International J.Math. Combin. Vol.1(2021), 79-84

The Doubly Connected Hub Number of Graph

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Abstract: For a given connected graph $G = (V, E)$, a hub set S of G is a set of vertices with the property that for any pair of vertices outside of S, there is a path between them with all intermediate vertices in S. The hub number $h(G)$ is then defined to be the size of a smallest hub set of G. A set S is a doubly connected hub set if both $\langle S \rangle$ and $\langle V(G) - S \rangle$ are connected. The cardinality of the minimum doubly connected hub set in G is the doubly connected hub number and is denoted by $h_{cc}(G)$. In this paper, the doubly connected hub number for several classes of graphs is computed, bounds in terms of other graph parameters are also determined.

Key Words: Hub number, connected hub number, domination number, doubly connected domination number, doubly connected hub number.

AMS(2010): 05C40, 05C69.

§1. Introduction

In this paper we are concerned with simple graphs, that have no loops and no multiple or directed edges. Let G be such a graph, and let p and q be the number of its vertices and edges, respectively. Then we say that G is an (p, q)-graph. For graph theoretic terminology, we refer to [2].

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

Consider the graphs that represent transportation networks, that is the vertices can be taken to be locations or destinations, and an edge exists between two vertices precisely when there is an "easy passage" between the corresponding locations. For example, a city's network of streets, with vertices representing intersections or other points of intersect, and edges road segments. We are connected with a certain kind of connectivity, specifically we want a set S such that any traffic between disparate points in our network passes solely through vertices in this set.

Suppose that $S \subseteq V(G)$ and let $u, v \in V(G)$. An S-path between u and v is a path where all intermediate vertices are from S. (This includes the degenerate cases where the path consists

[1]Received December 2, 2020, Accepted March 10, 2021.

of the single edge uv or a single vertex u if $u = v$, call such an S-path trivial.) A set $S \subseteq V(G)$ is a hub set of G if it has the property that, for any $u, v \in V(G) - S$, there is an S-path in G between u and v. The smallest size of a hub set in G is called a hub number of G, and is denoted by $h(G)$. A hub set S of a connected graph G is called a connected hub set if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected hub set of G is called the connected hub number of G and is denoted by $h_c(G)$ [9].

A subset S of a graph G is called a dominating set if each vertex of $V - S$ is adjacent to at least one vertex of S. The domination number of a graph G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set S of a connected graph G is called a doubly connected dominating set if the induced subgraphs $\langle S \rangle$ and $\langle V - S \rangle$ are connected. The minimum cardinality of a doubly connected dominating set of G is called the doubly connected domination number of G and is denoted by $\gamma_{cc}(G)$ [3].

A graph is acyclic if it has no cycles. A tree is a connected acyclic graph. A (p, q) graph is called unicyclic if it is connected and $p = q$.

A double star is the tree obtained from two disjoint stars $K_{1,n}$ and $K_{1,m}$ by connecting their centers.

For disjoint graphs G_1 and G_2, the join $G = G_1 + G_2$ is the graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. Let us denote by $G - v$ the graph obtained from G by removing the vertex $v \in V(G)$ and all edges incident to v.

A connected subgraph B of G is called a block if B has no cut-vertex and every subgraph $B' \subseteq G$ with $B \subset B'$ has at least one cut-vertex. A connected graph G is called a block graph if every block in G is complete. A vertex v of a graph G is called a simplicial vertex if every two vertices of $N_G(v)$ are adjacent in G.

We need the following to prove main results.

Theorem 1.1([1]) *For any connected graph G, $\gamma_{cc}(G) = p - t$, $p \geq 3$, where t is the maximal number of simplicial vertices in a block with largest number of vertices.*

§2. The Doubly Connected Hub Number of Graph

Definition 2.1 *Let G be a connected graph. A doubly connected hub set S of G is a subset of $V(G)$ such that any pair of vertices of $V - S$ are connected by a path, whose all intermediate vertices are in S and both $\langle S \rangle$ and $\langle V(G) - S \rangle$ are connected. The cardinality of the minimum doubly connected hub set in G is the doubly connected hub number and is denoted by $h_{cc}(G)$.*

It is clear that $h_{cc}(G)$ is well-defined for any connected graph G, since $V(G) - v$ is a doubly connected hub set. In all situations of interest, we will assume G to be connected.

It is obvious that any doubly connected hub set in a graph G is also a connected hub set and any connected hub set is also a hub set, and thus we obtain the obvious bound $h(G) \leq h_c(G) \leq h_{cc}(G)$.

It is obvious that the difference $h_{cc}(G) - h_c(G)$ can be arbitrarily large in a graph G. It can be easily checked that $h_c(K_{1,n}) = 1$, while $h_{cc}(K_{1,n}) = n$, $n \geq 3$.

We now proceed to compute $h_{cc}(G)$ for some standard graphs.

Remark 2.2 Notice that

(1) For any complete graph K_p, $h_{cc}(K_p) = h_c(K_p) = 0$;
(2) For any double star $S_{n,m}$, $h_{cc}(S_{n,m}) = n + m + 1$;
(3) For any cycle C_p, $h_{cc}(C_p) = h_c(C_p) = p - 3$;
(4) For any complete bipartite graph $K_{n,m}$,

$$h_{cc}(K_{n,m}) = \begin{cases} 0, \text{ if } m = n = 1 \ ; \\ 1, \text{ if } m = 1 \textit{ and } n = 2 \textit{ or } m = 2 \textit{ and } n \geq 2 \ ; \\ 2, \text{ if } n, m \geq 3. \end{cases}$$

(5) For the wheel $W_{1,n}$, $n \geq 3$, $h_{cc}(W_{1,n}) = 1$.

Observation 2.3 Let $G \cong K_{m_1,m_2,\cdots,m_k}$ be the complete k-partite graph, $k \geq 3$ with $m_1 \leq m_2 \leq \cdots \leq m_k$. Then,

$$h_{cc}(G) = \begin{cases} 0, \text{ if } m_i = 1, 1 \leq i \leq k \ ; \\ 1, \text{ if } m_i \leq 2 \textit{ and } m_j \geq 2, 1 \leq i \leq j \leq k \ ; \\ 2, \text{ if } m_1 \geq 3. \end{cases}$$

Observation 2.4 If G_1 and G_2 are disjoint connected graphs, then

$$h_{cc}(G_1 + G_2) = \begin{cases} 0, \text{ if } h_{cc}(G_1) = 0 \textit{ and } h_{cc}(G_2) = 0 \ ; \\ 1, \text{ if } h_{cc}(G_1) = 1 \textit{ or } h_{cc}(G_2) = 1 \ ; \\ 2, \text{ otherwise.} \end{cases}$$

Proposition 2.5 *Let S be a minimum doubly connected hub set of a graph G. Then,*

(1) *there is at most one support vertex in $V(G) - S$;*
(2) *there is at most one cut- vertex in $V(G) - S$;*
(3) *there is at most one pendant vertex in $V(G) - S$.*

Theorem 2.6 *For any connected graph G, $h_{cc}(G) \leq \gamma_{cc}(G)$. The bound sharp for $G \cong K_{1,n}$, $n \geq 3$.*

Proof Let S be a minimum doubly connected dominating set of G. Then both $\langle S \rangle$ and $\langle V(G) - S \rangle$ are connected and for any $v \in V - S$, there exists $u \in S$ such that $v \in N(u)$. Since $\langle S \rangle$ is connected, it follows that for any $v, w \in V - S$, there is a path between them with all intermediate vertices in S. Thus, S is doubly connected hub set. Thus, $h_{cc}(G) \leq \gamma_{cc}(G)$. □

We observe that $h_{cc}(G) = \gamma_{cc}(G) = p - 1$, $p \geq 4$ only for a tree with all supports adjacent to at least two pendant vertices, otherwise $\gamma_{cc}(G) - h_{cc}(G) = 1$.

Theorem 2.7 *If G is a block graph, then $h_{cc}(G) \leq p-t$, $p \geq 3$, where t is the maximal number of simplicial vertices in a block with largest number of vertices.*

Proof The proof follows from Theorem 1.1 and Theorem 2.6. □

Proposition 2.8 *If $h_{cc}(G) = h_c(G)$, then $h_{cc}(G) \leq p-2$.*

Proof For any connected graph G with $p \geq 3$ we have $h_c(G) \leq p-2$. Since $h_c(G) \leq h_{cc}(G)$ and by hypothesis, $h_c(G) = h_{cc}(G)$, we have $h_{cc}(G) \leq p-2$. □

Theorem 2.9 *For any unicyclic graph G, $h_{cc}(G) - h_c(G) = k-1$, $k \geq 1$, where k is the number of pendant vertices of G.*

Proof Let G be a unicyclic graph of order p and let u be the only common vertex between a cycle C_n and a tree T of G. Let D be the set of all pendant vertices of G such that $|D| = k$ and let S be a minimum connected hub set of G. Then $V - S$ is the set of all pendant vertices and any two adjacent vertices v and w of a cycle C_n of G, $v \neq u$ and $w \neq u$. That is, $|S| = p-(k+2)$. Let S' be a minimum doubly connected hub set of G. Then $S' = V(G) - \{v, w, w'\}$, where $\langle vww' \rangle = P_3$, $v \neq u$, $w \neq u$ and $w' \neq u$. That is, $|S'| = p-3$. Thus,

$$h_{cc}(G) - h_c(G) = p - 3 - [p - (k+2)] = k - 1.$$ □

Corollary 2.10 *For any unicyclic graph G, $h_c(G) = h_{cc}(G)$ if and only if G is contain at most one pendant vertex.*

Proof The Proof follows from Remark 2.2 and Theorem 2.9. □

Theorem 2.11 *The difference $h_{cc}(G) - h_{cc}(G-v)$ can be arbitrarily large.*

Proof Let $H \cong K_{1,k} + K_1$ and let G be a graph obtained by adding two pendant edges and two pendant vertices u and v to the vertices of degree $k+1$ of the graph H (see Figure 1). Then $V(G) - \{u, w'\}$ is minimum doubly connected hub set of G. Therefore, $h_{cc}(G) = k+2$. Also, the set $\{v, w\}$ is minimum doubly connected hub set of $G-u$. Hence, $h_{cc}(G-u) = 2$. Thus, $h_{cc}(G) - h_{cc}(G-v) = k$. □

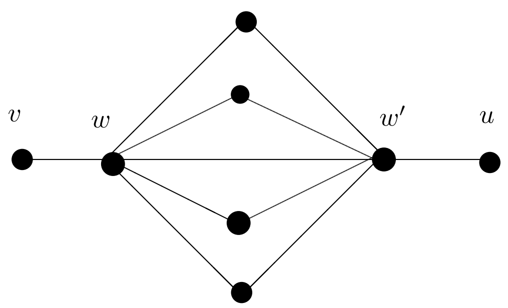

Figure 1

Theorem 2.12 *The difference $h_{cc}(G-v)-h_{cc}(G)$ can be arbitrarily large.*

Proof Let $G \cong P_n + K_1$ and let v be a vertex of K_1. Then clearly, $h_{cc}(G)=1$. Since $h_{cc}(G-v)=n-2$, it follows that $h_{cc}(G-v)-h_{cc}(G)=n-3$. □

§3. Bounds

Theorem 3.1 *For any connected (p,q) graph G with $p \geq 3$, $0 \leq h_{cc}(G) \leq p-1$, with equality of the lower bound for $G \cong K_p$ and equality of the upper bound for $G \cong K_{1,n}$, $n \geq 3$.*

Proof The inequality of the lower bound is obvious. Now we prove the inequality of the upper bound. Let S be a minimum doubly connected hub set of a connected graph G. We consider the following cases.

Case 1. G is a tree.

We consider the following subcases.

Subcase 1.1 Each support vertex of G is adjacent to at least two pendant vertices, $h_{cc}(G)=p-1$.

Subcase 1.2 There exists at least one support vertex of G is adjacent to one pendant vertex, $h_{cc}(G)=p-2$.

Case 2. G is not a tree, $h_{cc}(G) \leq p-2$.

Thus, from all the above cases, $h_{cc}(G) \leq p-1$. □

Theorem 3.2 *For any connected graph G, $h_{cc}(G) \leq p-\delta(G)$. The bound sharp for $G \cong K_{1,n}$, $n \geq 3$.*

Proof Suppose that $P: v_1v_2...v_k$ be the longest path in a graph G. Assume that $H = G - \{v_1, v_2, \cdots, v_\delta\}$. We claim that H is connected. Suppose to the contrary that H is disconnected. Let C be the component of H such that $v_{\delta(G)+1} \notin C$ and let $u_1u_2\cdots u_n$ be the longest path of C. Suppose $S_1 = v_1v_2\cdots v_\delta$ and $S_2 = u_1u_2\cdots u_m$ such that $l = d(S_1,S_2)$. Then we may assume v_ru_i if $l=1$ and $v_rx_1x_2\cdots x_{l-1}u_i$ when $l \geq 2$ is the shortest path between S_1 and S_2, where $1 \leq r \leq \delta$ and $1 \leq i \leq m$. Assume that j is the largest positive integer such that $u_ou_j \in E(G)$. We consider the following cases.

Case 1. $i \geq j$. Then $u_0\cdots u_iv_rv_{r+1}\cdots v_k$ if $l=1$ and $u_0\cdots u_ix_{l-1}\cdots x_1v_rv_{r+1}\cdots v_k$ when $l \geq 2$ is a path of G longer than P which is a contradiction.

Case 2. $i \leq j$. Then $u_{i+1}\cdots u_ju_0\cdots u_iv_rv_{r+1}\cdots v_k$ if $l=1$ and $u_{i+1}\cdots u_jx_{l-1}\cdots x_1v_rv_{r+1}\cdots v_k$ when $l \geq 2$ is a path of G longer than P which is a contradiction.

Thus, from all the above cases, H is connected. Since $\delta(G) > (G[v_1,\cdots,v_\delta])$, it follows that each v_i, $(1 \leq i \leq \delta)$ has at least one neighbor $V(G)-\{v_1,\cdots,v_\delta\}$, and hence $V(G)-\{v_1,\cdots,v_\delta\}$ is a hub set of G. Since $G[v_1,\cdots,v_\delta]$ is connected, $V(G)-\{v_1,\cdots,v_\delta\}$ is a doubly connected hub set of G. □

Theorem 3.3 *For any connected graph G, $h_{cc}(G) \geq p - \Delta(G) - 1$. The bound sharp for $G \cong C_p$, $p \geq 4$.*

Theorem 3.4 *For any (p, q) graph G with both G and $\overline{G}$ are connected, $h_{cc}(G) + h_{cc}(\overline{G}) \leq 2p - 1$.*

Form Theorem 3.2, $h_{cc}(G) \leq p - \delta(G)$ and $h_{cc}(\overline{G}) \leq p - \delta(\overline{G})$. We have,

$$\begin{aligned} h_{cc}(G) + h_{cc}(\overline{G}) &\leq 2p - (\delta(G) + \delta(\overline{G})) \\ &= 2p - (p - 1 - \Delta(\overline{G}) + \delta(\overline{G})) \\ &= p + 1 + (\Delta(\overline{G}) - \delta(\overline{G})). \end{aligned}$$

Since $\Delta(\overline{G}) - \delta(\overline{G})) \leq p - 2$, we have $h_{cc}(G) + h_{cc}(\overline{G}) \leq 2p - 1$.

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