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The Characterizations of Nonnull Inclined Curves in Lorentzian Space $\mathbf{L}^5$

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Abstract: In this paper, making use of the author's method appeared in [1] , we define nonnull inclined curve in L^5. We also give some new characterizations of these curves in Lorentzian 5-space L^5.

Key Words: Inclined curves, Lorentzian 5-space, Frenet frame.

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§1. Introduction

A helix is a curve, the tangent of which makes a constant angle with a fixed line. Standard screws, bolts and a double-stranded molecule of DNA are the most common examples for helices in the nature and structures. First, Lancret in 1802 gave the characterizations of this curve. He obtained that "a curve is a helix if and only if ratio of curvature k_1 to torsion k_2 is costant". In [5], Ç. Camcı, K. İlarslan, L. Kula and H. Hilmi Hacısalihoğlu studied generalized helices in E^n and N.Ekmekci, H. Hilmi Hacısalihoğlu investigated harmonic curvatures in Lorentzian space in [6]. Then A. Altın obtained helix in R^n_v in [4]. Recently, in [1,2,3] , T. A. Ali studied inclined curves and slant helices in E^5 and E^n.

In this study, by considering inclined curves in the Euclidean 5-space E^5 as given in [1], we investigate necessary and sufficient conditions to be inclined for a nonnull curve in Lorentzian 5-space L^5 and obtain some characterizations of nonnull inclined curve in terms of their curvatures.

§2. Preliminaries

Let $\alpha : I \subset R \to L^5$ be a regular curve in L^5. The curve α is spacelike if all of its velocity vectors are spacelike, and timelike and null can be defined similarly. If $\langle \alpha'(t), \alpha'(t)\rangle = \pm 1$, then α is called a unit speed curve, where $\langle , \rangle$ detones the scalar product of L^5.

Let $\alpha : I \subset R \to L^5$ be a regular curve in L^5 and $\psi = \{\alpha'(t), \alpha''(t), \alpha'''(t), \alpha^{iv}(t), \alpha^{v}(t)\}$ a maximal linear independent and nonnull set. An orthonormal system $\{V_1(t), V_2(t), V_3(t), V_4(t),$

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$V_5(t)\}$ can be obtained from ψ. This is called a Serre-Frenet frame at the point $\alpha(t)$.

Defiinition 2.1([4]) *Let α be a unit speed curve in L^5 and let the set $\{V_1(t), V_2(t), V_3(t), V_4(t), V_5(t)\}$ be the Serre-Frenet frame at the point $\alpha(t)$. Then, the following hold*

$$\begin{aligned} &V_1'(t) = \epsilon_1(t)k_1(t)V_2(t), \\ &\vdots \\ &V_i'(t) = -\epsilon_i(t)k_{i-1}(t)V_{i-1}(t) + \epsilon_i(t)k_i(t)V_{i+1}(t),\ 1 < i < 5 \\ &\vdots \\ &V_5'(t) = -\epsilon_5(t)k_4(t)V_4(t), \end{aligned} \tag{2.1}$$

where $\epsilon_i(t)$ denotes $\langle V_i(t), V_i(t)\rangle = \pm 1,$.

Definition 2.2([1]) *A unit speed curve $\alpha : I \to E^5$ is is said to be an inclined curve if its tangent T makes a constant angle with a fixed direction U.*

Theorem 2.1([1]) *Let $\alpha : I \to E^5$ be a unit speed curve regular curve in E^5. Then α is an inclined curve if and only if the function*

$$\left(\frac{k_1}{k_2}\right)^2 + \frac{1}{k_3^2}\left[\left(\frac{k_1}{k_2}\right)'\right]^2 + \frac{1}{k_4^2}\left[\frac{k_1k_3}{k_2} + \left[\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]^2$$

is a constant. Moreover, this constant agrees with $\tan^2\theta$, being θ the angle that makes T with the fixed direction U that determines α.

§3. Characterizations of Nonnull Inclined Curves in $\mathbf{L^5}$

In this section, we define nonnull inclined curves in L^5. We also give some new characterizations of these curves in Lorentzian 5-space L^5.

Definition 3.1 *A unit speed nonnull curve $\gamma : I \to L^5$ is called a nonnull inclined curve in L^5 if its first Frenet vector V_1 makes a constant angle with a fixed direction U.*

Theorem 3.1 *Let $\gamma : I \to L^5$ be a unit speed nonnull curve in Lorentzian space L^5. Then γ is a nonnull inclined curve if and only if the function*

$$\frac{\epsilon_5}{\epsilon_3}\sqrt{\epsilon_4\epsilon_5}\left(\frac{k_1}{k_2}\right)^2 + \left[\frac{1}{\epsilon_3k_3}\left(\frac{k_1}{k_2}\right)'\right]^2 + \left[\frac{1}{\epsilon_4k_4}\left[\frac{\epsilon_4k_1k_3}{k_2} + \left[\frac{1}{\epsilon_3k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]\right]^2$$

is constant.

Proof Let $\gamma : I \subset R \to L^5$ be a unit speed nonnull curve in L^5. Assume that γ is a nonnull inclined curve. Let U be the fixed direction which makes a costant angle ϕ with $V_{1.}$ Consider the differentiable functions $a_{i,}\ 1 \le i \le 5$,

$$U = \sum_{i=1}^{5} a_i(s)V_i(s), \quad s \in I, \tag{3.1}$$

that is

$$a_i =< V_i, U >, \quad 1 \le i \le 5.$$

Then the function $a_1(s) =< V_1(s), U >$ is constant, that is

$$a_1(s) =< V_1(s), U >= \begin{cases} \cos\phi = const, & \text{if } \gamma \text{ is a spacelike curve} \\ \cos h\phi = const, & \text{if } \gamma \text{ is a timelike curve} \end{cases} \tag{3.2}$$

By differentiation of (3.2) with respect to s and using the Frenet formula (2.1), we have

$$a_1'(s) = -\epsilon_1 k_1 a_2 = 0. \tag{3.3}$$

Then $a_2 = 0$ and therefore U is in the subspace $Sp\{V_1, V_3, V_4, V_5\}$. Because the vector field U is constant, a differentiation in (3.1) together (2.1) gives the following system of ordinary differential equation:

$$\begin{aligned} &\epsilon_2 k_1 a_1 - \epsilon_2 k_2 a_3 = 0, \\ &a_3' - \epsilon_3 k_3 a_4 = 0, \\ &a_4' + \epsilon_4 k_3 a_3 - \epsilon_4 k_4 a_5 = 0, \\ &a_5' + \epsilon_5 k_4 a_4 = 0. \end{aligned} \tag{3.4}$$

The first three equations in (3.4) lead to

$$\begin{aligned} a_3 &= \frac{k_1}{k_2} a_1, \\ a_4 &= \frac{1}{\epsilon_3 k_3} \left(\frac{k_1}{k_2}\right)' a_1, \\ a_5' &= \frac{1}{\epsilon_4 k_4} \left[\frac{\epsilon_4 k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3} \left(\frac{k_1}{k_2}\right)'\right]'\right] a_1 \end{aligned} \tag{3.5}$$

We do the change of variables:

$$t(s) = \int^s k_4(u) du \quad \frac{dt}{ds} = k_4(s).$$

In particular, and from equation (3.4), we have

$$a_4'(t) = \epsilon_4 a_5(t) - \epsilon_4 \left(\frac{k_3(t)}{k_4(t)}\right) a_3(t).$$

As the last equation of (3.4) yields

$$a_5''(t) + \epsilon_4 \epsilon_5 a_5(t) = \epsilon_4 \epsilon_5 \left(\frac{k_1(t) k_3(t)}{k_2(t) k_4(t)}\right) a_1. \tag{3.6}$$

The general solution of this equation is obtained

$$\begin{aligned} a_5(t) = [&\left(A - \int \frac{k_1(t)k_3(t)}{k_2(t)k_4(t)} \sin\sqrt{\epsilon_4\epsilon_5} t dt\right) \cos\sqrt{\epsilon_4\epsilon_5} t \\ &+ \left(B + \int \frac{k_1(t)k_3(t)}{k_2(t)k_4(t)} \cos\sqrt{\epsilon_4\epsilon_5} t dt\right) \sin\sqrt{\epsilon_4\epsilon_5} t] \sqrt{\epsilon_4\epsilon_5} a_1, \end{aligned} \tag{3.7}$$

where A and B are arbitrary constants. Then (3.7) takes the following form

$$a_5(s) = [\left(A - \int [\frac{k_1(s)k_3(s)}{k_2(s)} \sin \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds\right) \cos \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds$$

$$+ \left(B + \int [\frac{k_1(s)k_3(s)}{k_2(s)} \cos \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds\right) \sin \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds] \sqrt{\epsilon_4\epsilon_5} a_1. \tag{3.8}$$

From the last equation of (3.4), the function a_4 is given by

$$a_4(s) = [\left(A - \int [\frac{k_1(s)k_3(s)}{k_2(s)} \sin \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds\right) \sin \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds$$

$$- \left(B + \int [\frac{k_1(s)k_3(s)}{k_2(s)} \cos \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds\right) \cos \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds] \frac{\sqrt{\epsilon_4\epsilon_5}}{\epsilon_5} a_1. \tag{3.9}$$

From equation (3.9) with the first two equation in (3.4), leads to the following equation:

$$\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)' = [\left(A - \int [\frac{k_1k_3}{k_2} \sin \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds\right) \sin \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds$$

$$- \left(B + \int [\frac{k_1k_3}{k_2} \cos \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds\right) \cos \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds] \frac{\sqrt{\epsilon_4\epsilon_5}}{\epsilon_5}. \tag{3.10}$$

From equation (3.5), we have

$$\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]$$

$$= [\left(A - \int [\frac{k_1k_3}{k_2} \sin \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds\right) \cos \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds$$

$$+ \left(B + \int \frac{k_1k_3}{k_2} \cos \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds\right) \sin \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds] \sqrt{\epsilon_4\epsilon_5}. \tag{3.11}$$

The equation (3.10) can be written by

$$\frac{\epsilon_5}{\sqrt{\epsilon_4\epsilon_5}\epsilon_3}\frac{k_1}{k_2}\left(\frac{k_1}{k_2}\right)' = [\left(A - \int [\frac{k_1k_3}{k_2} \sin \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds\right) \sin \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds$$

$$- \left(B + \int [\frac{k_1k_3}{k_2} \cos \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds\right) \cos \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds] \frac{k_1k_3}{k_2}.$$

If we integrate the above equation, we have

$$\frac{\epsilon_5}{\sqrt{\epsilon_4\epsilon_5}\epsilon_3}\frac{k_1^2}{k_2^2} = C - \left(\sqrt{\epsilon_4\epsilon_5}\right)^2 [\left(A - \int [\frac{k_1k_3}{k_2} \sin \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds\right)^2$$

$$+ \left(B + \int [\frac{k_1k_3}{k_2} \cos \int \sqrt{\epsilon_4\epsilon_5} k_4(s)ds]ds\right)^2] \tag{3.12}$$

where C is constant of integration. From equation (3.10), (3.11) and (3.12), we get

$$\frac{\sqrt{\epsilon_4\epsilon_5}\epsilon_5}{\epsilon_3}\left(\frac{k_1}{k_2}\right)^2 + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]^2 + \left[\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]\right]^2 = C$$

(3.13)

Furthermore this constant C calculates as follows. From equation (3.13) together the three equations of (3.4), if the first Frenet vector V_1 is a timelike vector and V_i $(i = 2, 3, 4, 5)$ is a spacelike vector, we have

$$C = \frac{a_3^2 + a_4^2 + a_5^2}{a_1^2} = \frac{1 - a_1^2}{a_1^2} = -\tanh^2\phi$$

Similarly, if the second Frenet vector V_2 is a timelike vector and V_i $(i = 1, 3, 4, 5)$ is a spacelike vector, we have

$$C = \tan^2\phi$$

where we have used (2.1) and the fact that U is a unit vector field.

Conversely, assume that the condition (3.13) is satisfied for a curve γ. Let $\phi \in R$ be so that

$$C = \begin{cases} -\tanh^2\phi & \text{if } V_1 \text{ is a timelike curve} \\ \tan^2\phi & \text{if } V_1 \text{ is a spacelike curve} \end{cases}$$

Let define the unit vector U by if the first Frenet vector V_1 is a timelike vector

$$U = \cos h\phi \left[V_1 + \frac{k_1}{k_2}V_3 + \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)' V_4 + \frac{1}{k_4}\left[\frac{k_1k_3}{k_2} + \left[\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right] V_5 \right]$$

if V_1 is a timelike vector and

$$U = \cos\phi \left[V_1 + \frac{k_1}{k_2}V_3 + \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)' V_4 + \frac{1}{k_4}\left[\frac{k_1k_3}{k_2} + \left[\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right] V_5 \right]$$

if V_1 is a spacelike vector. By taking account (3.13), a differentiation of U gives that $\frac{dU}{ds} = 0$, which it means that U is a constant vector field.On the other hand, the scalar product between the first Frent vector V_1 and U is

$$< V_1(s), U >= \begin{cases} \cos\phi = const, & \text{if } V_1 \text{ is a spacelike curve} \\ \cos h\phi = const, & \text{if } V_1 \text{ is a timelike curve} \end{cases}.$$

Thus γ is a nonnull inclined curve. □

Theorem 3.2 *Let $\gamma : I \subset R \to L^5$ be a unit speed nonnull curve in Lorentzian 5-space L^5. Then γ is a nonnull inclined curve if and only if there exists a C^2-function f such that*

$$\begin{aligned} \epsilon_4 k_4 f(s) &= \epsilon_4 \frac{k_1k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]', \\ \frac{1}{k_4}\frac{d}{ds}f(s) &= -\frac{\sqrt{\epsilon_4\epsilon_5}}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'. \end{aligned} \quad (3.14)$$

Proof Assume that γ is a nonnull inclined curve. A differentiation of (3.13) gives

$$\frac{\sqrt{\epsilon_4\epsilon_5}\epsilon_5}{\epsilon_3}\left(\frac{k_1}{k_2}\right)\left(\frac{k_1}{k_2}\right)' + \frac{1}{\epsilon_3k_3}\left(\frac{k_1}{k_2}\right)'\left[\frac{1}{\epsilon_3k_3}\left(\frac{k_1}{k_2}\right)'\right]'$$

$$+\left\{\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2}+\left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]\right\}\left\{\left[\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2}+\left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]\right]'\right\}=0. \tag{3.15}$$

If we consider that $\epsilon_4 = \epsilon_5$, after some manipulations the equation (3.15) takes the following form:

$$\sqrt{\epsilon_4\epsilon_5}\frac{\epsilon_5 k_4}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'+\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2}+\left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]=0. \tag{3.16}$$

If we define $f = f(s)$ by

$$f(s)=\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2}+\left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right].$$

Then the equation (3.16) writes as

$$f'(s)=-\sqrt{\epsilon_4\epsilon_5}\frac{\epsilon_5 k_4}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'.$$

Conversely, if (3.14) holds, we define a unit constant vector U by, if the first Frenet vector V_1 is a timelike vector

$$U=\cos h\phi\left[V_1+\frac{k_1}{k_2}V_3+\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'V_4+\frac{1}{k_4}\left[\frac{k_1 k_3}{k_2}+\left[\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]V_5\right]$$

and if the first Frenet vector V_1 is a spacelike vector

$$U=\cos\phi\left[V_1+\frac{k_1}{k_2}V_3+\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'V_4+\frac{1}{k_4}\left[\frac{k_1 k_3}{k_2}+\left[\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]V_5\right].$$

We have that

$$< V_1(s), U >=\begin{cases}\cos\phi = const, & \text{if } V_1 \text{ is a spacelike curve}\\ \cos h\phi = const, & \text{if } V_1 \text{ is a timelike curve}\end{cases}.$$

that is γ is a nonnull inclined curve. □

Theorem 3.3 *Let $\gamma : I \subset R \to L^5$ be a unit speed nonnull curve in Lorentzian 5-space L^5. Then γ is a nonnull inclined curve if and only if the following condition is satisfied*

$$\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'=[\left(A-\int[\frac{k_1 k_3}{k_2}\sin\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right)\sin\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds$$
$$-\left(B+\int[\frac{k_1 k_3}{k_2}\cos\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right)\cos\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds]\frac{\sqrt{\epsilon_4\epsilon_5}}{\epsilon_5}. \tag{3.17}$$

for some constants A and B.

Proof Suppose that γ is a nonnull inclined curve. By using Theorem 3.2, let define $g(s)$ and $h(s)$ by

$$\psi(s)=\int^s k_4(u)du, \tag{3.18}$$

$$
\begin{aligned}
g(s) &= \frac{a_5}{a_1}\cos\sqrt{\epsilon_4\epsilon_5}\psi + \frac{\epsilon_5}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\sin\sqrt{\epsilon_4\epsilon_5}\psi + \int\left[\sqrt{\epsilon_4\epsilon_5}\frac{k_1k_3}{k_2}\sin\sqrt{\epsilon_4\epsilon_5}\psi\right]ds,\\
h(s) &= \frac{a_5}{a_1}\sin\sqrt{\epsilon_4\epsilon_5}\psi - \frac{\epsilon_5}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\cos\sqrt{\epsilon_4\epsilon_5}\psi + \int\left[\sqrt{\epsilon_4\epsilon_5}\frac{k_1k_3}{k_2}\cos\sqrt{\epsilon_4\epsilon_5}\psi\right]ds.
\end{aligned} \tag{3.19}
$$

If we differentiate equations (3.19) with respect to s and taking into account of (3.18), we obtain $\dfrac{dg}{ds} = 0$ and $\dfrac{dh}{ds} = 0$. Therefore, there exists constants A and B such that $g(s) = A$ and $h(s) = B$. By substituting into (3.19) and solving the resulting equations for $\dfrac{1}{\epsilon_3 k_3}\left(\dfrac{k_1}{k_2}\right)'$, we get

$$
\begin{aligned}
\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)' &= [\left(A - \int[\frac{k_1k_3}{k_2}\sin\sqrt{\epsilon_4\epsilon_5}\psi]ds\right)\sin\sqrt{\epsilon_4\epsilon_5}\psi\\
&\quad - \left(B + \int[\frac{k_1k_3}{k_2}\cos\sqrt{\epsilon_4\epsilon_5}\psi]ds\right)\cos\sqrt{\epsilon_4\epsilon_5}\psi]\frac{\sqrt{\epsilon_4\epsilon_5}}{\epsilon_5}.
\end{aligned} \tag{3.20}
$$

Conversely, suppose that (3.17) holds. In order to apply Theorem 3.2 , we define

$$
\begin{aligned}
&\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]\\
&= [\left(A - \int[\frac{k_1k_3}{k_2}\sin\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right)\cos\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds\\
&\quad + \left(B + \int\frac{k_1k_3}{k_2}\cos\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right)\sin\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds]\sqrt{\epsilon_4\epsilon_5}
\end{aligned} \tag{3.21}
$$

with $\psi(s) = \int^s k_{n-1}(u)du$, a direct differentiation of (3.17) gives

$$
\left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]' = \left[\frac{\epsilon_4 k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right] + \frac{\epsilon_4 k_1 k_3}{k_2}.
$$

This shows the left condition in (3.17). Furthermore, a straightforward computation leads to

$$
\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right] = -\frac{\epsilon_4 k_4}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)',
$$

which finish the proof. □

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