International J.Math. Combin. Vol.1(2016), 34-41

Surface Family with a Common Natural Geodesic Lift

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Abstract: In the present paper, we find a surface family possessing the natural lift of a given curve as a geodesic. We express necessary and sufficient conditions for the given curve such that its natural lift is a geodesic on any member of the surface family. We present a sufficient condition for ruled surfaces with the above property. Finally, we illustrate the method with some examples.

Key Words: Ruled surfaces, curve, geodesic, Frenet frame.

AMS(2010): 53A04, 53A05.

§1. Introduction

Curves and surfaces play an important role in differential geometry. In recent years, there is an ascending interest on finding surfaces possessing a given curve as a common curve instead of finding and characterizing curves on a given surface. In 2004, Wang et. al. [1] proposed a method to find surfaces having a given curve as a common geodesic. Kasap et. al. [2] generalized the marching-scale functions of Wang and obtained a larger family of surfaces. Li et. al. [3] derived the necessary and sufficient constraint for a line of curvature. Bayram et. al. [4] studied parametric surfaces which interpolate a given curve as a common asymptotic. Ergün et. al. [5] obtained a surface family from a given spacelike or timelike line of curvature in Minkowski 3-space.

Inspired with the above studies, we find a surface family possessing the natural lift of a given curve as a common geodesic. We obtain the sufficient condition for the resulting surface to be a ruled surface.

We start with presenting some background. A parametric curve $\alpha(s)$, $L_1 \leq s \leq L_2$, is a curve on a surface $P(s,t)$ in $\mathbb{R}^3$ that has a constant s or t-parameter value. In this paper, α' denotes the derivative of α with respect to arc length parameter s and we assume that α is a regular curve with $\alpha''(s) \neq 0$, $L_1 \leq s \leq L_2$. For every point of $\alpha(s)$, the set

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[1] Received May 12, 2015, Accepted February 8, 2016.

$\{T(s), N(s), B(s)\}$ is called the Frenet frame along $\alpha(s)$, where $T(s) = \alpha'(s)$, $N(s) = \frac{\alpha''}{\|\alpha''\|}$ and $B(s) = T(s) \times N(s)$ are the unit tangent, principal normal, and binormal vectors of the curve at the point $\alpha(s)$, respectively. Derivative formulas of the Frenet frame is governed by the relations

$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{1}$$

where $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = -\langle B'(s), N(s)\rangle$ are called the curvature and torsion of the curve $\alpha(s)$, respectively [6].

Let M be a surface in $\mathbb{R}^3$ and let $\alpha : I \longrightarrow M$ be a parameterized curve. α is called an integral curve of X if

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \ \text{(for all } t \in I),$$

where X is a smooth tangent vector field on M. We have

$$TM = \bigcup_{P \in M} T_P M = \chi(M),$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of tangent vector fields on M.

For any parameterized curve $\alpha : I \longrightarrow M$, $\overline{\alpha} : I \longrightarrow TM$ given by ([7])

$$\bar{\alpha}(s) = \left(\alpha(s), \alpha'(s)\right) = \alpha'(s)\,|_{\alpha(s)} \tag{2}$$

is called the *natural lift* of α on TM.

If a rigid body moves along a unit speed curve $\alpha(s)$, then the motion of the body consists of translation along α and rotation about α. The rotation is determined by an angular velocity vector ω which satisfies $T' = \omega \times T$, $N' = \omega \times N$ and $B' = \omega \times B$. The vector ω is called the *Darboux vector.* In terms of Frenet vectors T, N and B, Darboux vector is given by $\omega = \tau T + \kappa B$ [8]. Also, we have $\kappa = \|\omega\| \cos\theta$, $\tau = \|\omega\| \sin\theta$, where θ is the angle between the Darboux vector ω and binormal vector $B(s)$ of α. Observe that $\theta = \arctan\frac{\tau}{\kappa}$ $(Fig.\ 1)$.

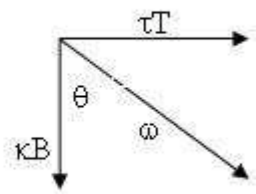

Fig.1 Darboux vector ω, tangent vector T and binormal vector B of α

Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be an arc length curve and $\bar{\alpha}(s)$, $L_1 \leq s \leq L_2$, be the natural lift of α. Then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{3}$$

where $\{T(s), N(s), B(s)\}$ and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ are the Frenet frames of the curves α and $\bar{\alpha}$, respectively, and θ is the angle between the Darboux vector and binormal vector of α.

§2. Surface Family with a Common Natural Geodesic Lift

Suppose we are given a 3-dimensional parametric curve $\alpha(s),\ L_1 \le s \le L_2$, in which s is the arc length and $\|\alpha''(s)\| \neq 0,\ L_1 \le s \le L_2$. Let $\bar{\alpha}(s),\ L_1 \le s \le L_2,$ be the involute of $\alpha(s)$.

Surface family that interpolates $\bar{\alpha}(s)$ as a common curve is given in the parametric form as

$$P(s,t) = \overline{\alpha}(s) + u(s,t)\bar{T}(s) + v(s,t)\bar{N}(s) + w(s,t)\bar{B}(s), \tag{4}$$

$L_1 \le s \le L_2,\ T_1 \le t \le T_2,$ where $u(s,t),\ v(s,t)$ and $w(s,t)$ are C^1 functions and are called *marching-scale functions* and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ is the Frenet frame of the curve $\bar{\alpha}$. Using Eqn. (3) we can express Eqn. (4) in terms of Frenet frame $\{T(s), N(s), B(s)\}$ of the curve α as

$$\begin{aligned} P(s,t) &= \overline{\alpha}(s) + (w(s,t)\sin\theta - v(s,t)\cos\theta)\,T(s) \\ &\quad + u(s,t)\,N(s) + (v(s,t)\sin\theta + w(s,t)\cos\theta)\,B(s), \end{aligned} \tag{5}$$

where $L_1 \le s \le L_2,\ T_1 \le t \le T_2$.

Remark 1 Observe that choosing different marching-scale functions yields different surfaces possessing $\bar{\alpha}(s)$ as a common curve.

Our goal is to find the necessary and sufficient conditions for which the curve $\overline{\alpha}(s)$ is isoparametric and geodesic on the surface $P(s,t)$. Firstly, as $\bar{\alpha}(s)$ is an isoparametric curve on the surface $P(s,t)$, there exists a parameter $t_0 \in [T_1, T_2]$ such that

$$u(s,t_0) = v(s,t_0) = w(s,t_0) \equiv 0,\ L_1 \le s \le L_2,\ T_1 \le t_0 \le T_2. \tag{6}$$

Secondly the curve $\bar{\alpha}$ is geodesic on the surface $P(s,t)$ if and only if along the curve the surface normal vector field $n(s,t_0)$ is parallel to the principal normal vector field $\bar{N}$ of the curve $\bar{\alpha}$. The normal vector of $P(s,t)$ can be written as

$$n(s,t) = \frac{\partial P(s,t)}{\partial s} \times \frac{\partial P(s,t)}{\partial t}.$$

By Eqns. (3) and (5), the normal vector along the curve $\overline{\alpha}$ can be expressed as

$$n(s,t_0)=\kappa\left[-\frac{\partial w}{\partial t}(s,t_0)\bar{N}(s)+\frac{\partial v}{\partial t}(s,t_0)\bar{B}(s)\right], \tag{7}$$

where κ is the curvature of the curve α. Since $\kappa(s)\neq 0,\ L_1\leq s\leq L_2$, the curve $\bar{\alpha}$ is a geodesic on the surface $P(s,t)$ if and only if

$$\frac{\partial w}{\partial t}(s,t_0)\neq 0,\ \frac{\partial v}{\partial t}(s,t_0)\equiv 0.$$

So, we can present:

Theorem 2 *Let $\alpha(s)$, $L_1\leq s\leq L_2$, be a unit speed curve with nonvanishing curvature and $\bar{\alpha}(s)$, $L_1\leq s\leq L_2$, be its natural lift. $\bar{\alpha}(s)$ is a geodesic on the surface (4) if and only if*

$$\begin{cases} u(s,t_0)=v(s,t_0)=w(s,t_0)\equiv 0, \\ \frac{\partial w}{\partial t}(s,t_0)\neq 0,\ \frac{\partial v}{\partial t}(s,t_0)\equiv 0, \end{cases} \tag{8}$$

where $L_1\leq s\leq L_2,\ T_1\leq t,\ t_0\leq T_2$ (t_0 fixed).

Corollary 3 *Let $\alpha(s)$, $L_1\leq s\leq L_2$, be a unit speed curve with nonvanishing curvature and $\bar{\alpha}(s)$, $L_1\leq s\leq L_2$, be its natural lift. If*

$$u(s,t)=w(s,t)=(t-t_0),\ v(s,t)\equiv 0, \tag{9}$$

where $L_1\leq s\leq L_2,\ T_1\leq t,t_0\leq T_2$ (t_0 fixed) then (4) is a ruled surface and $\bar{\alpha}$ is a geodesic on it.

§3. Examples

Example 1 Let $\alpha(s)=\left(\frac{4}{5}\cos s,1-\sin s,-\frac{3}{5}\cos s\right)$ be a unit speed curve. Then, it is easy to show that

$$\begin{aligned} T(s) &= \left(-\frac{4}{5}\sin s,-\cos s,\frac{3}{5}\sin s\right), \\ N(s) &= \left(-\frac{4}{5}\cos s,\sin s,\frac{3}{5}\cos s\right), \\ B(s) &= \left(-\frac{3}{5},0,-\frac{4}{5}\right), \\ \kappa &= 1,\ \tau=0,\ \theta=0. \end{aligned}$$

We have

$$\bar{\alpha}(s)=\left(-\frac{4}{5}\sin s,-\cos s,\frac{3}{5}\sin s\right)$$

as the natural lift of α with Frenet vectors

$$\begin{aligned}\bar{T}(s) &= \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right),\\ \bar{N}(s) &= \left(\frac{4}{5}\sin s, \cos s, -\frac{3}{5}\sin s\right),\\ \bar{B}(s) &= \left(-\frac{3}{5}, 0, -\frac{4}{5}\right).\end{aligned}$$

If we choose $u(s,t) = w(s,t) = t,\ v(s,t) \equiv 0$, then Eqn. (9) is satisfied and we get the ruled surface

$$\begin{aligned}P_1(s,t) &= \bar{\alpha}(s) + t\left[\bar{T}(s) + \bar{B}(s)\right]\\ &= \Big(-\frac{4}{5}(\sin s + t\cos s) - \frac{3}{5}t, t\sin s - \cos s,\\ &\quad \frac{3}{5}(\sin s + t\cos s) - \frac{4}{5}t\Big),\end{aligned}$$

$-2 < s \leq 2,\ -1 \leq t \leq 1$, possessing $\bar{\alpha}$ as a geodesic such as those shown in Fig.2.

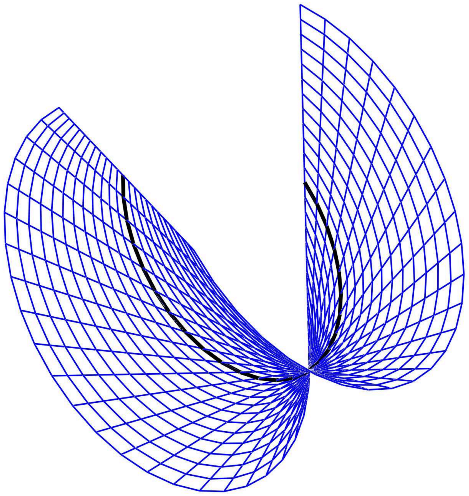

Fig.2 Ruled surface $P_1(s;t)$ as a member of the surface family and its common natural geodesic lift $\overline{\alpha}$

For the same curve, if we choose $u(s,t) = e^{2t} - 1,\ v(s,t) \equiv 0,\ w(s,t) = t$, then Eqn. (8) is satisfied and we obtain the surface

$$\begin{aligned}P_2(s,t) &= \bar{\alpha}(s) + \left(e^{2t} - 1\right)\bar{T}(s) + t\bar{B}(s)\\ &= \Big(-\frac{4}{5}\left(\left(e^{2t} - 1\right)\cos s + \sin s\right) - \frac{3}{5}t, \left(e^{2t} - 1\right)\sin s - \cos s,\\ &\quad \frac{3}{5}\left(\left(e^{2t} - 1\right)\cos s + \sin s\right) - \frac{4}{5}t\Big),\end{aligned}$$

where $-3 < s \leq 3,\ -1 \leq t \leq 1$ interpolating $\bar{\alpha}$ as the natural geodesic lift $(Fig.\ 3)$.

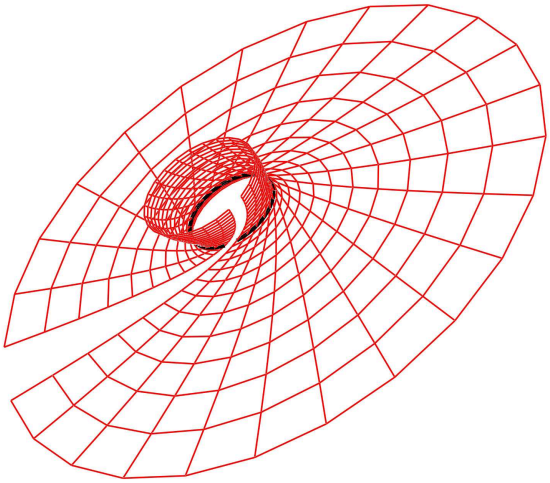

Fig.3 $P_2(s;t)$ as a member of the surface family and its common natural geodesic lift $\overline{\alpha}$

Example 2 Let $\alpha(s) = \left(\frac{\sqrt{3}}{2}\sin s, \frac{s}{2}, \frac{\sqrt{3}}{2}\cos s\right)$ be an arc length helix. One can show that

$$\begin{aligned} T(s) &= \left(\frac{\sqrt{3}}{2}\cos s, \frac{1}{2}, -\frac{\sqrt{3}}{2}\sin s\right), \\ N(s) &= (-\sin s, 0, -\cos s), \\ B(s) &= \left(-\frac{1}{2}\cos s, \frac{\sqrt{3}}{2}, \frac{1}{2}\sin s\right), \\ \kappa &= \frac{\sqrt{3}}{2},\ \tau = \frac{1}{2},\ \theta = \frac{\pi}{6}. \end{aligned}$$

We obtain

$$\bar{\alpha}(s) = \left(\frac{\sqrt{3}}{2}\cos s, \frac{1}{2}, -\frac{\sqrt{3}}{2}\sin s\right)$$

as the natural lift of α with Frenet vectors

$$\begin{aligned} \bar{T}(s) &= (-\sin s, 0, -\cos s), \\ \bar{N}(s) &= (-\cos s, 0, \sin s), \\ \bar{B}(s) &= (0, 1, 0). \end{aligned}$$

Choosing marching scale functions as $u(s,t) = s^2 t,\ v(s,t) \equiv 0,\ w(s,t) = \sin t$ we get the surface

$$\begin{aligned} P_3(s,t) &= \bar{\alpha}(s) + s^2 t \bar{T}(s) + \sin t \bar{B}(s) \\ &= \left(\frac{\sqrt{3}}{2}\cos s - s^2 t \sin s, \frac{1}{2} + \sin t, -\frac{\sqrt{3}}{2}\sin s - s^2 t \cos s \right) \end{aligned}$$

satisfying Eqn. (8) possessing $\bar{\alpha}$ as a common natural geodesic lift $(Fig.\ 4)$.

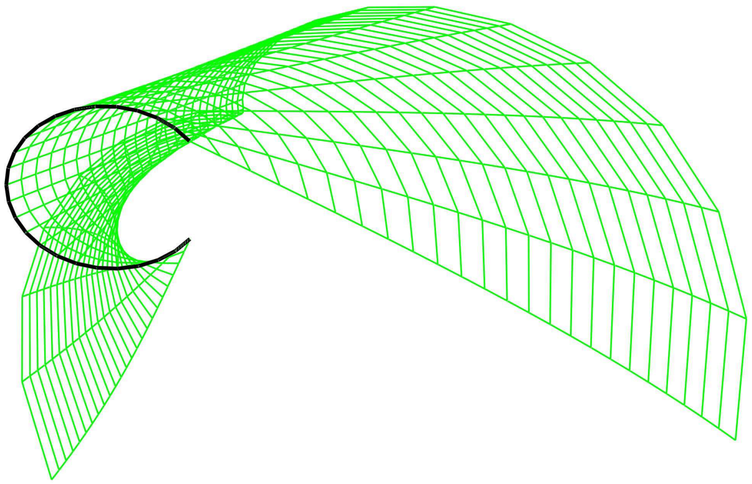

Fig.4 $P_3(s;t)$ as a member of the surface family and its common natural geodesic lift $\overline{\alpha}$

If we let $u(s,t) = s\tan t,\ v(s,t) = (\cos t) - 1,\ w(s,t) = s\sin t$, then Eqn. (8) is satisfied and we have

$$\begin{aligned} P_4(s,t) &= \bar{\alpha}(s) + s\tan t \bar{T}(s) + (\cos t - 1)\bar{N}(s) + s\sin t \bar{B}(s) \\ &= \left(\frac{\sqrt{3}}{2}\cos s - s(\tan t)\sin s + \cos s(1-\cos t), \frac{1}{2} + s\sin t, \right. \\ &\qquad \left. -\frac{\sqrt{3}}{2}\sin s - s(\tan t)\cos s + \sin s(\cos t - 1) \right), \end{aligned}$$

$0 < s \leq 3,\ 0 \leq t \leq 1$, as a member of the surface family possessing $\bar{\alpha}$ as a common natural geodesic lift shown in Fig.5.

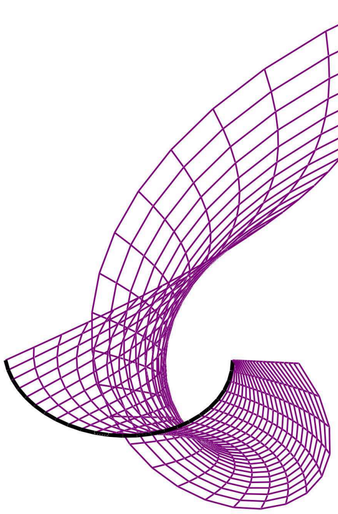

Fig.5 $P_4(s,t)$ as a member of the surface family and its common natural geodesic lift $\bar{\alpha}$.

Acknowledgments

The second author would like to thank TUBITAK (The Scientific and Technological Research Council of Turkey) for their financial supports during his doctorate studies.

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