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The Crossing Number of $K_{1,5,n}$

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Abstract: In this paper, we determine the crossing number of the complete tripartite graph $K_{1,5,n}$ for any integer $n \geq 1$, related with Smarandache 2-manifolds on spheres.

Key Words: Good drawings, complete tripartite graphs, crossing number.

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§1. Introduction

The *crossing number* $cr(G)$ of a graph G is the smallest crossing number among all drawings of G in the plane. It is well known that the crossing number of graph is attained only in good drawings of the graph related with *map geometries*, i.e., *Smarandache* 2*-manifolds* (see [8] for details), which are those drawings where no edges cross itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges intersect in a common point. Let ϕ be a good drawing of the graph G, we denote the number of crossings in this drawing of G by cr_ϕ(G).

The investigation on the crossing number of a graph is a classical and however very difficult problem (for example, see [3]). Garey and Johnson [4] have proved that the problem to determine the crossing number of a graph is NP-complete. Because of its difficulty, presently we only know the crossing number of some classes of special graphs, for example: the complete graphs with small number of vertices ([15]), the complete bipartite graph of less number of vertices in one bipartite partition ([7],[15]), certain generalized Peterson graphs ([12]), and some Cartesian product graphs of two circuits([2],[11]-[14]), of path and stars ([9]).

The crossing numbers of complete bipartite graphs $K_{m,n}$ were computed by D.J.Kleitman [7], for the case $m \leq 6$. He proved that

$$cr(K_{m,n}) = Z(m,n), \text{if } m \leq 6, \quad \text{where} \quad Z(m,n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor.$$

On the crossing number of the complete tripartite graphs, as far as the authors know, there

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only are the following two results: Kouhei Asano [1] proved that

$$cr(K_{1,3,n}) = Z(4,n) + \lfloor \frac{n}{2} \rfloor, \text{and } cr(K_{2,3,n}) = Z(5,n) + n;$$

and Huang [5] recently proves that $cr(K_{1,4,n}) = n(n-1)$.

In this paper, using Kleitman's theorem, we determine the crossing number of complete tripartite graph $K_{1,5,n}$ for any integer $n \geq 1$. The main result of this paper is the following theorem.

Theorem 1 (the main result) *For any integer $n \geq 1$,*

$$cr(K_{1,5,n}) = Z(6,n) + 4\lfloor \frac{n}{2} \rfloor$$

We now explain some notations. Let G be a graph with vertex set V and edge set E. If $A \subseteq E$ (or $A \subseteq V$), we use $G\langle A \rangle$ to denote the subgraph of G induced by A; if G is known from the context, we simply write $\langle A \rangle$ instead of $G\langle A \rangle$. For two mutually disjoint subsets X and Y of V, we use E_{XY} to denote all the edges of G incident with a vertex in X and a vertex in Y. For a vertex v, E_v denotes all the edges of G incident with v.

Let A and B be two sets of edges of a graph G. If ϕ is a good drawing of G, we denote $cr_\phi(A,B)$ by the number of all crossings whose two crossed edges are respectively in A and in B. Especially, $cr_\phi(A,A)$ will be denoted by $cr_\phi(A)$. If G has the edge set E, the two signs $cr_\phi(G)$ and $cr_\phi(E)$ are essential the same.

The following formulas, which can be shown easily, are usually used in the proofs of our lemmas and theorem.

$$\begin{aligned} cr_\phi(A \cup B) &= cr_\phi(A) + cr_\phi(B) + cr_\phi(A,B) \\ cr_\phi(A, B \cup C) &= cr_\phi(A,B) + cr_\phi(A,C), \end{aligned} \tag{1}$$

where A, B and C are mutually disjoint subsets of E.

In the next section we shall give some lemmas, and then prove our theorem in the last one.

§2. Some Lemmas

Lemma 2.1 *Let G be a complete bipartite graph $K_{m,n}$ with the edge set E and the vertex bipartition (Y,Z), where $Y = \{y_1, \cdots, y_m\}$, and $Z = \{z_1, \cdots, z_n\}$. If ϕ is any good drawing of G, then*

$$(n-2)cr_\phi(E) = \sum_{i=1}^{n} cr_\phi(E \setminus E_{z_i}).$$

Proof The conclusion follows from the fact that in the drawing of $K_{m,n}$, there are n drawings of $K_{m,n-1}$, and each crossing occurs in $(n-2)$ of them. □

Lemma 2.2 *Let G be a complete tripartite graph $K_{s,m,n}$ with the edge set E and the vertex tripartition (X,Y,Z), where $X = \{x_1, \cdots, x_s\}$, $Y = \{y_1, \cdots, y_m\}$, and $Z = \{z_1, \cdots, z_n\}$. If ϕ is any good drawing of G, then we have*

(i) $\sum\limits_{i=1}^{n} cr_\phi(E \setminus E_{z_i}) = (n-2)cr_\phi(E) + \sum\limits_{i=1}^{n} cr_\phi(E_{XY}, E_{z_i}) + 2cr_\phi(E_{XY})$;

(ii) $\sum\limits_{i=1}^{m} cr_\phi(E \setminus E_{y_i}) = (m-2)cr_\phi(E) + \sum\limits_{i=1}^{m} cr_\phi(E_{XZ}, E_{y_i}) + 2cr_\phi(E_{XZ})$

Proof We only prove (i), because (ii) is analogous by the symmetry of the vertex tripartition of G. Using the formula (1), we have

$$\begin{aligned} cr_\phi(E) &= cr_\phi(E_{XY} \cup E_{XZ} \cup E_{YZ}) \\ &= cr_\phi(E_{XY}) + cr_\phi(E_{XZ} \cup E_{YZ}) + cr_\phi(E_{XY}, E_{XZ} \cup E_{YZ}) \\ &= cr_\phi(E_{XY}) + cr_\phi(E_{XZ} \cup E_{YZ}) + \sum_{i=1}^{n} cr_\phi(E_{XY}, E_{z_i}) \end{aligned} \quad (2)$$

Since $\langle E_{XZ} \cup E_{YZ} \rangle$ is isomorphic to the complete bipartite graph $K_{s+m,n}$ with the vertex bipartition $(X \cup Y, Z)$, it follows from by Lemma 2.1 that

$$(n-2)cr_\phi(E_{XY} \cup E_{YZ}) = \sum_{i=1}^{n} cr_\phi\Big((E_{XZ} \cup E_{YZ}) \setminus E_{z_i}\Big) \quad (3)$$

On the other hand, using the formula (1) again we have

$$\begin{aligned} cr_\phi(E \setminus E_{z_i}) &= cr_\phi\Big((E_{XY} \cup E_{XZ} \cup E_{YZ}\Big) \setminus E_{z_i}\Big) \\ &= cr_\phi(E_{XY}) + cr_\phi\Big((E_{XZ} \cup E_{YZ}) \setminus E(z_i)\Big) \\ &\quad + cr_\phi\Big(E_{XY}, (E_{XZ} \cup E_{YZ}) \setminus E_{z_i}\Big) \\ &= cr_\phi(E_{XY}) + cr_\phi\Big((E_{XZ} \cup E_{YZ}) \setminus E_{z_i}\Big) \\ &\quad + \sum_{j=1}^{n} cr_\phi(E_{XY}, E_{z_j}) - cr_\phi(E_{XY}, E_{z_i}), \end{aligned}$$

namely, we have

$$cr_\phi(E \setminus E_{z_i}) = cr_\phi(E_{XY}) + cr_\phi\Big((E_{XZ} \cup E_{YZ}) \setminus E_{z_i}\Big) + \sum_{j=1}^{n} cr_\phi(E_{XY}, E_{z_j}) - cr_\phi(E_{XY}, E_{z_i}) \quad (4)$$

Taking sum for i on the two sides of (4) above, we obtain that

$$\begin{aligned}\sum_{i=1}^{n} cr_\phi(E\setminus E_{z_i}) &= ncr_\phi(E_{XY})+\sum_{i=1}^{n} cr_\phi\Big((E_{XZ}\cup E_{YZ})\setminus E_{z_i}\Big)\\ &\quad +\sum_{i=1}^{n}\Big(\sum_{j=1}^{n} cr_\phi(E_{XY},E_{z_j})-cr_\phi(E_{XY},E_{z_i})\Big)\\ &= ncr_\phi(E_{XY})+\sum_{i=1}^{n} cr_\phi\Big((E_{XZ}\cup E_{YZ})\setminus E_{z_i}\Big)+(n-1)\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_i})\\ &= ncr_\phi(E_{XY})+(n-2)cr_\phi(E_{XY}\cup E_{YZ})\\ &\quad +(n-1)\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_i}) \qquad \text{(by (3) above)}\\ &= 2cr_\phi(E_{XY})+\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_i})\\ &\quad +(n-2)\Big(cr_\phi(E_{XY})+cr_\phi(E_{XZ}\cup E_{YZ})+\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_i})\Big)\\ &= 2cr_\phi(E_{XY})+\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_i})+(n-2)cr_\phi(E) \qquad \text{(by (2) above)}\end{aligned}$$

This proves the lemma. □

Note that in Lemma 2 above, if X is a set containing a single vertex x, then E_{XY} is the set of edges incident to x, and thus $cr_\phi(E_{XY})=0$ by any good drawing ϕ.

Lemma 2.3 *Let G be a complete tripartite graph $K_{1,5,n}$ with the edge set E and the vertex tripartition (X,Y,Z), where $X=\{x\}$, $Y=\{y_1,\cdots,y_5\}$, and $Z=\{z_1,\cdots,z_n\}$. If ϕ is a good drawing of G satisfying that $cr_\phi(E)=Z(6,n)+4\left[\frac{n}{2}\right]-a$ for some a. Then we have*

(1) if $n=2k$, then $\sum\limits_{i=1}^{5} cr_\phi(E_{XZ},E_{y_i})\geq 2k^2-2k+3a$;

(2) if $n=2k+1$, then $\sum\limits_{i=1}^{5} cr_\phi(E_{XZ},E_{y_i})\geq 2k^2-4+3a$.

Proof Let e_i denote the edge xy_i for $1\leq i\leq 5$, and f_j denote the edge xz_j for $1\leq j\leq n$. Without loss of generality, assume that under the drawing ϕ, the reverse clock order of these five edges e_i $(1\leq i\leq 5)$ around x is: $e_1\to e_2\to e_3\to e_4\to e_5$. These five edges form five angles: $\alpha_i=\angle e_ixe_{i+1}$, where $1\leq i\leq 5$ and the indices are read module 5. We see that in the plane R^2, there exists a circle neighbor $N(x,\varepsilon)=\{s\in R^2:||s-x||<\varepsilon\}$, where ε is a sufficiently small positive number, such that for any other edge e of $K_{1,5,n}$ not incident with x, e can not be located in $N(x,\varepsilon)$. Since the graph $K_{1,5,n}$ has still n edges f_j that are incident to x $(1\leq j\leq n)$, let A_i denote the set of all those edges f_j, each of which lies in the angle α_i (see the Fig. 1 in the next page). Clearly, we have that $|A_1|+|A_2|+|A_3|+|A_4|=n$.

In the following, associated with the drawing ϕ of G, we shall produce five new graphs G_i, together with their respective good drawing ϕ_i $(1\leq i\leq 5)$, where each G_i is isomorphic to the complete bipartite graph $K_{5,n+1}$. We shall heavily illustrate how to obtain the graph G_1 and its drawing ϕ_1, for the rest cases the method is analogous.

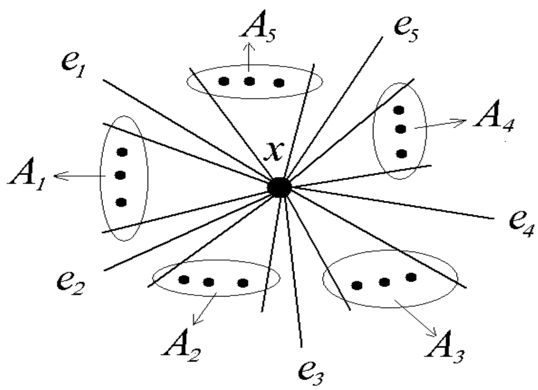

Fig. 1

First, we delete all edges in E_{y_1} and the vertex y_1 from G, and then remove the part of e_i lying in $N(x, \varepsilon)$ for $2 \leq i \leq 5$ (not remove the vertex x); add a new vertex z_{n+1} in some location of $e_4 \cap N(x, \varepsilon)$. Now we connected z_{n+1} to x and y_i $(i = 2, 3, 4, 5)$ by the following way: connect z_{n+1} to x and y_4 respectively along the original two sections of e_4; connect z_{n+1} to y_3 by first traversing through α_3 (near to x) and then along the original section of e_3 lying out $N(x, \varepsilon)$; connect z_{n+1} to y_2 by successively traversing through α_3 and α_2 (near to x) and then along the original section of e_2 lying out $N(x, \varepsilon)$; connect z_{n+1} to y_5 by first traversing through α_4 (near to x) and then along the original section of e_5 lying out $N(x, \varepsilon)$. Then we obtain the graph G_1 with its a good drawing ϕ_1. Obviously, G_1 is isomorphic to $K_{5,n+1}$. The following figure 2 helps us to understand the obtained graph G_1 and its drawing ϕ_1, where the dotted line denote the way how z_{n+1} is connected to x and y_i $(2 \leq i \leq 5)$.

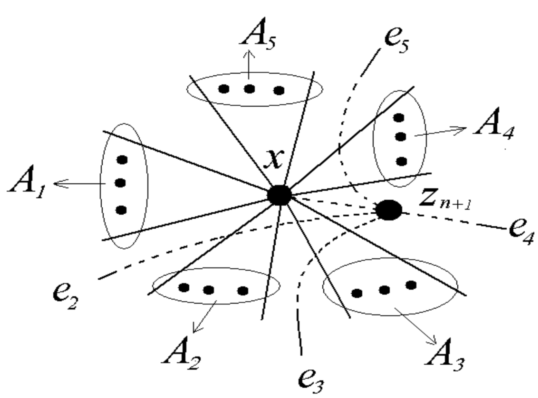

Fig. 2

Then it is not difficult to see that

$$cr_{\phi_1}(G_1) = cr_{\phi}\Big(E \setminus E_{y_1}\Big) + |A_2| + 2|A_3| + |A_4|. \tag{4}$$

By the symmetry of y_i, we can analogously easily obtain the graphs G_i and its goods drawings ϕ_i for $2 \leq i \leq 5$. For example, the graph G_2, together with its good drawing ϕ_2, is displayed in the following figure 3.

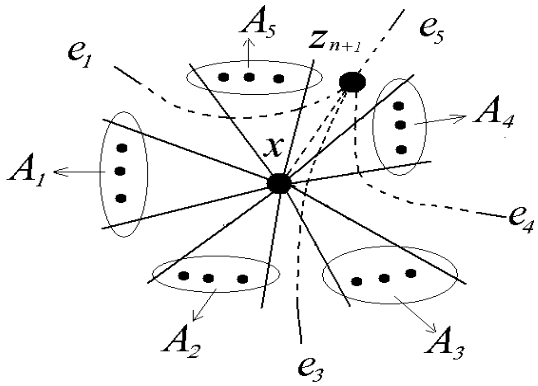

Fig. 3

Similarly, for ϕ_2, ϕ_3, ϕ_4 and ϕ_5, we have respectively the following equalities :

$$cr_{\phi_2}(G_2) = cr_\phi(E \setminus E_{y_2}) + |A_3| + 2|A_4| + |A_5| \tag{5}$$

$$cr_{\phi_3}(G_3) = cr_\phi(E \setminus E_{y_3}) + |A_1| + 2|A_5| + |A_4| \tag{6}$$

$$cr_{\phi_4}(G_4) = cr_\phi(E \setminus E_{y_4}) + |A_2| + 2|A_1| + |A_5| \tag{7}$$

$$cr_{\phi_5}(G_5) = cr_\phi(E \setminus E_{y_5}) + |A_1| + 2|A_2| + |A_3| \tag{8}$$

Since each G_i $(1 \leq i \leq 5)$ is isomorphic to the complete graph $K_{5,n+1}$, we get that $cr_{\phi_i}(G_i) \geq Z(5, n+1)$. Therefore, by (4)–(8) above, we have

$$\begin{aligned}
5Z(5,n+1) &\leq \sum_{i=1}^{5} cr_{\phi_i}(G_i) \\
&= \sum_{i=1}^{5} cr_\phi(E \setminus E_{y_i}) + 4\sum_{i=1}^{5} |A_i| \\
&= \sum_{i=1}^{5} cr_\phi(E \setminus E_{y_i}) + 4n \\
&= 3cr_\phi(E) + \sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) + 2cr_\phi(E_{XY}) + 4n \quad \text{(by Lemma ?? (2))} \\
&= 3cr_\phi(E) + \sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) + 4n \quad \text{(because } cr_\phi(E_{XY}) = 0\text{)}
\end{aligned}$$

So it follows that

$$\begin{aligned}\sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) &\geq 5Z(5,n+1) - 3cr_\phi(E) - 4n \\ &= 5Z(5,n+1) - 3\Big(Z(6,n) + 4\left[\frac{n}{2}\right] - a\Big) - 4n \\ &= \begin{cases} 2k^2 - 2k + 3a, & \text{when } n = 2k; \\ \\ 2k^2 - 4 + 3a, & \text{when } n = 2k+1 \end{cases}\end{aligned}$$

This proves the lemma. □

Lemma 2.4 *Let G be the complete tripartite graph $K_{1,5,n}$ with the edge set E and the vertex tripartition (X, Y, Z), where $X = \{x\}$, $Y = \{y_1, \cdots, y_5\}$, and $Z = \{z_1, \cdots, z_n\}$. Assume that $n = 2k+1$, where $k \geq 0$. If ϕ is a good drawing of G satisfying that $cr_\phi(E \setminus E_{z_j}) = Z(6, n-1) + 4\left[\frac{n-1}{2}\right]$ for any $1 \leq j \leq n$, then $cr_\phi(E) \neq Z(6,n) + 4\left[\frac{n}{2}\right] - 1$.*

Proof Assume to contrary that $cr_\phi(E) = Z(6,n) + 4\left[\frac{n}{2}\right] - 1$. By using the formula (2) in the proof of lemma 2.2, we have

$$Z(6,n) + 4\left[\frac{n}{2}\right] - 1 = cr_\phi(E) = cr_\phi(E_{XZ}) + cr_\phi(E_{XY} \cup E_{YZ}) + \sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}).$$

Since $\langle E_{XY} \cup E_{YZ}\rangle$ is isomorphic to the complete bipartite graph $K_{5,n+1}$, we have that $cr_\phi(E_{XY} \cup E_{YZ}) \geq Z(5, n+1)$. Noting that $cr_\phi(E_{XZ}) = 0$, we thus have

$$\sum_{i=1}^{5} cr_\phi(E_{XY}, E_{y_i}) \leq Z(6,n) + 4\left[\frac{n}{2}\right] - 1 - Z(5,n+1) = 2k^2 - 1$$

On the other hand, by our assumption that $cr_\phi(E) = Z(6,n) + 4\left[\frac{n}{2}\right] - 1$, and that $n = 2k+1$, with the help of Lemma 2.3(ii) we have $\sum\limits_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) \geq 2k^2 - 1$. This implies that

$$\sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) = 2k^2 - 1 \tag{9}$$

Since $\langle E \setminus E_{z_j}\rangle$ is isomorphic to the complete tripartite graph $K_{1,m,n-1}$ with the vertex tripartition $(X, Y, Z \setminus \{z_j\})$, applying the formula (2) in the proof of Lemma 2.2 to the graph $\langle E \setminus E_{z_j}\rangle$, we have

$$cr_\phi(E \setminus E_{z_j}) = cr_\phi\Big(E_{X(Z\setminus\{z_j\})}\Big) + cr_\phi\Big(E_{XY} \cup E_{Y(Z\setminus\{z_j\})}\Big) + \sum_{i=1}^{5} cr_\phi\Big(E_{X(Z\setminus\{z_j\})}, E'_{y_i}\Big),$$

where $E'_{y_i} = E_{X\{y_i\}} \cup E_{(Z\setminus\{z_j\})\{y_i\}}$.

Since $\langle E_{XY} \cup E_{Y(Z\setminus\{z_j\})}\rangle$ is isomorphic to the complete bipartite graph $K_{5,n}$, $cr_\phi\Big(E_{XY} \cup E_{Y(Z\setminus\{z_j\})}\Big) \geq Z(5,n)$. Again, since $E_{X(Z\setminus\{z_j\})}$ is the set of edges incident to x, we have that

$cr_\phi\Big(E_{X(Z\setminus\{z_j\})}\Big) = 0$ by the good drawing ϕ. Therefore we have

$$\begin{aligned}\sum_{i=1}^{5} cr_\phi\Big(E_{X(Z\setminus\{z_j\})}, E'_{y_i}\Big) &= cr_\phi(E\setminus E_{z_j}) - cr_\phi\Big(E_{X(Z\setminus\{z_j\})}\Big) - cr_\phi\Big(E_{XY}\cup E_{Y(Z\setminus\{z_j\})}\Big) \\ &= cr_\phi(E\setminus E_{z_j}) - cr_\phi\Big(E_{XY}\cup E_{Y(Z\setminus\{z_j\})}\Big) \\ &\le Z(6,n-1) + 4\left[\frac{n-1}{2}\right] - Z(5,n) \\ &= 2k^2 - 2k\end{aligned}$$

That is to say, we have

$$\sum_{i=1}^{5} cr_\phi\Big(E_{X(Z\setminus\{z_j\})}, E'_{y_i}\Big) \le 2k^2 - 2k \tag{10}$$

Because $E_{X\{z_j\}} \cup E_{\{z_j\}\{y_i\}}$ is the set of edges incident to z_j, $cr_\phi(E_{X\{z_j\}}, E_{\{z_j\}\{y_i\}}) = 0$ by the good drawing ϕ. Note that $E'_{y_i} = E_{y_i}\setminus E_{\{z_j\}\{y_i\}}$. Hence, we have

$$\begin{aligned}cr_\phi(E_{XZ}, E_{y_i}) &= cr_\phi(E_{XZ}, E'_{y_i}) + cr_\phi(E_{X(Z\setminus\{z_j\})}, E_{\{z_j\}\{y_i\}}) \\ &= \Big(cr_\phi(E_{X(Z\setminus\{z_j\})}, E'_{y_i}) + cr_\phi(E_{X\{z_j\}}, E'_{y_i})\Big) + cr_\phi(E_{X(Z\setminus\{z_j\})}, E_{\{z_j\}\{y_i\}}) \\ &= cr_\phi(E_{X(Z\setminus\{z_j\})}, E'_{y_i}) + cr_\phi(E_{X\{z_j\}}, E_{y_i}\setminus E_{\{z_j\}\{y_i\}}) \\ &\quad + cr_\phi(E_{X(Z\setminus\{z_j\})}, E_{\{z_j\}\{y_i\}}) \\ &= cr_\phi(E_{X(Z\setminus\{z_j\})}, E'_{y_i}) + cr_\phi(E_{X\{z_j\}}, E_{y_i}) - cr_\phi(E_{X\{z_j\}}, E_{\{z_j\}\{y_i\}}) \\ &\quad + cr_\phi(E_{XZ}, E_{\{z_j\}\{y_i\}}) - cr_\phi(E_{X\{z_j\}}, E_{\{z_j\}\{y_i\}}) \\ &= cr_\phi(E_{X(Z\setminus\{z_j\})}, E'_{y_i}) + cr_\phi(E_{X\{z_j\}}), E_{y_i}) + cr_\phi(E_{XZ}, E_{\{z_j\}\{y_i\}})\end{aligned}$$

Taking sum for i on two sides of the last equality above, we have

$$\begin{aligned}\sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) &= \sum_{i=1}^{5} cr_\phi(E_{X(Z\setminus\{z_j\})}, E'_{y_i}) + \sum_{i=1}^{5} cr_\phi(E_{X\{z_j\}}, E_{y_i}) \\ &\quad + \sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{\{z_j\}\{y_i\}})\end{aligned}$$

Combining with (9) and (10) above, we then obtain that

$$2k^2 - 1 \le 2k^2 - 2k + \sum_{i=1}^{5} cr_\phi(E_{X\{z_j\}}, E_{y_i}) + \sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{\{z_j\}\{y_i\}}) \tag{11}$$

Again, taking sum for j on the two sides of the inequality (11) above, and noticing $n = 2k+1$, we get that

$$\begin{aligned}
\sum_{j=1}^{n}(2k^2-1) &\leq \sum_{j=1}^{n}(2k^2-2k)+\sum_{j=1}^{n}\sum_{i=1}^{5} cr_\phi(E_{X\{z_j\}},E_{y_i})+\sum_{j=1}^{n}\sum_{i=1}^{5} cr_\phi(E_{XZ},E_{\{z_j\}\{y_i\}})\\
&= (2k+1)(2k^2-2k)+\sum_{i=1}^{5}\Big(\sum_{j=1}^{n} cr_\phi(E_{X\{z_j\}},E_{y_i})\Big)\\
&\quad +\sum_{i=1}^{5}\Big(\sum_{j=1}^{n} cr_\phi(E_{XZ},E_{\{z_j\}\{y_i\}})\Big)\\
&= (2k+1)(2k^2-2k)+\sum_{i=1}^{5} cr_\phi(E_{XZ},E_{y_i})+\sum_{i=1}^{5} cr_\phi(E_{XZ},E_{Z\{y_i\}})\\
&= (2k+1)(2k^2-2k)+\sum_{i=1}^{5} cr_\phi(E_{XZ},E_{y_i})\\
&\quad +\sum_{i=1}^{5}\Big(cr_\phi(E_{XZ},E_{y_i})-cr_\phi(E_{XZ},E_{X\{y_i\}}\Big) \qquad (\text{because } E_{Z\{y_i\}}=E_{y_i}\setminus E_{X\{y_i\}})\\
&= (2k+1)(2k^2-2k)+2\sum_{i=1}^{5} cr_\phi(E_{XZ},E_{y_i})
\end{aligned}$$

(This is because $E_{XZ}\cup E_{X\{y_i\}}$ is the set of edges incident to x, by the good drawing ϕ , $cr_\phi(E_{XZ},E_{Z\{y_i\}})=0$ for any $1\leq i\leq 5$)

$$= (2k+1)(2k^2-2k)+2(2k^2-1) \qquad (\text{by (9) above})$$

Therefore, it follows that $(2k+1)(2k-1)\leq 2(2k^2-1)$. This is a contradiction for any real number k, and proving the conclusion. □

§3. Proof of Theorem 1

Let the complete tripartite graph $K_{1,5,n}$ having the edge set E and the vertex tripartition (X,Y,Z), where $X=\{x\}$, $Y=\{y_1,\cdots,y_5\}$, and $Z=\{z_1,\cdots,z_n\}$. To show that $cr(K_{1,5,n})\leq Z(6,n)+4\left[\frac{n}{2}\right]$, we consider a drawing of $K_{1,5,n}$ as a immersion into R^2, satisfying the following:

(1) $\phi(x)=(0,1)$;

(2) $\phi(y_i)=(0,(-1)^i i), i=1,2, \phi(y_3)=(\varepsilon,-2), \phi(y_4)=(\varepsilon,3), \phi(y_5)=(2\varepsilon,4)$,where ε is a sufficiently small positive;

(3) $\phi(z_j)=\Big((-1)^j\left[\frac{j+1}{2}\right],0\Big)$.

For example, a drawing of $K_{1,5,5}$ on the plane is shown in the Fig.4. It is not difficult to see that $cr_\phi(E)=Z(6,n)+4\left[\frac{n}{2}\right]$. This thus shows that $cr(K_{1,5,n})\leq Z(6,n)+4\left[\frac{n}{2}\right]$. In order to prove the theorem, we only need to prove the conclusion that $cr_\phi(K_{1,5,n})\geq Z(6,n)+4\left[\frac{n}{2}\right]$ for any good drawing ϕ. Assume to contrary that there is a good drawing ϕ of $K_{1,5,n}$ satisfying $cr_\phi(K_{1,5,n}) = Z(6,n)+4\left[\frac{n}{2}\right]-a$, where $a\geq 1$. We now consider the following two cases, according to as n is even or odd.

Claim 1 *The desired conclusion is true when $n\,(=2k)$ is even.*

Subproof By our assumption that $cr_\phi(K_{1,5,n}) = Z(6,n) + 4\left[\frac{n}{2}\right] - a$, it then follows from Lemma 2.3(i) that

$$\sum_{i=1}^{5} cr_\phi(E_{xz}, E_{y_i}) \geq 2k^2 - 2k + 3a.$$

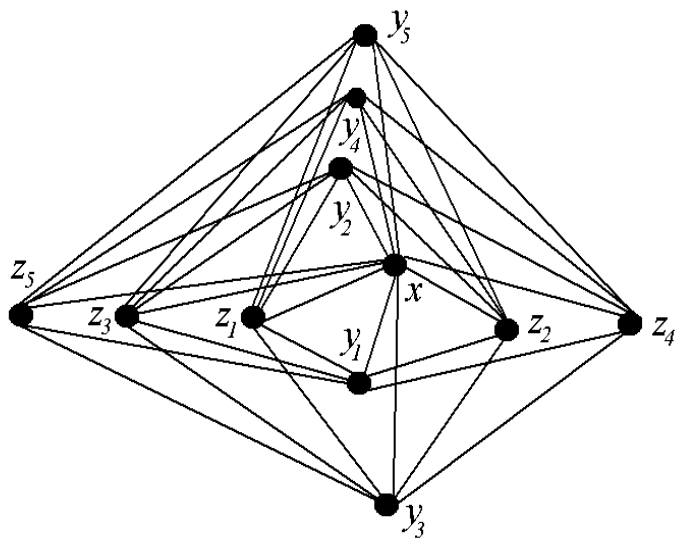

Fig. 4

Note that $cr_\phi(E_{XZ}) = 0$ by the good drawing ϕ. Since $\langle E_{XZ} \cup E_{YZ}\rangle$ is isomorphic to the complete bipartite graph $K_{5,n+1}$ with the vertex bipartition $(Y, X \cup Z)$, we have that $cr_\phi(E_{XY} \cup E_{YZ}) \geq Z(5, n+1)$. Using the formulas (2) in the proof of Lemma 2.2, we get that

$$\begin{aligned} Z(6,n) + 4\left[\frac{n}{2}\right] - a &= cr_\phi(E) \\ &= cr_\phi(E_{XZ}) + cr_\phi(E_{XY} \cup E_{YZ}) + \sum_{i=1}^{5} cr_\phi(E_{XZ}, E_{y_i}) \\ &\geq Z(5, n+1) + \sum_{i=1}^{5}(E_{XZ}, E_{y_i}) \end{aligned}$$

Therefore, $\sum\limits_{i=1}^{5} cr_\phi(E_{XZ}, E(y_i)) \leq 2k^2 - 2k - a$. So, we get that $2k^2 - 2k + 3a \leq 2k^2 - 2k - a$, namely, $a \leq 0$. This contradicts to the hypothesis that $a \geq 1$, proving the claim.

Claim 2 *The desired conclusion is true when* $n\,(= 2k+1)$ *is odd.*

Subproof. Since n is odd, by Lemma 2.3(ii) we first have

$$\sum_{i=1}^{5} cr_\phi(E_{XZ}, E(y_i)) \geq 2k^2 - 4 + 3a$$

Similarly, using the formulas (2) in the proof of Lemma 2.2, we get that

$$\begin{aligned} Z(6,n)+4\left[\frac{n}{2}\right]-a &= cr_\phi(E) \\ &\geq Z(5,n+1)+\sum_{i=1}^{5} cr_\phi(E_{XZ},E_{y_i}), \end{aligned}$$

which follows that $\sum\limits_{i=1}^{5}(E_{XZ},E_{y_i}) \leq 2k^2-a$. Hence, we get that $2k^2-4+3a \leq 2k^2-a$, namely $a \leq 1$. Since $a \geq 1$ by our assumption, this implies that $a=1$, and thus it must be that

$$cr_\phi(E) = Z(6,n)+4\left[\frac{n}{2}\right]-1 \tag{12}$$

Again, with the help of the formula (1), we have

$$\begin{aligned} cr_\phi(E) &= cr_\phi(E_{XY}\cup E_{XZ}\cup E_{YZ}) \\ &= cr_\phi(E_{XY})+cr_\phi(E_{XZ}\cup E_{YZ})+cr_\phi(E_{XY},E_{XZ}\cup E_{YZ}) \\ &= cr_\phi(E_{XY})+cr_\phi(E_{XZ}\cup E_{YZ})+\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_i}) \end{aligned}$$

Since $\langle E_{XZ}\cup E_{YZ}\rangle$ is isomorphic to the complete bipartite graph $K_{6,n}$ with the vertex bipartition $(X\cup Y,Z)$, it has that $cr_\phi(E_{XZ}\cup E_{YZ}) \geq Z(6,n)$. Noting that $cr_\phi(E_{XY})=0$ by the good drawing of ϕ, we thus have

$$Z(6,n)+4\left[\frac{n}{2}\right]-1 = cr_\phi(E) \geq Z(6,n)+\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_i}),$$

which follows that

$$\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_i}) \leq Z(6,n)+4\left[\frac{n}{2}\right]-1-Z(6,n) = 4k-1 \tag{13}$$

Combining with Lemma 2.2(i), we have

$$\begin{aligned} \sum_{i=1}^{n} cr_\phi(E\setminus E_{z_i}) &= 2cr_\phi(E_{XY})+\sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_j})+(n-2)cr_\phi(E) \\ &= \sum_{i=1}^{n} cr_\phi(E_{XY},E_{z_j})+(n-2)cr_\phi(E) \quad (\text{because } cr_\phi(E_{XY})=0) \\ &\leq 4k-1+(n-2)\Big(Z(6,n)+4\left[\frac{n}{2}\right]-1\Big) \quad (\text{by (12) and (13) above}) \\ &= n\Big(Z(6,n-1)+4\left[\frac{n-1}{2}\right]\Big) \qquad (\text{because } n=2k+1\ ) \end{aligned}$$

That is to say, we have

$$\sum_{i=1}^{n} cr_\phi(E\setminus E_{z_i}) \leq n\Big(Z(6,n-1)+4\left[\frac{n-1}{2}\right]\Big) \tag{14}$$

On the other hand, since, for any $1 \le i \le n$, $\langle E \setminus E(z_i)\rangle$ is isomorphic to the complete tripartite graph $K_{1,5,n-1}$, and since $n-1$ is even, it follows from the truth of Claim 1 that $cr_\phi(E \setminus E_{z_i}) \ge Z(6, n-1) + 4\left[\frac{n-1}{2}\right]$ for any $1 \le i \le n$. Combined with (14) above, it must happen that $cr_\phi(E \setminus E_{z_i}) = Z(6, n-1) + 4\left[\frac{n-1}{2}\right]$ for any $1 \le i \le n$. This, together with n being odd and (12) above, contradicts Lemma 2.4, proving this claim.

Therefore, the proof of Theorem 1 is finished. □

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