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The Signed Product Cordial for Corona Between Paths and Fourth Power of Cycles

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Abstract: A graph $G = (V, E)$ is called signed product cordial if it is possible to label the vertex by the function $f : V \to \{-1, 1\}$ and label the edges by $f^* : E \to \{-1, 1\}$, where $f^*(uv) = f(u).f(v)$, $u, v \in V$ so that $|v_{-1} - v_1| \leqslant 1$ and $|e_{-1} - e_1| \leqslant 1$. In this paper,some new results on signed product cordial labeling are proposed. The necessary and sufficient conditions of signed product cordial for corona between paths and fourth power of cycles are presented.

Key Words: Corona graph, fourth power, second power, signed product cordial graph, Smarandachely signed product cordial labeling.

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§1. Introduction

The graph theory is a mathematical subfield of discrete mathematics. One area of graph theory of considerable recent research is that of graph labeling. The concept of graph labeling was introduced during the sixties' of the last century by Rosa [14]. Labeling methods are used for a wide range of applications in different subjects including coding theory, computer science and communication networks. Yegnanarayanan [16] explores some of the interesting applications of graph labeling With advancement in technology the occurrence of more complex networking system is seen to have emerged. Many researches have been working with different types of labeling graphs [5,8,11]. In 1954 Harray introduced S-cordiality [12]. An excellent reference for this purpose is the survey written by Gallian [9]. The original concept of cordial graphs is due to Cahit[6]. Cordial Labeling finds its application in Automated Routing algorithms, Communications relevant Adhoc Networks and many others[10].

For more details about the graph labeling, the reader can refer to [2-4,13]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1([15]) *If the vertices of the graph are assigned values subject to certain conditions is known as graph labeling. Following three are the common features of any graph labeling*

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problem:

(1) *a set of numbers from which vertex labels are assigned;*
(2) *a rule that assigns a value to each edge;*
(3) *a condition that these values must satisfy.*

The present work is targeted to discuss one such labeling known as signed product cordial labeling defined as follows:

Definition 1.2 *Let $G = (V, E)$ be a graph and let $f : V \to \{-1, 1\}$ be a labeling of its vertices, and let the induced edge labeling $f^* : E \to \{-1, 1\}$ be given by $f^*(e) = (f(u).f(v))$, where $e = uv$ and $u, v \in V$. Let v_{-1} and v_1 be the numbers of vertices that are labeled by -1 and 1, respectively, and let e_{-1} and e_1 be the corresponding numbers of edges. Such a labeling is called signed product cordial if both $|v_{-1} - v_1| \leqslant 1$ and $|e_{-1} - e_1| \leqslant 1$ hold. On the other hand, if $|v_{-1} - v_1| \geq 2$ or $|e_{-1} - e_1| \geq 2$, such a labeling is called Smarandachely signed product cordial labeling.*

The concept of signed product cordial labeling was introduced by Baskar Babujee [1].

Definition 1.3 *The corona $G_1 \bigodot G_2$ of two graphs G_1 (with n_1 vertices, m_1 edges) and G_2 (with n_2 vertices, m_2 edges) is defined as the graph obtained by taking one copy of G_1 and copies of G_2, and then joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.*

Definition 1.4 *The fourth power of a paths P_n, denoted by P_n^4, is $P_n \bigcup J$, where J is the set of all edges of the form edges v_iv_j such that $2 \leq d(v_iv_j) \leq 4$ and $i < j$ where $d(v_iv_j)$ is the shortest distance from v_i to v_j.*

§2. Terminologies and Notations

A path with m vertices and $m - 1$ edges is denoted by P_m. Also, a cycle with n vertices and n edges, denoted by C_n, and its fourth power C_n^4 has n vertices and $4n - 9$ edges. We let L_{4r} denote the labeling $(-1)_2 11$ $(-1)_2 11 \cdots (-1)_2 11$ (repeated r-times), Let L'_{4r} denote the labeling $(-1)11(-1)$ $(-1)11(-1) \cdots (-1)11(-1)$ (repeated r-times). The labeling $11(-1)_2$ $11(-1)_2 \cdots 11(-1)_2$ (repeated r-times) and labeling $1(-1)_2 1$ $1(-1)_2 1 \cdots 1(-1)_2 1$ (repeated r-times) are written S_{4r} and S'_{4r}. Let M_r denote the labeling $(-1)1$ $(-1)1 \cdots (-1)1$, zero-one repeated $r$times if r is even and $(-1)1$ $(-1)1 \cdots (-1)1(-1)$ if r is odd; for example, $M_6 = (-1)1(-1)1(-1)1$ and $M_5 = (-1)1(-1)1(-1)$. We let M'_r denote the labeling $1(-1)1(-1) \cdots 1(-1)$.

Sometimes, we modify the labeling M_r or M'_r by adding symbols at one end or the other (or both). Also, L_{4r} (or L'_{4r}) with extra labeling from right or left (or both sides). If L is a labeling for a path p_m and M is a labeling for fourth power of path C_n, then we use the notation $[L; M]$ to represent the labeling of the corona $P_m \bigodot C_n^4$. Additional notation that we use is the following: for a given labeling of the corona $P_m \bigodot C_n^4$, we let v_i and e_i (for $i = -1, 1$) be the numbers of vertices and edges, respectively, that are labeled by i of the corona $P_m \bigodot C_n^4$, and let x_i and a_i be the corresponding quantities for p_m, and we let y_i and b_i be those for C_n^4, which are connected with vertices labeled (-1) of P_m.

Similarly, let y'_i and b'_i for C_n^4 which are connected with vertices labeled 1 of P_m. It is easy to verify that $v_{-1} = x_{-1} + x_{-1}y_{-1} + x_1y'_{-1}$, $v_1 = x_1 + x_{-1}y_1 + x_1y'_1$, $e_{-1} = a_{-1} + x_{-1}b_{-1} + x_1b'_{-1} + x_{-1}y_{-1} + x_1y'_1$ and $e_1 = a_1 + x_{-1}b_1 + x_1b'_1 + x_{-1}(x_{-1}y_1) + x_1y'_{-1}$. Thus, $v_{-1} - v_1 = (x_{-1} - x_1) + x_{-1}(y_{-1} - y_1) + x_1(y'_{-1} - y'_1)$ and $e_{-1} - e_1 = (a_{-1} - a_1) + x_{-1}(b_{-1} - b_1) + x_1(b'_{-1} - b'_1) + x_{-1}(y_{-1} - y_1) - x_1(y'_{-1} - y'_1)$. When it comes to the proof, we only need to show that, for each specified combination of labeling, $|v_{-1} - v_1| \le 1$ and $|e_{-1} - e_1| \le 1$.

§3. Main Results

In this section, we study The signed product cordial for corona between paths and fourth power of cycles.

Lemma 3.1 *The corona* $P_m \odot c_3^4$ *is signed product cordial if and only if* $m \neq 1$.

Proof Since $C_3^4 \equiv P_3^4$, $P_m \odot c_3^4$ is signed product cordial [9]. □

Lemma 3.2 *If* $n \equiv 0 (mod\ 4)$, *then* $P_m \odot c_n^4$ *is signed product cordial for all* $m \ge 1$.

Proof Suppose that $n = 4s$, where $s \ge 2$. The following cases will be examined.

Case 1. Suppose that $m = 1$. Then we label the vertices of $P_1 \odot c_{4s}^4$ by $[(-1); S'_{4s}]$. Therefore $x_{-1} = 1, x_1 = 0, a_{-1} = a_1 = 0, y_{-1} = y_1 = 2s, b_{-1} = 8s - 4, b_1 = 8s - 5$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 1$. As an example, Figure 1 illustrates $P_1 \odot c_8^4$. Hence, $P_1 \odot c_{4s}^4$ is signed product cordial.

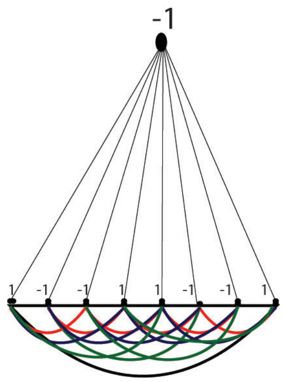

Figure 1 $P_1 \odot c_8^4$

Case 2. Suppose that $m = 2$. Then we label the vertices of $P_2 \odot c_{4s}^4$ by $[(-1)1; S'_{4s}, (-1)_3 1_3 M_{4S-6}]$. Therefore $x_{-1} = x_1 = 1, a_{-1} = 1, a_1 = 0, y_{-1} = y_1 = 2s, b_{-1} = 8s-4, b_1 = 8s-5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = 8s - 5, b'_1 = 8s - 4$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. As an example, Figure (2) illustrates $P_2 \odot c_8^4$. Hence, $P_2 \odot c_{4s}^4$ is signed product cordial.

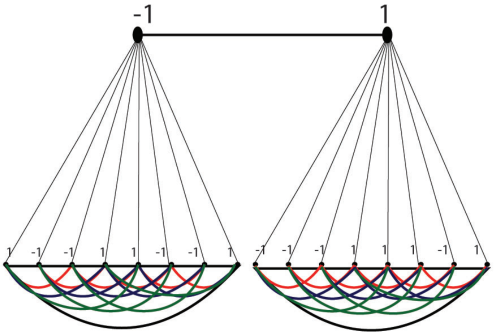

Figure 2 $P_2 \bigodot c_8^4$

Case 3. Suppose that $m = 3$. Then we label the vertices of $P_3 \bigodot c_{4s}^4$ by $[(-1)(-1)1; S'_{4S}, S'_{4S}, (-1)_3 1_3 M_{4S-6}]$. Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = a_1 = 1, y_{-1} = y_1 = 2s, b_{-1} = 8s-4, b_1 = 8s-5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = 8s-5, b'_1 = 8s-4$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 1$. As an example, Figure (3) illustrates $P_3 \bigodot c_8^4$. Hence, $P_3 \bigodot c_{4s}^4$ is signed product cordial.

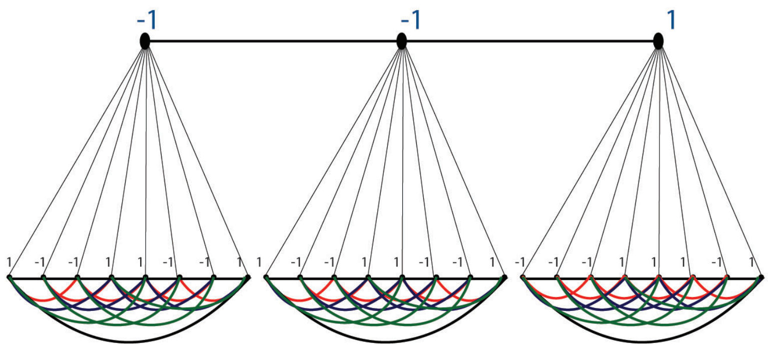

Figure 3 $P_3 \bigodot c_8^4$

Case 4. $m = 0(mod\ 4)$. Suppose that $m = 4r$, where $r \geq 1$. Then we label the vertices of $P_{4r} \bigodot c_{4s}^4$ by $[L_{4r}; S'_{4s}, S'_{4s}, 1_3 M_{4s-6}(-1)_3, 1_3 M_{4s-6}(-1)_3, \cdots, (r - time)]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 2r-1, a_1 = 2r, y_{-1} = y_1 = 2s, b_{-1} = 8s-4, b_1 = 8s-5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = 8s-5, b'_1 = 8s-4$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. As an example, Figure (4) illustrates $P_4 \bigodot c_8^4$. Hence, $P_{4r} \bigodot c_{4s}^4$ is signed product cordial.

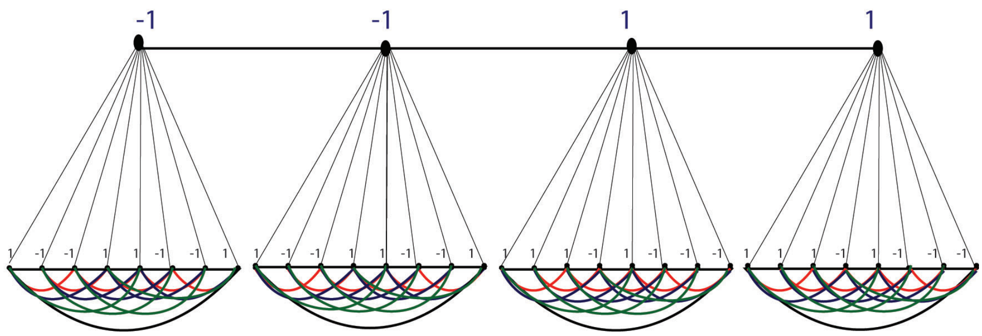

Figure 4 $P_4 \bigodot c_8^4$

Case 5. Suppose that $m = 4r + 1$, where $r \geq 1$. Then we label the vertices of $P_{4r+1} \bigodot c_{4s}^4$ by $[L_{4r}(-1); S'_{4s}, S'_{4s}, 1_3M_{4s-6}(-1)_3, 1_3M_{4s-6}(-1)_3, \cdots, (r - time)s'_{4s}$. Therefore $x_{-1} = 2r + 1, x_1 = 2r, a_{-1} = a_1 = 2r, y_{-1} = y_1 = 2s, b_{-1} = 8s - 4, b_1 = 8s - 5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = 8s - 5, b'_1 = 8s - 4$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 1$. Hence, $P_{4r+1} \bigodot c_{4s}^4$ is signed product cordial.

Case 6. Suppose that $m = 4r + 2$, where $r \geq 1$. Then we label the vertices of $P_{4r+2} \bigodot C_{4s}^4$ by $[L_{4r}(-1)1; S'_{4s}, S'_{4s}, 1_3M_{4s-6}(-1)_3, 1_3M_{4s-6}(-1)_3, \cdots, (r - time)S'_{4s}, 1_3M_{4s-6}(-1)_3]$. Therefore $x_{-1} = x_1 = 2r + 1, a_{-1} = 2r + 1, a_1 = 2r, y_{-1} = y_1 = 2s, b_{-1} = 8s - 4, b_1 = 8s - 5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = 8s - 5, b'_1 = 8s - 4$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_{4r+2} \bigodot c_{4s}^4$ is signed product cordial.

Case 7. Suppose that $m = 4r + 3$, where $r \geq 1$. Then we label the vertices of $P_{4r+3} \bigodot C_{4s}^4$ by $[L_{4r}1(-1)(-1); S'_{4s}, S'_{4s}, 1_3M_{4s-6}(-1)_3, 1_3M_{4s-6}(-1)_3, \cdots, (r - time)\ 1_3M_{4s-6}(-1)_3, S'_{4s}, S'_{4s}]$. Therefore $x_{-1} = 2r + 2, x_1 = 2r + 1, a_{-1} = 2r, a_1 = 2r + 2, y_{-1} = y_1 = 2s, b_{-1} = 8s - 4, b_1 = 8s - 5, y'_{-1} = y'_1 = 2s$ and $b'_{-1} = 8s - 5, b'_1 = 8s - 4$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = -1$. Hence, $P_{4r+3} \bigodot c_{4s}^4$ is signed product cordial. □

Lemma 3.3 *If $n \equiv 1(mod\ 4)$, then $P_m \bigodot c_n^4$signed product cordial for all $m \geq 1$.*

Proof Suppose that $n = 4s + 1$, where $s \geq 2$. The following cases will be examined.

Case 1. Suppose that $m = 1$. Then we label the vertices of $P_1 \bigodot c_{4s+1}^4$ by $[(-1); 1_3S_{4s-4}(-1)_2]$. Therefore $x_{-1} = 1, x_1 = 0, a_{-1} = a_1 = 0, y_{-1} = 2s, y_1 = 2s + 1, b_{-1} = 8s - 3, b_1 = 8s - 2$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_1 \bigodot c_{4s+1}^4$ is signed product cordial.

Case 2. Suppose that $m = 2$. We label the vertices of $P_2 \bigodot c_{4s+1}^4$ by $[(-1)1; L_{4s}, 1_3S_{4s-4}(-1)_2]$. Therefore $x_{-1} = x_1 = 1, a_{-1} = 1, a_1 = 0, y_{-1} = 2s + 1, y_1 = 2s, b_{-1} = 8s - 2, b_1 = 8s - 3, y'_{-1} = 2s, y'_1 = 2s + 1$ and $b'_{-1} = 8s - 3, b'_1 = 8s - 2$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Hence, $P_2 \bigodot c_{4s+1}^4$ is signed product cordial.

Case 3. Suppose that $m = 3$. Then we label the vertices of $P_3 \bigodot c_{4s+1}^4$ by

$$[(-1)(-1)1; 1(-1)_3S'_{4s-4}1, 1(-1)_3S'_{4s-4}1, 1_3S_{4s-4}(-1)_2].$$

Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = a_1 = 1, y_{-1} = 2s, y_1 = 2s + 1, b_{-1} = 8s - 3, b_1 = 8s - 2, y'_{-1} = 2s + 1, y'_1 = 2s$ and $b'_{-1} = 8s - 3, b'_1 = 8s - 2$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_3 \bigodot c_{4s+1}^4$ is signed product cordial.

Case 4. Suppose that $m = 4r$, where $r \geq 1$. Then we label the vertices of $P_{4r} \bigodot C_{4s+1}^4$ by $[L_{4r}; L_{4s}(-1), L_{4s}(-1), S'_{4s}1, S'_{4s}1, ..., (r - time)]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 2r - 1, a_1 = 2r, y_{-1} = 2S + 1, y_1 = 2S, b_{-1} = 8S - 2, b_1 = 8S - 3, y'_{-1} = 2S, y'_1 = 2S + 1$ and $b'_{-1} = 8s - 2, b'_1 = 8s - 3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Hence, $P_{4r} \bigodot c_{4s+1}^4$ is signed product cordial.

Case 5. Suppose that $m = 4r + 1$, where $r \geq 1$. Then we label the vertices of $P_{4r+1} \bigodot c_{4s+1}^4$ by $[S_{4r}(-1); S'_{4s}1, S'_{4s}1, L_{4s}(-1), L_{4s}(-1), \cdots, (r - time)\ s_{4s}1$. Therefore $x_{-1} = 2r + 1, x_1 =$

$2r, a_{-1} = 2r-1, a_1 = 2r+1, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = 8s-2, b_1 = 8s-3, y'_{-1} = 2s, y'_1 = 2s+1, b'_{-1} = 8s-2, b'_1 = 8s-3, y''_{-1} = 2s, y''_1 = 2s+1$ and $b''_{-1} = 8s-2, b''_1 = 8s-3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+1} \bigodot c^4_{4s+1}$ is signed product cordial.

Case 6. Suppose that $m = 4r+2$, where $r \geq 1$. Then we label the vertices of $P_{4r+2} \bigodot c^4_{4s+1}$ by $[L_{4r}(-1)1; L_{4s}(-1), L_{4s}(-1), S'_{4s}1, S'_{4s}1, \cdots, (r-time), L_{4s}(-1), S'_{4s}1]$. Therefore $x_{-1} = x_1 = 2r+1, a_{-1} = 2r+1, a_1 = 2r, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = 8s-2, b_1 = 8s-3, y'_{-1} = 2s, y'_1 = 2s+1$ and $b'_{-1} = 8s-2, b'_1 = 8s-3$. It follows that $v_{-1}-v_1 = 0$ and $e_{-1}-e_1 = 1$. Hence, $P_{4r+2} \bigodot c^4_{4s+1}$ is signed product cordial.

Case 7. Suppose that $m = 4r+3$, where $r \geq 1$. Then we label the vertices of $P_{4r+3} \bigodot C^4_{4s+1}$ by $[L_{4r}1(-1)(-1); L_{4s}(-1), L_{4s}(-1), S'_{4s}1, S'_{4s}1, \cdots, (r-time)\ S'_{4s}1, L_{4s}(-1), s_{4s}1]$. Therefore $x_{-1} = 2r+2, x_1 = 2r+1, a_{-1} = 2r, a_1 = 2r+2, y_{-1} = 2s+1, y_1 = 2s, b_{-1} = 8s-2, b_1 = 8s-3, y'_{-1} = 2s, y'_1 = 2s+1, b'_{-1} = 8s-2, b_1 = 8s-3, y''_{-1} = 2s, y''_1 = 2s+1$ and $b''_{-1} = 8s-2, b''_1 = 8s-3$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+3} \bigodot c^4_{4s+1}$ is signed product cordial. □

Lemma 3.4 *If $n \equiv 2(mod\ 4)$, then $P_m \bigodot c^4_n$signed product cordial for all $m \geq 1$.*

Proof Suppose that $n = 4s+2$, where $s \geq 2$. The following cases will be examined.

Case 1. Suppose that $m = 1$. Then we label the vertices of $P_1 \bigodot C^4_{4s+2}$ by $[(-1); (-1)_3 1_3 S_{4s-4}]$. Therefore $x_{-1} = 1, x_1 = 0, a_{-1} = a_1 = 0, y_{-1} = y_1 = 2s+1, b_{-1} = 8s, b_1 = 8s-1$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 1$. Hence, $P_1 \bigodot c^4_{4s+2}$ is signed product cordial.

Case 2. Suppose that $m = 2$. Then we label the vertices of $P_2 \bigodot c^4_{4s+2}$ by

$$[(-1)1; (-1)_3 1_3 L'_{4s-4}, (-1)_3 M'_2 1_3 M_{4s-6}].$$

Therefore $x_{-1} = x_1 = 1, a_{-1} = 1, a_1 = 0, y_{-1} = y_1 = 2s+1, b_{-1} = 8s, b_1 = 8s-1, y'_{-1} = y'_1 = 2s+1$ and $b'_{-1} = 8s-1, b'_1 = 8s$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 1$. Hence, $P_2 \bigodot c^4_{4s+2}$ is signed product cordial.

Case 3. Suppose that $m = 3$. Then we label the vertices of $P_3 \bigodot C^4_{4s+2}$ by

$$[(-1)(-1)1; (-1)_3 1_3 L'_{4s-4}, (-1)_3 1_3 L'_{4s-4}, (-1)_3 M'_2 1_3 M_{4s-6}].$$

Therefore $x_{-1} = 2, x_1 = 1, a_{-1} = a_1 = 1, y_{-1} = y_1 = 2s+1, b_{-1} = 8s, b_1 = 8s-1, y'_{-1} = y'_1 = 2s+1$ and $b'_{-1} = 8s-1, b'_1 = 8s$. It follows that $v_{-1} - v_1 = 1$ and $e_{-1} - e_1 = 1$. Hence, $P_3 \bigodot c^4_{4s+2}$ is signed product cordial.

Case 4. Suppose that $m = 4r$, where $r \geq 1$. Then we label the vertices of $P_{4r} \bigodot c^4_{4s+2}$ by $[L_{4r}; (-1)_3 1_3 L'_{4s-4}, (-1)_3 1_3 L'_{4s-4}, (-1)_3 M'_2 1_3 M_{4s-6},$
$(-1)_3 M'_2 1_3 M_{4s-6}, \cdots, (r-time)]$. Therefore $x_{-1} = x_1 = 2r, a_{-1} = 2r-1, a_1 = 2r, y_{-1} = y_1 = 2s+1, b_{-1} = 8s, b_1 = 8s-1, y'_{-1} = y'_1 = 2s+1$ and $b'_{-1} = 8s-1, b'_1 = 8s$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = -1$. Hence, $P_{4r} \bigodot c^4_{4s+2}$ is signed product cordial.

Case 5. Suppose that $m = 4r+1$, where $r \geq 1$. Then we label the vertices of $P_{4r+1} \bigodot c^4_{4s+2}$

by $[L_{4r}(-1);(-1)_3 1_3 L'_{4s-4},(-1)_3 1_3 L'_{4s-4},(-1)_3 M'_2 1_3\ M_{4s-6},(-1)_3 M'_2 1_3 M_{4s-6},\cdots,(r-time)$ $(-1)_3 1_3 L'_{4s-4}$. Therefore $x_{-1}=2r+1, x_1=2r, a_{-1}=a_1=2r, y_{-1}=y_1=2s+1, b_{-1}=8s, b_1=8s-1, y'_{-1}=y'_1=2s+1$ and $b'_{-1}=8s-1, b'_1=8s$. It follows that $v_{-1}-v_1=1$ and $e_{-1}-e_1=1$. Hence, $P_{4r+1}\bigodot c^4_{4s+2}$ is signed product cordial.

Case 6. Suppose that $m=4r+2$, where $r\geq 1$. Then we label the vertices of $P_{4r+2}\bigodot c^4_{4s+2}$ by

$$[L_{4r}(-1)1;(-1)_3 1_3 L'_{4s-4},(-1)_3 1_3 L'_{4s-4},(-1)_3 M'_2 1_3 M_{4s-6},(-1)_3 M'_2 1_3 M_{4s-6},$$
$$\cdots,(r-time)(-1)_3 1_3 L'_{4s-4},(-1)_3 M'_2 1_3 M_{4s-6}].$$

Therefore $x_{-1}=x_1=2r+1, a_{-1}=2r+1, a_1=2r, y_{-1}=y_1=2s+1, b_{-1}=8s, b_1=8s-1, y'_{-1}=y'_1=2s+1$ and $b'_{-1}=8s-1, b'_1=8s$. It follows that $v_{-1}-v_1=0$ and $e_{-1}-e_1=1$. Hence, $P_{4r+2}\bigodot c^4_{4s+2}$ is signed product cordial.

Case 7. Suppose that $m=4r+3$, where $r\geq 1$. Then we label the vertices of $P_{4r+3}\bigodot c^4_{4s+2}$ by $[L_{4r}1(-1)(-1);(-1)_3 1_3 L'_{4s-4},(-1)_3 1_3 L'_{4s-4},(-1)_3 M'_2 1_3\ M_{4s-6},(-1)_3 M'_2 1_3 M_{4s-6},\cdots,(r-time)$, $(-1)_3 M'_2 1_3 M_{4s-6},(-1)_3 1_3 L'_{4s-4}$, $(-1)_3 1_3 L'_{4s-4}]$. Therefore $x_{-1}=2r+2, x_1=2r+1, a_{-1}=2r, a_1=2r+2, y_{-1}=y_1=2s+1, b_{-1}=8s, b_1=8s-1, y'_{-1}=y'_1=2s+1$ and $b'_{-1}=8s-1, b'_1=8s$. It follows that $v_{-1}-v_1=1$ and $e_{-1}-e_1=-1$. Hence, $P_{4r+3}\bigodot c^4_{4s+2}$ is signed product cordial. □

Lemma 3.5 *If $n\equiv 3(mod\ 4)$, then $P_m\bigodot c^4_n$ signed product cordial for all $m\geq 1$.*

Proof Suppose that $n=4s+3$, where $s\geq 2$. The following cases will be examined.

Case 1. Suppose that $m=2$. We label the vertices of $P_2\bigodot c^4_{4s+3}$ by $[(-1)1;M_{4s+3},S'_{4s}1(-1)1]$. Therefore $x_{-1}=x_1=1, a_{-1}=1, a_1=0, y_{-1}=2s+2, y_1=2s+1, b_{-1}=8s+2, b_1=8s+1, y'_{-1}=2s+1, y'_1=2s+2$ and $b'_{-1}=8s+2, b'_1=8s+1$. It follows that $v_{-1}-v_1=0$ and $e_{-1}-e_1=1$. Hence, $P_2\bigodot c^4_{4s+3}$ is signed product cordial.

Case 2. Suppose that $m=4r$, where $r\geq 1$. Then we label the vertices of $P_{4r}\bigodot C^4_{4s+3}$ by $[L_{4r};M_{4s+3},M_{4s+3},S'_{4s}1(-1)1,S'_{4s}1(-1)1,...,(r-time)]$. Therefore $x_{-1}=x_1=2r, a_{-1}=2r-1, a_1=2r, y_{-1}=2s+2, y_1=2s+1, b_{-1}=8s+2, b_1=8s+1, y'_{-1}=2s+1, y'_1=2s+2$ and $b'_{-1}=8s+2, b'_1=8s+1$. It follows that $v_{-1}-v_1=0$ and $e_{-1}-e_1=-1$. Hence, $P_{4r}\bigodot c^4_{4s+3}$ is signed product cordial.

Case 3. Suppose that $m=4r+1$, where $r\geq 1$. Then we label the vertices of $P_{4r+1}\bigodot c^4_{4s+3}$ by $[S_{4r}(-1);S'_{4s}1(-1)1,S'_{4s}1(-1)1,M_{4s+3},M_{4s+3},,\cdots,(r-time),s'_{4s}1(-1)1$. Therefore $x_{-1}=2r+1, x_1=2r, a_{-1}=2r-1, a_1=2r+1, y_{-1}=2s+2, y_1=2s+1, b_{-1}=8s+2, b_1=8s+1, y'_{-1}=2s+1, y'_1=2s+2, b'_{-1}=8s+2, b'_1=8s+1, y''_{-1}=2s+1, y'_1=2s+2$ and $b''_{-1}=8s+2, b''_1=8s+1$. It follows that $v_{-1}-v_1=0$ and $e_{-1}-e_1=0$. Hence, $P_{4r+1}\bigodot c^4_{4s+3}$ is signed product cordial.

Case 4. Suppose that $m=4r+2$, where $r\geq 1$. Then we label the vertices of $P_{4r+2}\bigodot c^4_{4s+3}$ by $[L_{4r}(-1)1;M_{4s+3},M_{4s+3},S'_{4s}1(-1)1,S'_{4s}1(-1)1,\cdots,(r-time),M_{4s+3},S'_{4s}1(-1)1]$. Therefore $x_{-1}=x_1=2r+1, a_{-1}=2r+1, a_1=2r, y_{-1}=2s+2, y_1=2s+1, b_{-1}=8s+2, b_1=8s+1, y'_{-1}=2s+1, y'_1=2s+2$ and $b'_{-1}=8s+2, b'_1=8s+1$. It follows that $v_{-1}-v_1=0$ and

$e_{-1} - e_1 = 1$. Hence, $P_{4r+2} \bigodot c^4_{4s+3}$ is signed product cordial.

Case 5. Suppose that $m = 4r + 3$, where $r \geq 1$. Then we label the vertices of $P_{4r+3} \bigodot c^4_{4s+3}$ by

$$[L_{4r}1(-1)(-1); M_{4s+3}, M_{4s+3}, S'_{4s}1(-1)1, S'_{4s}1(-1)1, \cdots, (r-time), S'_{4s}1(-1)1, M_{4s+3}, S_{4s}1(-1)1].$$

Therefore $x_{-1} = 2r + 2, x_1 = 2r + 1, a_{-1} = 2r, a_1 = 2r + 2, y_{-1} = 2s + 2, y_1 = 2s + 1, b_{-1} = 8s+2, b_1 = 8s+1, y'_{-1} = 2s+1, y'_1 = 2s+2, b'_{-1} = 8s+2, b_1 = 8s+1, y''_{-1} = 2s+1, y''_1 = 2s+2$ and $b''_{-1} = 8s + 2, b''_1 = 8s + 1$. It follows that $v_{-1} - v_1 = 0$ and $e_{-1} - e_1 = 0$. Hence, $P_{4r+3} \bigodot c^4_{4s+3}$ is signed product cordial. □

As a consequence of all lemmas mentioned above we conclude that the corona between paths and fourth power of cycles is signed product cordial for all m, n if and only if $m \geq 1$, $n \geq 7$ except $(m, n) = (1, 7)$ or $(3, 7)$ and also if $n = 3$ for all $m \neq 1$, i.e., the conclusion following.

Theorem 3.5 *$P_m \bigodot c^4_n$ is signed product cordial for all m, n if and only if $m \geq 1$, $n \geq 7$ except $(m, n) = (1, 7)$ or $(3, 7)$.*

In [5] it is proved that the corona $P_m \bigodot c^4_3$ is signed product cordial if and only if $m \neq 1$. Certainly, we get a general result on the signed product cordial of the corona between paths and fourth power of cycles in this paper.

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