multipartite digraph, one has the following Lemma 2.2.

**Lemma** 2.2 *Let $D$ be a locally semicomplete multipartite digraph. Let $x \in V(D)$ and $v_1$, $v_2$ be two distinct vertices in $N_D^+(x)$ (or $N_D^-(x)$), then each of the following results holds.*

(1) *either $v_1$ and $v_2$ belong to the same partite set in which there exists no arc between $v_1$ and $v_2$ or there exists at least an arc between $v_1$ and $v_2$;*

(2) *If $v_1$ and $v_2$ belong to the same partite set, then $N_D^-(v_1) = N_D^-(v_2)$ and $N_D^+(v_1) = N_D^+(v_2)$.*

Applying the similar method used in [4] we can prove the following Lemma 2.3.

**Lemma** 2.3 *Let $D$ be a locally semicomplete multipartite digraph. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two vertex disjoint closed trails. If $A(\mathcal{E}_1, \mathcal{E}_2) \neq \emptyset$ and $A(\mathcal{E}_2, \mathcal{E}_1) \neq \emptyset$, then there exists a closed trail $\mathcal{E}$ of $D$ such that $V(\mathcal{E}) = V(\mathcal{E}_1) \cup V(\mathcal{E}_2)$.*

*Proof* Consider the bipartite digraph $B$ with partitions $V(\mathcal{E}_1)$, $V(\mathcal{E}_2)$ and arcs $A(\mathcal{E}_1, \mathcal{E}_2) \cup A(\mathcal{E}_2, \mathcal{E}_1)$. We have two distinct cases.

**Case 1.** Every vertex of $B$ has positive in- and out- degree. Then, clearly, $B$ contains a closed trail $\mathcal{E}$ that connects $\mathcal{E}_1$ and $\mathcal{E}_2$, so $\mathcal{E} \cup \mathcal{E}_1 \cup \mathcal{E}_2$ is the desired trail.

**Case 2.** There exists one vertex of $B$ which has no positive in-degree or out-degree.

Let $v_1 v_2 \ldots v_h v_1$ be a spanning trail of $\mathcal{E}_1$ and let $w_1 w_2 \ldots w_k w_1$ be a spanning trail of $\mathcal{E}_2$. Without loss of generality, assume that there is a vertex, say $x$, of $\mathcal{E}_1$ such that $x$ has no out-neighbors in $\mathcal{E}_2$. As $A(\mathcal{E}_1, \mathcal{E}_2) \neq \emptyset$, there exists another vertex, say $v_s$ of $\mathcal{E}_1$ with at least one out-neighbor in $\mathcal{E}_2$. Assume $(v_s, w_t) \in A(\mathcal{E}_1, \mathcal{E}_2)$.

Now consider the out neighbors $v_{s+1}$ and $w_t$ of $v_s$. If $v_{s+1}$ and $w_t$ are in the same partite set $V_i$. By Lemma 2.2(2), $(w_{t-1}, v_{s+1}) \in A(\mathcal{E}_2, \mathcal{E}_1)$, thus, $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \{(w_{t-1}, v_{s+1}), (v_s, w_t)\} \setminus \{(w_{t-1}, w_t), (v_s, v_{s+1})\}$ is the desired trail. Otherwise, by Lemma 2.2(1), there exists an arc between $w_t$ and $v_{s+1}$. If $(w_t, v_{s+1}) \in A(\mathcal{E}_2, \mathcal{E}_1)$, then $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \{(w_t, v_{s+1}), (v_s, w_t)\} \setminus \{(v_s, v_{s+1})\}$ is the desired trail. If $(w_t, v_{s+1}) \notin A(\mathcal{E}_2, \mathcal{E}_1)$, then $(v_{s+1}, w_t) \in A(\mathcal{E}_2, \mathcal{E}_1)$.

Next we consider two out neighbors $v_{s+2}$ and $w_t$ of $v_{s+1}$, by the similar discussion as considering $v_{s+1}$ and $w_t$, one has that either $D$ contains a closed trail $\mathcal{E}$ such that $V(\mathcal{E}) = V(\mathcal{E}_1) \bigcup V(\mathcal{E}_2)$ or there exists an arc $(v_{s+2}, w_t) \in A(\mathcal{E}_2, \mathcal{E}_1)$. If the former holds, the theorem is proved. Otherwise, $(v_{s+2}, w_t) \in A(\mathcal{E}_2, \mathcal{E}_1)$.

Repeatedly using this procedures, one of the following two results holds: there exists a closed trail $\mathcal{E}$ of D such that $V(\mathcal{E}) = V(\mathcal{E}_1) \bigcup V(\mathcal{E}_2)$ or every vertex in $V(\mathcal{E}_1)$ has the out neighbor $w_t$. Since $x$ has no out-neighbors in $\mathcal{E}_2$, the result that every vertex in $V(\mathcal{E}_1)$ has the out neighbor $w_t$ can not happen, thus, there exists a closed trail $\mathcal{E}$ of D such that $V(\mathcal{E}) = V(\mathcal{E}_1) \cup V(\mathcal{E}_2)$. $\square$

By the proof of Lemma 2.3, one has the following Lemma 2.4.

**Lemma** 2.4 *Let $D$ be a locally semicomplete multipartite digraph and $\mathcal{E}_1, \mathcal{E}_2$ be two vertex disjoint closed trails. Let $(x, y) \in A(\mathcal{E}_1, \mathcal{E}_2)$ and $A(\mathcal{E}_2, \mathcal{E}_1) = \emptyset$, then $(v, y) \in A(\mathcal{E}_1, \mathcal{E}_2)$ for each $v \in V(\mathcal{E}_1)$.*

**Theorem** 2.5 *Let $D$ be a locally semicomplete multipartite digraph. Then $D$ is supereulerian if and only if it is strong and has an eulerian factor.*

*Proof* If $D$ is supereulerian, equivalently, there is a spanning eulerian subdigraph in $D$, then clearly it is strong and has an eulerian factor which consists of the union of arc disjoint cycles spanning $V(D)$.

Now assume that $D$ is strong and has an eulerian factor $\mathcal{C}$, we will show that $D$ has a spanning eulerian subdigraph. The following is a procedure to produce a spanning eulerian subdigraph of $D$ for a given eulerian factor.

Form a minimal collection of vertex disjoint closed trails by merging those trails of $\mathcal{C}$ having common vertices. For any two closed trails $\mathcal{E}, \mathcal{F}$ in the collection with $A(\mathcal{E}, \mathcal{F}) \neq \emptyset$ and $A(\mathcal{F}, \mathcal{E}) \neq \emptyset$, join $\mathcal{E}$ and $\mathcal{F}$ into a closed trail as in Lemma 2.3.

Let $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_t$ be the collection of closed trails of $D$ obtained after the first step is no more applicable. Note that all the trails have at least two vertices, and for any two distinct trails $\mathcal{E}_i$ and $\mathcal{E}_j$, at least one of $A(\mathcal{E}_i, \mathcal{E}_j)$ and $A(\mathcal{E}_j, \mathcal{E}_i)$ is empty. Let $D'$ be the digraph with the set of vertices $\{a_1, \ldots, a_t\}$ and the set of arcs $\{(a_i, a_j) | A(\mathcal{E}_i, \mathcal{E}_j) \neq \emptyset \text{ for } 1 \leq i, j \leq t\}$. Note that $D'$ has no 2-circuit. By the fact that $D$ is strong, $D'$ is strong. Let $f : \{a_i, i \in [t]\} \to \{\mathcal{E}_i, i \in [t]\}$ be a bijective map such that $f(a_i) = \mathcal{E}_i$ for each $i \in [t]$. We has the following claim.

**Claim 1.** $D'$ has a hamiltonian (directed) circuit.

**Proof of Claim 1.** Let $C$ be a circuit of $D'$ with the maximum number of vertices. If $V(D') = V(C)$, then $C$ is our desired circuit. So suppose by contradiction that $|V(C)| < t$. Since $D'$ is strong, there exists a dipath $Q$ with only two ends in $C$ and $Q$ has at least three vertices. Let $Q$ be chosen such that the length of the shortest dipath $P$ in $C$ between the endpoints of $Q$ is minimum. Let $x, v, y$ be the first, second, and last vertex of $Q$ (to see Fig.1). Let $P = \langle x, x_1, \cdots, y\rangle$ and $Q = \langle x, v, \cdots, y\rangle$.

Since $C$ is a circuit with the maximum length, $y$ can not be $x_1$ nor a vertex $x$. In deed, if $y = x_1$, then circuit, gotten from $C$ by replacing $P$ with $Q$, has larger length than that of $C$ which is a contradiction.

If $y = x$, the out-neighbor of $x$ in $C$ is denoted by $x_1$. We will show that $(x_1, v) \in A(D')$. In fact, we consider $v$ and $x_1$. Since $(x, v), (x, x_1) \in D'$ and $D'$ has no 2-circuits, by Lemma **??**, there exist $\varepsilon_1 \in V(f(x)), \varepsilon_2 \in V(f(x_1))$ and $\varepsilon_3 \in V(f(v))$ such that $(\varepsilon_1, \varepsilon_2), (\varepsilon_1, \varepsilon_3) \in A(D)$. By the definition of the locally semicomplete multipartite digraph, either $\varepsilon_2$ and $\varepsilon_3$ belong to the same partite set or there exists an arc between $\varepsilon_2$ and $\varepsilon_3$ in $D$. If $\varepsilon_2$ and $\varepsilon_3$ belong to the same partite set in $D$. Since $f(v)$, as a closed trail of $D$, has at least two vertices and there exists a vertex, say $\varepsilon_4 \in V(f(v))$ such that $(\varepsilon_4, \varepsilon_3) \in A(D)$. By Lemma 2.2(2), $(\varepsilon_4, \varepsilon_2) \in A(D)$ which implies that $(v, x_1) \in A(D')$, then $C_1 := C \cup \{(x, v), (v, x_1)\} - \{(x, x_1)\}$ is the circuit of $D'$ and $|C_1| \geq |C|$ which contradicts the choice of $C$. Now assume that there exists an arc between

$\varepsilon_2$ and $\varepsilon_3$ in $D$. If $(\varepsilon_3, \varepsilon_2) \in A(D)$, it implies that $(v, x_1) \in A(D')$ which is a contradiction. So $(\varepsilon_2, \varepsilon_3) \in A(D)$, it implies that $(x_1, v) \in A(D')$.

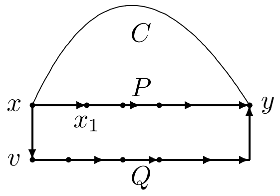

**Fig.1** Circuit $C$ with the dipath $Q$

By the same reason as considering $v$ and $x_1$, then consider $v$ and the out-neighbor, say $x_2$, of $x_1$ in $C \cap D'$. We get that $(x_2, v) \in A(D')$. Repeatedly using this procedures, we can derive that $(t, v) \in A(D')$ for every $t \in V(C)$. Let $z$ be the in-neighbor of $x$ in $C$. Note that $y = x$. By $(z, v) \in A(D')$, then $C_1 := C \cup Q \cup \{(z, v)\} - \{(z, x), (x, v)\}$ is the circuit of $D'$ with $|C_1| \geq |C|$ which contradicts the choice of $C$. As a result, $y$ can not equal $x_1$ nor a vertex $x$.

Since $(x, v)$ and $(x, x_1)$ are distinct arcs of $D'$, there exists $t_1, t_2 \in V(f(x))$, $z_1 \in V(f(v))$ and $z_2 \in V(f(x_1))$ such that $(t_1, z_1) \in A(f(x), f(v))$ and $(t_2, z_2) \in A(f(x), f(x_1))$ in $D$. By Lemma 2.4 and no 2-circuit in $D'$, $(t_1, z_2) \in A(f(x), f(x_1))$ in $D$. Since $z_1$ and $z_2$ are the out-neighbors of $t_1$ in $D$. By Lemma 2.2, either $z_1$ and $z_2$ are in the the same partite set or there exists an arc between $z_1$ and $z_2$ in $D$. The later case can not happen. In fact, the arc between $z_1$ and $z_2$ in $D$ implies that there exists an arc between $v$ and $x_1$ in $D'$. If $(v, x_1) \in D'$, then $C \bigcup \{(x, v), (v, x_1)\} - \{(x, x_1)\}$ is the circuit with the length longer than that of $C$, it contradicts with the choice of $C$. If $(x_1, v) \in D'$, then it contradicts with the shortest dipath $P$ in $C$. As a result, there exists no arc between $v$ and $x_1$ in $D'$. Now assume that $z_1$ and $z_2$ are in the same partite set in $D$, since $f(v)$ contains at least two vertices and $f(v)$ is a closed trail of $D$, there exists $z_3 \in f(v)$ such that $(z_1, z_3) \in A(f(v)) \subseteq D$, by Lemma 2.2(2), $(z_2, z_3) \in A(D)$, it implies that $(x_1, v) \in A(D')$ which is a contradiction. The proof of Claim is finished.

By Claim 1, $D'$ has a hamiltonian circuit $C$. Without loss of generality, let $C = a_1 a_2 \cdots a_t a_1$. By Lemma 2.4 and no 2-circuits in $D'$, there exists $w_i \in \mathcal{E}_i$ for each $i \in [t]$ such that $(v_{i-1}, w_i) \in A(\mathcal{E}_{i-1}, \mathcal{E}_i)$ for each $v_{i-1} \in V(\mathcal{E}_{i-1})$ and $2 \leq i \leq t$, and $(v_t, w_1) \in A(\mathcal{E}_t, \mathcal{E}_1)$ for each $v_t \in V(\mathcal{E}_t)$. As a result, $w_1 w_2 \cdots w_t w_1$ is a circuit of $D$. Then $D$ is supereulerian with a eulerian digraph $(\bigcup_{i=1}^{t} \mathcal{E}_i) \cup \{w_1 w_2 \cdots w_t w_1\}$. □

From Theorem 2.5 and Lemma 2.1, we can derive the following Theorem 2.6.

**Theorem** 2.6 *Let $D$ be a strong locally semicomplete multipartite digraph. If $\lambda(D) \geq \alpha(D)$, then $D$ is supereulerian.*

Since a semicomplete multipartite digraph is the strong locally semicomplete multipartite digraph, thus the following known result can be gotten directly.

**Corollary** 2.7([4]) *Let $D$ be a semicomplete multipartite digraph. If $\lambda(D) \geq \alpha(D)$, then $D$ is supereulerian.*

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