

k-Product Cordial Labeling of Cone Graphs

K. Jeya Daisy¹, R. Santrin Sabibha², P. Jeyanthi³ and Maged Z. Youssef⁴

1. Department of Mathematics, Holy Cross College, Nagercoil, Tamilnadu, India

2. Research scholar, Register No. 20212072092001, Manonmaniam Sundaranar University, Tirunelveli, Tamilnadu, India

3. Research Centre, Department of Mathematics, Govindammal Aditanar College for Women, Tiruchendur 628215, Tamilnadu, India

4. Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University, Riyadh 11623, Saudi Arabia and also Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Cairo, Egypt

E-mail: jeyadaisy@yahoo.com, sanithazhi@gmail.com
jeyajeyanthi@rediffmail.com, mzyoussef11566@yahoo.com

Abstract: Let f be a map from $V(G)$ to $\{0, 1, \dots, k-1\}$ where k is an integer, $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(\text{mod } k)$. f is called a k -product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, \dots, k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x ($x = 0, 1, \dots, k-1$). In this paper, we prove that the graphs such as 1-cone $C_n + K_1$ and double cone DC_n admit 5-product cordial labeling. Also, we show that the double cone DC_n does not admit 4-product cordial labeling.

Key Words: Cordial labeling, product cordial labeling, k -product cordial labeling, 4-product cordial graph, 5-product cordial graph, Smarandachely k -product cordial labeling.

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§1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [4]. While studying graph theory, one that has gained a lot of popularity during the last 60 years is the concept of labelings of graphs due to its wide range of applications. Labeling is a function that allocates the elements of a graph to real numbers, usually positive integers. In 1967, Rosa [15] published a pioneering paper on graph labeling problems. Thereafter many types of graph labeling techniques have been studied by several authors. Gallian [3] in his survey beautifully classified them into graceful labeling and harmonious labelings, variations of graceful labelings, variations of harmonious labelings, magic type labelings, anti-magic type labelings and miscellaneous labelings. Cordial labeling is a weaker version of graceful and harmonious labeling was introduced by Cahit [2]. Let f be a

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function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1.

In 2004, Sundaram et al. [17] extended the concept of cordial labeling and defined product cordial labeling as follows: Let f be a function from $V(G)$ to $\{0, 1\}$. For each edge uv , assign the label $f(u)f(v)$. Then f is called product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with $i(i = 0, 1)$. Sundaram et al. [17] proved that the wheels are not product cordial. Many researchers have shown interest on this topic and showed that several classes of graphs admit product cordial labeling. An interested reader can refer to [3].

Followed by this, Ponraj et al. [14] further extended the concept of product cordial labeling and introduced a new labeling called k -product cordial labeling [14]. Let f be a map from $V(G)$ to $\{0, 1, \dots, k - 1\}$ where k is an integer, $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(\text{mod } k)$. f is called a k -product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, \dots, k - 1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x ($x = 0, 1, \dots, k - 1$), and otherwise, f is called a Smarandachely k -product cordial labeling if there is an integer pair $\{i, j\} \subset \{0, 1, \dots, k - 1\}$ such that $|v_f(i) - v_f(j)| > 1$ or $|e_f(i) - e_f(j)| > 1$. For k -product cordial labeling, they proved that k -product cordial labeling of stars and bistars further they studied the 4-product cordial labeling behavior of paths, complete graphs and combs. Jeyanthi and Maheswari [12] proved that W_n if $n \equiv 1(\text{mod } 3)$ is 3-product cordial graph. In [13] Ponraj et al. proved that wheel $W_n = C_n + K_1$ is 4-Product Cordial if and only if $n=5$ or 9. For further results on 3-product and 4-product cordial labeling one can refer to [3]. Inspired by the concept of k -product cordial labeling and the results in [14], we further studied on k -product cordial labeling and established that the following graphs admit k -product cordial labeling: union of graphs [5]; fan and double fan graphs [6]; powers of paths [7]; the maximum number of edges in a 4-product cordial graph of order p is $4\lceil \frac{p-1}{4} \rceil \lfloor \frac{p-1}{4} \rfloor + 3$ [8]; Napier bridge graphs [9]; paths [10] and product of graphs [11]. In this paper we find some new results on k -product cordial labeling.

We recall the following definitions to prove our main results.

Definition 1.1([1]) *The graph $C_n + K_1$ is called as 1-cone. It is also called as wheel.*

Definition 1.2([16]) *The graph $C_n + \overline{K_2}$ is called as a double cone denoted by DC_n .*

§2. Main Results

In this section, we exhibit that the graphs 1-cone $C_n + K_1$ and double cone DC_n admit 5-product cordial labeling. Also we show that the double cone DC_n does not admit 4-product cordial labeling.

Theorem 2.1 *The 1-cone $C_n + K_1$ is a 5-product cordial graph if and only if $n \equiv 1, 2$ or $3(\text{mod } 5)$ for $n > 3$.*

Proof Let the vertex set and the edge set of $C_n + K_1$ be $V(C_n + K_1) = \{v, v_i; 1 \leq i \leq n\}$ and $E(C_n + K_1) = \{(v, v_i); 1 \leq i \leq n\} \cup \{(v_i, v_{i+1}); 1 \leq i \leq n-1\} \cup \{(v_n, v_1)\}$ respectively. Let us consider the following six cases. Define $f : V(C_n + K_1) \rightarrow \{0, 1, 2, 3, 4\}$ as follows:

Case 1. If $n \equiv 1(\text{mod } 5)$ for $n > 3$, then $f(v) = 4, f(v_n) = 1, f(v_i) = 0; 1 \leq i \leq \lfloor \frac{n}{5} \rfloor$.

Subcase 1.1 If n is even, let $i = \lfloor \frac{n}{5} \rfloor + j, 1 \leq j \leq 4 \lfloor \frac{n}{5} \rfloor$.

$$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1, 6(\text{mod } 8), \\ 1 & \text{if } j \equiv 2, 5(\text{mod } 8), \\ 2 & \text{if } j \equiv 3, 7(\text{mod } 8), \\ 3 & \text{if } j \equiv 4, 0(\text{mod } 8). \end{cases}$$

Subcase 1.2 If n is odd, let $i = \lfloor \frac{n}{5} \rfloor + j, 1 \leq j \leq 4 \lfloor \frac{n}{5} \rfloor$.

$$\text{For } n = 11, f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, 6(\text{mod } 8), \\ 4 & \text{if } j \equiv 2, 5(\text{mod } 8), \\ 2 & \text{if } j \equiv 3, 7(\text{mod } 8), \\ 3 & \text{if } j \equiv 4, 0(\text{mod } 8). \end{cases}$$

$$\text{For } n > 11, f(v_i) = \begin{cases} 2 & \text{if } j \equiv 3(\text{mod } 4), \\ 3 & \text{if } j \equiv 0(\text{mod } 4). \end{cases}$$

$$\text{For } 1 \leq j \leq 8, f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1(\text{mod } 4), \\ 4 & \text{if } j \equiv 2(\text{mod } 4). \end{cases}$$

$$\text{For } 4 \lfloor \frac{n}{5} \rfloor - 7 \leq j \leq 4 \lfloor \frac{n}{5} \rfloor, f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1(\text{mod } 4), \\ 1 & \text{if } j \equiv 2(\text{mod } 4). \end{cases}$$

$$\text{For } 9 \leq j \leq 4 \lfloor \frac{n}{5} \rfloor - 8, f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1, 6(\text{mod } 8), \\ 1 & \text{if } j \equiv 2, 5(\text{mod } 8). \end{cases}$$

From the above cases we get

$$\begin{aligned} v_f(0) + 1 &= v_f(1) = v_f(2) + 1 = v_f(3) + 1 = v_f(4) = \lfloor \frac{n}{5} \rfloor + 1, \\ e_f(0) &= e_f(1) + 1 = e_f(2) + 1 = e_f(3) + 1 = e_f(4) = 2 \lfloor \frac{n}{5} \rfloor + 1. \end{aligned}$$

Case 2. $n \equiv 2(\text{mod } 5)$ for $n > 3$.

Subcase 2.1 n is odd.

We assign labels to the vertices v and v_i ($1 \leq i \leq n-1$) as in Subcase 1.1, then assign 2 to v_n .

Subcase 2.2 n is even.

We assign labels to the vertices v and v_i ($1 \leq i \leq n-1$) as in Subcase 1.2, then assign 2

to v_n . From this label we get

$$\begin{aligned} v_f(0) + 1 = v_f(1) = v_f(2) = v_f(3) + 1 = v_f(4) &= \left\lfloor \frac{n}{5} \right\rfloor + 1, \\ e_f(0) = e_f(1) + 1 = e_f(2) = e_f(3) = e_f(4) &= 2 \left\lfloor \frac{n}{5} \right\rfloor + 1. \end{aligned}$$

Case 3. If $n \equiv 3(\text{mod } 5)$ for $n > 3$, then $f(v) = 4$, $f(v_{n-2}) = 1$, $f(v_{n-1}) = 2$, $f(v_n) = 3$, $f(v_i) = 0$; $1 \leq i \leq \left\lfloor \frac{n}{5} \right\rfloor - 1$, $f(v_{\left\lfloor \frac{n}{5} \right\rfloor+2}) = 2$, $f(v_{\left\lfloor \frac{n}{5} \right\rfloor+3}) = 3$, $f(v_{\left\lfloor \frac{n}{5} \right\rfloor+4}) = 0$ and for $i = \left\lfloor \frac{n}{5} \right\rfloor + 4 + j$, $1 \leq j \leq 4 \left\lfloor \frac{n}{5} \right\rfloor - 4$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, 6(\text{mod } 8), \\ 4 & \text{if } j \equiv 2, 5(\text{mod } 8), \\ 2 & \text{if } j \equiv 3, 7(\text{mod } 8), \\ 3 & \text{if } j \equiv 4, 0(\text{mod } 8). \end{cases}$$

If n is even,

$$f(v_{\left\lfloor \frac{n}{5} \right\rfloor}) = 1, \quad f(v_{\left\lfloor \frac{n}{5} \right\rfloor+1}) = 4;$$

if n is odd,

$$f(v_{\left\lfloor \frac{n}{5} \right\rfloor}) = 4, \quad f(v_{\left\lfloor \frac{n}{5} \right\rfloor+1}) = 1.$$

Then, we have

$$v_f(0) + 1 = v_f(1) = v_f(2) = v_f(3) = v_f(4) = \left\lfloor \frac{n}{5} \right\rfloor + 1$$

and for $n = 8$,

$$e_f(0) + 1 = e_f(1) + 1 = e_f(2) + 1 = e_f(3) = e_f(4) + 1 = 2 \left\lfloor \frac{n}{5} \right\rfloor + 2,$$

for $n > 8$,

$$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) + 1 = e_f(4) + 1 = 2 \left\lfloor \frac{n}{5} \right\rfloor + 2.$$

Case 4. $n \equiv 4(\text{mod } 5)$ for $n > 3$.

Let $n = 5t + 4$, then $|V(C_n + K_1)| = 5t + 5$ and $|E(C_n + K_1)| = 10t + 8$. Thus, $v_f(i) = t + 1$ ($i = 0, 1, 2, 3, 4$) and $e_f(i) = 2t + 1$ or $2t + 2$ ($i = 0, 1, 2, 3, 4$). Clearly, $f(v) \neq 0$. If $v_f(0) = t + 1$, then $e_f(0) > 2t + 2$ for $t \geq 0$. Therefore, $|e_f(0) - e_f(j)| > 1$ for some $j = 1, 2, 3, 4$. Hence, $C_n + K_1$ is not a 5-product cordial graph if $n \equiv 4(\text{mod } 5)$.

Case 5. $n \equiv 0(\text{mod } 5)$ for $n > 3$.

Let $n = 5t$, then $|V(C_n + K_1)| = 5t + 1$ and $|E(C_n + K_1)| = 10t$. Thus, $v_f(i) = t$ or $t + 1$ ($i = 0, 1, 2, 3, 4$) and $e_f(i) = 2t$ ($i = 0, 1, 2, 3, 4$). Clearly, $f(v) \neq 0$. If $v_f(0) = t$ or $t + 1$, then $e_f(0) > 2t$ for $t \geq 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for some $j = 1, 2, 3, 4$. Hence, $C_n + K_1$ is not a 5-product cordial graph if $n \equiv 0(\text{mod } 5)$.

Case 6. If $n = 3$, then $|V(C_3 + K_1)| = 4$ and $|E(C_3 + K_1)| = 6$. Thus, $v_f(i) = 0$ or 1

($i = 0, 1, 2, 3, 4$) and $e_f(i) = 1$ or 2 ($i = 0, 1, 2, 3, 4$). If $v_f(0) = 0$, then $e_f(0) = 0$. If $v_f(0) = 1$, then $e_f(0) = 3$. Therefore, $|e_f(0) - e_f(j)| > 1$ for some $j = 1, 2, 3, 4$. Hence, $C_3 + K_1$ is not a 5-product cordial graph. \square

An example of 5-product cordial labeling of $C_{11} + K_1$ is shown in Figure 1.

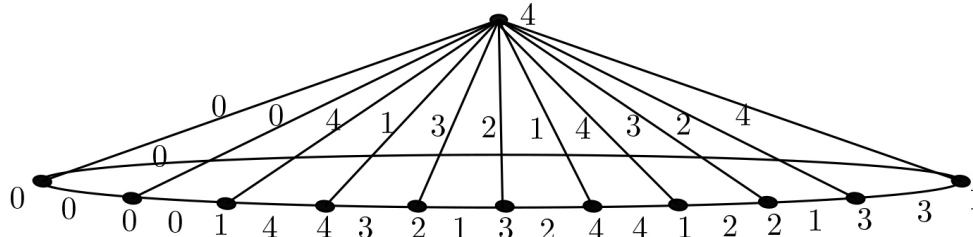


Figure 1. 5-product cordial labeling of $C_{11} + K_1$

Theorem 2.2 *The double cone DC_n is not a 4-product cordial graph for all $n \geq 3$.*

Proof Let the vertex set and the edge set of DC_n be $V(DC_n) = \{u, v, v_i; 1 \leq i \leq n\}$ and $E(DC_n) = \{(u, v_i), (v, v_i); 1 \leq i \leq n\} \cup \{(v_i, v_{i+1}); 1 \leq i \leq n - 1\} \cup \{(v_1, v_n)\}$ respectively. We assume that DC_n is a 4-product cordial graph with a 4-product cordial labeling f on DC_n . Let us consider the following four cases.

Case 1. If $n \equiv 0(mod 4)$ for $n > 3$, let $n = 4t$, then $|V(DC_n)| = 4t + 2$ and $|E(DC_n)| = 12t$. Thus, $v_f(i) = t$ or $t + 1$ ($i = 0, 1, 2, 3$) and $e_f(i) = 3t$ ($i = 0, 1, 2, 3$). Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. If $v_f(0) = t$ or $t + 1$, then $e_f(0) > 3t$ for $t \geq 1$. We get a contradiction to f is a 4-product cordial labeling. Hence, DC_n is not a 4-product cordial graph if $n \equiv 0(mod 4)$.

Case 2. If $n \equiv 1(mod 4)$ for $n > 3$, let $n = 4t + 1$, then $|V(DC_n)| = 4t + 3$ and $|E(DC_n)| = 12t + 3$. Thus, $v_f(i) = t$ or $t + 1$ ($i = 0, 1, 2, 3$) and $e_f(i) = 3t$ or $3t + 1$ ($i = 0, 1, 2, 3$). Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. Obviously, $v_f(0) = t$. Otherwise $e_f(0) > 3t + 1$ is not possible. We assign 0 to the vertices of the cycle in such a way that $e_f(0) = 3t + 1$. Then, $v_f(2) = t + 1$. Clearly, $f(v) \neq 2$, $f(u) \neq 2$ and 2 must be assigned inconsecutively. Otherwise $e_f(0) > 3t + 1$ is not possible. Then, $4t + 2 \leq e_f(2) \leq 4t + 4$ for $t \geq 1$. We get a contradiction to f is a 4-product cordial labeling. Hence, DC_n is not a 4-product cordial graph if $n \equiv 1(mod 4)$ for $n > 3$.

Case 3. If $n \equiv 2(mod 4)$ for $n > 3$, let $n = 4t + 2$, then $|V(DC_n)| = 4t + 4$ and $|E(DC_n)| = 12t + 6$. Thus, $v_f(i) = t + 1$ ($i = 0, 1, 2, 3$) and $e_f(i) = 3t + 1$ or $3t + 2$ ($i = 0, 1, 2, 3$). Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. Obviously, $v_f(0) = t + 1$ then $e_f(0) > 3t + 2$ for $t \geq 1$. We get a contradiction to f is a 4-product cordial labeling. Hence, DC_n is not a 4-product cordial graph if $n \equiv 2(mod 4)$ for $n > 3$.

Case 4. If $n \equiv 3(mod 4)$ for $n \geq 3$, let $n = 4t + 3$, then $|V(DC_n)| = 4t + 5$ and $|E(DC_n)| = 12t + 9$. Thus, $v_f(i) = t + 1$ or $t + 2$ ($i = 0, 1, 2, 3$) and $e_f(i) = 3t + 2$ or $3t + 3$ ($i = 0, 1, 2, 3$). Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. If $v_f(0) = t + 1$ or $t + 2$, then $e_f(0) > 3t + 3$ for $t \geq 0$. We get a contradiction to f is a 4-product cordial labeling. Hence, DC_n is not a 4-product cordial graph if $n \equiv 3(mod 4)$ for $n \geq 3$. \square

Theorem 2.3 *The double cone DC_n is a 5-product cordial graph if and only if $n \equiv 1$ or $2 \pmod{5}$ for $n \geq 3$.*

Proof Let the vertex set and the edge set of DC_n be $V(DC_n) = \{u, v, v_i; 1 \leq i \leq n\}$ and $E(DC_n) = \{(u, v_i), (v, v_i); 1 \leq i \leq n\} \cup \{(v_i, v_{i+1}); 1 \leq i \leq n-1\} \cup \{(v_1, v_n)\}$ respectively. Let us consider the following five cases.

Define $f : V(DC_n) \rightarrow \{0, 1, 2, 3, 4\}$ as follows:

Case 1. If $n \equiv 1 \pmod{5}$ for $n > 3$, then $f(u) = 3, f(v) = 4, f(v_i) = 0; 1 \leq i \leq \lfloor \frac{n}{5} \rfloor$.

Subcase 1.1 If n is even, then $f(v_{n-4}) = 4, f(v_{n-3}) = 1, f(v_{n-2}) = 2, f(v_{n-1}) = 3, f(v_n) = 1$.

Let $i = \lfloor \frac{n}{5} \rfloor + j, 1 \leq j \leq 4 \lfloor \frac{n}{5} \rfloor - 4$.

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, 6 \pmod{8}, \\ 4 & \text{if } j \equiv 2, 5 \pmod{8}, \\ 2 & \text{if } j \equiv 3, 7 \pmod{8}, \\ 3 & \text{if } j \equiv 4, 0 \pmod{8}. \end{cases}$$

From this label we get

$$v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) = v_f(4) = \lfloor \frac{n}{5} \rfloor + 1.$$

For $n = 6$,

$$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) = e_f(4) = 3 \lfloor \frac{n}{5} \rfloor + 1.$$

For $n > 6$,

$$e_f(0) = e_f(1) + 1 = e_f(2) = e_f(3) + 1 = e_f(4) = 3 \lfloor \frac{n}{5} \rfloor + 1.$$

Subcase 1.2 If n is odd, let $i = \lfloor \frac{n}{5} \rfloor + j, 1 \leq j \leq 4 \lfloor \frac{n}{5} \rfloor + 1$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, 6 \pmod{8} \\ 4 & \text{if } j \equiv 2, 5 \pmod{8} \\ 2 & \text{if } j \equiv 3, 7 \pmod{8} \\ 3 & \text{if } j \equiv 4, 0 \pmod{8}. \end{cases}$$

Then, we have

$$v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) = v_f(4) = \lfloor \frac{n}{5} \rfloor + 1,$$

$$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) = e_f(4) = 3 \lfloor \frac{n}{5} \rfloor + 1.$$

Case 2. If $n \equiv 2 \pmod{5}$ for $n > 3$, then $f(u) = 3$, $f(v) = 4$, $f(v_{n-1}) = 1$, $f(v_n) = 2$, $f(v_i) = 0$; $1 \leq i \leq \lfloor \frac{n}{5} \rfloor - 1$ and $f(v_{\lfloor \frac{n}{5} \rfloor}) = 4$, $f(v_{\lfloor \frac{n}{5} \rfloor + 1}) = 1$, $f(v_{\lfloor \frac{n}{5} \rfloor + 2}) = 2$, $f(v_{\lfloor \frac{n}{5} \rfloor + 3}) = 3$, $f(v_{\lfloor \frac{n}{5} \rfloor + 4}) = 0$.

Subcase 2.1 If n is odd, let $i = \lfloor \frac{n}{5} \rfloor + 4 + j$, $1 \leq j \leq 4 \lfloor \frac{n}{5} \rfloor - 4$ and

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, 6 \pmod{8}, \\ 4 & \text{if } j \equiv 2, 5 \pmod{8}, \\ 2 & \text{if } j \equiv 3, 7 \pmod{8}, \\ 3 & \text{if } j \equiv 4, 0 \pmod{8}. \end{cases}$$

Subcase 2.2 If n is even, let $i = \lfloor \frac{n}{5} \rfloor + 4 + j$; $1 \leq j \leq 4 \lfloor \frac{n}{5} \rfloor - 4$,

$$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1, 6 \pmod{8}, \\ 1 & \text{if } j \equiv 2, 5 \pmod{8}, \\ 2 & \text{if } j \equiv 3, 7 \pmod{8}, \\ 3 & \text{if } j \equiv 4, 0 \pmod{8}. \end{cases}$$

Then, we have

$$v_f(0) + 1 = v_f(1) = v_f(2) = v_f(3) = v_f(4) = \lfloor \frac{n}{5} \rfloor + 1.$$

For $n = 7$,

$$e_f(0) + 1 = e_f(1) + 1 = e_f(2) + 1 = e_f(3) = e_f(4) + 1 = 3 \lfloor \frac{n}{5} \rfloor + 2.$$

For $n > 7$,

$$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) + 1 = e_f(4) + 1 = 3 \lfloor \frac{n}{5} \rfloor + 2.$$

Case 3. If $n \equiv 3 \pmod{5}$ for $n \geq 3$, let $n = 5t + 3$, then $|V(DC_n)| = 5t + 5$ and $|E(DC_n)| = 15t + 9$. Thus, $v_f(i) = t + 1$ ($i = 0, 1, 2, 3, 4$) and $e_f(i) = 3t + 1$ or $3t + 2$ ($i = 0, 1, 2, 3, 4$). Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. If $v_f(0) = t + 1$, then $e_f(0) > 3t + 2$ for $t \geq 0$. Hence, DC_n is not a 5-product cordial graph if $n \equiv 3 \pmod{5}$ for $n \geq 3$.

Case 4. If $n \equiv 4 \pmod{5}$ for $n > 3$, let $n = 5t + 4$, then $|V(DC_n)| = 5t + 6$ and $|E(DC_n)| = 15t + 12$. Thus, $v_f(i) = t + 1$ or $t + 2$ ($i = 0, 1, 2, 3, 4$) and $e_f(i) = 3t + 2$ or $3t + 3$ ($i = 0, 1, 2, 3, 4$). Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. If $v_f(0) = t + 1$ or $t + 2$, then $e_f(0) > 3t + 3$ for $t \geq 0$. Hence, DC_n is not a 5-product cordial graph if $n \equiv 4 \pmod{5}$.

Case 5. If $n \equiv 0 \pmod{5}$ for $n > 3$, let $n = 5t$, then $|V(DC_n)| = 5t + 2$ and $|E(DC_n)| = 15t$. Thus, $v_f(i) = t$ or $t + 1$ ($i = 0, 1, 2, 3, 4$) and $e_f(i) = 3t$ ($i = 0, 1, 2, 3, 4$). Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. If $v_f(0) = t$ or $t + 1$, then $e_f(0) > 3t$ for $t > 0$. Hence, DC_n is not a 5-product cordial graph if $n \equiv 0 \pmod{5}$ for $n > 3$. \square

An example of 5-product cordial labeling of DC_{12} is shown in Figure 2.

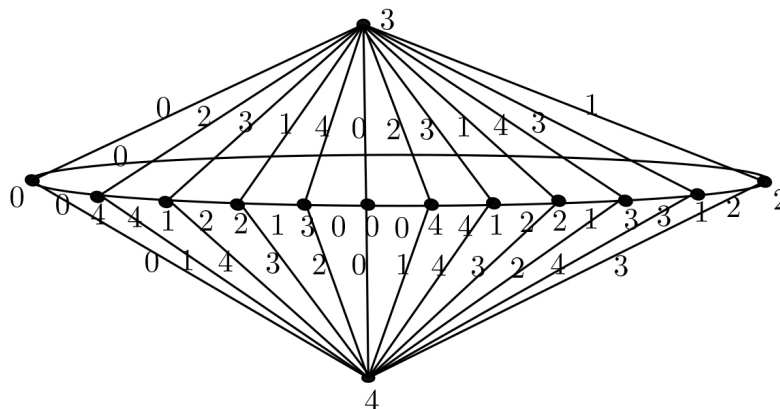


Figure 2. 5-product cordial labeling of DC_{12}

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