International J.Math. Combin. Vol.1 (2007), No.1, 110-116

The Crossing Number of the Join of C_m and P_n

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Abstract: In this paper, the crossing numbers of $P_m \vee P_n$, $C_m \vee P_n$ and $C_m \vee C_n$ are determined for arbitrary integers $m, n \geq 1$, which are related with parallel bundles in planar map geometries, i.e., Smarandache spherical geometries.

Keywords: Crossing number, join of graphs, path, cycle.

AMS(2000): O5C25, O5C62.

§1. Introduction

Let G be a simple and undirected graph with vertex set V and edge set E. The *crossing number* $cr(G)$ of the graph G is the minimum number of pairwise intersections of edges in all drawings of G in a plane, which are related with parallel bundles in *planar map geometries*, i.e., *Smarandache spherical geometries* (see [6]-[7] for details). It is well known that the crossing number of a graph is attained only in *good* drawings , means that no edge crosses itself, no two edges cross more than once, no two edges incident with the same vertex cross, no more than two edges cross at a point of the plane, and no edge meets a vertex which is not one of its endpoints. It is easy to see that a drawing with the minimum number of crossings (an *optimal drawing*) is always a good drawing. Let D be a good drawing of the graph G, we denote the number of crossings in D by $cr_D(G)$. Let A and B be disjoint edge subsets of G. We denote by $cr_D(A, B)$ the number of crossings between edges of A and B, and by $cr_D(A)$ the number of crossings whose two crossed edges are both in A. Let H be a subgraph of G, the restricted drawing $D|_H$ is said to be a *subdrawing* of H. As for more on the theory of crossing number, we refer readers to [1] and [2]. In this paper, we also use the term *region* in non-planar drawings. In this case, crossings are considered to be vertices of the map.

Let G_1 and G_2 be two disjoint graphs. The *union* of G_1 and G_2, denoted by $G_1 + G_2$, has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, and the *join* of G_1 and G_2 is obtained by adjoining every vertex of G_1 to every vertex of G_2 in $G_1 + G_2$ which is denoted by $G_1 \vee G_2$ (see [3]).

Let $K_{m,n}$ denote the complete bipartite graph on sets of m and n vertices, P_n the path of length n and C_m the cycle with m vertices.

From these definitions, following results are well-known.

[1]Received August 6, 2007. Accepted September 10, 2007

[2]Supported by the key project of the Education Department of Hunan Province of China (05A037)

Proposition 1.1 *Let G_1 be a graph homeomorphic to G_2. Then $cr(G_1) = cr(G_2)$.*

Proposition 1.2 *If G_1 is a subgraph of G_2, then $cr(G_1) \leq cr(G_2)$.*

Proposition 1.3 *Let D be a good drawing of a graph G. If A, B and C are three mutually disjoint edge subsets of G, then we have*

(1) $cr_D(A \cup B) = cr_D(A) + cr_D(A, B) + cr_D(B)$;

(2) $cr_D(A \cup B, C) = cr_D(A, C) + cr_D(B, C)$.

Proposition 1.4([4]) *If G has n vertices and m edges with $n \geq 3$, then $cr(G) \geq m - 3n + 6$.*

Computing the crossing number of graphs is a classical problem , and yet it is also an elusive one. In fact, Garey and Johnson in [5] have proved that to determine the crossing number of graphs is NP-complete in general. At present, the classes of graphs whose crossing numbers have been determined are very scarce.

On the crossing number of the complete bipartite graphs $K_{m,n}$, Zarankiewicz gave a drawing of $K_{m,n}$ in [8] which demonstrates that

$$cr(K_{m,n}) \leq Z(m,n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor.$$

and conjectured $cr(K_{m,n}) = Z(m,n)$, Which is called the *Zarankiewicz conjecture*. More precisely, Kleitman proved in [9] that if $m \leq 6$ and $m \leq n$, $cr(K_{m,n}) = Z(m,n)$.

As we known, results for the join of graphs are fewer, particularly, Bogdan Oporowski proved $cr(C_3 \vee C_5) = 6$ in [4]. Based on this, we begin to consider the crossing numbers of the join of P_m and P_n, C_m and P_n, C_m and C_n, and get the following theorems which consist of these main results in this paper.

Theorem A *If $m \geq 1, n \geq 1$ and $min\{m, n\} \leq 5$, then*

$$cr(P_m \vee P_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m+1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor.$$

If $m \geq 3, n \geq 1$ and $min\{m, n+1\} \leq 6$, then

$$cr(C_m \vee P_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor + 1$$

and if $m \geq 3, n \geq 3$, $min\{m, n\} \leq 6$, then

$$cr(C_m \vee C_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2.$$

Theorem B *If the Zarankiewicz conjecture is held for $m \geq 7$ and $m \leq n$, then if $m \geq 1, n \geq 1$, $min\{m, n\} \geq 6$,*

$$cr(P_m \vee P_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m+1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor;$$

if $m \geq 3, n \geq 1$, $min\{m, n+1\} \geq 7$,

$$cr(C_m \vee P_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor + 1$$

and if $m \geq 3, n \geq 3, \quad min\{m,n\} \geq 7$,

$$cr(C_m \vee C_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2.$$

§2. Some Lemmas

Lemma 2.1 (1) *There exists a good drawing* D_1 *of* $P_m \vee P_n$ *for given integers* $m \geq 1$ *and* $n \geq 1$ *such that*

$$cr_{D_1}(P_m \vee P_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m+1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor;$$

(2) *There exists a good drawing* D_2 *of* $C_m \vee P_n$ *for given integers* $m \geq 3$ *and* $n \geq 1$ *such that*

$$cr_{D_2}(C_m \vee P_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor + 1;$$

(3) *There exists a good drawing* D_3 *of* $C_m \vee C_n$ *for given integers* $m \geq 3$ *and* $n \geq 3$ *such that*

$$cr_{D_3}(C_m \vee C_n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2.$$

Proof By Fig.2.1-Fig.2.3, the conclusions are immediately held. □

Lemma 2.2 $\quad cr(C_3 \vee C_3) = 3$.

Proof From Lemma 2.1(3), $cr(C_3 \vee C_3) \leq 3$. We know $C_3 \vee C_3$ has 6 vertices and 15 edges, then $cr(C_3 \vee C_3) \geq 15 - 3 \times 6 + 6 = 3$. Therefore the conclusion is held. □

In the following Lemmas, let G be a connected graph with $V(G) = \{x_1, x_2, \ldots, x_n \ (n \geq 3)\}$ and C_m a cycle with $V(C_m) = \{y_1, y_2, \ldots, y_m\}$. Then we know that $V(C_m \vee G) = V(C_m) \cup V(G)$ and $E(C_m \vee G) = E(C_m) \cup E(G) \cup E^*$, here $E^* = \{x_i y_j | i = 1, 2, \ldots, n; \ j = 1, 2, \ldots, m\}$.

Lemma 2.3 *For any good drawing* D *of* $C_m \vee G$,

$$cr(C_m \vee G) \geq cr_D(E^*) \geq cr(K_{m,n}).$$

Proof Since the edge-induced subgraph of E^* is $K_{m,n}$, the conclusion is evident. □

Lemma 2.4 *Let* ϕ *be an optimal drawing of* $C_m \vee G$. *Then* $cr_\phi(E(C_m)) = 0$.

Proof We assume there exists an optimal drawing ϕ of $C_m \vee G$ such that $cr_\phi(E(C_m)) \neq 0$. Then $m \geq 4$ and there exist two crossed edges $e, f \in E(C_m)$. We assume that $e = y_i y_j$, $f = y_k y_l$, where i, j, k, l are distinct. For convenience, we denote the crossing between e and f by v. Since C_m is 2-connected, there exist two paths P_1 and P_2 connected y_i and y_k, y_j and y_l, respectively and $P_i (i = 1, 2)$ does not pass v. In the following, we shall produce a new good drawing ϕ' of $C_m \vee G$ (see Fig.2.2 below).

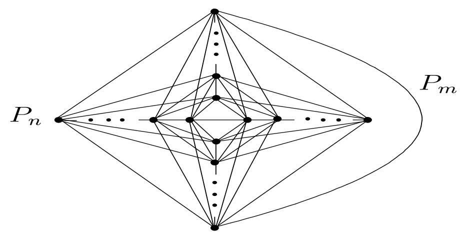

Fig.2.1

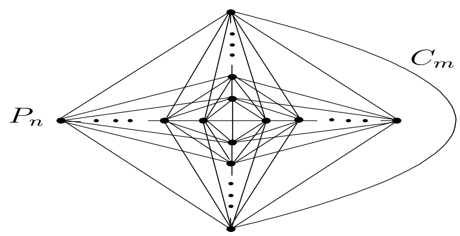

Fig.2.2

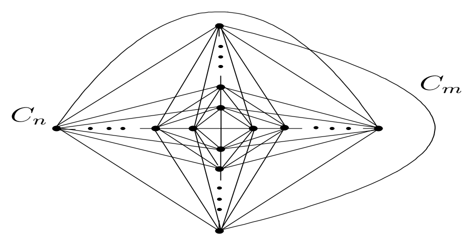

Fig.2.3

At first, we connect y_i to y_l sufficiently close to the section between y_i and v of e and the section between y_l and v of f, then we get a new edge $e' = y_i y_l$. Analogously, we can get another new edge $f' = y_j y_k$. Secondly, we delete two original edges e and f. In this way, we produce a new good drawing ϕ' of $C_m \vee G$ such that the crossing v in ϕ is deleted in ϕ', the other crossings in ϕ are not changed in ϕ' and there is no new crossing occurring in ϕ', then we get that $cr_{\phi'}(C_m \vee G) = cr_\phi(C_m \vee G) - 1$, contradicts to that ϕ is an optimal drawing. □

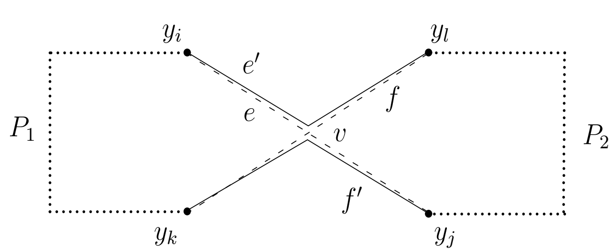

Fig.2.2

Lemma 2.5 *Let ϕ be a good drawing of $C_m \vee G$ such that $cr_\phi(E(C_m)) = 0$, $cr_\phi(E(C_m), E(G)) =$*

0 *and* $cr_\phi(E(C_m), E^*) \le 1$.

(1) *If* $cr_\phi(E(C_m), E^*) = 0$, *then* $cr_\phi(C_m \vee G) \ge \frac{1}{2}n(n-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$;

(2) *If* $cr_\phi(E(C_m), E^*) = 1$, *then* $cr_\phi(C_m \vee G) \ge \frac{1}{2}(n-1)(n-2)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 1$.

Proof Since $cr_\phi(E(C_m)) = 0$, the subdrawing $\phi|_{C_m}$ divides the plane into two regions. As $cr_\phi(E(C_m), E(G)) = 0$ and G is connected, any vertex x_i of G lies in the same region, say the finite region. For convenience, let $E^i = \{x_iy_j | j = 1, 2, \ldots, m\}$ for $i = 1, 2, \ldots, n$. Then $cr_\phi(E^i) = 0$. Since $E^* = \cup_{i=1}^n E^i$, we find that $cr_\phi(E^*) = \sum_{1\le i<k\le n} cr_\phi(E^i, E^k)$.

(i) Since $cr_\phi(E(C_m), E^*) = 0$, then for any $i = 1, 2, \ldots, n$, x_iy_j does not cross any edge in $E(C_m)$. For any integers i, k, $1 \le i < k \le n$, x_i must be connected to each y_j ($j = 1, 2, \ldots, m$), these m edges connecting x_i to all $y_j \in V(C_m)$ which divide the finite region into m subregions, we know that x_k lies in one of these subregions. Thus the m edges connecting x_k to y_j must cross the edges adjacent to x_i at least $\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ times (see Fig.2.3 below). Then $cr_\phi(E^i, E^k) \ge \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. So $cr_\phi(C_m \vee G) \ge cr_\phi(E^*) = \sum_{1\le i<k\le n} cr_\phi(E^i, E^k) \ge \frac{1}{2}n(n-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. Our conclusion (1) is held.

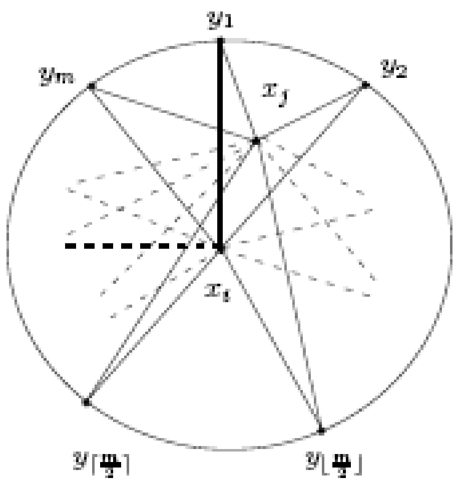

Fig.2.3

(ii) Since $cr_\phi(E(C_m), E^*) = 1$, there exists only one $k \in \{1, 2, \ldots, n\}$. Without loss of generality, we assume that $k = n$ such that for some $j \in \{1, 2, \ldots, m\}$, x_ny_j crosses exactly one edge in $E(C_m)$. For any integer $i = 1, 2, \ldots, n-1$, x_iy_j does not cross any edge in $E(C_m)$. Similar to (i), for $1 \le i < k \le n-1$, $cr_\phi(E^i, E^k) \ge \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. Then $cr_\phi(C_m \vee G) \ge cr_\phi(E^*) + 1 \ge \sum_{1\le i<k\le n-1} cr_\phi(E^i, E^k) + 1 \ge \frac{1}{2}(n-1)(n-2)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 1$. Our conclusion (2) is held too. □

§3. Proofs

Proof of Theorem A

(1) If $n = 1$, $P_m \vee P_1$ is a planar graph, the conclusion is held.

If $n \geq 2$, from Lemma 2.1(1) we know $cr(P_m \vee P_n) \leq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m+1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n+1}{2}\rfloor$. Since P_n is connected, combining with Lemma 2.3, $cr(P_m \vee P_n) \geq cr(K_{m+1,n+1})$. For $min\{m,n\} \leq 5$, $cr(K_{m+1,n+1}) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m+1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n+1}{2}\rfloor$. Then $cr(P_m \vee P_n) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m+1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n+1}{2}\rfloor$. So the conclusion is held.

(2) From Lemma 2.1(2), we know that $cr(C_m \vee P_n) \leq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n+1}{2}\rfloor + 1$.

If $n = 1$, $cr(C_m \vee P_1) \leq 1$, and $C_m \vee P_1$ has a subgraph which is homeomorphic to K_5, then the conclusion is held.

If $n \geq 2$, since P_n is connected, combining with Lemma 2.3 and $min\{m, n+1\} \leq 6$, $cr(C_m \vee P_n) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n+1}{2}\rfloor$. We assume there exists an optimal drawing ϕ such that $cr_\phi(C_m \vee P_n) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n+1}{2}\rfloor$. By Lemma 2.3 and $min\{m, n+1\} \leq 6$, $cr_\phi(E^*) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n+1}{2}\rfloor$. While

$$\begin{aligned} cr_\phi(C_m \vee P_n) &= cr_\phi(E(C_m)) + cr_\phi(E(P_n)) + cr_\phi(E^*) \\ &+ cr_\phi(E(C_m), E(P_n)) + cr_\phi(E(C_m), E^*) + cr_\phi(E(P_n), E^*), \end{aligned}$$

we get $cr_\phi(E(C_m)) = 0$, $cr_\phi(E(C_m), E(P_n)) = 0$ and $cr_\phi(E(C_m), E^*) = 0$, combining with Lemma 2.5(1), $cr_\phi(C_m \vee P_n) \geq \frac{1}{2}n(n+1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. It is easy to check that $\frac{1}{2}n(n+1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor > \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n+1}{2}\rfloor$ for integers $m \geq 3$ and $n \geq 2$, a contradiction. Thus the conclusion is held.

(3) By Lemma 2.2, we have determined the crossing number of $C_3 \vee C_3$. Without loss of generality, we can assume $n \geq 4$ in the following arguments.

From Lemma 2.1(3) we know that $cr(C_m \vee C_n) \leq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2$. Since C_n is connected, by Lemma 2.3 and $min\{m,n\} \leq 6$, $cr(C_m \vee C_n) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$. We assume there exists an optimal drawing φ such that

$$\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor \leq cr_\varphi(C_m \vee C_n) \leq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 1.$$

By Lemma 2.3 and $min\{m,n\} \leq 6$, $cr_\varphi(E^*) \geq \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$.
By Lemma 2.4, $cr_\varphi(E(C_m)) = 0$ and $cr_\varphi(E(C_n)) = 0$. As $cr_\varphi(C_m \vee C_n) = cr_\varphi(E(C_m)) + cr_\varphi(E(C_n)) + cr_\varphi(E^*) + cr_\varphi(E(C_m), E(C_n)) + cr_\varphi(E(C_m), E^*) + cr_\varphi(E(C_n), E^*)$, we get that

$$cr_\varphi(E(C_m), E(C_n)) \leq 1, \quad cr_\varphi(E(C_m), E^*) \leq 1.$$

If $cr_\varphi(E(C_m), E(C_n)) = 1$, since C_m and C_n are vertex-disjoint cycles, then they cross at least twice, also a contradiction. So $cr_\varphi(E(C_m), E(C_n)) = 0$.

If $cr_\varphi(E(C_m), E^*) = 0$, by Lemma 2.5(1), $cr_\phi(C_m \vee C_n) \geq \frac{1}{2}n(n-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$. It is easy to check that $\frac{1}{2}n(n-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor > \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 1$ for integers $m \geq 3$ and $n \geq 4$, a contradiction.

If $cr_\varphi(E(C_m), E^*) = 1$, by Lemma 2.5(2), $cr_\phi(C_m \vee C_n) \geq \frac{1}{2}(n-1)(n-2)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 1$, it is also easy to check that $\frac{1}{2}(n-1)(n-2)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 1 > \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 1$ for $m \geq 3$ and $n \geq 4$, a contradiction too. So the conclusion is held.

This completes the proof of Theorem A. □

Proof of Theorem B

If the Zarankiewicz conjecture is held for integers $m \geq 7$ and $m \leq n$, then the crossing number of $K_{m,n}$ is $Z(m,n)$ for $m \geq 7$ and $m \leq n$, so the proof of Theorem B is analogous to the proof of Theorem A. □

Notice that these drawings D_1, D_2 and D_3 in Fig.2.1 − 2.3 are optimal drawings of $P_m \vee P_n$ for integers $m \geq 1$ and $n \geq 1$, $C_m \vee P_n$ for integers $m \geq 3$ and $C_m \vee C_n$ for integers $m \geq 3$ and $n \geq 3$, respectively.

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