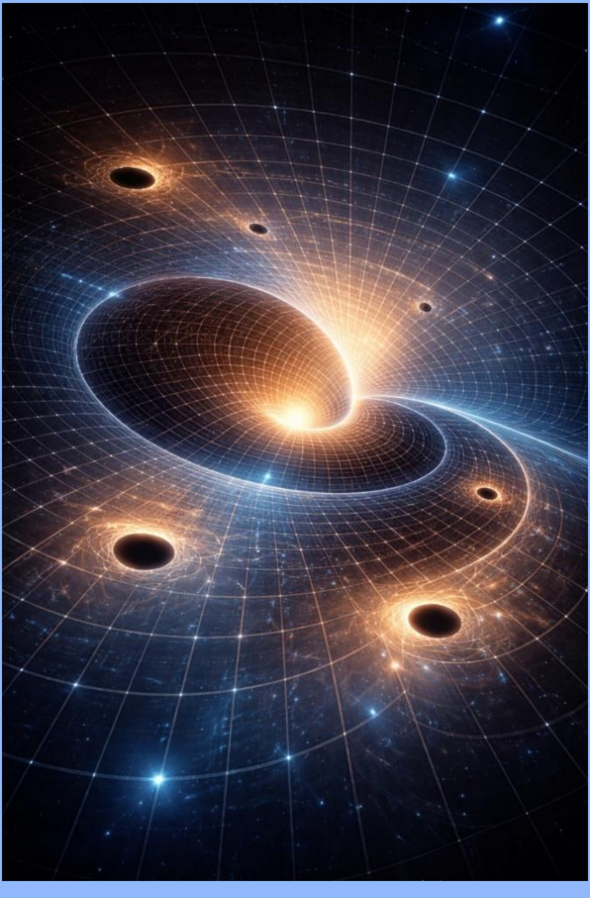
Florentin Smarandache

INFINITESIMAL PUNCTURES 1

Infinitesimally Punctured Geometry

Foundations of Weak–Strong Manifolds and Distributional Curvature



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What if singularities are not physical infinities, but signs of an incomplete geometry?

FLORENTIN SMARANDACHE

INFINITESIMAL PUNCTURES 1

INFINITESIMALLY PUNCTURED GEOMETRY

INFINITESIMAL PUNCTURES series

1 INFINITESIMALLY PUNCTURED GEOMETRY

2 INFINITESIMALLY PUNCTURED PHYSICS

3 INFINITESIMALLY PUNCTURED STRUCTURES

The Infinitesimally Punctured Wave (IPW), Infinitesimally Punctured Surface (IPSu), Infinitesimally Punctured Space (IPSp), Infinitesimally Punctured Manifold (IPM), and in general Infinitesimally Punctured Quantum Physics (IPQP) were introduced and developed by Florentin Smarandache in 2019 and respectively in 2025-2026.

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The development of this book was significantly enhanced by an integrated suite of advanced AI technologies, each playing a role in the manuscript's evolution:

- **Lumo AI**: Facilitated multilingual drafting and the cohesive integration of intricate mathematical concepts.
- **SciSpace**: Enabled streamlined literature searches and precise citation handling.
- **Perplexity**: Provided rapid access to foundational definitions and pertinent research findings.
- **Elicit**: Assisted in the structured formulation of research inquiries and the selection of appropriate datasets.
- **Claude 3.5 Sonnet (Anthropic)**: Supported the detailed drafting of mathematical proofs and the optimization of logical structures.
- **Wolfram Alpha**: Utilized for rigorous symbolic computation, the validation of algebraic formulas, and the creation of technical examples.
- **ChatGPT 5.2**: Provided essential support for iterative revisions, linguistic polishing, and bibliographic formatting.
- **Gemini**: Instrumental in verifying interdisciplinary terminology and refining descriptive captions for figures.
- **Figurelabs**: Employed to design and generate high-quality scientific diagrams and illustrations.

By combining these innovative platforms, I was able to conduct comprehensive literature reviews, ensure the mathematical integrity of the work through rigorous validation, and achieve a clear, multilingual narrative that defines the final character of this volume.

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SERIES PREFACE
THE INFINITESIMAL PUNCTURE PROGRAM

The presence of singularities and ultraviolet divergences in modern theoretical physics has long been regarded as a signal that our most successful mathematical frameworks become unreliable at extreme scales. Traditionally, these pathologies are treated as technical problems to be regulated, renormalized, or bypassed through increasingly elaborate dynamical modifications.

The present series adopts a different viewpoint. It asks whether the root of these difficulties lies not primarily in the dynamical laws, but in the geometric ontology upon which those laws are formulated. In particular, it questions the assumption that matter must be represented as point-like entities inserted into an otherwise smooth spacetime manifold.

The central proposal is that matter and physical attributes arise as intrinsic geometric defects—measure-zero punctures—within spacetime itself. Curvature, charge, and quantum behavior are interpreted as distributionally supported features of geometry rather than externally imposed sources. In this perspective, spacetime is not merely a passive stage but a structured entity whose internal architecture encodes what is conventionally called matter.

The series develops this idea systematically. The first volume establishes the mathematical foundations of infinitesimally punctured manifolds. The second derives dynamical laws from a hybrid variational principle. The third formulates a unified structural framework capable of accommodating multiple coexisting geometric regimes.

Together, these volumes aim to articulate a coherent geometric paradigm in which singularities are replaced by structure, and physical entities are reinterpreted as manifestations of spacetime's internal organization.

TERMINOLOGY

Puncture
A closed, measure-zero subset of a manifold carrying defect-supported geometric structure.

Defect
Synonym of puncture when emphasizing physical interpretation.

Singular Point
Not used. Replaced by *puncture*.

Blow-Up
Classical divergence of curvature invariants; replaced by defect-supported curvature.

Weak Geometry
Smooth geometry on M_{IP}.

Strong Geometry
Distributional geometry supported on (P).

Regularisation
Approximation of a puncture by smooth metric families.

Bound State
Discrete eigenvalue induced by operator domain choice.

Holonomy
Parallel-transport-induced group element around a defect.

Spectral Flow
Shift of eigenvalues induced by puncture parameters.

Geometry-First Principle
Dynamics must emerge from geometric structure, not precede it.

Structural
Refers to definitions and relations independent of dynamical laws.

SYMBOL GLOSSARY

A. Sets, Spaces, and Manifolds

Symbol	*Meaning*
$\boldsymbol{M}$	Smooth n-dimensional manifold
$\boldsymbol{P \subset M}$	Closed measure-zero defect set (puncture set)
$\boldsymbol{M_{\mathrm{IP}}}$	Infinitesimally punctured manifold, $M \setminus P$
$\boldsymbol{N_\varepsilon(P)}$	ε-neighbourhood of the defect set
$\boldsymbol{\partial M}$	Boundary of a manifold (not used for punctures)
$\boldsymbol{K \subset M}$	Compact subset

B. Metrics, Connections, and Geometry

Symbol	*Meaning*
$\boldsymbol{g_{\mu\nu}}$	Metric tensor
$\boldsymbol{g_\varepsilon}$	Regularising family of smooth metrics
$\boldsymbol{\nabla}$	Levi-Civita connection
$\boldsymbol{\nabla^{(0)}}$	Smooth connection on M_{IP}
$\boldsymbol{\Gamma^{\alpha}{}_{\beta\gamma}}$	Connection coefficients
$\boldsymbol{\Delta^{\alpha}{}_{\beta\gamma}}$	Defect-supported part of the connection
$\boldsymbol{R^{\alpha}{}_{\beta\mu\nu}}$	Riemann curvature tensor
$\boldsymbol{R^{(0)}}$	Smooth curvature part
$\boldsymbol{R_P}$	Defect-supported curvature
$\boldsymbol{R_T}$	Smooth (“Truth”) curvature
$\boldsymbol{R_F}$	Defect (“Falsity”) curvature
$\boldsymbol{R_I}$	Transition (“Indeterminacy”) curvature
$\boldsymbol{K_{\mu\nu}}$	Extrinsic curvature
$\boldsymbol{\delta_P}$	Dirac-type distribution supported on P

C. Measures and Integrability

Symbol	***Meaning***
$d\mu_g$	Metric volume measure
r	Geodesic distance to the defect
α	Local divergence exponent
k	Codimension of the defect

D. Functional Analysis

Symbol	***Meaning***
$L^2(M_{\mathrm{IP}})$	Square-integrable functions on the punctured manifold
$H^1(M_{\mathrm{IP}})$	First Sobolev space on the punctured manifold
Δ	Laplace–Beltrami operator
Δ_λ	Self-adjoint extension of the Laplacian
$\mathcal{D}_0$	Initial operator domain (typically $C_c^\infty(M_{\mathrm{IP}})$)
Δ^*	Adjoint of Δ
$n_\pm$	Deficiency indices $\dim\,\ker\,(\Delta^* \mp i)$
$R_\lambda(z)$	Resolvent $(\Delta_\lambda - z)^{-1}$
G_z	Green function at spectral parameter z
$\Gamma_1,\ \Gamma_2$	Boundary maps used in the theory of self-adjoint extensions

E. Spectral and Wave Quantities

Symbol	***Meaning***
$\sigma(\Delta)$	Spectrum of the (self-adjoint) Laplacian
σ_{ess}	Essential spectrum
E	Eigenvalue
k	Wave number (or momentum magnitude)
$\delta_0(k)$	s-wave phase shift
$K(t,x,y)$	Heat kernel (fundamental solution of $\partial_t - \Delta$)

$\boldsymbol{\psi}$	Eigenfunction or wavefunction
$\boldsymbol{u(t,x)}$	Time-dependent wave field

F. Topology and Holonomy

Symbol	*Meaning*
$\boldsymbol{\pi_1(M)}$	Fundamental group of the manifold
$\boldsymbol{\gamma}$	Loop (closed curve) in M
$\boldsymbol{\mathcal{P}}\mathbf{exp}$	Path-ordered exponential (used for holonomy)
$\boldsymbol{\delta}$	Deficit angle associated with a conical defect
$\boldsymbol{\alpha}$	Holonomy parameter (often $\alpha = \delta/2\pi$)

G. Indices and Conventions

Symbol	*Meaning*
Greek indices $\boldsymbol{(\mu, \nu, \alpha, \dots)}$	Spacetime indices (summed via Einstein convention)
Latin indices $\boldsymbol{(i, j, \dots)}$	Spatial indices
Einstein summation	Repeated indices are summed over their range
Signature	Mostly-plus metric signature $(- + + + \cdots)$

H. Neutrosophic Structural Components

Symbol	*Meaning*
$\boldsymbol{T}$	Truth component
$\boldsymbol{I}$	Indeterminacy component
$\boldsymbol{F}$	Falsity component

FOREWORD TO INFINITESIMALLY PUNCTURED GEOMETRY

This volume is devoted exclusively to the geometric and analytic foundations of infinitesimally punctured manifolds. Its purpose is not to propose new physical laws, introduce new forces, or reformulate existing field equations. Rather, it aims to develop a precise and self-consistent mathematical framework in which singular behavior is replaced by finite, distributionally supported geometric structure.

For more than a century, singularities have appeared at the deepest levels of theoretical physics. They arise in general relativity as curvature blow-ups, in quantum field theory as ultraviolet divergences, and in quantum mechanics as pointlike idealizations of particles. Although a wide array of techniques—regularization, renormalization, cutoff procedures, and modified dynamics—have been devised to manage these infinities, the singularities themselves persist as structural features of the underlying models. Their ubiquity suggests that they may not merely represent technical shortcomings, but rather indicate a deeper mismatch between physical ontology and geometric representation.

The guiding idea of this volume is simple: if a closed, measure-zero subset is removed from a smooth manifold and curvature is allowed to concentrate there in a controlled, integrable manner, then many classical singularities may be reinterpreted as geometric defects rather than pathologies. In this perspective, what appears as a divergence in a smooth framework becomes a finite, well-defined distributional object supported on an infinitesimal puncture.

The resulting geometric object is neither a manifold with boundary nor a manifold with holes in the usual topological sense. It is a hybrid structure in which two regimes coexist:

- a weak regime, where ordinary smooth differential geometry holds, and
- a strong regime, where geometric quantities may possess distributional components supported on the puncture set.

This coexistence is not an ad hoc modification but a structural feature of the geometry itself. Smoothness is not abandoned; it is localized. Singular behavior is not eliminated; it is reorganized into a finite, geometrically meaningful form.

The book develops this framework step by step. After motivating the need for a new geometric viewpoint, it introduces infinitesimally punctured manifolds, distributional curvature, integrability criteria, and a codimension-based classification of defects. Weak and strong geometric regimes are formulated and their compatibility conditions established. The analytic consequences of puncturing are then explored through operator theory, self-adjoint extensions, spectral analysis, and wave propagation. Throughout, low-dimensional toy models provide explicit realizations of the abstract constructions and demonstrate how purely geometric defects generate observable analytic effects.

No variational principles, field equations, or cosmological models appear in this volume. Those developments are intentionally deferred. The present work is concerned solely with structure, not dynamics. Its aim is more modest and more fundamental: to show that a consistent, rigorous geometry exists in which singularities are replaced by finite geometric structure, and in which distributional curvature is not pathological but natural.

In this sense, the volume may be read as a preparatory text. It establishes the mathematical backbone for subsequent books in the series and also serves as a standalone reference on infinitesimally punctured geometry. Readers should not expect answers to all physical questions here. Instead, they will find the groundwork upon which such answers may eventually be built.

If the central claim of this book can be summarized in a single sentence, it is this:

Singularities need not be cured by modifying physical laws; they can be replaced by geometry.

Note. *Some of the ideas presented here have already been (and will continue to be) the subject of scientific articles and communications. In this volume, to make the reading easier and accessible beyond a strictly academic audience, I have stripped the exposition of citations and references. A bibliography can be found at the end of the third volume.*

CHAPTER 1
WHY A NEW GEOMETRY?

The singularities that appear in modern theoretical physics are not necessarily physical infinities. They may instead signal a structural mismatch between the ontology we assume (point-like particles) and the geometric framework we use (smooth manifolds).

> This volume develops a minimal geometric shift: rather than inserting point sources into a smooth continuum, we model matter as intrinsic geometric defects of spacetime itself.

1.1 The Triumphs (and the Cracks) of Modern Physics

Modern theoretical physics rests on three pillars:

- **General Relativity (GR)**
- **Quantum Field Theory (QFT)**
- **Quantum Mechanics (QM)**

Each is empirically successful and mathematically powerful, yet each encounters structural tension at extreme regimes.

Area	Standard Formalism	What Works	Where It Breaks Down
General Relativity	Smooth Lorentzian manifold $(M, g_{\mu\nu})$ with Einstein's equation	Perihelion precession, light bending, gravitational waves, cosmology	Curvature invariants (e.g. Kretschmann scalar) diverge at black-hole centers and the Big Bang
Quantum Field Theory	Fields $\phi(x)$ on fixed background; perturbative expansion	High-precision cross sections, running couplings	Ultraviolet divergences; infinite self-energies
Quantum Mechanics	Wavefunction $\psi(x,t)$ with Schrödinger equation	Interference, tunneling, spectra	Measurement problem; no geometric ontology of particles

All three frameworks share a structural assumption: **Matter is inserted into a pre-existing smooth spacetime as a zero-dimensional object or distributional source.** Mathematically, this insertion produces divergences.

The question we raise is not:

> "How can we better regulate these infinities?"

but rather:

> "Why do they arise in the first place?"

1.2 Existing Remedies — and Their Shared Assumption

Over the past decades, numerous approaches have addressed ultraviolet divergences or singularities.

Approach	**Core Idea**	**Strength**	**Limitation**
Hard UV cutoffs	Impose momentum cutoff Λ	Finite integrals	Breaks Lorentz invariance
String theory	Replace particles by 1-D strings	Soft short-distance behavior	Requires extra structure; no direct evidence
Noncommutative geometry	$[x^\mu, x^\nu] = i\theta^{\mu\nu}$	Minimal area	Unitarity/causality issues
Higher-derivative gravity	Add R^2 terms	Improved UV behavior	Ghost modes
Effective field theory	Absorb divergences	Predictive at low energy	No explanation of singularity origin

All of these modify the dynamics but **none modifies the ontology of matter**; they preserve the assumption that fundamental entities are point-like. This book instead asks:

> **What if particles are not inserted into spacetime, but arise as structural defects of spacetime?**

1.3 Infinitesimally Punctured Manifolds

1.3.1 Definition

> Let (M) be a smooth n-dimensional differentiable manifold equipped with a metric g. Let $P \subset M$ be a closed subset of Lebesgue measure zero.
>
> We define the **infinitesimally punctured manifold**:

$$M_{\mathrm{IP}} \coloneqq M \setminus P \qquad \text{(I.1)}$$

> Two geometric regimes coexist:
>
> - **Weak regime:** smooth geometry on M_{IP}
> - **Strong regime:** defect-supported geometry on P

> **Axiom I.1 (Structural Coexistence)**
>
> Any tensor field that is smooth on M_{IP} may acquire an additional distributional component supported on P.
>
> The defect set is not "empty space removed"; it is a locus where the geometric regime changes.

1.3.2 Distributional Curvature

Let $R^{\alpha}{}_{\beta\mu\nu}$ denote the Riemann tensor of g. On M_{IP} it is smooth. Across P it may decompose as

$$R^{\alpha}{}_{\beta\mu\nu} = R^{\alpha}{}_{\beta\mu\nu}\mid_{\mathrm{smooth}} + R^{\alpha}{}_{\beta\mu\nu}\mid_{P} \qquad \text{(I.2)}$$

where

- $R^{\alpha}{}_{\beta\mu\nu}\mid_{\mathrm{smooth}}$ is smooth on M_{IP}.
- $R^{\alpha}{}_{\beta\mu\nu}\mid_{P}$ is a distribution supported on P.

The essential requirement is

$$\int_K \mid R^{\alpha}{}_{\beta\mu\nu} \mid \; dV < \infty \text{ for every compact } K \subset M \qquad \text{(I.3)}$$

Thus curvature remains **integrable** even if distributional. This replaces curvature blow-up with finite, defect-supported curvature.

1.3.3 Physical Interpretation of a Puncture

Physical Quantity	Geometric Realisation
Mass	Integrated defect-supported Einstein tensor
Charge	Holonomy of a $U(1)$ connection around P
Spin	Winding number of normal bundle
Quantum indeterminacy	Geometry of infinitesimal neighbourhood $N_{\varepsilon}(P)$

Define the ε-neighbourhood:

$$N_{\varepsilon}(P) = \{\, x \in M \mid d_g(x, P) < \varepsilon \,\} \qquad \text{(I.4)}$$

This region interpolates between weak and strong regimes.

A puncture is therefore a finite, geometrically well-defined carrier of physical attributes—not a singularity.

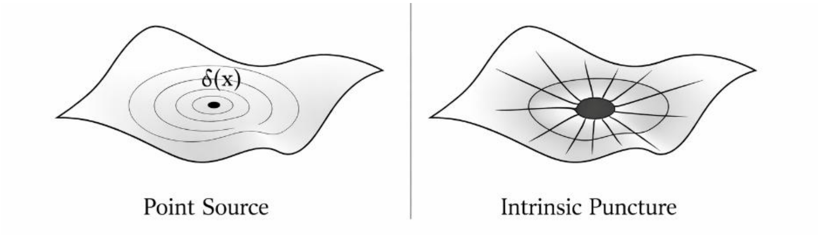


Figure I.1.1 — Point Source vs Puncture

1.4 Neutrosophic Decomposition

Neutrosophic logic proposes three independent components:

- Truth (T)
- Indeterminacy (I)
- Falsity (F)

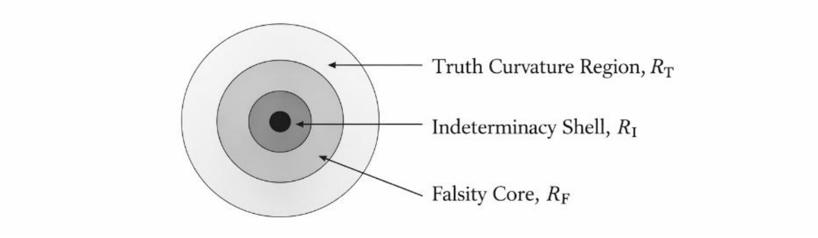


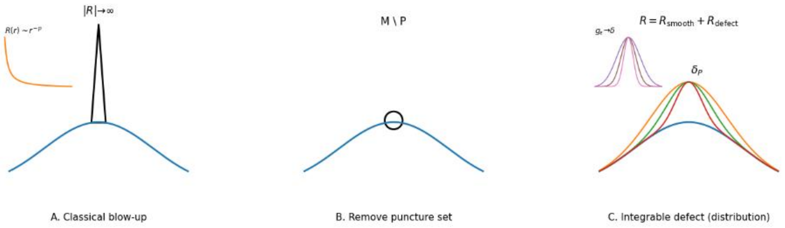
Figure I.1.2 — Neutrosophic Structural Decomposition

In geometric language:

- R_T: smooth curvature on M_{IP}
- R_F: defect-supported curvature on P
- R_I: transition curvature in $N_\varepsilon(P)$

We write

$$R = R_T + R_F + R_I \qquad \text{(I.5)}$$



A. Classical blow-up

B. Remove puncture set

C. Integrable defect (distribution)

Figure I.1.3 — From Singularity to Integrable Defect

1.5 Relation to Neutrosophic Geometry and Smarandache Structures

The threefold decomposition of geometric objects into smooth, defect-supported, and transition components is structurally analogous to the neutrosophic triad (Truth–Indeterminacy–Falsity). In the present volume this correspondence is used solely as an interpretive guide. No neutrosophic algebraic operations, neutrosophic metrics, or multi-space constructions are assumed.

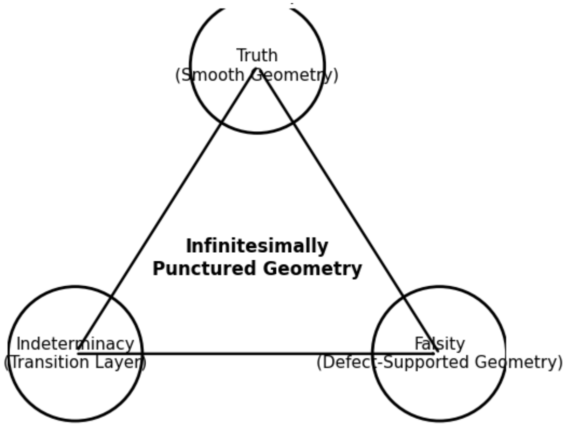


Figure I.1.4 — Neutrosophic Structural Analogy (Geometric Interpretation)

The purpose of this analogy is to highlight that infinitesimally punctured geometry naturally exhibits a multi-regime structure in which mutually incompatible geometric behaviors coexist on the same underlying set.

A systematic development of Smarandache multi-space geometry and neutrosophic differential geometry is deferred to the third book in the series, *Infinitesimally Punctured Structures*.

1.6 Concluding Perspective

For over a century, singularities have marked the limits of our theories. They may not signal a breakdown of physical law; they may signal a mismatch between

- **Zero-dimensional ontology**, and
- **Smooth geometric background.**

Infinitesimally Punctured Physics proposes a minimal shift:

> **Matter is not inserted into geometry. Matter is geometry—concentrated, structured, distributionally encoded.**

> This volume provides the geometric foundation for that claim. The dynamical consequences will unfold in subsequent volumes.

CHAPTER 2
INFINITESIMALLY PUNCTURED MANIFOLDS

A smooth manifold minus a closed measure-zero set is not merely a manifold with holes. It is a hybrid geometric object carrying two coexisting regimes: smooth geometry and defect-supported geometry.

> This chapter formalises that statement.

2.1 From a Smooth Manifold to Its Punctured Counterpart

2.1.1 Basic Setting

Let

- M be a smooth, connected, Hausdorff, second-countable n-dimensional manifold.
- g a C^∞ Riemannian or Lorentzian metric on M.
- $P \subset M$ a closed subset of Lebesgue measure zero.

We define the **infinitesimally punctured manifold**

$$M_{\mathrm{IP}} := M \setminus P \qquad \text{(I.6)}$$

and refer to the triple

$$(M, P, g) \qquad \text{(I.7)}$$

as an **infinitesimally punctured geometric structure**.

> **Definition I.1 (Infinitesimally Punctured Manifold)**
>
> An infinitesimally punctured manifold (IPM) is the pair

$$\left(M_{\mathrm{IP}}, g\restriction_{M_{\mathrm{IP}}}\right) \qquad \text{(I.8)}$$

> together with the data of the defect set P. Its geometry consists of
>
> - a **smooth regime** on M_{IP}, and
> - a **defect regime** supported on P.

Remark 1.

Because P has measure zero,

$$\int_M f\, d\,\mu_g = \int_{M_{\mathrm{IP}}} f\, d\,\mu_g \text{for all } f \in L^1(M) \qquad \text{(I.9)}$$

so removing P does not change integrals of regular functions, although it may affect topology and distributional structure.

Remark 2.

An infinitesimally punctured manifold is not a manifold with boundary; no boundary conditions are imposed a priori.

2.2 Measure-Zero Sets and Topological Impact

Measure-zero does **not** imply topological triviality.

Example 1 (Punctured Plane)

Take $M = \mathbb{R}^2$ and $P = \{0\}$. Then

$$\pi_1(M_{\mathrm{IP}}) \cong \mathbb{Z} \qquad \text{(I.10)}$$

so deleting a single point changes simple connectivity.

Proposition I.1 (Codimension-2 Defects and Holonomy)

Let P be a closed embedded submanifold of codimension 2. If M is simply connected, then

$$\pi_1(M_{\mathrm{IP}}) \neq 0 \qquad \text{(I.11)}$$

Consequently, loops encircling P may carry non-trivial holonomy. This fact underlies

- charge interpretation,
- Aharonov–Bohm-type phase shifts,
- gauge-like structures.

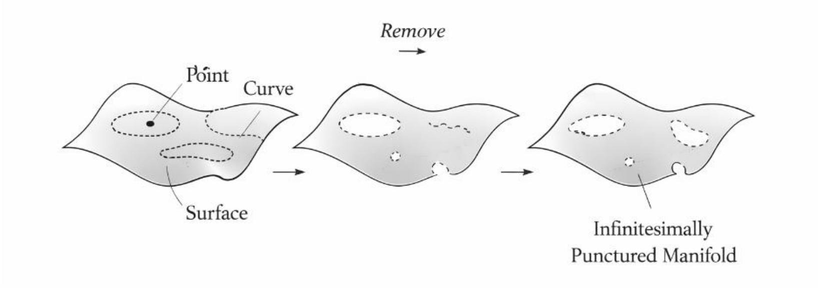


Figure I.2.1 — Removing a Measure-Zero Set

2.3 Distributional Geometry on M_{IP}

The central structural idea is that curvature may become distributional when supported on P.

2.3.1 Test Functions and Distributions

Let $C_c^\infty(M)$ denote smooth compactly supported functions. A **scalar distribution** T is a continuous linear functional

$$T: C_c^\infty(M) \to \mathbb{R} \qquad \text{(I.12)}$$

If T is supported on P,

$$\text{supp}\,(T) \subset P \qquad \text{(I.13)}$$

Such distributions can be written as finite linear combinations of Dirac-type distributions (and their derivatives).

2.3.2 Distributional Curvature

Let ∇ be the Levi-Civita connection of g. We allow the connection to split as

$$\nabla = \nabla^{(0)} + \Gamma_P \qquad \text{(I.14)}$$

where

- $\nabla^{(0)}$ is smooth on M_{IP},
- Γ_P is a tensor-valued distribution supported on P.

The curvature tensor then decomposes as

$$R = R^{(0)} + R_P \qquad \text{(I.15)}$$

with R_P distribution-supported.

2.4 Integrability and Finite Curvature

We now state the key integrability condition.

Lemma I.1 (Finite-Curvature Criterion)

> Consider a defect of codimension k whose curvature locally behaves as

$$|R| \sim r^{-\alpha} \qquad \text{(I.16)}$$

> where r is the geodesic distance from P. Then R is locally integrable near P **iff**

$$\alpha < k. \qquad \text{(I.17)}$$

> **Interpretation (4-dimensional spacetime)**
>
> - A point defect has codimension 4.
> - The worst allowed divergence is r^{-3}.

Thus the classical Schwarzschild invariant divergence $\sim r^{-6}$ is non-integrable.

The punctured formalism replaces it with integrable, defect-supported curvature, removing the blow-up while preserving total mass.

2.5 Classification by Codimension

Codimension	Typical Defect	Distributional Object
1	Thin shell	Surface delta
2	Cosmic string	Line delta
3	Worldline (in 4-D spacetime)	Point delta
$> n$	Topologically inert	No curvature support

Only defects with codimension $\leq n$ can influence curvature.

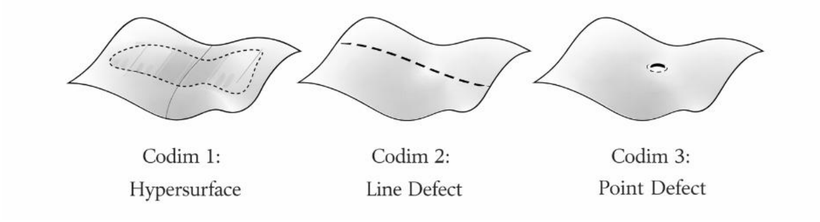


Figure I.2.2 — Codimension Classification

2.6 Indeterminate Transition Layer

Around each defect define the **ε-neighbourhood**

$$N_\varepsilon(P) = \{\, x \in M \mid d_g(x, P) < \varepsilon \,\} \qquad \text{(I.18)}$$

Inside $N_\varepsilon(P)$ we may introduce a regularised metric family

$$g_\varepsilon \;\longrightarrow\; g(\text{distributionally as } \varepsilon \to 0) \qquad \text{(I.19)}$$

This layer

- smooths the defect at finite ε,
- converges to distributional curvature, and
- provides the geometric origin of the **Indeterminacy** sector introduced in Chapter 1.

> In this volume g_ε is purely a regularisation; it carries no independent dynamics yet.

2.7 Holonomy Around a Defect

Let γ be a closed loop in $M \setminus P$ that encircles a defect set P of codimension 2. Because the loop is contractible in the punctured manifold but not contractible in the full space including P, parallel transport of a vector along γ need not return the vector to its original orientation.

Parallel transport around γ defines a linear map on the tangent space,

$$V^\mu \mapsto \left(\mathcal{P}\exp \oint_\gamma \Gamma\right)^\mu{}_\nu V^\nu, \qquad (I.20)$$

where Γ denotes the affine connection and $\mathcal{P}$ indicates path ordering.

If the defect-supported curvature $R_P$vanishes, the connection is locally flat near P, and the holonomy is trivial. Conversely, if

$$R_P \neq 0,$$

then the holonomy is nontrivial: vectors transported around the defect acquire a finite transformation even though the curvature vanishes pointwise away from P. This phenomenon expresses a central feature of infinitesimally punctured geometry:

Curvature concentrated on a measure-zero set manifests globally through holonomy rather than locally through smooth curvature fields.

2.7.1 Conical Defect (Two-Dimensional Case)

Consider a two-dimensional punctured plane with metric

$$ds^2 = dr^2 + (1-\alpha)^2 r^2 d\theta^2, \theta \in [0,2\pi),$$

corresponding to a conical geometry with deficit angle

$$\delta = 2\pi\alpha.$$

The Gaussian curvature vanishes for $r \neq 0$, yet is distributionally supported at the origin. Parallel transport of a vector once around the origin produces a pure rotation,

$$\text{Holonomy} = \text{Rotation}(\delta), \qquad (I.21)$$

meaning that the vector is rotated by angle δupon completing the loop.

2.7.2 Structural Interpretation

Holonomy provides a geometric invariant of the defect that does not depend on the detailed shape of the loop, but only on its winding around P. In this sense, holonomy plays the role of a **defect charge**: it encodes global geometric information associated with the puncture. This observation has several important implications:

- Codimension 2 defects are topologically detectable even when smooth curvature vanishes off the defect.
- Defect-supported curvature produces observable global effects without requiring extended curvature fields.
- Holonomy serves as a bridge between local distributional geometry and global topology.

> Thus, in infinitesimally punctured manifolds, holonomy replaces curvature blow-up as the primary indicator of concentrated geometric structure.

CHAPTER 3
WEAK–STRONG GEOMETRY AND THE STRUCTURE OF DEFECTS

On an infinitesimally punctured manifold, geometry does not disappear at the defect set; it changes regime.

> This chapter formalises the coexistence of smooth and defect-supported geometry and clarifies how distributional curvature arises from controlled geometric jumps.

The coexistence of weak and strong geometric regimes on a single underlying set may be viewed as a geometric instance of a Smarandache-type structure, in which multiple axiomatic systems operate simultaneously. In this volume, the coexistence is formulated entirely within classical differential geometry and distribution theory.

3.1 The Weak Regime

Let *(M, P, g)* be as defined in Chapter 2. On the punctured region

$$M_{\mathrm{IP}} = M \setminus P \qquad \text{(I.22)}$$

the metric g is smooth, and its Levi-Civita connection satisfies

$$\nabla_\lambda^{(0)} g_{\mu\nu} = 0 \qquad \text{(I.23)}$$

The (smooth) Riemann tensor reads

$$\begin{aligned} R^{(0)\alpha}{}_{\beta\mu\nu} &= \partial_\mu \Gamma^{(0)\alpha}{}_{\beta\nu} - \partial_\nu \Gamma^{(0)\alpha}{}_{\beta\mu} \\ &+ \Gamma^{(0)\alpha}{}_{\sigma\mu}\, \Gamma^{(0)\sigma}{}_{\beta\nu} - \Gamma^{(0)\alpha}{}_{\sigma\nu}\, \Gamma^{(0)\sigma}{}_{\beta\mu} \end{aligned} \qquad \text{(I.24)}$$

All classical differential-geometric identities hold on M_{IP}, including the (second) Bianchi identity

$$\nabla^{(0)}_{[\lambda} R^{(0)\alpha}{}_{|\beta|\mu\nu]} = 0 \qquad \text{(I.25)}$$

This is the **weak regime**: ordinary smooth geometry away from defects.

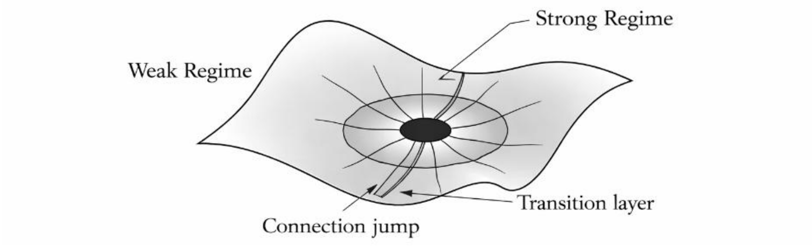


Figure I.3.1 — Weak vs Strong Regimes

3.2 The Strong Regime: Defect-Supported Structure

The defect set P carries geometric data not described by smooth tensors.

Definition I.2 (Defect-Supported Tensor)

> A tensor-valued distribution T_P is **defect-supported** iff

$$\mathrm{supp}(T_P) \subset P \qquad \text{(I.26)}$$

> Such tensors represent curvature or connection contributions concentrated on P.

Connection Decomposition

> We allow the affine connection to split as

$$\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{(0)\alpha}{}_{\beta\gamma} + \Delta^{\alpha}{}_{\beta\gamma} \qquad \text{(I.27)}$$

> where
>
> - $\Gamma^{(0)}$ is smooth on M_{IP},
> - Δ is a distribution supported on P.
>
> The curvature then splits

$$R^{\alpha}{}_{\beta\mu\nu} = R^{(0)\alpha}{}_{\beta\mu\nu} + R_P{}^{\alpha}{}_{\beta\mu\nu} \qquad \text{(I.28)}$$

> with

$$R_P = \nabla^{(0)}\Delta + \Delta \wedge \Delta \qquad \text{(I.29)}$$

> interpreted distributionally.

3.3 Jump Geometry and Codimension Structure

When P is an embedded submanifold, distributional curvature arises from controlled jumps in the connection.

3.3.1 Codimension 1: Hypersurface Defects

Let P be a smooth hypersurface with unit normal n^{μ}. Define the extrinsic curvature

$$K_{\mu\nu} = \nabla_{\mu}^{(0)} n_{\nu} \qquad \text{(I.30)}$$

If the metric is continuous but its first derivative has a finite jump across P, curvature acquires a surface-delta term

$$R_P \sim \delta_P\,[K] \qquad \text{(I.31)}$$

where $[K]$ denotes the jump in extrinsic curvature. No stress-energy tensor is introduced here.

3.3.2 Codimension 2: Conical Defects

In 2-D polar coordinates consider

$$ds^2 = dr^2 + (1-\delta)^2 r^2 d\theta^2 \qquad \text{(I.32)}$$

The Gaussian curvature vanishes for $r > 0$, but

$$R_P = 2\pi\delta\ \delta^{(2)}(r) \qquad \text{(I.33)}$$

so the curvature is distribution-supported at the origin. Holonomy around the defect equals the deficit angle δ.

3.3.3 Codimension 3 or Higher

For isolated worldlines in 4-D spacetime curvature may contain point-supported distributions

$$R_P \sim m\,\delta^{(4)}(x - x_0) \qquad \text{(I.34)}$$

provided the integrability condition (I.17) is satisfied. Higher-codimension sets that violate integrability contribute no finite curvature.

3.4 Structural Compatibility Conditions

Weak and strong regimes cannot be arbitrary.

Axiom I.2 (Metric Continuity)

The metric g is continuous across P:

$$\lim_{x \to P} g_{\mu\nu}(x) = g_{\mu\nu} \mid_P \qquad \text{(I.35)}$$

Discontinuities in the metric itself would generate non-integrable curvature.

Axiom I.3 (Controlled Connection Jump)

The jump in the connection must produce curvature satisfying the integrability condition

$$\int_K \mid R \mid < \infty \text{ for all compact } K \qquad \text{(I.36)}$$

These axioms guarantee that defects are geometrically finite.

3.5 Neutrosophic Structural Decomposition (Geometric Form)

We now reinterpret the curvature split (I.28) in the triadic language introduced in Chapter 1.

$$R_T := R^{(0)} \qquad \text{(I.37)}$$

$$R_F := R_P \qquad \text{(I.38)}$$

$$R_I := \lim_{\varepsilon \to 0} \left(R(g_\varepsilon) - R^{(0)} - R_P\right) \qquad \text{(I.39)}$$

where g_ε is the regularising family from (I.19). Thus:

$$R = R_T + R_F + R_I \qquad \text{(I.40)}$$

3.6 Low-Dimensional Illustrative Examples

The abstract notions introduced in the preceding sections—weak and strong regimes, defect-supported curvature, and holonomy—become especially transparent in low-dimensional settings. In such cases, explicit calculations can be performed and the geometric effects of puncturing can be visualized directly. These examples are not intended as physical models, but as structural prototypes.

3.6.1 Punctured Circle

Let

$$M = S^1$$

and remove a single point $P \subset S^1$. The resulting space $S^1 \setminus P$ is topologically an open interval, but it retains memory of the removed point through its global structure.

Geodesics on $S^1 \setminus P$ remain smooth everywhere except at the defect, where the affine parameterization may fail to extend continuously. Consider parallel transport of a vector once around the circle, avoiding the puncture. The resulting holonomy is

$$\text{Holonomy} = e^{i\alpha}, \qquad (I.41)$$

where the parameter α encodes an angular mismatch associated with the defect.

Interpretation.

Although no smooth curvature exists on $S^1 \setminus P$, the puncture induces a nontrivial global effect: a vector transported around the circle acquires a phase. Thus, even in one dimension, a point defect can be detected through holonomy.

This model serves as the simplest illustration of:

- Geometry without local curvature,
- Yet with nontrivial global structure.

3.6.2 Punctured Plane

Consider the two-dimensional metric (I.32)

$$ds^2 = dr^2 + (1-\delta)^2 r^2 d\theta^2,$$

which describes a conical geometry with deficit angle δ. The puncture lies at $r = 0$.

For $r \neq 0$, the Gaussian curvature vanishes identically. Nevertheless, parallel transport of a vector around a loop encircling the origin yields

$$\text{Holonomy} = \text{Rotation}(\delta), (I.42)$$

i.e., a rotation by angle δ.

Interpretation.

The entire geometric effect of the defect is encoded in a single number, the deficit angle. Curvature is concentrated at the puncture in a distributional sense, while the surrounding region remains flat.

This example illustrates:

- Codimension 2 defects produce holonomy.
- Global geometric effects arise without smooth curvature fields.

3.6.3 Punctured Three-Space

A line defect along the z-axis in three-dimensional space may be modeled by

$$ds^2 = dr^2 + (1 - \delta)^2 r^2 d\theta^2 + dz^2, \qquad (I.43)$$

where (r, θ, z) are cylindrical coordinates.

For $r \neq 0$, the space is locally flat. At $r = 0$, curvature is concentrated on the line and appears as a line-supported delta distribution.

Interpretation.

This geometry represents the three-dimensional analogue of the conical plane: each transverse plane exhibits the same deficit angle δ, while the defect extends along a line.

Consequences:

- Curvature vanishes everywhere in the weak regime.
- Holonomy around the defect is nontrivial.
- The defect is detectable through loops encircling the axis.

> **Unifying Perspective**
>
> Across one, two, and three dimensions, a common pattern emerges:
>
> - Punctures introduce no smooth curvature away from the defect.
> - Defect-supported curvature manifests through holonomy and global geometric effects.
> - Geometry alone encodes information traditionally attributed to localized sources.
>
> These toy models provide concrete evidence that infinitesimally punctured manifolds possess intrinsic geometric content even in the absence of dynamics.

CHAPTER 4
GEOMETRIC REGULARISATION OF CLASSICAL SINGULARITIES

4.1 The Classical Singularity Problem

In classical general relativity, metrics such as the Schwarzschild metric exhibit curvature invariants diverging at

$$r = 0 \qquad \text{(I.44)}$$

For example, the Kretschmann scalar behaves as

$$K \sim r^{-6} \qquad \text{(I.45)}$$

This divergence is non-integrable in four-dimensional spacetime.

In the punctured framework we reinterpret the singular point as

$$P = \{ r = 0 \} \qquad \text{(I.46)}$$

and **remove** it from the manifold.

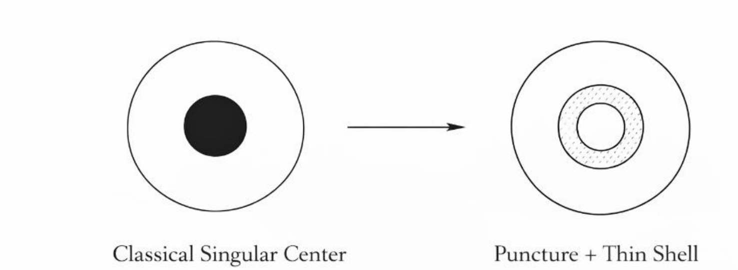


Figure I.4.1 — Schwarzschild Core Replacement

4.2 Removal vs. Blow-Up

Key structural shift

Classical approach	**Punctured approach**
Geometry defined everywhere	Geometry defined only on $M \setminus P$
Divergence occurs at a point	Curvature at P replaced by a distribution

The metric is smooth for

$$r > 0 \qquad \text{(I.47)}$$

We do **not** attempt to extend it through $r = 0$.

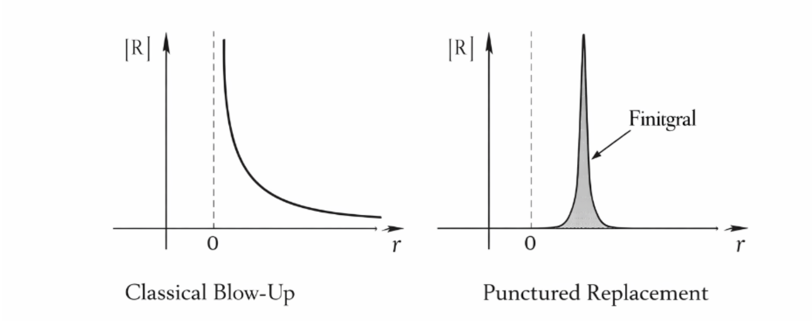


Figure I.4.2 — Classical Blow-Up vs Punctured Replacement

4.3 Distributional Replacement

Instead of the divergent invariant K, define

$$R = R^{(0)} + R_P \qquad \text{(I.48)}$$

where

- $R^{(0)}$ is smooth for $r > 0$,
- R_P is supported at $r = 0$.

The total curvature satisfies

$$\int_K | R |< \infty \qquad \text{(I.49)}$$

for all compact regions K. Hence the geometry becomes finite in the distributional sense.

4.4 Regularising Families

Let g_ε be a family of smooth metrics such that

$$g_\varepsilon \to g \text{ for } r > 0 \qquad \text{(I.50)}$$

and near $r = 0$ the core is smoothed over a radius ε.

Assume

$$\sup_{\varepsilon} \int_K | R(g_\varepsilon) | < \infty \qquad \text{(I.51)}$$

Then

$$R(g_\varepsilon) \to R^{(0)} + R_P \text{distributionally} \qquad \text{(I.52)}$$

Thus defect-supported curvature is obtained as the limit of smooth approximations.

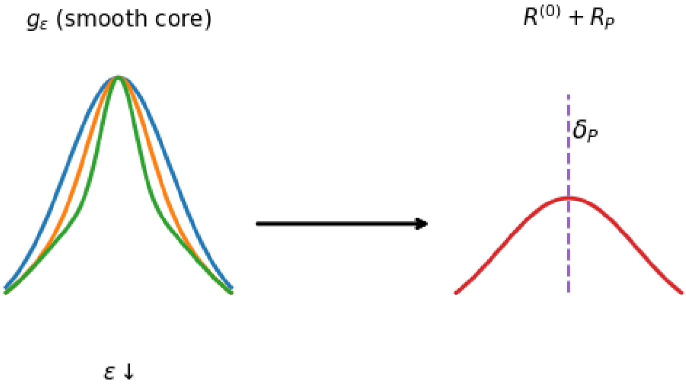


Figure I.4.3 — Regularising Families and Distributional Limit

4.5 Conical Regularisation Example

Consider a 2-D metric

$$ds_\varepsilon^2 = dr^2 + f_\varepsilon(r)^2 \, d\theta^2 \qquad \text{(I.53)}$$

with

$$f_\varepsilon(r) = \begin{cases} r, & r > \varepsilon \\ (1-\delta)\, r, & r \ll \varepsilon \end{cases} \qquad \text{(I.54)}$$

Then the Gaussian curvature concentrates near $r = 0$, and

$$\int R(g_\varepsilon) \to 2\pi\,\delta \qquad \text{(I.55)}$$

producing a conical delta curvature.

4.6 Structural Principle of Finite Replacement

> **Theorem I.2 (Finite Replacement Principle)**
>
> *Let a smooth metric exhibit curvature blow-up confined to a closed measure-zero set* P*. If there exists a regularising family* g_ε *satisfying (I.51), then the limit geometry is an infinitesimally punctured manifold with defect-supported curvature. No curvature invariant diverges in the distributional sense.*

The theorem is purely geometric; it does **not** depend on Einstein's field equations.

> This chapter's purpose was to show that many classical singular geometries become finite and integrable when interpreted as punctured manifolds with distributional curvature.

CHAPTER 5
OPERATOR THEORY ON INFINITESIMALLY PUNCTURED MANIFOLDS

Removing a measure-zero set does not change integrals of smooth functions, but it does change operator domains. This is where puncturing becomes analytically non-trivial.

5.1 Functional Setting

Let

$$M_{\mathrm{IP}} = M \setminus P \qquad \text{(I.56)}$$

where P is closed and of Lebesgue measure zero.

Let g be smooth on M_{IP}.

Define the Hilbert space:

$$L^2(M_{\mathrm{IP}}) = \left\{ f \mid \int_{M_{\mathrm{IP}}} |f|^2 \, d\mu_g < \infty \right\} \qquad \text{(I.57)}$$

Since P has measure zero,

$$L^2(M_{\mathrm{IP}}) \cong L^2(M) \qquad \text{(I.58)}$$

as Hilbert spaces, **but** the domains of differential operators differ.

5.2 The Laplace–Beltrami Operator

Define the Laplacian on smooth compactly supported functions:

$$\Delta f = \nabla^{\mu}\nabla_{\mu} f \qquad \text{(I.59)}$$

Initial domain:

$$\mathcal{D}_0 = C_c^{\infty}(M_{\mathrm{IP}}) \qquad \text{(I.60)}$$

The operator is symmetric on $L^2(M_{\mathrm{IP}})$.

***The central question:* Is it essentially self-adjoint?**

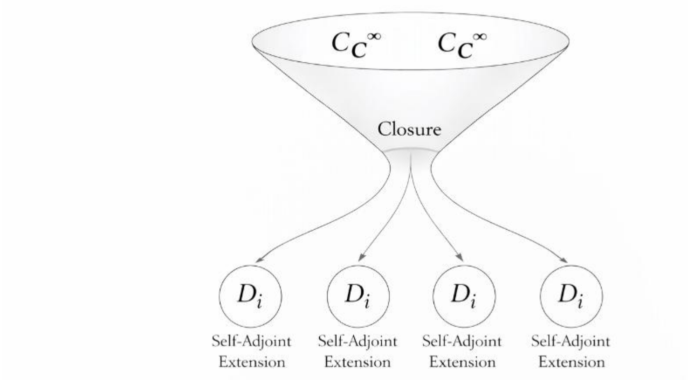


Figure I.5.1 — Operator Domains

5.3 Essential Self-Adjointness and Defects

Removing a set can destroy essential self-adjointness; the outcome depends critically on the codimension of the removed set.

> **Proposition I.3 (Codimension Criterion)**
>
> Let M be complete and smooth, and remove a closed submanifold P.
>
> *If*

$$\mathrm{codim}(P) \geq 4 \qquad (\text{I}.61)$$

> *then* the Laplacian on $C_c^\infty(M_{\text{IP}})$ is essentially self-adjoint.
>
> *If*

$$\mathrm{codim}(P) \leq 3 \qquad (\text{I}.62)$$

> *then* self-adjoint extensions may exist.

> **Interpretation**
>
> - High-codimension defects are analytically invisible.
> - Low-codimension defects produce new boundary conditions—this mirrors the geometric codimension classification of Chapter 2.

5.4 Self-Adjoint Extensions

Let $\bar{\Delta}$ denote the closure of Δ. If the deficiency indices are non-zero,

$$n_{\pm} = \dim\ \ker\,(\Delta^{*} \mp i) \qquad \text{(I.63)}$$

then there exists a family of self-adjoint extensions parametrised by unitary maps

$$U\colon \ker(\Delta^{*} + i) \longrightarrow \ker(\Delta^{*} - i) \qquad \text{(I.64)}$$

Each extension corresponds to a choice of **defect boundary condition**; no physical interpretation is imposed at this stage.

5.5 Model Case: Punctured $\mathbb{R}^3$

Take

$$M = \mathbb{R}^3, P = \{0\} \qquad \text{(I.65)}$$

so

$$M_{\mathrm{IP}} = \mathbb{R}^3 \setminus \{0\} \qquad \text{(I.66)}$$

The Laplacian admits a one-parameter family of self-adjoint extensions. Near $r = 0$ functions behave as

$$f(r) \ \sim\ A\,\frac{1}{r} + B \qquad \text{(I.67)}$$

The extension fixes a linear relation

$$B = \lambda A \qquad \text{(I.68)}$$

where the parameter λ encodes the defect behaviour. This phenomenon arises purely from removing a point.

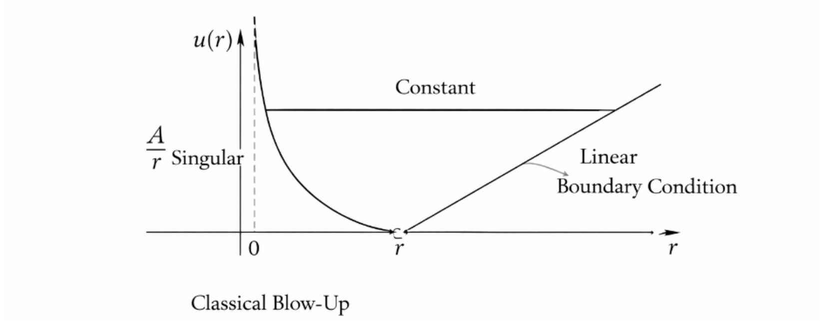


Figure I.5.2 — Radial Behavior Near Defect

5.6 Sobolev Spaces on Punctured Domains

Define

$$H^1(M_{\mathrm{IP}}) = \{ f \in L^2 \mid \nabla f \in L^2 \} \qquad \text{(I.69)}$$

> **Key distinction:**
>
> - If $\operatorname{codim}(P) \geq 2$, smooth compactly supported functions are dense in H^1.
> - If $\operatorname{codim}(P) = 1$, trace spaces appear, so hypersurface defects behave differently from point defects.

5.7 Boundary Triples and Defect Maps

Self-adjoint extensions can be described using boundary maps. For radial functions near a point defect:

$$\Gamma_1 f = \lim_{r \to 0} r\, f(r) \qquad \text{(I.70)}$$

$$\Gamma_2 f = \lim_{r \to 0} \left(f(r) - \frac{\Gamma_1 f}{r} \right) \qquad \text{(I.71)}$$

Extensions correspond to linear relations

$$\Gamma_2 f = \lambda\, \Gamma_1 f \qquad \text{(I.72)}$$

No particular choice of extension is preferred at this stage.

5.8 Spectral Consequences

Different self-adjoint extensions alter the spectrum of the Laplacian. For certain values of λ negative eigenvalues may appear, leading to:

> - Shifted continuous spectrum,
> - Emergence of bound states,
> - Modified scattering phase shifts.

CHAPTER 6

SPECTRAL AND WAVE STRUCTURE ON PUNCTURED MANIFOLDS

> Removing a measure-zero set does not change the Hilbert space, but it can change
>
> - the operator domain,
> - the resolvent,
> - the spectrum,
> - the scattering matrix.
>
> This chapter formalises those effects.

6.1 Spectral Preliminaries

Let

$$M_{\mathrm{IP}} = M \setminus P \qquad (\text{I.73})$$

and let Δ_λ denote a self-adjoint extension of the Laplacian described in Chapter 5. The spectral theorem gives

$$\Delta_\lambda = \int_{\sigma(\Delta_\lambda)} \mu \, d\, E_\mu \qquad (\text{I.74})$$

where E_μ is the spectral measure.

***Key question:* How does puncturing change $\boldsymbol{\sigma(\Delta)}$?**

6.2 Essential Spectrum Stability

> **Theorem I.3 (Stability of Essential Spectrum)**
>
> If P is compact and of measure zero, then

$$\sigma_{\mathrm{ess}}(\Delta_\lambda) = \sigma_{\mathrm{ess}}(\Delta) \qquad (\text{I.75})$$

> where Δ is the Laplacian on the unpunctured manifold.

Consequences

- Continuous (essential) spectrum is stable.
- Only the discrete spectrum may change.
- Defects behave like finite-rank perturbations.

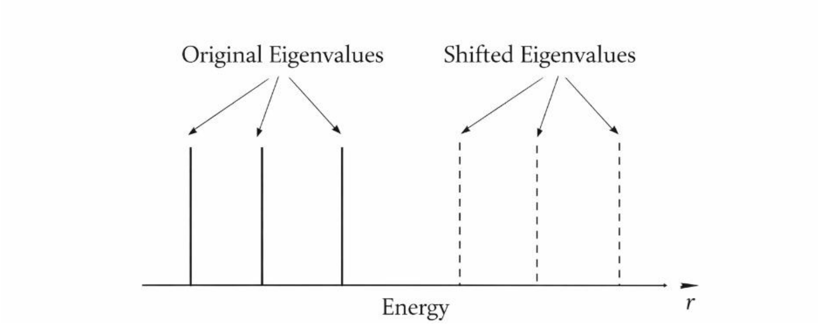


Figure I.6.1 — Spectral Shift

6.3 Discrete Eigenvalues from Point Defects

Consider

$$M = \mathbb{R}^3, P = \{0\} \qquad \text{(I.76)}$$

For certain extension parameters $\lambda < 0$,

$$\Delta_\lambda \text{ admits a negative eigenvalue } E < 0 \qquad \text{(I.77)}$$

This corresponds analytically to a bound state, even though no external potential was added. The bound state emerges solely from the domain choice—a principal analytic consequence of puncturing.

6.4 Resolvent Difference Formula

Define the resolvent

$$R_\lambda(z) = (\Delta_\lambda - z)^{-1} \qquad \text{(I.78)}$$

For codimension-3 point defects in $\mathbb{R}^3$ the resolvent satisfies

$$R_\lambda(z) = R_0(z) + \frac{1}{\alpha(z) - \lambda} \mid G_z\rangle\langle G_{\bar{z}} \mid \qquad \text{(I.79)}$$

where

- R_0 is the free resolvent,
- G_z is the Green function at the origin,
- $\alpha(z)$ encodes the renormalised singular behaviour.

Thus the defect acts as a **rank-one perturbation**.

6.5 Scattering and Phase Shift

Let $k^2 > 0$. Scattering states satisfy

$$(-\Delta_\lambda - k^2)\psi = 0 \qquad \text{(I.80)}$$

In three dimensions with a point defect, the s-wave phase shift obeys

$$k\cot\,\delta_0(k) = -\frac{1}{a} \qquad \text{(I.81)}$$

where a depends on the extension parameter λ.

Consequently:

- The defect modifies the phase shift,
- No classical potential is required.

Analytically this is identical to zero-range (contact) scattering.

6.6 Holonomy and Spectral Flow

For codimension-2 defects (e.g., conical geometry) the holonomy angle δ modifies eigenmodes:

$$\psi_m(\theta) = e^{i(m+\alpha)\theta} \qquad \text{(I.82)}$$

with

$$\alpha = \frac{\delta}{2\pi} \qquad \text{(I.83)}$$

The angular-momentum spectrum shifts

$$m \in \mathbb{Z} \longrightarrow m + \alpha \qquad \text{(I.84)}$$

so the defect induces **spectral flow**, linking the geometry of Chapter 3 to spectral theory.

6.7 Wave Propagation

Consider the wave equation

$$\partial_t^2 u - \Delta_\lambda u = 0 \qquad \text{(I.85)}$$

The associated energy functional is

$$E[u] = \frac{1}{2} \int (|\, \partial_t u \,|^2 + |\, \nabla u \,|^2) \qquad \text{(I.86)}$$

Energy remains conserved for any self-adjoint extension, but:

- Local behaviour near the defect differs,
- Reflected and transmitted components depend on λ.

Defects act as scattering centres without any classical potential.

6.8 Heat Kernel Modification

The heat kernel for the extended Laplacian is

$$K_\lambda(t, x, y) = e^{t\Delta_\lambda}(x, y) \qquad \text{(I.87)}$$

For small t the asymptotic expansion acquires a defect term:

$$\mathrm{Tr}\,(e^{t\Delta_\lambda}) = \mathrm{Tr}\,(e^{t\Delta}) + C_\lambda + O(\sqrt{t}) \qquad \text{(I.88)}$$

> The constant C_λ encodes the contribution of the puncture, showing that spectral geometry can detect defects.

CHAPTER 7
LOW DIMENSIONAL TOY MODELS

> Toy models serve three purposes:
>
> 1. Verify abstract theorems explicitly.
> 2. Exhibit spectral and geometric effects concretely.
> 3. Prepare intuition for later physical interpretation.

7.1 The Punctured Circle

Let:

$$M = S^1, P = \{\theta = 0\} \qquad \text{(I.89)}$$

Then:

$$M_{\mathrm{IP}} = (0,2\pi) \qquad \text{(I.90)}$$

The Laplacian is:

$$\Delta = -\frac{d^2}{d\theta^2} \qquad \text{(I.91)}$$

Initial domain:

$$C_c^\infty(0,2\pi) \qquad \text{(I.92)}$$

This operator is not essentially self-adjoint.

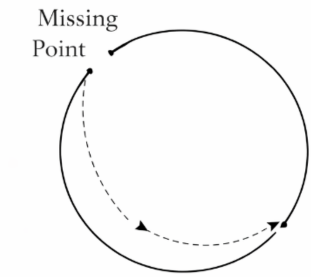


Figure I.7.1 — Punctured Circle. *Circle with missing point and identification condition.*

Self-Adjoint Extensions

Boundary values at the puncture define

$$\psi(2\pi) = e^{i\alpha}\, \psi(0) \qquad \text{(I.93)}$$

$$\psi'(2\pi) = e^{i\alpha}\, \psi'(0) \qquad \text{(I.94)}$$

with parameter $\alpha \in [0,2\pi)$.

Spectrum

Eigenfunctions:

$$\psi_n(\theta) = e^{i(n+\alpha/2\pi)\theta} \qquad \text{(I.95)}$$

Eigenvalues:

$$\lambda_n = \left(n + \frac{\alpha}{2\pi}\right)^2 \qquad \text{(I.96)}$$

Thus the puncture shifts angular momentum – the simplest spectral-flow model.

7.2 The Conical Plane

Let:

$$ds^2 = dr^2 + (1-\delta)^2\, r^2\, d\theta^2 \qquad \text{(I.97)}$$

with $\theta \in [0,2\pi)$.

Curvature (distributional):

$$R = 2\pi\, \delta\, \delta^{(2)}(r) \qquad \text{(I.98)}$$

Laplacian

$$\Delta = \frac{1}{r}\partial_r(r\,\partial_r) + \frac{1}{(1-\delta)^2 r^2}\partial_\theta^2 \qquad \text{(I.99)}$$

Separate variables:

$$\psi(r,\theta) = R_m(r)\, e^{im\theta} \qquad \text{(I.100)}$$

Radial equation:

$$R_m'' + \frac{1}{r}R_m' - \frac{m^2}{(1-\delta)^2 r^2}R_m + k^2 R_m = 0 \qquad \text{(I.101)}$$

Solutions are Bessel functions of order

$$\nu_m = \frac{|\, m\, |}{1-\delta} \qquad \text{(I.102)}$$

Hence the deficit angle rescales the angular quantum number.

Spectral Consequence

Angular spectrum becomes

$$m \longrightarrow \frac{m}{1-\delta} \qquad \text{(I.103)}$$

matching the holonomy analysis of Chapter 3.

Geometry directly modifies the spectrum.

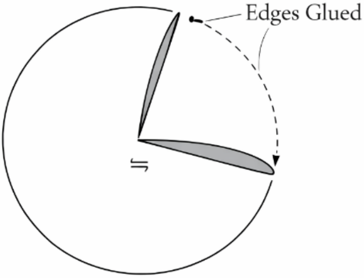


Figure I.7.2* — *Conical Plane. *Top view of wedge removed and edges glued.*

7.3 Point Defect in $\mathbb{R}^3$

Let:

$$M = \mathbb{R}^3, P = \{0\} \qquad \text{(I.104)}$$

Radial Laplacian (s-wave sector):

$$\Delta = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \qquad \text{(I.105)}$$

Near $r = 0$ functions behave as

$$\psi(r) \ \sim \ \frac{A}{r} + B \qquad \text{(I.106)}$$

Extension condition:

$$B = \lambda A \qquad \text{(I.107)}$$

> **Bound State**
>
> For $\lambda < 0$ the operator admits a single negative eigenvalue

$$E = -\frac{1}{\lambda^2} \qquad \text{(I.108)}$$

No potential was introduced; the bound state emerges from the puncture.

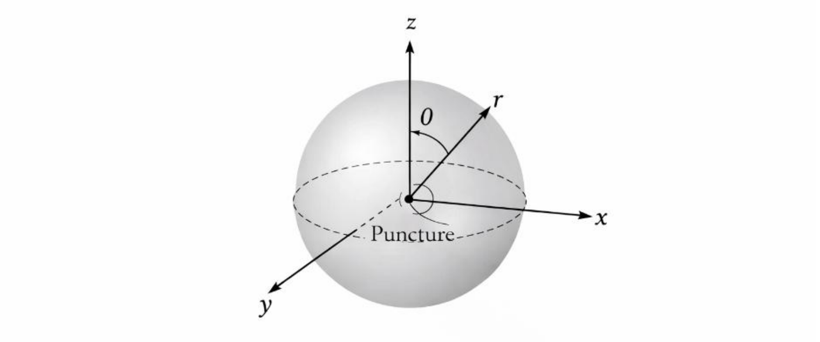


Figure I.7.3 — 3D Point Defect. *Radial coordinate system with puncture.*

7.4 Wave Scattering in 2-D Conical Geometry

Incoming plane wave

$$\psi_{\text{in}} = e^{ikx} \qquad \text{(I.109)}$$

Asymptotic form

$$\psi \sim e^{ikx} + f(\theta)\,\frac{e^{ikr}}{\sqrt{r}} \qquad \text{(I.110)}$$

The scattering amplitude $f(\theta)$ depends on the deficit angle δ. Although the curvature vanishes for $r > 0$, the global conical geometry **modifies scattering** – a purely geometric scattering effect.

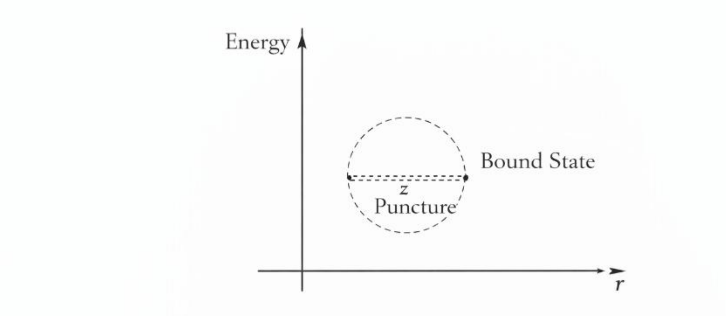


***Figure 7.4* — Bound State Emergence.** *Potential-free bound state sketch.*

7.5 Heat Kernel on the Punctured Plane

For a 2-D conical defect the heat-trace expansion reads

$$\mathrm{Tr}\,(e^{t\Delta}) = \frac{\text{Area}}{4\pi t} + \frac{\delta}{12\,(1-\delta)} + O(t) \qquad \text{(I.111)}$$

The constant term encodes the **defect contribution**, showing that the heat trace detects curvature concentrated at the puncture.

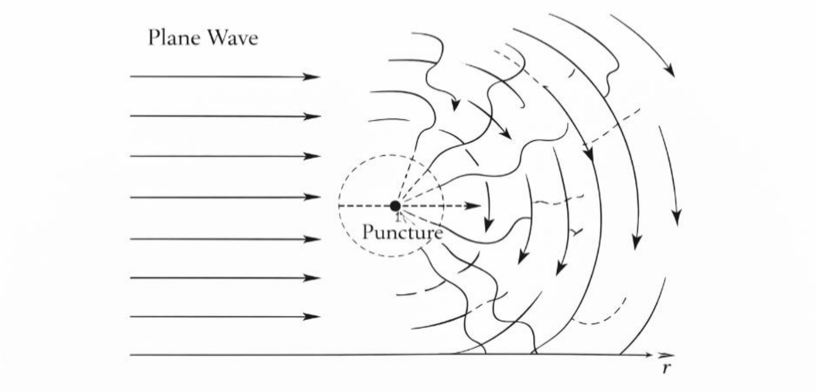


***Figure I.7.5* — Scattering Pattern.** *Incoming plane wave distorted by defect.*

7.6 Concluding Remarks on the Toy Models

The low-dimensional examples developed in this chapter demonstrate, in explicit and solvable settings, the central claim of this volume: punctures are not passive absences but active geometric structures with concrete analytic and spectral consequences.

Without introducing external potentials, additional fields, or modified dynamics, the mere presence of a defect alters operator domains, shifts spectra, generates bound states, modifies scattering, and contributes to heat kernel coefficients.

Across one-, two-, and three-dimensional models, a common pattern emerges. Geometry alone encodes information traditionally attributed to sources. Spectral flow on the punctured circle, angular rescaling on the conical plane, bound states in punctured three-space, and purely geometric scattering all arise from the same structural mechanism: the replacement of singular points by infinitesimal punctures carrying defect-supported curvature.

These toy models serve as concrete proofs of concept. They show that the abstract framework of infinitesimally punctured manifolds is not merely formal, but produces verifiable mathematical effects already at the level of linear operators and wave propagation. In this sense, they bridge the gap between foundational geometry and physical interpretation.

With these examples in place, the volume has completed its task: to establish that **singularities may be replaced by finite geometric structure**, and that such structure possesses intrinsic analytic content. The next step is to investigate how this structure evolves dynamically—a question taken up in the following volume.

CONCLUSION

This volume has developed the geometric and analytic foundations of infinitesimally punctured manifolds. Starting from the observation that classical singularities arise when point-like entities are forced into smooth geometric frameworks, we have proposed an alternative structural viewpoint in which matter and physical attributes are represented as intrinsic geometric defects—closed, measure-zero subsets of spacetime carrying distributionally supported curvature.

Within this framework, singular behavior is not eliminated by modifying dynamical laws, but replaced by finite geometric structure. The introduction of weak and strong regimes permits smooth differential geometry to coexist with defect-supported distributions on the same underlying set. Curvature blow-ups are reinterpreted as integrable concentrations, and regularising metric families provide a precise mathematical mechanism for passing from smooth approximations to punctured limits.

A central theme of the book has been that geometry alone already encodes a rich analytic content. Operator theory on punctured manifolds reveals that the mere presence of defects alters domains, self-adjointness properties, and spectral behavior. Bound states, spectral shifts, and scattering effects arise without the introduction of external potentials. Low-dimensional toy models make these consequences explicit and demonstrate that punctures possess observable mathematical signatures even in the absence of dynamics.

No field equations have been postulated in this volume. No variational principles have been invoked. This restraint is deliberate. The aim has been to establish that a coherent geometric structure exists in which singularities are replaced by defects and in which distributional curvature is not pathological but natural. Only once such a structure is firmly in place does it become meaningful to ask how it evolves.

The next volume, *Infinitesimally Punctured Physics*, addresses this question. Building on the framework developed here, it introduces a hybrid variational principle in which smooth curvature, defect-supported curvature, and an indeterminate transition sector enter on equal footing.

Variation of this action yields a master field equation that generalizes Einstein's theory and promotes defect density to a genuine geometric source. Within this setting, mass, charge, and spin acquire geometric interpretations, and quantum behavior is reformulated in terms of Infinitesimally Punctured Quantum Physics, where quantum objects are modeled as dense ensembles of infinitesimal constituents.

Thus, the present volume provides the structural bedrock upon which dynamics will be constructed. It establishes that singularities can be replaced by geometry, and that geometry alone already carries nontrivial analytic and spectral content. The transition from structure to law is the task of the volume that follows.

OPEN PROBLEMS

> The geometric framework developed in this volume raises several natural questions whose resolution lies beyond the present foundational treatment. We record a selection of problems that appear particularly significant.

1. Uniqueness of Defect Structures

To what extent is the distributional curvature supported on a defect set uniquely determined by regularising metric families? Given two distinct regularisations converging to the same punctured manifold, under what conditions do they produce identical defect-supported tensors?

2. Higher-Codimension Integrability

While integrability criteria have been established for curvature behavior near defects of fixed codimension, a systematic classification of admissible divergence profiles remains open. In particular, can one characterize all defect-supported curvature distributions compatible with global geometric constraints?

3. Coupling to Additional Geometric Structures

The present volume has restricted attention to Riemannian and pseudo-Riemannian geometry. How does the punctured framework extend to manifolds equipped with additional structures—such as symplectic forms, spin bundles, or gauge connections—and how do defects interact with these structures?

4. Global Topological Constraints

The relationship between defect sets and global topology requires further investigation. How do punctures alter characteristic classes, index theorems, and spectral invariants in higher dimensions? Can defect-supported curvature contribute nontrivially to global topological quantities?

5. Canonical Choice of Self-Adjoint Extension

In low codimension, the Laplace–Beltrami operator admits families of self-adjoint extensions. Is there a geometrically distinguished extension associated naturally with a given defect structure, or must additional principles be invoked?

6. Emergence of Dynamical Laws

The present work has deliberately refrained from introducing variational principles or field equations. A fundamental open problem is whether the weak–strong geometric decomposition admits a natural action functional whose variation yields consistent dynamical laws without reintroducing classical singularities.

> These problems delineate the boundary between geometry and physics within the punctured framework. The next volume addresses several of them by introducing a hybrid variational principle and promoting defect density to a dynamical quantity. The transition from structural possibility to dynamical necessity forms the central theme of *Infinitesimally Punctured Physics*.

LIST OF DEFINITIONS

Definition I.1 — Infinitesimally Punctured Manifold

An infinitesimally punctured manifold is a pair (M_{IP}, g) with

$$M_{\mathrm{IP}} = M \setminus P,$$

where $P \subset M$ is closed and of measure zero, together with the data of the defect set P.

Definition I.2 — Defect-Supported Tensor

A tensor-valued distribution whose support is contained in the defect set P.

Definition I.3 — Weak Regime

The smooth geometric regime defined on M_{IP} where classical differential geometry holds.

Definition I.4 — Strong Regime

The defect-supported geometric regime on P carrying distributional curvature.

Definition I.5 — Regularising Metric Family

A family of smooth metrics g_ε converging distributionally to a punctured metric.

Definition I.6 — Self-Adjoint Extension

A self-adjoint operator whose restriction to $C_c^\infty(M \setminus P)$ coincides with the minimal Laplacian.

Definition I.7 — Transition Layer

The ε-neighbourhood $N_\varepsilon(P)$ in which geometry interpolates between the weak and strong regimes.

Definition I.8 — Neutrosophic Structural Decomposition

The splitting

$$R = R_T + R_F + R_I$$

into smooth (R_T), defect-supported (R_F), and transition (R_I) curvature component.

THEOREMS AND LEMMAS

Lemma I.1 — Finite-Curvature Criterion

A defect of codimension k whose curvature behaves locally as

$$| R | \sim r^{-\alpha}$$

is locally integrable near the defect **iff**

$$\alpha < k.$$

Proposition I.1 — Codimension-2 Defects and Holonomy

If P is a closed embedded submanifold of codimension 2 in a simply-connected manifold M, then

$$\pi_1(M \setminus P) \neq 0,$$

and loops encircling P may carry non-trivial holonomy.

Proposition I.3 — Codimension Criterion for Essential Self-Adjointness

Let M be complete and smooth and let P be a closed submanifold.
If

$$\text{codim}\,(P) \geq 4,$$

then the Laplacian on $C_c^\infty(M \setminus P)$ is essentially self-adjoint.
If

$$\text{codim}\,(P) \leq 3,$$

then self-adjoint extensions may exist.

Theorem I.2 — Finite Replacement Principle

Let a smooth metric exhibit curvature blow-up confined to a closed measure-zero set P. If a regularising family g_ε exists with uniformly bounded curvature integrals, then the limit geometry is an infinitesimally punctured manifold with defect-supported curvature and **no** distributional divergence.

Theorem I.3 — Stability of the Essential Spectrum

For a compact measure-zero defect set P, the essential spectrum of any self-adjoint extension of the Laplacian on $M \setminus P$ coincides with the essential spectrum of the Laplacian on the unpunctured manifold M:

$$\sigma_{\text{ess}}(\Delta_\lambda) = \sigma_{\text{ess}}(\Delta).$$

INDEX TERMS

J
jump conditions (connection)

K
Kretschmann scalar

L
Laplace–Beltrami operator
Lebesgue measure
limit in distribution sense
local integrability

M
manifold
measure-zero set
metric continuity
metric regularisation

N
Neutrosophic decomposition
Neutrosophic triad
normal bundle

O
operator domain
operator closure
operator extension

P
parallel transport
phase shift
point defect
puncture
punctured domain
punctured manifold

Q
quantum interpretation

R
rank-one perturbation
regularisation
Riemann tensor
resolvent operator

S
scattering
self-adjoint extension
singularity
Sobolev space
spectral flow
spectral stability
strong regime
smooth regime

T
test function
topological defect
topology
transition layer

U
ultraviolet divergence

V
variational principle

W
wave equation
weak regime
weak–strong structure

APPENDIX A
ANALYTICAL BACKGROUND FOR INFINITESIMALLY PUNCTURED GEOMETRY

> The material collected in this appendix summarizes the minimal analytical framework required for the internal consistency of infinitesimally punctured manifolds. No new analytical structures are introduced. All constructions rely on standard results from distribution theory, Sobolev spaces, and the theory of self-adjoint extensions of singular differential operators.

Complete proofs may be found in classical references such as L. Hörmander, *The Analysis of Linear Partial Differential Operators*, and R. A. Adams & J. J. F. Fournier, *Sobolev Spaces*.

A.1 Test Functions and Distributions

Let M be a smooth manifold. Denote by

$$C_c^\infty(M) = \{\varphi: M \to \mathbb{C} \mid \varphi \text{ smooth with compact support}\}.$$

Definition A.1 (Distribution).

A distribution on M is a continuous linear functional

$$T: C_c^\infty(M) \to \mathbb{C}.$$

The space of all distributions is denoted by $\mathcal{D}'(M)$.

Example A.1 (Dirac Distribution).

For $p \in M$,

$$\langle \delta_p, \varphi \rangle = \varphi(p), \forall\, \varphi \in C_c^\infty(M).$$

The support of δ_p is $\{p\}$; it is the prototype of a puncture-supported object.

A.2 Weak Derivatives

Let $f \in L^1_{\text{loc}}(M)$.

Definition A.2 (Weak Derivative).

A function $g \in L^1_{\text{loc}}(M)$ is the weak derivative of f with respect to the coordinate x^i if

$$\int_M f\ \partial_i \varphi\, d\mu - \int_M g\ \varphi\, d\mu = 0, \forall\ \varphi \in C_c^\infty(M).$$

Weak derivatives allow differentiation across nonsmooth or punctured regions.

Remark.

If f is smooth on $M \setminus P$ and exhibits finite jumps across a discrete set P, its weak derivative contains delta-supported contributions. This observation underlies the punctured derivative operators used in Chapter 2.

A.3 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^n$ be open.

Definition A.3 (First-Order Sobolev Space).

$$H^1(\Omega) = \{u \in L^2(\Omega) \mid \nabla u \in L^2(\Omega)\},$$

with norm

$$\| u \|_{H^1}^2 = \int_\Omega |\ u\ |^2\ \ dx + \int_\Omega |\ \nabla u\ |^2\ \ dx.$$

Relevance to IP Geometry.

If $\psi \in H^1(M_{\text{IP}})$, then

- the associated energy integral is finite,
- jump behavior near punctures is controlled,
- distributional corrections are admissible.

A.4 Distributional Laplacian with Point Support

Let Δ denote the Euclidean Laplacian on $\mathbb{R}^n$.

If f is smooth on $\mathbb{R}^n \setminus \{p\}$ and has a Green-type singularity at p, then

$$\Delta f = (\Delta f)_{\text{reg}} + C\ \delta_p. \qquad \text{(A.1)}$$

Example A.2 (Fundamental Solution).

For $n > 2$,

$$G(x) = \frac{1}{(n-2)\,\omega_n} \mid x \mid^{2-n}, \Delta G = \delta_0,$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$.

A.5 Integrable Singularities

Let

$$R(x) \sim \mid x \mid^{-k}, x \to 0.$$

Proposition A.1 (Local L^1 Criterion).

$$R \in L^1_{\text{loc}}(\mathbb{R}^n) \Leftrightarrow k < n.$$

This criterion guarantees that curvature concentrations considered in Chapter 4 define distributions.

A.6 Self-Adjoint Extensions of Punctured Operators

Let

$$H = -\Delta + \beta\,\delta_p, \beta \in \mathbb{R},$$

initially defined on $C_c^\infty(\mathbb{R}^n \setminus \{p\})$.

Associated quadratic form:

$$Q[f] = \int_{\mathbb{R}^n} \mid \nabla f \mid^2 \; dx + \beta \;\mid f(p) \mid^2, f \in H^1(\mathbb{R}^n).$$

Proposition A.2.

The form Q is closed and bounded below; therefore a unique self-adjoint operator exists via the Friedrichs extension. This ensures that punctured Hamiltonians used in Chapter 5 generate unitary evolution.

A.7 Distributional Curvature

Let g be smooth on $M \setminus P$ with P closed and of measure zero. If $g \in H^1_{\text{loc}}(M)$, then the Christoffel symbols belong to $L^2_{\text{loc}}(M)$ and curvature components define distributions:

$$R_{\mu\nu\rho\sigma} \in \mathcal{D}'(M).$$

A.8 Structural Summary

An infinitesimally punctured manifold is mathematically consistent provided:

- P is closed and measure zero,
- Metric and curvature are locally integrable,
- Distributional extensions are well defined,
- Relevant operators admit self-adjoint realizations.

> The analytical machinery required by infinitesimally punctured geometry lies entirely within standard functional analysis and weak differential geometry. The framework does **not** introduce exotic objects; it reorganizes classical tools to interpret singular behavior as finite geometric structure.

APPENDIX B
EXERCISES & THOUGHT EXPERIMENTS

Chapter 1 — Motivation, Punctures, and Structural Decomposition

Exercise B.1.1 — Puncture Set and Measure-Zero

Let

$$M = \mathbb{R}^2, P = \{0\}.$$

1. Show that P is closed and has Lebesgue measure zero.
2. Show that

$$M_{\mathrm{IP}} = M \setminus P$$

is connected but **not** simply connected.

3. Compute

$$\pi_1\ (M_{\mathrm{IP}}).$$

Exercise B.1.2 — Neighborhood of a Puncture

Let $P = \{0\} \subset \mathbb{R}^n$ and define

$$N_\varepsilon(P) = \{\, x \mid x < \varepsilon \,\}.$$

1. Show $N_\varepsilon(P)$ is open and has finite measure for each $\varepsilon > 0$.
2. Show

$$\mu\left(N_\varepsilon(P)\right) \to 0 \text{ as } \varepsilon \to 0.$$

3. Interpret this limit in the context of the "transition layer" $N_\varepsilon(P)$.

Thought Experiment B.1.3 — Singularity vs Structural Incompleteness

A classical curvature invariant diverges at a point p. Rather than interpreting this as a physical infinity, declare p to be part of a defect set P supporting distributional curvature.

What minimal properties must this defect-supported contribution satisfy for the geometry to remain consistent (e.g., integrability, well-defined action on test functions, compatibility with smooth geometry off P)?

Chapter 2 — Infinitesimally Punctured Manifolds and Integrability

Exercise B.2.1 — Integrable Singularity Criterion (Solution)

In $\mathbb{R}^n$ define

$$R(r) = r^{-k}, r = | x |.$$

Compute

$$I(\varepsilon) = \int_{|x|<\varepsilon} R(r)\, dx$$

and determine precisely for which k the integral converges as $\varepsilon \downarrow 0$.

Solution (sketch). Using spherical coordinates ($dx = r^{n-1} dr\, d\Omega$),

$$I(\varepsilon) \propto \int_0^{\varepsilon} r^{n-1-k}\, dr,$$

which converges iff $n - k > 0$, i.e. $k < n$.

Exercise B.2.2 — A Weak Derivative with a Jump

Define

$$f(x) = \begin{cases} x, & x < 0, \\ 2x, & x > 0. \end{cases}$$

1. Compute the classical derivative away from 0.
2. Compute the weak derivative f' on $\mathbb{R}$.
3. Show explicitly that a Dirac-delta term appears.

Hint: Use Definition A.2 and integrate by parts on $(-\infty, 0) \cup (0, \infty)$.

Thought Experiment B.2.3 — Puncture vs Removed Point

Compare

- (i) the space $\mathbb{R}^2 \setminus \{0\}$, and
- (ii) $\mathbb{R}^2$ equipped with a defect-supported distribution at 0.

Discuss topology, geodesic behavior, and how "physical attributes" might be encoded without introducing divergent fields.

Chapter 3 — Weak–Strong Geometry and Defect-Supported Curvature

Exercise B.3.1 — Conical Defect and Total Curvature

Consider the conical metric

$$ds^2 = dr^2 + (1-\delta)^2\, r^2\, d\theta^2, \theta \in [0,2\pi).$$

1. Show that curvature vanishes for $r > 0$.
2. Show that the **total** curvature equals $2\pi\delta$ when interpreted distributionally.

Hint: Use Gauss–Bonnet on a punctured disk.

Exercise B.3.2 — Holonomy Around a Defect

For the same conical metric, compute the holonomy of parallel transport around a circle $r =$ const. Show it corresponds to a rotation by the deficit angle δ.

Thought Experiment B.3.3 — What Counts as "Strong" Structure?

Give examples of defect-supported data that are

- topological (holonomy class) versus
- analytic (delta curvature).

Which aspects are invariant under smooth regularisation families g_ε?

Chapter 4 — Geometric Regularisation of Classical Singularities

Exercise B.4.1 — Curvature Integrability for a Regularised Core (Solution)

Assume the regularised scalar curvature

$$R_\varepsilon(r) = \frac{1}{(r+\varepsilon)^3}, r \geq 0,\ \varepsilon > 0.$$

Compute

$$I(a,\varepsilon) = \int_0^a r^2\ R_\varepsilon(r)\, dr$$

and show it is finite for each $\varepsilon > 0$. Analyse the limit $\varepsilon \to 0$.

Solution (sketch). Direct computation yields a logarithmic term

$$\log\left(\frac{a+\varepsilon}{\varepsilon}\right),$$

finite for $\varepsilon > 0$ but diverging as $\varepsilon \to 0$, consistent with concentration into a distribution.

Exercise B.4.2 — Distributional Limit (Conceptual)

Let $R_\varepsilon(r)$ be as above. Describe (without computing constants) how one would show that R_ε converges **in the sense of distributions** to a defect-supported term plus a regular part. Specify what test functions are used and what must be bounded.

Thought Experiment B.4.3 — What Is "Inside" a Puncture?

If curvature is supported on P, what does it mean geometrically to speak of an "interior'' of P? Discuss three viewpoints:

1. Absence of manifold points,
2. Regime change (weak ↔ strong),
3. Limiting idealisation of a finite core $N_\varepsilon(P)$.

Chapter 5 — Operator Domains and Self-Adjoint Extensions

Exercise B.5.1 — Symmetry of the Punctured Laplacian

Let Δ be the Laplacian on $\mathbb{R}^n \setminus \{0\}$ with domain $C_c^\infty(\mathbb{R}^n \setminus \{0\})$. Show that Δ is symmetric on L^2.

Exercise B.5.2 — Radial Asymptotics in $\mathbb{R}^3 \setminus \{0\}$

Assume an s-wave function satisfies

$$\psi(r) \sim \frac{A}{r} + B, r \to 0.$$

Explain why such behavior lies outside $H^1(\mathbb{R}^3)$ unless constrained.

1. State a boundary relation $B = \lambda A$ that selects a self-adjoint extension.

Thought Experiment B.5.3 — Geometric Meaning of Extension Parameters

Self-adjoint extension parameters are often viewed as “coupling constants.’’ In the punctured framework, propose a **purely geometric** interpretation of such parameters (e.g., defect strength, core regularisation class).

Chapter 6 — Spectrum, Scattering, and Heat-Kernel Signatures

Exercise B.6.1 — Spectral Shift on a Punctured Circle

Using the boundary condition

$$\psi(2\pi) = e^{i\alpha}\, \psi(0),$$

derive the eigenvalues

$$\lambda_n = \left(n + \frac{\alpha}{2\pi}\right)^2.$$

Exercise B.6.2 — Phase Shift for a Point Defect (Qualitative)

Explain why a point defect in $\mathbb{R}^3$ can produce a non-trivial s-wave phase shift **without** introducing any classical potential. Identify where the defect enters the analysis.

Exercise B.6.3 — Heat-Kernel Constant Term

For a 2-D conical defect, the heat-trace expansion contains a constant term depending on δ. Explain why such a term indicates that the spectrum detects curvature concentrated at the defect.

Chapter 7 — Low-Dimensional Toy Models

Exercise B.7.1 — Bound State from an Extension Parameter

For $\mathbb{R}^3 \setminus \{0\}$, assume a self-adjoint extension leads to a negative eigenvalue $E < 0$. Show that **dimensional analysis alone** implies

$$E \propto -\lambda^{-2},$$

where λ is a length-scale parameter associated with the extension.

Thought Experiment B.7.2 — "Potential-Free" Localization

In standard quantum mechanics, bound states are associated with potentials. In the punctured framework, localisation arises from operator-domain structure. Discuss what experimental or conceptual distinctions might exist between these mechanisms, purely at the level of wave propagation and scattering.

Graduate-Level Extension

Problem G1 — Spectral Theory of Punctured Laplacians

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and

$$P = \{p_1, \dots, p_N\} \subset \Omega.$$

Consider operators realised via self-adjoint extensions associated with point defects.

1. Using quadratic-form methods, show existence of a self-adjoint realisation and characterise its domain in terms of boundary maps at each p_j.
2. Prove the spectrum is discrete and accumulates only at $+\infty$.
3. Discuss how extension parameters influence the existence of negative eigenvalues.

Hint: Use the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and boundedness of point evaluation in dimensions $n \leq 3$.

Problem G2 — Distributional Curvature and Geometric Charges

Let $M = \mathbb{R}^2$ and consider a conical defect at the origin with deficit angle δ.

1. Show that curvature vanishes away from the origin.
2. Show that the integrated curvature over any disk containing the origin equals $2\pi\delta$.
3. Interpret $2\pi\delta$ as a geometric “charge’’ associated with the puncture.

This problem provides a purely geometric analogue of source terms without invoking field equations.

> The exercises in this appendix emphasize a single theme: punctures alter geometry and analysis through topology, distributional curvature, and operator domains. Even before dynamics is introduced, the framework generates explicit spectral, scattering, and integrability effects. These problems prepare the reader for the transition in the next volume, where puncture-supported structure is promoted from a static geometric ingredient to a dynamical quantity.

The *Infinitesimal Punctures* Series

The *Infinitesimal Punctures* series develops a geometric framework in which singularities and point-like sources are replaced by measure-zero defects carrying distributional structure. Instead of inserting matter into spacetime as external entities, physical attributes are interpreted as intrinsic features of geometry. The series progresses from foundational definitions, through dynamical formulations, to a unified structural and meta-geometric formalism.

1. *Infinitesimally Punctured Geometry*

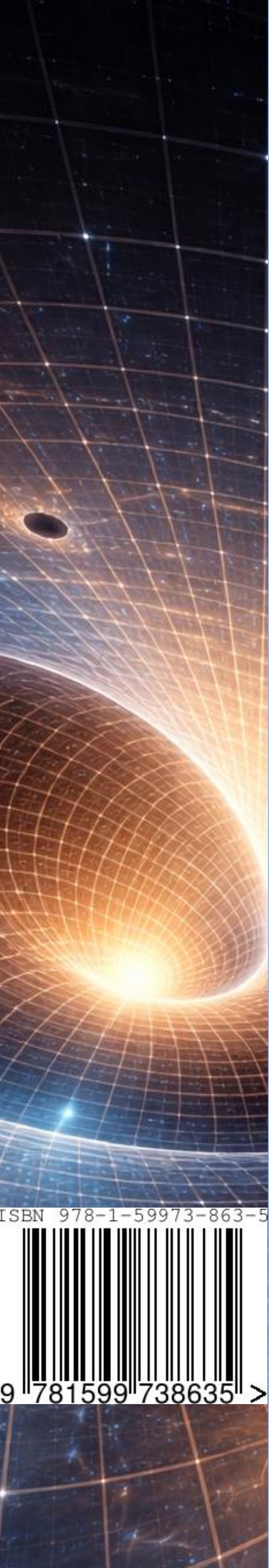
This volume establishes the mathematical foundations of infinitesimally punctured manifolds. It introduces weak–strong geometric regimes, distributional curvature, integrability criteria, and operator theory on punctured domains. Singularities are reinterpreted as finite geometric structures, and explicit low-dimensional models demonstrate the analytic and spectral consequences of punctures.

2. *Infinitesimally Punctured Physics*

The second volume develops the dynamical laws governing punctured spacetime. A hybrid variational principle leads to a master field equation in which smooth curvature, defect-supported curvature, and an indeterminate transition sector enter on equal footing. Mass, charge, and quantum behavior acquire geometric interpretations, and applications include regularised black holes, modified cosmology, and geometric views of dark matter and dark energy.

3. *Infinitesimally Punctured Structures*

The final volume formulates a unified Smarandache–Neutrosophic structural framework for multi-regime geometry. It develops S-MultiSpace and S-MultiStructure geometry, hybrid connections, generalized curvature, and variational principles for topological matter. The volume provides a meta-geometric language in which multiple geometric regimes coexist within a single coherent structure.



For more than a century, singularities and ultraviolet divergences have stood at the frontiers of modern theoretical physics, marking points where our most successful theories cease to be mathematically well defined.

Infinitesimal Punctures proposes a structural shift in perspective: instead of inserting point-like sources into smooth manifolds, matter and physical attributes are interpreted as intrinsic geometric defects—measure-zero punctures—within spacetime itself. In this framework, curvature, charge, and quantum behavior arise not as external additions but as distributionally supported features of geometry.

The series develops this idea systematically, moving from foundational geometry, through dynamical physical laws, to a unified S-MultiSpace and S-MultiStructure structural formalism.

The Infinitesimally Punctured Wave (IPW), Infinitesimally Punctured Surface (IPSu), Infinitesimally Punctured Space (IPSp), Infinitesimally Punctured Manifold (IPM), and in general Infinitesimally Punctured Quantum Physics (IPQP) were introduced and developed by Florentin Smarandache in 2019 and respectively in 2025-2026.

The ‘infinitesimal distance’ (which is virtual and theoretical) was later extended by the author to a ‘very tiny real distance’ (which is practical), allowing a wave to be ‘broken’ in a real sense at any point.

This volume establishes the geometric and analytic foundations of infinitesimally punctured manifolds. It introduces the weak–strong decomposition of geometry, in which smooth differential structure coexists with defect-supported distributional curvature on measure-zero sets. Classical singularities are reinterpreted as punctures carrying finite, integrable curvature, replacing divergent invariants with well-defined distributional objects.

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