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The (a, d)-Ascending Subgraph Decomposition

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Abstract: Let G be a graph of size q and a, n, d be positive integers for which $\frac{n}{2}[2a + (n-1)d] \leq q < \left(\frac{n+1}{2}\right)[2a + nd]$. Then G is said to have (a, d)-ascending subgraph decomposition ((a, d)-ASD) if the edge set of G can be partitioned into n-non-empty sets generating subgraphs $G_1, G_2, G_3, \ldots, G_n$ with out isolating vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq n-1$ and $|E(G_i)| = a + (i-1)d$. In this paper, we find (a, d)-ASD for $K_n, K_{m,n}$ and for product graphs.

Key Words: ASD, (a, d)-ASD, Smarandachely (P, Q)-decomposition, Smarandachely (a, d)-decomposition.

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§1. Introduction

By a graph we mean a finite undirected graph without loops or multiple edges. A wheel on p vertices is denoted by W_p. A path of length t is denoted by P_{t+1}. A graph obtained from two graphs G_1 and G_2 by taking one copy of G_1 (which has p-vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2 is denoted by $G_1 \odot G_2$. Terms not defined here are used in the sense of Harary [4]. Throughout this paper $G \subset H$ means G is a subgraph of H.

Let $G = (V, E)$ be a simple graph of order p and size q. If $G_1, G_2, \ldots, G_n$ are edge disjoint subgraphs of G such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_n)$, then $\{G_1, G_2, \ldots, G_n\}$ is said to be a decomposition of G.

The concept of ASD was introduced by Alavi et al. [1]. The graph G of size q where $\begin{pmatrix} n+1 \\ 2 \end{pmatrix} \leq q < \begin{pmatrix} n+2 \\ 2 \end{pmatrix}$, is said to have an ascending subgraph decomposition (ASD) if

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G can be decomposed into n-subgraphs $G_1, G_2, \ldots, G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \le i \le n-1$. We generalize the concept of ASD as follows:

Definition 1.1 *A graph G has a Smarandachely (P,Q)-decomposition for graphical properties P and Q, $P \subset Q$ if the edge set $E(G)$ can be partitioned into non-empty sets generating subgraphs $H \in P$ without isolating vertices such that each such H is isomorphic to a proper subgraph of $J \in Q$. In particular, we define a Smarandachely (a,d)- decomposition is a Smarandachely (P,Q)-decomposition, where $P = \{G_j/|E(G_j)| = a + (j-1)d\}$ and $Q = P = \{G_{j+1}/G_j \in P$ and $|E(G_{j+1})| = a + jd\}$ into subgraphs $G_1, G_2, \ldots, G_n$.*

In other words G is a simple graph of size q and a, n, d are positive integers for which $\frac{n}{2}[2a + (n-1)d] \le q < \left(\frac{n+1}{2}\right)[2a + nd]$. Then (a,d)-ascending subgraph decomposition $((a,d) - ASD)$ of G is the edge disjoint decomposition of G into subgraphs $G_1, G_2, \ldots, G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \le i \le n-1$ and $|E(G_i)| = a + (i-1)d$. The following theorems will be useful in proving certain results in Section 2.

Theorem 1.2([1]) *Let G be a graph of size q, where $\binom{n+1}{2} \le q < \binom{n+2}{2}$ for some positive integers n, such that G has an ascending subgraph decomposition $G_1, G_2, \ldots, G_n$ such that G_i has size i for $1 \le i \le n-1$ and G_n has size $q - \binom{n}{2}$.*

Theorem 1.3([2]) *$C_n \times C_n$ is decomposed into two Hamilton cycles if n is odd.*

Theorem 1.4([2]) *K_n is (i) decomposed into $\frac{n}{2}$-Hamilton cycles if n is odd. (ii) decomposed into $\left\lfloor \frac{n+1}{2} \right\rfloor$-Hamilton cycles and a 1-factors if n is even.*

§2. Main Results

Definition 2.1 *Let G be a graph of size q and a, n, d be positive integers for which $\left(\frac{n}{2}\right)[2a+(n-1)d] \le q < \left(\frac{n+1}{2}\right)[2a+nd]$. Then G is said to have (a,d)- ascending subgraph decomposition $((a,d)-ASD)$ if the edge set of G can be partitioned into n non-empty sets generating subgraphs $G_1, G_2, \ldots, G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \le i \le n-1$ and $|E(G_i)| = a + (i-1)d$.*

Remark 2.2 From the above definition, the usual ASD of G coincides with $(1,1)$-ASD of G.

Example 2.3 Consider the Graph G.

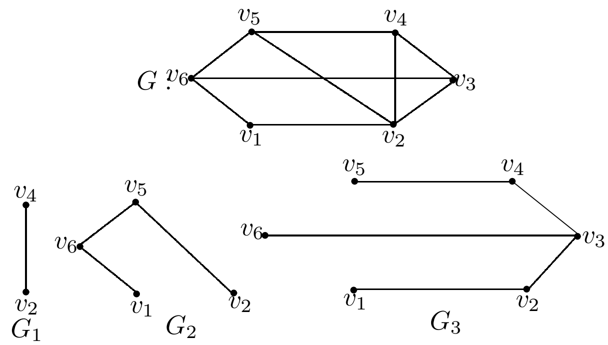


Fig.2.1

Clearly, $\{G_1, G_2, G_3\}$ is a $(1, 2)$-ASD of G.

Theorem 2.4 *Let G be a graph of size q, where $\binom{n}{2}[2a+(n-1)d] \leq q < \binom{n+1}{2}[2a+nd]$ for some positive integer n, such that G has (a, d)- ASD, then G has an (a, d)-ASD $G_1, G_2, \ldots, G_n$ such that G_i has size $a+(i-1)d$ for $1 \leq i \leq n-1$ and G_n has size $q-\binom{n-1}{2}[2a+(n-2)d]$.*

The following number theoretical result will be useful for proving further results.

Lemma 2.5 *Given that the numbers $a, a+d, a+2d, \ldots, a+(n-1)d$ are in A.P $(a, d \in Z)$. Then their sum is*

(i) $S_n = (a-d)n + d\binom{n+1}{2}$ *if* $d \leq a$ *and*

(ii) $S_n = a\binom{n+1}{2} + (d-a)\binom{n}{2}$ *if* $d \geq a$.

§3. (a, d)-ASD on Complete Graphs and Complete Bipartite Graphs

Theorem 3.1 *K_{n+1} has (a, d)-ASD if and only if $a = 1, d = 1$.*

Proof Suppose the graph K_{n+1} has (a, d)-ASD $G_1, G_2, \ldots, G_n$ with $|E(G_i)| = a+(i-1)d$, for $1 \leq i \leq n$.

By (ii) of Lemma 2.5, $q(K_{n+1}) = a\binom{n+1}{2} + (d-a)\binom{n}{2}$. Also since $q(K_{n+1}) = \binom{n+1}{2}$, we have $a = 1$ and $d = 1$. □

As it was mentioned in [3] that the complete graph K_{n+1} with $(n+1)$ vertices could easily be proved to have a star ASD and a path ASD, The converse follows.

Theorem 3.2 *$K_{n,n}$ has (a, d)-ASD, $d \geq a$ if and only if $a = 1$ and $d = 2$.*

Proof Suppose the graph $K_{n,n}$ admits $(a,d)-ASD$, $d \geq a$. If the graph $K_{n,n}$ admits $(a,d)-ASD\ G_1, G_2, \ldots, Gn$ then by (ii) of Lemma 2.5, we have $|E(K_{n,n})| = a\begin{pmatrix} n+1 \\ 2 \end{pmatrix} + (d-a)\begin{pmatrix} n \\ 2 \end{pmatrix}$.

Also, $|E(K_{n,n})| = n^2 = \begin{pmatrix} n+1 \\ 2 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix}$, so we have $a = 1$ and $d = 2$.

Conversely, suppose $a = 1, d = 2$.

Case (i) When n is even, $n = 2k, k \in Z^+$.

Then $K_{n,n}$ can be decomposed into k-hamilton cycles $H_1, H_2, \ldots, H_k$. Now, decompose the hamilton cycles H_i into paths G_i and $G_{n-(i-1)}$ of length $2i-1$ and $2n-(2i-1)$ for $1 \leq i \leq k$. Clearly, $\{G_1, G_2, \ldots, G_n\}$ is the required (1,2)-ASD of $K_{n,n}$.

Case (ii) When n is odd, $n = 2k+1, k \in Z^+$.

Let (X, Y) be the bipartition of $K_{n,n}$, where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. Define $H_1 = \{(x_n, y_j) : j = n-2\}$. For $2 \leq i \leq n-1$, define H_i by $H_{n+1-i} = \{(x_i, y_j) : j = 2i-2$ to $i+n-2\} \cup \{(x_j, y_{i+j-2}) : j = i+1$ to $n\}$, where addition is taken module n with residues $1, 2, 3, \ldots, n$ instead of the usual residues $0, 1, 2, \ldots, n-1$. $H_n = \{(x_1, y_k) : k = 1, 2, \ldots, n\} \cup \{(x_{j+1}, y_j) : 1 \leq j \leq n-1\}$. Clearly, $\{H_1, H_2, \ldots, H_n\}$ is a $(1,2)-ASD$ of $K_{n,n}$. □

Example 3.3 Consider the graph $K_{7,7}$. Let (X, Y) be the bipartition of $K_{7,7}$ where $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$.

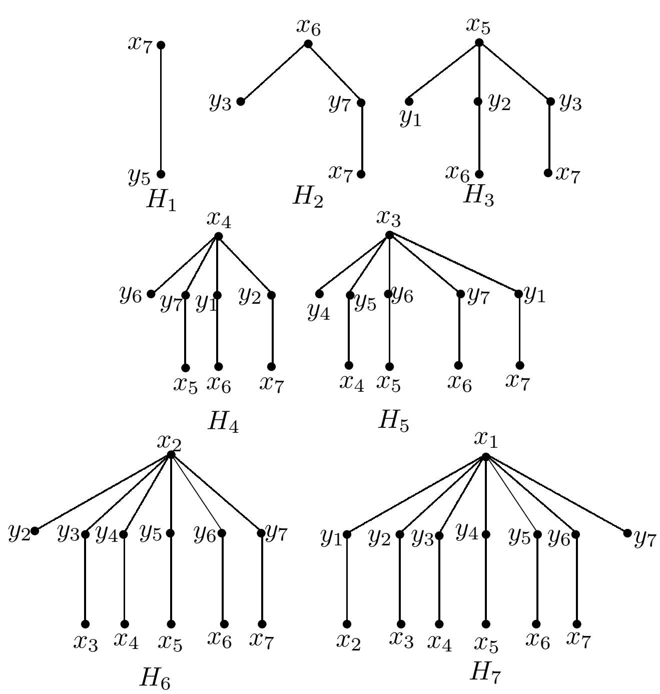


Fig. 3.1

Clearly, $\{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$ is $a(1,2) - ASD$ of $K_{7,7}$.

Theorem 3.4 *$K_{n,n}(n > 1)$ admits $(a,d)-ASD, d < a$ if and only if $n = 2a-1$ and $d = 1, a > 1$.*

Proof Suppose the graph $K_{n,n}(n > 1)$ admits $(a,d) - ASD$ where $d < a$, then by (i) Lemma 2.5, we have $|E(K_{n,n})| = (a-d)n + d\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$. Also, $|E(K_{n,n})| = n^2$. Therefore, $n^2 = (a-d)n + d\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$ and so $(2-d)n^2 = (2a-d)n$. Then $n = \frac{2a-d}{2-d}$. Since, $n > 1$, $a > d$, we have $2 - d > 0$. Then $d = 1$ and $a > 1$. Hence $n = 2a - 1$.

Conversely, Suppose $n = 2a-1, d = 1$ and $a > 1$. Let (X,Y) be the bipartition of $K_{n,n}$ where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$.

Define $T_{n-j-1} = \{(x_j, y_i) : 1 \le i \le n\} \cup \{(y_{i-j+1}, x_i) : \frac{n+2j+1}{2} \le i \le n\}$ where $1 \leqslant j \le \frac{n-1}{2}$ and $T_j = \{(x+n-j+1, y_i) : 1 \le i \le \frac{n-1}{2} + j\}$ where $1 \le j \le \frac{n-1}{2}$. Clearly, $\{T_1, T_2, \ldots, T_n\}$ is the required $(a,1) - ASD$ of $K_{n,n}$. □

Example 3.5 Consider the graph $K_{5,5}$. Let (X,Y) be the bipartition of $K_{5,5}$ where $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$. Clearly, $\{T_1, T_2, T_3, T_4, T_5\}$ is a $(3,1) - ASD$ of $K_{5,5}$.

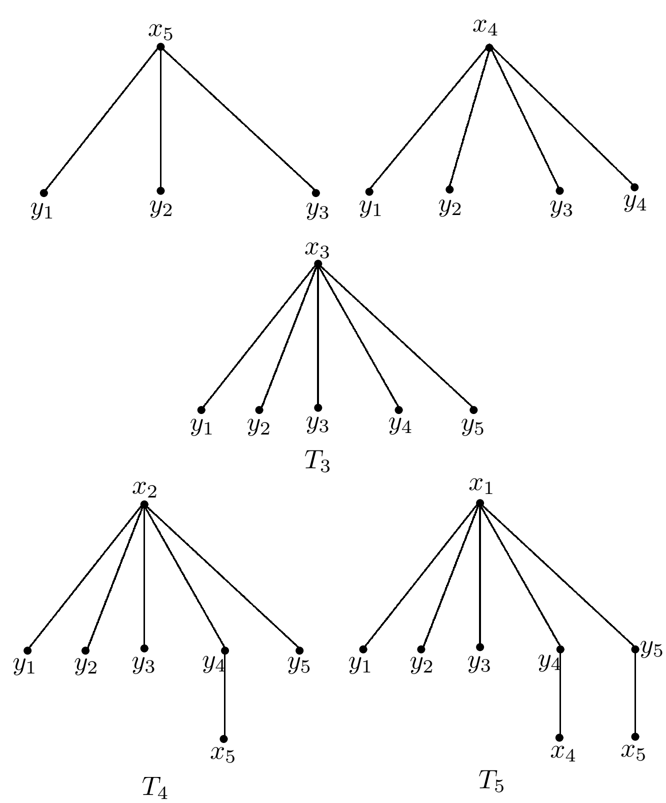


Fig. 3.2

§4. $(a,d)-ASD$ on Product Graphs

In this section, we prove some product graphs admit $(a,d)-ASD$.

Theorem 4.1 *$C_n \times C_n (n > 3)$ has $(2,4)-ASD$ when n is odd.*

Proof Note that $|E(C_n \times C_n)| = 2n^2$ and $|V(C_n \times C_n)| = n^2$. By Theorem 1.2, The graph $C_n \times C_n$ (n-odd) can be decomposed into two Hamilton cycles C_1 and C_2 of length n^2 respectively.

Case (i) When $n = 2k+1, k \equiv 1(mod\ 2)$.

Let $P_1 = C_1 - (v,x)$ and $P_2 = C_2 - (v,y)$ where $v,x,y \in V(C_n \times C_n)$ and $x \neq y$. First, define $P_1 = (xvy)$ when $k = 3$, decompose the path P_1 into paths P_i of length $(4i-2), 6 \leq i \leq 7$ and decompose the path P_2 into paths P_i of length $(4i-2), 2 \leq i \leq 5$. For, $k > 4$, decompose the path P_1 into paths P_i of length $(4i-2)$, where $2 \leq i \leq k - \lfloor \frac{k}{2} \rfloor - 1$ and $2\left(2 - \left\lfloor \frac{k}{2} \right\rfloor\right) + \left\lfloor \frac{k}{2} \right\rfloor + 1 \leq i \leq n$. Also decompose the path P_2 into paths P_i of length $(4i-2)$, where $\left(k - \left\lfloor \frac{k}{2} \right\rfloor\right) \leq i \leq 2\left(k - \left\lfloor \frac{k}{2} \right\rfloor\right) + \left\lfloor \frac{k}{2} \right\rfloor$. This is possible because of

$$\begin{aligned}
\mathcal{L}(P_1^1) &= \sum_{j=1}^{k-\lfloor \frac{k}{2} \rfloor - 2} (2+4j) + \sum_{j=2+\left(2\left(k-\lfloor \frac{k}{2} \rfloor\right)+k-\lfloor \frac{k}{2} \rfloor\right)4}^{n-1} (2+4j) \\
&= \frac{\left(k - \lfloor \frac{k}{2} \rfloor - 2\right)}{2} \left(12 + \left(\left(k - \left\lfloor \frac{k}{2} \right\rfloor - 2\right) - 1\right)4\right) \\
&\quad + \frac{\left(\lfloor \frac{k}{2} \rfloor + 1\right)}{2} \left(2\left(2 + \left(2\left(k - \left\lfloor \frac{k}{2} \right\rfloor\right) + \left\lfloor \frac{k}{2} \right\rfloor\right)4\right) + 4\left\lfloor \frac{k}{2} \right\rfloor\right) \\
&= 2\left(k - \left\lfloor \frac{k}{2} \right\rfloor - 2\right)\left(k - \left\lfloor \frac{k}{2} \right\rfloor\right) + \left(\frac{\lfloor \frac{k}{2} \rfloor + 1}{2}\right)\left(4 + 16k - 4\left\lfloor \frac{k}{2} \right\rfloor\right) \\
&= 2k^2 - 4k - 4k\left\lfloor \frac{k}{2} \right\rfloor + 4\left\lfloor \frac{k}{2} \right\rfloor + 2\left\lfloor \frac{k}{2} \right\rfloor^2 + 2 + 8k + 8k\left\lfloor \frac{k}{2} \right\rfloor - 2\left\lfloor \frac{k}{2} \right\rfloor^2 \\
&= 2k^2 + 4k + 2 + 4k\left\lfloor \frac{k}{2} \right\rfloor + 4\left\lfloor \frac{k}{2} \right\rfloor \\
&= 2k^2 + 4k + 2 + 2k(k-1) + 2(k-1) \\
&= 4k^2 + 4k \\
&= (2k+1)^2 - 1 = n^2 - 1 \\
\mathcal{L}(P_2') &= \left(\frac{k+1}{2}\right)\left(2\left(2 + \left(k - \left\lfloor \frac{k}{2} \right\rfloor - 1\right)4\right) + 4k\right) \\
&= (k+1)\left(6k - 2\left(2\left\lfloor \frac{k}{2} \right\rfloor + 1\right)\right) \\
&= (k+1)(6k-2k) = (2k+1)^2 - 1 = n^2 - 1.
\end{aligned}$$

From the above construction, clearly, $\{P_1, P_2, \ldots, P_n\}$ is a $(2,4)-ASD$ of $C_n \times C_n$.

Case (ii) When $n = 2k+1, k \equiv 0(mod\ 2)$.

Let $P_1' = C_1 - (v,x)$ and $P_2' = C_2 - (v,y)$ where $v,x,y \in V(C_n \times C_n)$ and $x \neq y$. First define $P_1 = (xvy)$, then decompose the path P_1' into paths P_2 of length 6 and P_j of length $(2+4j), 4 \leq j \leq n-1$ and $j = 0,1(mod\ 4)$ and also decompose the path P_2' into paths P_j of length $(2+4j), 2 \leq j \leq n-1$ and $j = 2,3(mod\ 4)$. This is possible, since

$$\begin{aligned}
\mathcal{L}(P_1') &= 6 + \sum_{\substack{j=4 \\ j\equiv 0,1(mod\ 4)}}^{n-1} (2+4j) \\
&= 6 + 2\sum_{\substack{j=4 \\ j\equiv 0,1(mod\ 4)}}^{n-1} 1 + 4\sum_{\substack{j=4 \\ j\equiv 0,1(mod\ 4)}}^{n-1} j \\
&= 6 + 2\sum_{\substack{j=4 \\ j\equiv 0,1(mod\ 4)}}^{2k} 1 + 4\sum_{\substack{j=4 \\ j\equiv 0,1(mod\ 4)}}^{2k} j \\
&= 6 + 2(k-1) + (4k^2+2k-4) = (2k+1)^2 - 1 = n^2 - 1
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}(P_2') &= \sum_{\substack{j=2 \\ j\equiv 2,3(mod\ 4)}}^{n-1} (2+4j) \\
&= 2\sum_{\substack{j=2 \\ j\equiv 2,3(mod\ 4)}}^{2k} 1 + 4\sum_{\substack{j=2 \\ j\equiv 2,3(mod\ 4)}}^{2k} j \\
&= 2k + 4\sum_{\substack{j=2 \\ j\equiv 2,3(mod\ 4)}}^{2k} j \\
&= 2k + (2k+4k^2) = (2k+1)^2 - 1 = n^2 - 1.
\end{aligned}$$

As in the case clearly, $\{P_1, P_2, \ldots, P_n\}$ is a $(2,4)-ASD$ of $C_n \times C_n$. □

Theorem 4.2 *$P_{n+1} \times P_{n+1}$ with size $q = 2n(n+1)$ admits $(4,4)-ASD$.*

Proof Let $G = P_{n+1} \times P_{n+1}$. Define $W_{i,j} = (u_i, v_j)$, where $1 \leq i,j \leq n+1$ and also define $V(G) = \{W_{i,j} : 1 \leq i,j \leq n+1\}$, $|E(G)| = 2(n^2+n)$.

Case (i) $n \equiv 3(mod\ 4), n = 4m-1 (m \in Z^+)$.

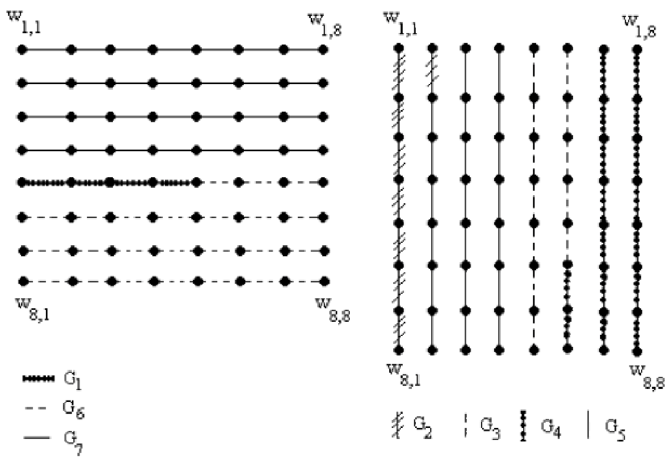
First define, $G_n = \{(W_{i,j}, V_{i,j+1}) : 1 \leq i \leq 4, 1 \leq j \leq n\}$ and define for $1 \leq k \leq \frac{n-3}{4}$.

$$\begin{aligned}
G_k &= \{(W_{i,j}, V_{i,j+1}) : i = 4k+1, 1 \leq j \leq 4k\} \\
G_{n-k} &= \{(W_{i,j}, W_{i,j+1}) : i = 4k+1, 4k+1 \leq j \leq n \text{ and} \\
&\qquad 4k+2 \leq i \leq 4(k+1), 1 \leq j \leq n\}
\end{aligned}$$

Also, define for $1 \leq \mathcal{L} \leq \dfrac{n+1}{4}$ and $k = \dfrac{n-3}{4}$.

$$G_{\mathcal{L}+k} = \{(W_{i,j}, V_{i+1,j}) : j = 4\mathcal{L}-3, 1 \le i \le n \text{ and}$$
$$j = 4\mathcal{L}-2, 1 \le i \le 4\mathcal{L}-3\}$$
$$G_{n-(\mathcal{L}+k)} = \{(W_{i,j}, W_{i+1,}) : 4\mathcal{L}-2 \le i \le n, j = 4\mathcal{L}-2 \text{ and}$$
$$1 \le i \le n, 4\mathcal{L}-1 \le j \le 4\mathcal{L}\}$$

Clearly, $\{G_1, G_2, \ldots, G_n\}$ is a $(4,4)-ASD$ of $P_{n+1} \times P_{n+1}$ (See Fig. 4.1).



$P_8 \times P_8$

Fig. 4.1

Case (ii) $n \equiv 0(mod\ 4), n = 4m(m \in Z^+)$.

First define, $G_n = \{(W_{i,j}, W_{i,j+1}) : 1 \le i \le 4, 1 \le j \le n\}$ and define for $1 \le k \le \frac{n-4}{4}$.

$$G_k = \{(W_{i,j}, W_{i,j+1}) : i = 4k+1, 1 \le j \le 4k\}$$
$$G_{n-k} = \{(W_{i,j}, W_{i,j+1}) : i = 4k+1, 4k+1 \le j \le n \text{ and}$$
$$4k+2 \le i \le 4(k+1), 1 \le j \le n\}$$

Define for $1 \le \mathcal{L} \le \frac{n-4}{4}$ and $p = \frac{n-4}{4}$.

$$G_{\mathcal{L}+p+1} = \{(W_{i,j}, W_{i+1,j}) : j = 4\mathcal{L}, 1 \le i \le n \text{ and}$$
$$j = 4\mathcal{L}+1, 1 \le i \le 4\mathcal{L}\}$$
$$G_{n-(\mathcal{L}+p+1)} = \{(W_{i,j}, W_{i+1,j}) : 4\mathcal{L}+1 \le i \le n, j = 4\mathcal{L}+1 \text{ and}$$
$$1 \le i \le n, 4\mathcal{L}+2 \le j \le 4l+3\}$$
$$G_{(p+1)} = \{(W_{i,j}, W_{i+1,j}) : i = n+1, 1 \le j \le n\} \text{ and}$$
$$G_{n-(p+1)} = \{(W_{i,j}, W_{i+1,j}) : 1 \le i \le n, 1 \le j \le 3\}$$

Finally define $G_{n/2} = \{(W_{i,j}, W_{i+1,j}) : 1 \le i \le n, n \le j \le n+1\}$. From the above construction clearly, $\{G_1, G_2, \ldots, G_n\}$ is a $(4,4) - ASD$ of $P_{n+1} \times P_{n+1}$ (See Fig. 4.2).

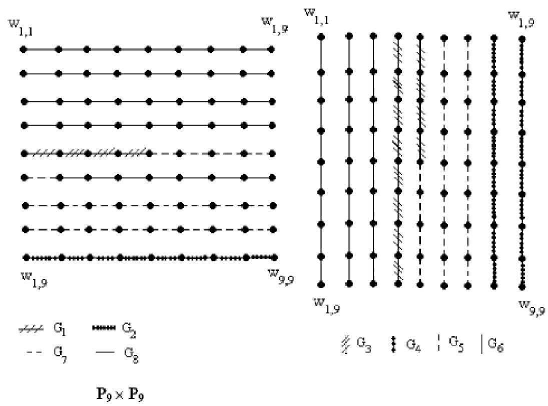


Fig 4.2

Case (iii) $n \equiv 1(mod\ 4), n = 4m+1 (m \in Z^+)$.

First define,

$$\begin{aligned} G_n = &\{(W_{i,j+1}W_{i,j}W_{i+1,j}) : i = 1, j = 1\} \\ &\cup \{(W_{i,i-1}W_{i,i}W_{i,i+1}W_{i-1,i}W_{i,i}W_{i+1,i}) : 2 \le i \le n\} \\ &\cup \{(W_{i,j-1}W_{i,j}W_{i-1,j}) : i = n+1, j = n+1\} \end{aligned}$$

Define for $1 \le r \le \dfrac{n-5}{2}$

$$\begin{aligned} G_{n-2r} = &\{(W_{i,j+1}W_{i,j}W_{i+1,j}) : i = 1, j = 2r+1\} \\ &\cup \{(W_{i,j-1}W_{i,j}W_{i,j+1}W_{i-1,j}W_{i,j}W_{i+1,j}) : 2 \le i \le n-2r \text{ and } j = 2r+i\} \\ &\cup \{(W_{i,j}W_{i+1,j}W_{i+1,j-1}) : i = n-2r \text{ and } j = n+1\} \end{aligned}$$

Also, define for $r = \dfrac{n-3}{2}$,

$$\begin{aligned} G'_2 = &\{(W_{i,j}W_{i,j+1}W_{i,j}W_{i+1,j}) : i = 3, j = 2r+1\} \\ &\cup \{(W_{i,j}W_{i+1,j}W_{i+1,j-1}) : i = n-2r, j = n+1\} \\ G'_3 = &\{(W_{i,j+1}W_{i,j}W_{i+1,j}) : i = 1, j = 2r+1\} \\ &\cup \{(W_{i,j-1}W_{i,j}W_{i,j+1}W_{i-1,j}W_{i,j}W_{i+1,j}) : i = 2, j = 2r+i\} \end{aligned}$$

Define for $1 \le k \le \dfrac{n-3}{2}$

$$\begin{aligned} G'_{n-2k-1} = &\{(W_{i+1,j}W_{i,j}W_{i,j+1}) : i = 1, j = 2k+1\} \\ &\cup \{(W_{i-1,j}W_{i,j}W_{i+1,j}W_{i,j-1}W_{i,j}W_{i,j+1}) : i = 2k+j \text{ and } \\ &\quad 2 \le j \le n-2k-2\} \\ &\cup \{(W_{i,j}W_{i,j+1}W_{i-1,j+1}) : i = n+1, j = n-2k-2\} \end{aligned}$$

Define

$C_1 = (W_{1,n}, W_{2,n}, W_{2,n+1}, W_{1,n+1}, W_{1,n})$,

$C_2 = (W_{n,1}, W_{n+1,n}, W_{n+1,2}, W_{n,2}, W_{n,1})$ and

$M = \{(W_{i,j}, W_{i,j+1}) : i = 1, n+1 \text{ and } j \equiv 0(mod\ 2)\}$

$\cup\{(W_{i,j}, W_{i+1,j}) : j = 1, n+1 \text{ and } i \equiv 0(mod\ 2)\}$.

Let $G_{n-1} = G'_{n-1} \cup C_1$ and $G_{n-3} = G'_{n-3} \cup C_2$. Define $G_1 = M_0, G_2 = G'_2 \cup M_1, G_3 = G'_3 \cup M_2$ and $G_{n-2k+1} = G'_{n-2k-1} \cup M_k$, where $3 \le k \le \frac{n-3}{2}$ and $M_i \cong 4K_2$ are suitably chosen from M in order to form $G_1, G_2, \ldots, G_n$ as $(4,4) - ASD$ (See Fig 4.3).

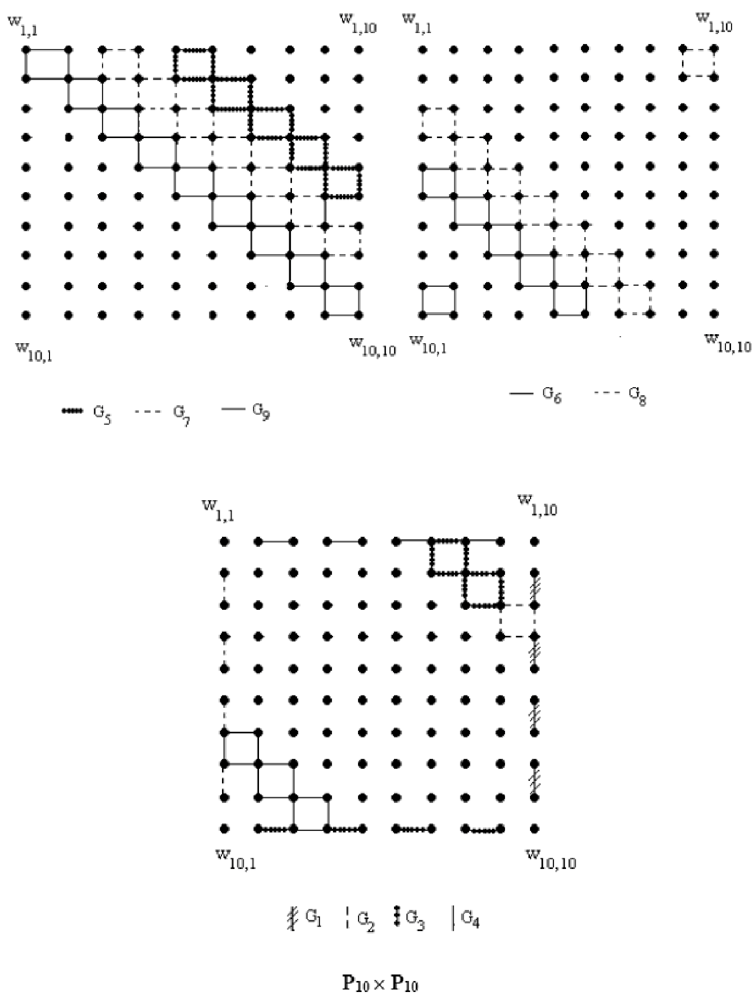


Fig. 4.3

Case (iv) $n \equiv 2(mod\ 4), n = 4\mathcal{L} + 2(\mathcal{L} \in Z^+)$.

For $1 \le m \le \mathcal{L}$ and $m \equiv 1(mod\ 2)$, define

$$\begin{aligned} G_m &= \{(W_{i,j}, W_{i+1,j}) : 4m-3 \le i \le 4m-2, n+2-m \le j \le n+1\} \\ &\cup \{(W_{i,j}, W_{i+1,j}) : 4m+1 \le i \le 4m-2, n+2-m \le j \le n+1\} \text{ and} \\ G_{n-(m-1)} &= \{(W_{i,j}, W_{i+1,j}) : 4m-3 \le i \le 4m-2 \text{ and } 1 \le j \le n+1-m\} \\ &\cup \{(W_{i,j}, W_{i+1,j}) : 4m+1 \le i \le 4m+2 \text{ and } 1 \le j \le n+1-m\}. \end{aligned}$$

For $1 \le m \le \mathcal{L}$ and $m \equiv 0(mod\ 2)$, define

$$\begin{aligned} G_m &= \{(W_{i,j}, W_{i+1,j}) : 4m-5 \le i \le 4m-4, n+m-2 \le j \le n+1\} \\ &\cup \{(W_{i,j}, W_{i+1,j}) : 4m-1 \le i \le 4m, n+m-2 \le j \le n+1\} \text{ and} \\ G_{n-(m-1)} &= \{(W_{i,j}, W_{i+1,j}) : 4m-5 \le i \le 4m-4 \text{ and } 1 \le j \le n+m-3\} \\ &\cup \{(W_{i,j}, W_{i+1,j}) : 4m-1 \le i \le 4m \text{ and } 1 \le j \le n+m-3\}. \end{aligned}$$

For $1 \le m \le \mathcal{L}$ and $m \equiv 1(mod\ 2)$, define

$$\begin{aligned} G_{m+\mathcal{L}} &= \{(W_{i,j}, W_{i,j+1}) : n-m-\mathcal{L}+2 \le i \le n+1 \text{ and} \\ &\quad 4m-3 \le j \le 4m-2\} \\ &\cup \{(W_{i,j}, W_{i,j+1}) : n-m-\mathcal{L}+2 \le i \le n+1 \text{ and} \\ &\quad 4m+1 \le j \le 4m+2\} \text{ and} \\ G_{n-(m+\mathcal{L}+1)} &= \{(W_{i,j}, W_{i+1,j}) : 1 \le i \le n-m-\mathcal{L}+1 \text{ and } 4m-3 \le j \le 4m-2\} \\ &\cup \{(W_{i,j}, W_{i+1,j}) : 1 \le i \le n-m-\mathcal{L}+1 \text{ and } 4m+1 \le i \le 4m+2\} \end{aligned}$$

and for $1 \le m \le \mathcal{L}$ and $m \equiv 0(mod\ 2)$,

$$\begin{aligned} G_{m+\mathcal{L}} &= \{(W_{i,j}, W_{i,j+1}) : n-m-\mathcal{L}+3 \le i \le n+1 \text{ and } 4m-5 \le j \le 4m-4\} \\ &\cup \{(W_{i,j}, W_{i,j+1}) : n-m-\mathcal{L}+3 \le i \le n+1 \text{ and } 4m-1 \le j \le 4m\} \text{ and} \\ G_{n-(m+\mathcal{L}+1)} &= \{(W_{i,j}, W_{i,j+1}) : 1 \le i \le n-m-\mathcal{L}-2 \text{ and } 4m-5 \le j \le 4m-4\} \\ &\cup \{(W_{i,j}, W_{i,j+1}) : 1 \le i \le n-m-\mathcal{L}+2 \text{ and } 4m-1 \le j \le 4m\}. \end{aligned}$$

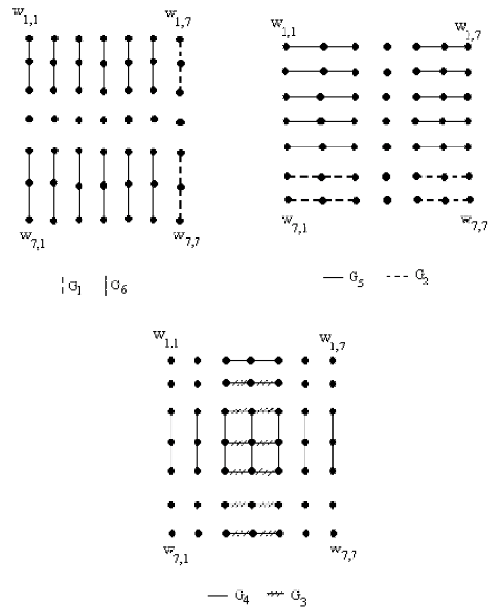
When $\mathcal{L}$ is even, define

$$\begin{aligned} G_{(n/2)} &= \{(W_{i,j}, W_{i,j+1}) : 2 \le i \le n+1, n-1 \le j \le n\} \text{ and} \\ G_{(n/2)+1} &= \{(W_{i,j}, W_{i+1,j}) : n-1 \le i \le n, 1 \le j \le n\} \\ &\cup \{(W_{i,j}, W_{i,j+1}) : i = 1, n-1 \le j \le n\}. \end{aligned}$$

When $\mathcal{L}$ is odd, define

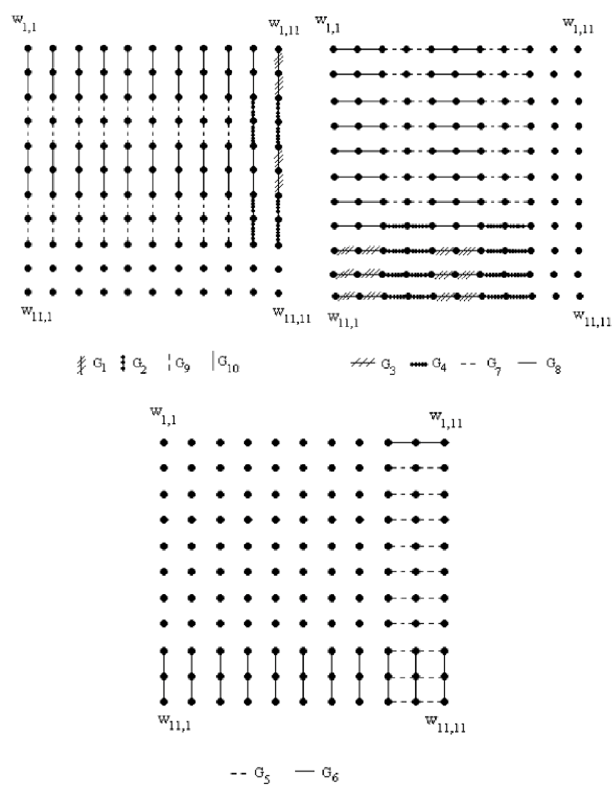
$$\begin{aligned} G_{(n/2)} &= \{(W_{i,j}, W_{i,j+1}) : 2 \le i \le n+1, n-3 \le j \le n-2\} \text{ and} \\ G_{(n/2)+1} &= \{(W_{i,j}, W_{i+1,j}) : n-3 \le i \le n-2 \text{ and } 1 \le j \le n+1\}. \end{aligned}$$

From the above construction clearly, $\{G_1, G_2, \ldots, G_n\}$ is a $(4,4)-ASD$ of $P_{n+1} \times P_{n+1}$. See Fig. 4.4(a) and Fig. 4.4(b). □



$P_7 \times P_7$

Fig. 4.4(a)



$P_{11} \times P_{11}$

Fig. 4.4(b)

§5. $(a,d)-ASD$ on Some Special Graphs

In this section $(a,d)-ASD$ is established for some special graphs like wheel, Carona and a special type in caterpillar.

Theorem 5.1 *$W_{n^2+1} = K_1 + C_{n^2} (n \geq 3)$ has $(a,d)-ASD$, $d \geq a$ if and only if $a = 2$ and $d = 4$.*

Proof Suppose W_{n^2+1} has $(a,d)-ASD$, $d \geq a$, By (ii) of Lemma 2.5, $|E(W_{n^2+1})| = a\begin{pmatrix} n+1 \\ 2 \end{pmatrix} + (d-a)\begin{pmatrix} n \\ 2 \end{pmatrix}$, also we have $|E(W_{n^2+1})| = 2n^2$.

From the above relations, we have $a = 2$ and $d = 4$. Conversely, let $V(W_{n^2+1}) = \{u_1, v_1, v_2, \ldots, v_{n^2}\}$. Define $G_1 = (u_1, v_1) \cup (v_1, v_2)$ and

$$G_i = \left\{((u_i, v_j) \cup (v_j, v_{j+1})) : \sum_{k=1}^{i-1}(2k-1) \leq j \leq \sum_{k=1}^{i}(2k-1)\right\}.$$

for $2 \leq i \leq n$. Where addition is taken modulo n^2 with residues $1, 2, 3, \ldots, n^2$ instead of the usual residues $0, 1, 2, \ldots, n^2-1$. Then clearly, $G_i \subseteq G_{i+1}, 1 \leq i \leq n-1$ and $|E(Gi)| = 2(2i-1)$ for $1 \leq i \leq n$. Hence, $\{G_1, G_2, \ldots, G_n\}$ is a $(2,4)-ASD$ of W_{n^2+1}. □

Example 5.2 A decomposition of W_{n^2+1}, where $n = 3$ into $(2,4)-ASD$ is illustrated in Fig. 5.1. Clearly, $\{G_1, G_2, G_3\}$ is a $(2,4)-ASD$.

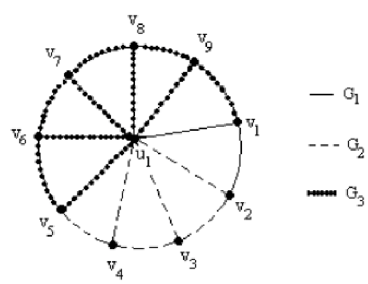


Fig. 5.1

Definition 5.3 *Let $T = S(v_1, v_2, \ldots, v_{n-1}, v_n, v_{n+1})$ be a caterpillar where v_i means n leaves attached to each vertex and v_{n+1} means no leaf attached to the last vertex.*

Theorem 5.4 *The caterpillar $T = S(v_0, v_1, v_2, \ldots, v_{n-1}, v_n)$ has an $(a,d)-ASD$, $(d \geq a)$ if and only if $a = 2$ and $d = 2$.*

Proof Suppose T admits (a,d) - ASD ($d \geq a$) By (ii) of Lemma 2.5, $|E(T)| = a\begin{pmatrix} n+1 \\ 2 \end{pmatrix} + (d-a)\begin{pmatrix} n \\ 2 \end{pmatrix}$. Also, $|E(T)| = (n+1)n = n^2+1 = 2\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$. From the above two relations, we have $a = 2$ and $d = 2$.

Conversely, suppose $a = 2$, $d = 2$. Let

$$V(G) = \{v_1, v_2, \ldots, v_n, v_{n+1}\} \cup \left\{v_1^{(k)}, v_2^{(k)}, \ldots, v_n^{(k)} : 1 \leq k \leq n\right\},$$

where v_i are vertices on the path P_n and $v_j^{(k)}(1 \leq k \leq n)$ are the vertices of the star at each $v_j(1 \leq j \leq n)$. Define for $1 \leq k \leq n, T_k = \{(v_k, v_{k+1})\} \cup \left\{\left(v_k, v_j^{(k)}\right) : 1 \leq j \leq n\right\}$.

Case (i) When n is odd, $n = 2m + 1$.

Decompose T_k for $k \equiv 0, 1(mod\ 2)$ into G_m and $G_{n-(m-1)}, 1 \leq m \leq \frac{n-1}{2}$. Where

$$G_m = \{(v_{2k}, v_{2k+1})\} \bigcup \left\{\left(v_{k+1}, v_j^{(k+1)}\right) : n - (2k-2) \leq j \leq n\right\}$$

and

$$G_{n-(m-1)} = \left\{\left(v_{k+1}, v_j^{(k)}\right) : 1 \leq j \leq n-(2k-1)\right\} \bigcup \{(v_{2k-1}, v_{2k})\} \bigcup \left\{\left(v_k, v_j^{(k)}\right) : 1 \leq j \leq n\right\}.$$

Define $G_{\frac{n+1}{2}} = \{(v_n, v_{n+1})\} \cup \left\{\left(v_n, v_j^{(n)}\right) : 1 \leq j \leq n\right\}$. Clearly $G_i \subseteq G_{i+1}, 1 \leq i \leq n-1$ and $|E(G_i)| = 2i, 1 \leq i \leq n$. Hence $\{G_1, G_2, \ldots, G_n\}$ is a $(2, 2) - ASD$ of T.

Case (ii) When n is even, $n = 2m$.

Decompose T_k for $k \equiv 0, 1(mod\ 4)$ into G_m and $G_{n-(m-1)}, 1 \leq m \leq \frac{n}{2}$ as in Case (i). Clearly $G_i \subseteq G_{i+1}, 1 \leq i \leq n-1$. Hence $\{G_1, G_2, \dot{;} G_n\}$ is a $(2, 2) - ASD$ of T. □

Corollary 5.5 *The corona $C_n \odot nK_1$ has $(a, d) - ASD$, $(d \geq a)$ if and only if $a = 2$ and $d = 2$.*

Proof By taking $v_{n+1} = v_1$ in $T = S(v_1, v_2, \ldots, v_n, v_{n+1})$. We have $T = C_n \odot nK_1$. □

References

[1] Y. Alavi, A.J. Boals, G. Chartrand, P.Erdos and O.R. Oellermann, The ascending Subgraph decomposition problem, *Congressus Numerantium*, **58**(1987), 7-14.

[2] J. Bosak, Decomposition of graphs, *Mathematics and its applications*, (East European series) Vol 47, Kluwer Academic Publications, 1980.

[3] R.J. Faudree, A. Gyarfas and R.H. Schelp, Graphs which have an ascending subgraph decomposition, *Congressus Numerantium*, **59**(1987), 49-54.

[4] F. Harary, *Graph Theory*, Addition - Wesley Publishing company Inc, USA,1969.