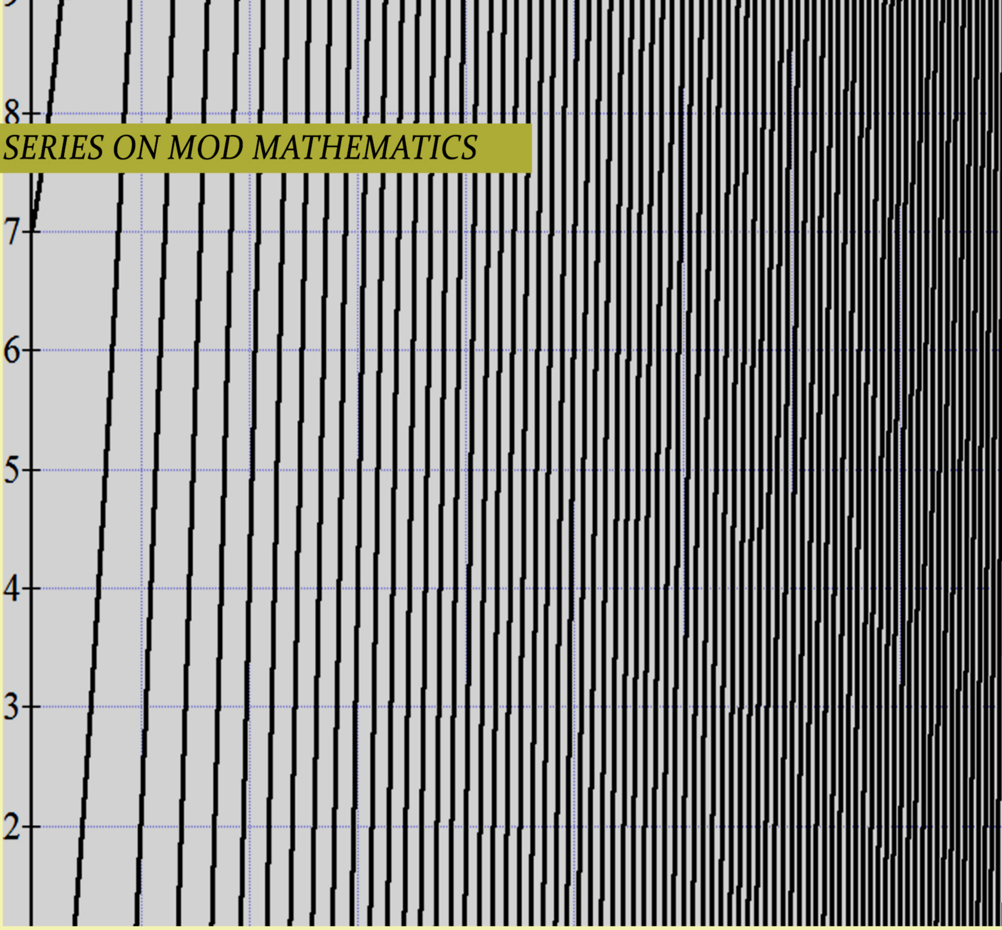

SERIES ON MOD MATHEMATICS

MOD PLANES

A NEW DIMENSION TO MODULO THEORY

W.B.VASANTHA KANDASAMY
K. ILANTHENRAL
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MOD Planes:
A New Dimension to Modulo Theory

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CONTENTS

PREFACE

In this book for the first time authors study mod planes using modulo intervals [0, m); $2 \leq m \leq \infty$. These planes unlike the real plane have only one quadrant so the study is carried out in a compact space but infinite in dimension. We have given seven mod planes viz real mod planes (mod real plane) finite complex mod plane, neutrosophic mod plane, fuzzy mod plane, (or mod fuzzy plane), mod dual number plane, mod special dual like number plane and mod special quasi dual number plane. These mod planes unlike real plane or complex plane or neutrosophic plane or dual number plane or special dual like number plane or special quasi dual number plane are infinite in numbers.

Further for the first time we give a plane structure to the fuzzy product set $[0, 1) \times [0, 1)$; where 1 is not included; this is defined as the mod fuzzy plane. Several properties are derived.

This study is new, interesting and innovative. Many open problems are proposed. Authors are sure these mod planes will give a new paradigm in mathematics.

We wish to acknowledge Dr. K Kandasamy for his sustained support and encouragement in the writing of this book.

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Chapter One

INTRODUCTION

In this book authors define the new notion of several types of MOD planes, using reals, complex numbers, neutrosophic numbers, dual numbers and so on. Such study is very new and in this study we show we have a MOD transformation from R to the MOD real plane and so on.

We give the references needed for this study.

In the first place we call intervals of the form [0, m) where $m \in N$ to be the MOD interval or small interval. The term MOD interval is used mainly to signify the small interval [0, m) that can represent $(-\infty, \infty)$. If m = 1 we call [0, 1) to be the MOD fuzzy interval.

We define $R_n(n) = \{(a, b) \mid a, b \in [0, m)\}$ to be the MOD real plane.

If m = 1 we call $R_n(1)$ as the MOD fuzzy plane.

The definitions are made and basic structures are given and these MOD planes extended to MOD polynomials and so on. For study of intervals [0, m) refer [20-1].

For neutrosophic numbers please refer [3, 4].

For the concept of finite complex numbers refer [15]. The notion of dual number can be had from [16].

The new notion of special quasi dual numbers refer [18].

The concept of special dual like numbers and their properties refer [17]. For decimal polynomials [5-10].

We have only mentioned about the books of reference. Several open problems are suggested in this book for the researchers.

It is important that the authors name these structures as MOD intervals or MOD planes mainly because they are small when compared to the real line or real plane. The term MOD is used mainly to show it is derived using MOD operation.

Chapter Two

THE MOD REAL PLANES

In this chapter authors for the first time introduce a new notion called "MOD real plane". We call this newly defined plane as MOD real plane mainly because it is a very small plane in size but enjoys certain features which is very special and very different from the general real plane.

Recall the authors have defined the notion of semi open squares of modulo integers [23]. Several algebraic properties about them were studied [23]. Here we take [0, m) the semi open interval, $1 \leq m < \infty$ and find $[0, m) \times [0, m)$; this represents a semi open square which we choose to call as the MOD real plane.

This is given the following representation.

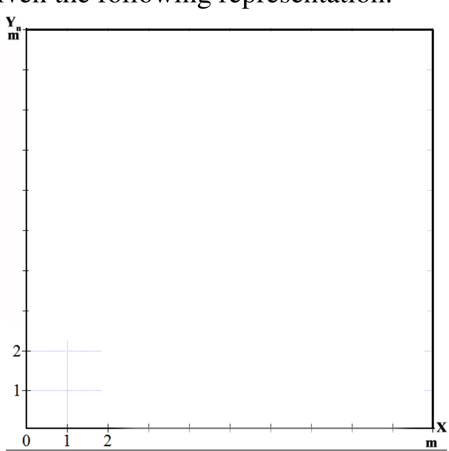

Figure 2.1

The point m on the line is not included. Further the dotted line clearly shows that the boundary is not included.

Further the X-axis is termed as X_n to show it is the MOD X - axis and Y_n the MOD Y-axis of the MOD real plane $R_n(m)$.

So this MOD real plane is denoted by $R_n(m)$. In fact we have infinite number of MOD planes as $1 \leq m < \infty$.

Thus throughout this book $R_n(m)$ will denote the MOD real plane where m shows the MOD interval [0, m) is used.

We will give examples of MOD planes.

Example 2.1: Let $R_n(5)$ be the MOD real plane relative to the interval [0, 5).

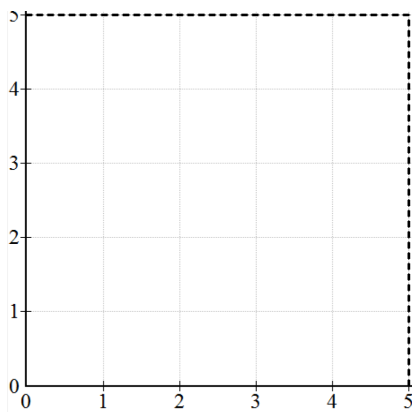

Figure 2.2

Any point P of (x,y) is such that $x, y \in [0, 5)$.

Example 2.2: Let $R_n(6)$ be the MOD real plane relative to interval [0, 6).

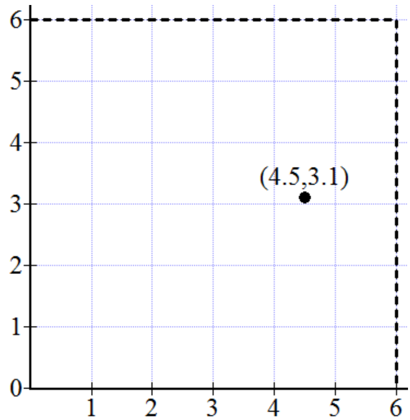

Figure 2.3

P(4.5, 3.1) is a point on the MOD plane $R_n(6)$.

Clearly P(8.5,7) $\notin$ $R_n(6)$ however

P(8.5 – 6, 7 – 6) = (2.5, 1)$\in$ $R_n(6)$.

Thus any point in the real plane R say (–17.2, 9.701) can be marked as (0.8, 3.701).

Thus every point in R has a corresponding point on R(1, 6) in the real plane is marked as (1, 0) and so on.

That is why we call this MOD plane, for any point in the real plane be it in any quadrant we can find its place and position in the MOD plane $R_n(m)$ which has only one quadrant.

(9, –17.01) in R is mapped on to (3, 0.99) in the MOD plane $R_n(6)$.

Let (–10.3, –12.7) $\in$ R, now the corresponding point in $R_n(6)$ is (1.7, 5.3) $\in$ $R_n(6)$.

Thus for a point in R there exist one point in $R_n(m)$.

It is important to keep on record for several points (or infinite number of points) in R, we may have only one point in $R_n(m)$.

We see (14, 9), (8, 15) and (20, 21) so on in R are mapped to (2, 3) only in the MOD plane $R_n(6)$.

That is why as the real plane R can be accommodated in $R_n(m)$; we call $R_n(m)$ the MOD real plane or small real plane.

Because of the fact Z_n cannot be ordered as a group or ring we cannot always find the modulo distance between any 2 points in $R_n(m)$ and even if it exist it is not well defined.

Example 2.3: Let $R_n(7)$ be the MOD plane on the interval [0, 7).

We see if (3, 2) is point in $R_n(7)$ it can be mapped on to infinite number of points in R.

For (10, 9), (17, 9), (24, 9), (31,9), …, (7n+3, 9); $n \in Z$ are all mapped on to (3,2).

Also all points are of the form (3, 9), (3, 16), (3, 23), (3, 30), (3, 37), …, (3, 7n + 2) ($n \in Z$) in R are mapped on to (3, 2) in $R_n(7)$.

Thus to each point in $R_n(7)$ there exist infinite number of points in R.

On the contrary to these infinite set of points in R we have only one point mapped in $R_n(7)$.

Also the points (10, 9), (17, 16), (10, 16), (17, 9),(10, 23) (17, 23), (24, 16), (24, 9) (24, 23) and so on is also an infinite collection which is mapped on to (3, 2).

Likewise to each point in $R_n(m)$ we have infinitely many points associated with it in R.

Let $(-3, -1) \in$ R then $(-3, -1)$ is mapped on to $(4, 6)$ in $R_n(7)$.

In view of this we have the following theorem.

THEOREM 2.1: *Let R be the real plane and $R_n(m)$ be the MOD real plane of the interval [0, m).*

To infinite number of points in R we have a unique point in $R_n(m)$.

The proof is direct as is evident from the examples.

Next we study about the concept of distance in $R_n(m)$.

We see in $R_n(13)$ the modulo distance between P(2, 1) and Q(12, 12) in the MOD real plane of $R_n(13)$ is zero.

$$
\begin{aligned}
PQ^2 &= (2-12)^2 + (1-12)^2 \\
&= (2+1)^2 + (1+1)^2 \\
&= 9 + 4 \\
&= 0 \bmod 13.
\end{aligned}
$$

So a distance concept cannot be defined in a MOD plane only the modulo distance.

For consider the MOD real plane $R_n(13)$.

Let P(2, 1) and Q(12, 12) be in $R_n(13)$ they are plotted in the following MOD plane.

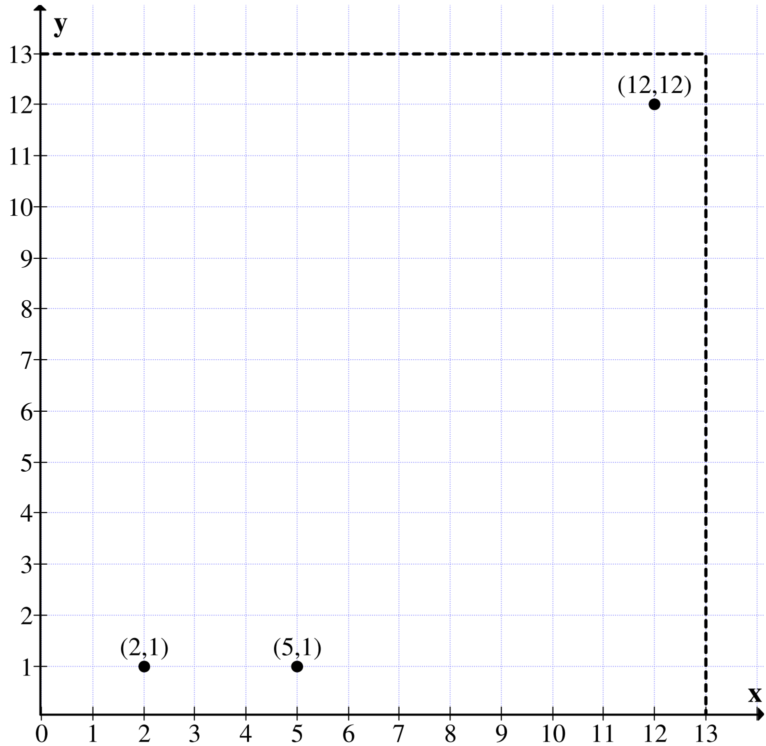

Figure 2.4

One may think that is the largest distance but on the contrary it is zero.

Consider S(5, 1) and 0 = (0, 0) $\in$ $R_n(13)$.

The distance

$$
\begin{aligned}
0_nS^2 &= (5-0)^2 + (1-0)^2 \\
&= 25 + 1 \\
&\equiv 0 \pmod{13}.
\end{aligned}
$$

So this distance is also zero in $R_n(13)$.

So a natural open conjecture is to find or define on $R_n(m)$ a special MOD distance.

Example 2.4: Let $R_n(8)$ be the MOD real plane of [0, 8).

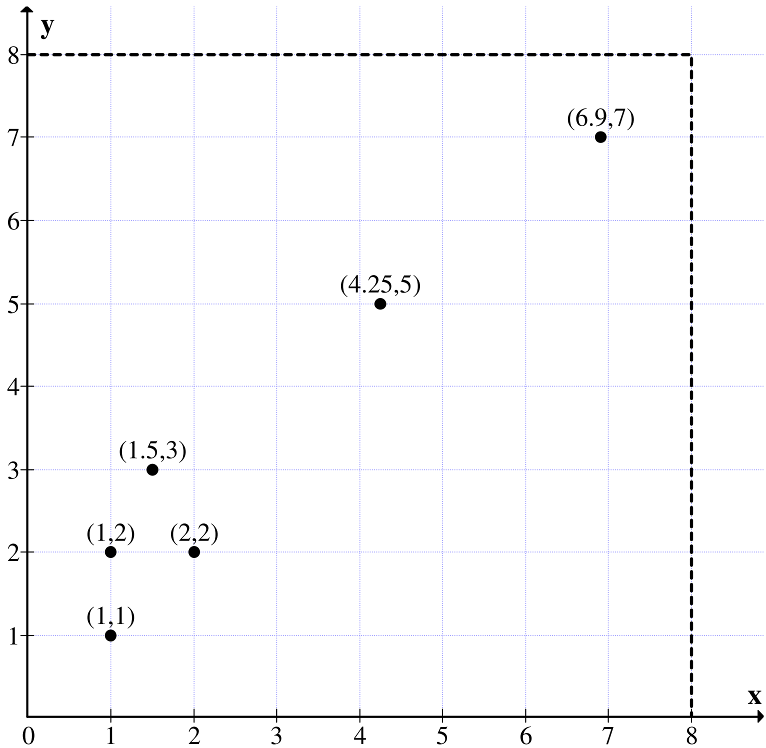

Figure 2.5

We see we have MOD real points marked in the MOD real plane.

However we are not in a position to place any distance concept.

We see how far taking the square root or cube root or n^{th} root of a number in [0, n) works out.

For the present for any $x \in [0,m)$ we define $\sqrt{x}$ to be as in case of reals. Also for x^2, it is mod m if $x \in [0, m)$.

We will see if

$$3.21 \in [0, 8)$$

then

$$(3.21)^2 = 2.3041 \in [0, 8);$$
$$\sqrt{3.21} = 1.791647287.$$

We see for

$$4 \in [0,8);\ 4^2 = 0 \pmod 8 \text{ but } \sqrt{4} = 2.$$

However finding for every $x \in [0,m)$, x^{-1} is a challenging question in the MOD real plane.

Next natural question will be; how many such MOD real planes exists?

The answer is infinite for every non zero integer $n \in Z^+$ we have a MOD real plane associated with it.

Next we show how the MOD real plane $R_n(m)$ is actually got by mapping all the other three quadrants into the first quadrant plane which is not as large as the real plane but on the smaller scale yet it is infinite in structure.

That is if $R_n(m)$ is the MOD real plane on $[0, m)$ and R the real plane then any $(x, y) \in R$ is mapped on to

$$(t = x \pmod m,\ s = y \pmod m))$$

if x and y are in the first quadrant.

If (x, y) is in the second quadrant then $x < 0$ and $y > 0$ so that (x, y) is mapped to

$$(m - t \equiv x \pmod m,\ s \equiv y \pmod m)).$$

On similar lines the points in other quadrants are also mapped to $R_n(m)$.

We will first illustrate this by an example.

Example 2.5: Let $R_n(10)$ be the MOD real plane related to the interval [0, 10).

Let P = (27, 48.33152) ∈ R, the first quadrant of R. Then P is mapped to (7, 8.33152) in $R_n(10)$.

Let Q = (–3.7201, 16.9) be the element in the second quadrant of R.

This point is mapped to (6.2799, 6.9) in the MOD real plane $R_n(10)$.

Let M = (–78.00391, –26.0007) be in the 3rd quadrant of R. M is mapped on to (1.99609, 3.9993) in the MOD real plane $R_n(10)$.

Let N = (25.0051, –12.921) be a point in the 4th quadrant of the real plane R. N is mapped on to (5.0051, 7.079) in the MOD real plane $R_n(10)$.

This is the way mappings are made. Thus all the three quadrants of the real plane is mapped on to the MOD real plane.

Next work will be to find how functions in the MOD real plane behave.

If we have a function in real plane then we study its appearance in the MOD real plane.

Example 2.6: Let $R_n(11)$ be the MOD real plane on the semi open interval [0, 11).

Let y = x be the function in the real plane, it is given in Figure 2.6.

The same function is given in the MOD plane in Figure 2.7.

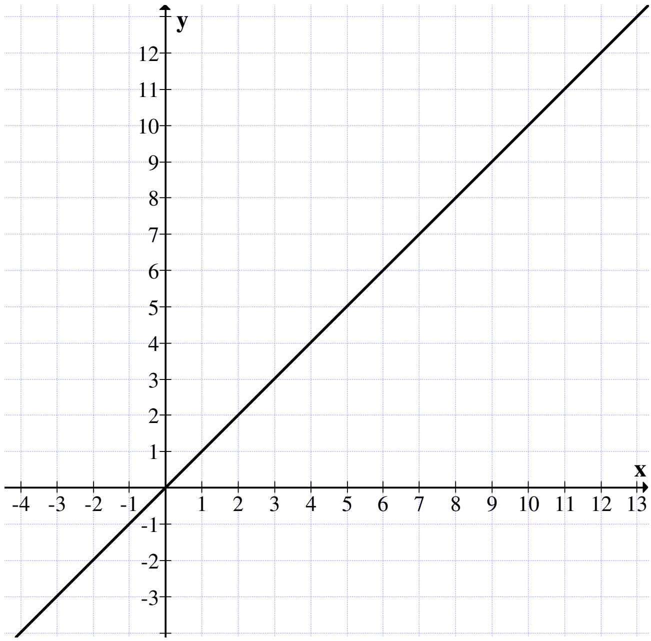

Figure 2.6

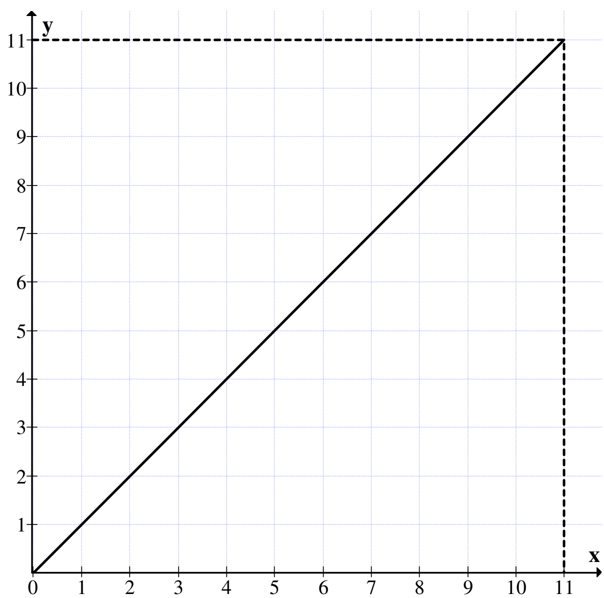

Figure 2.7

So this function y = x is the same in the first quadrant of the real plane and the MOD real plane.

Let us consider the function y = 2x, how does the function gets mapped in the real plane and the MOD real plane.

We know y = 2x in the real plane is given in the following:

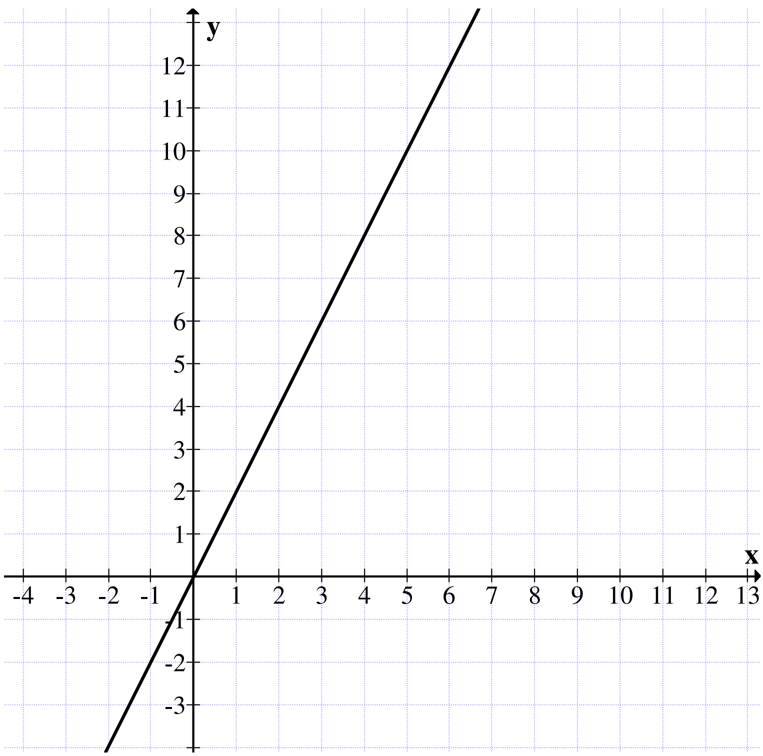

Figure 2.8

y = 2x in the MOD real plane $R_n(11)$ is as follows:

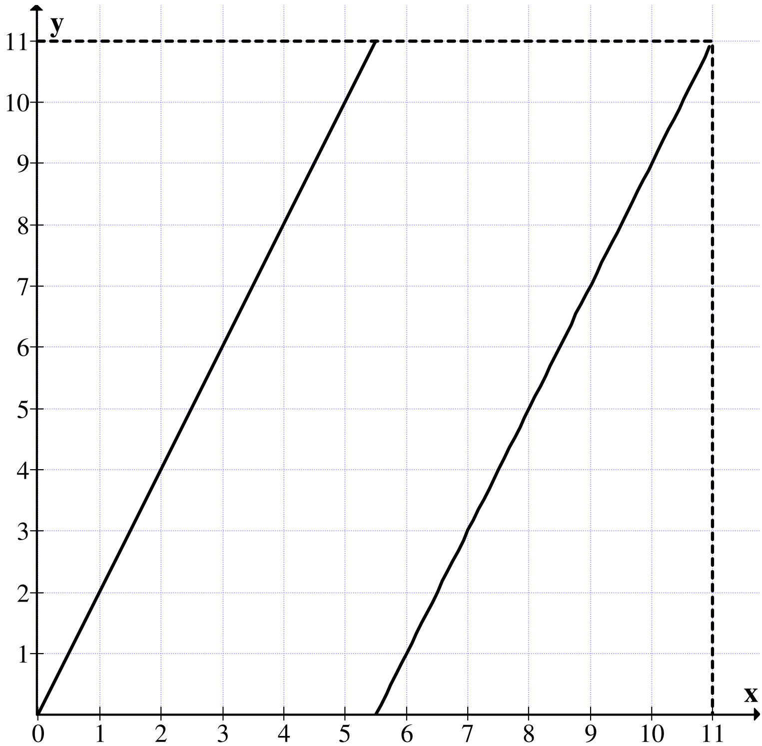

Figure 2.9

when

$$x = 5.5,\ y = 2 \times 5.5 = 0 \pmod{11},$$

when

$$x = 6,\ y = 2 \times 6 = 1 \pmod{11}$$

and when

$$x = 7,\ y = 2 \times 7 = 3 \pmod{11}.$$

Thus in both the representations the function y = 2x is distinct.

Thus functions defined on MOD real plane behaves differently from the functions defined in the real plane.

We will illustrate this situation by some examples.

Example 2.7: Let $R_n(12)$ be the MOD real plane associated with the interval.

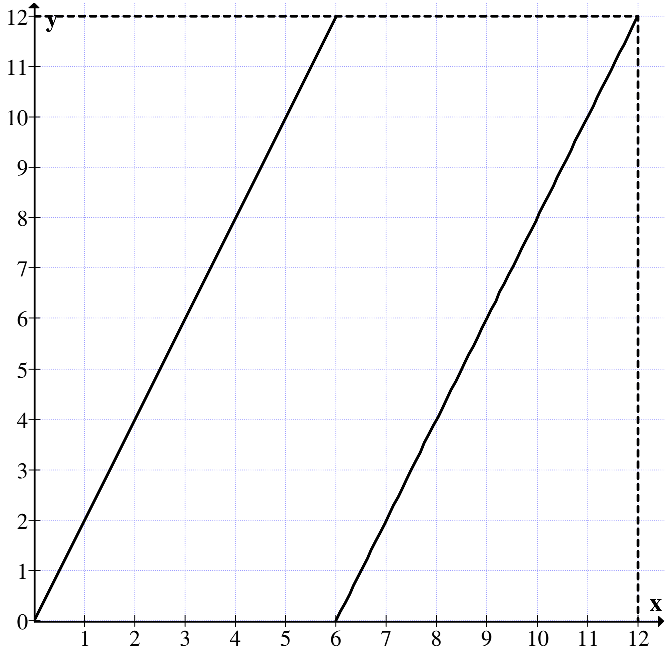

Figure 2.10

Let us consider the function y = 2x.

We see two MOD parallel lines and the points are also the same as far as the Y_n coordinate is concerned.

Now consider y = 4x in the real and the MOD real plane $R_n(12)$.

In the real plane, y = 4x graph is as follows:

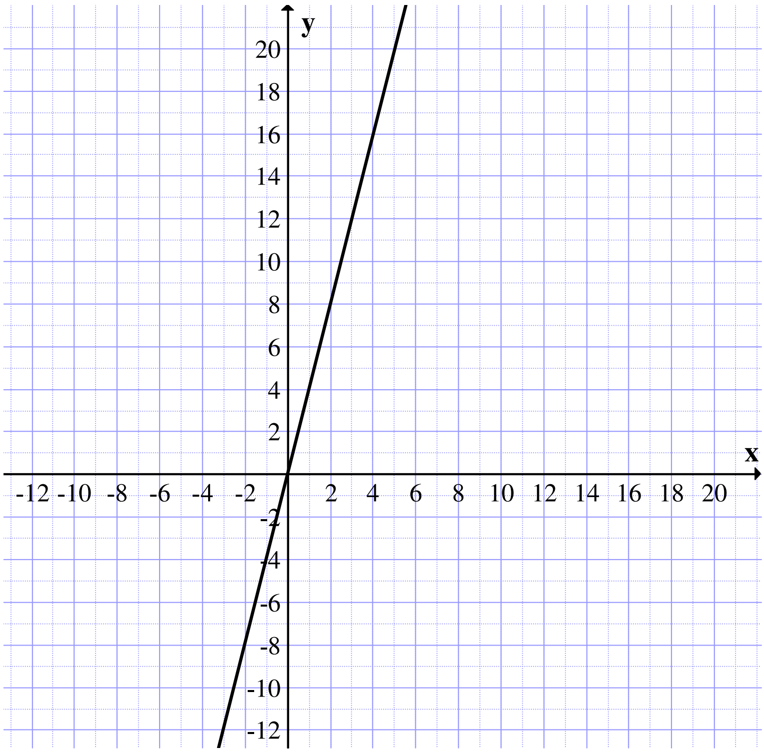

Figure 2.11

In the MOD plane $R_n(12)$, y = 4x is as follows:

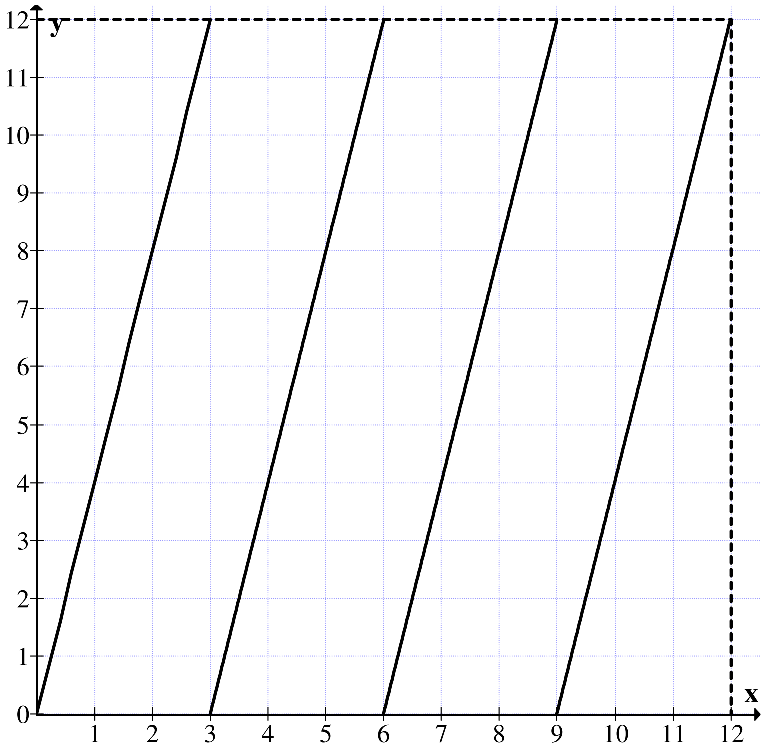

Figure 2.12

Thus we see y = 4x has the above representation in the MOD real plane associated with [0, 12).

Such study is interesting, we have not given difficult or complicated functions, that will be done in due course of time.

Now we have said that there exist infinite number of MOD real planes $R_n(m)$ associated with [0,m); $m \in Z^+$.

We see the smallest MOD real plane is $R_n(1)$ associated with the interval [0, 1), infact it is also known as the fuzzy MOD real plane described in the following;

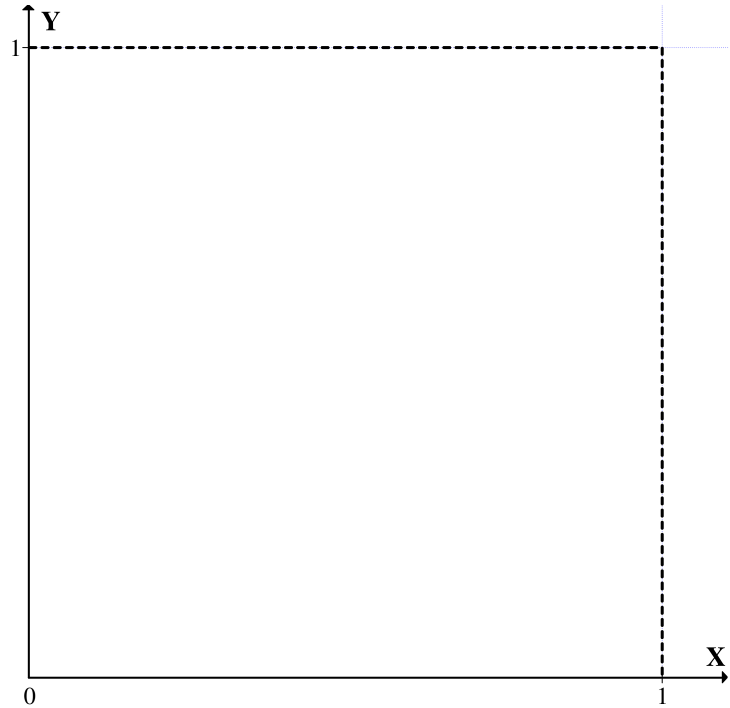

Figure 2.13

The function y = x can not have a representation as y = 1. x is zero.

Let y = 0.5x, then the representation of this in the fuzzy MOD real plane is given in Figure 2.14.

Now we find the graph of y = 0.2x in the fuzzy MOD real plane.

We see in case of y = 0.5x the maximum value y can get is 0.4999999…

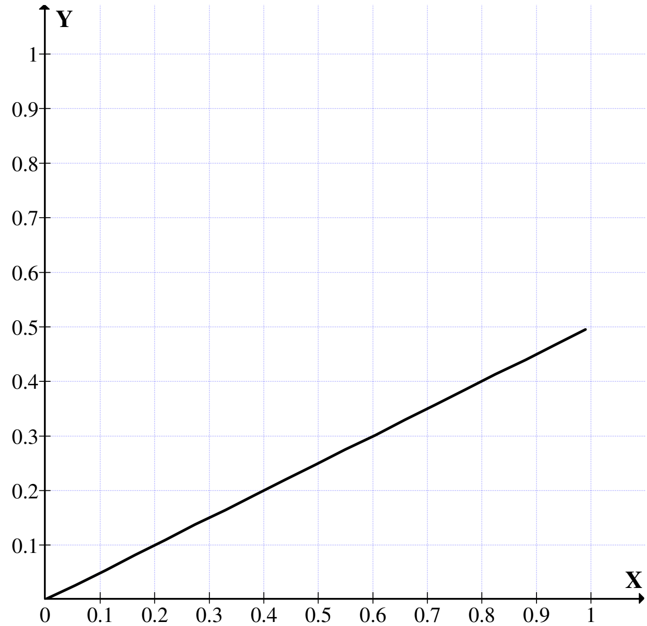

Figure 2.14

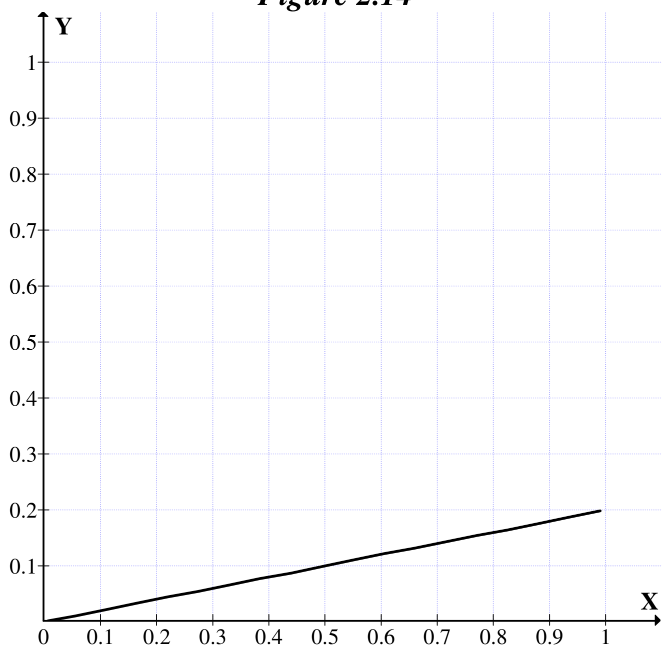

Figure 2.15

Likewise for the function y = 0.2x. The maximum value y can get is 0.199….

So these functions behave in a unique way in the fuzzy MOD real plane.

We will show how these two functions behave in a MOD real plane which is not fuzzy.

This will be illustrated by examples.

Example 2.8: Let us consider the MOD real plane $R_n(2)$ for the interval [0, 2).

Consider the function y = 0.5.

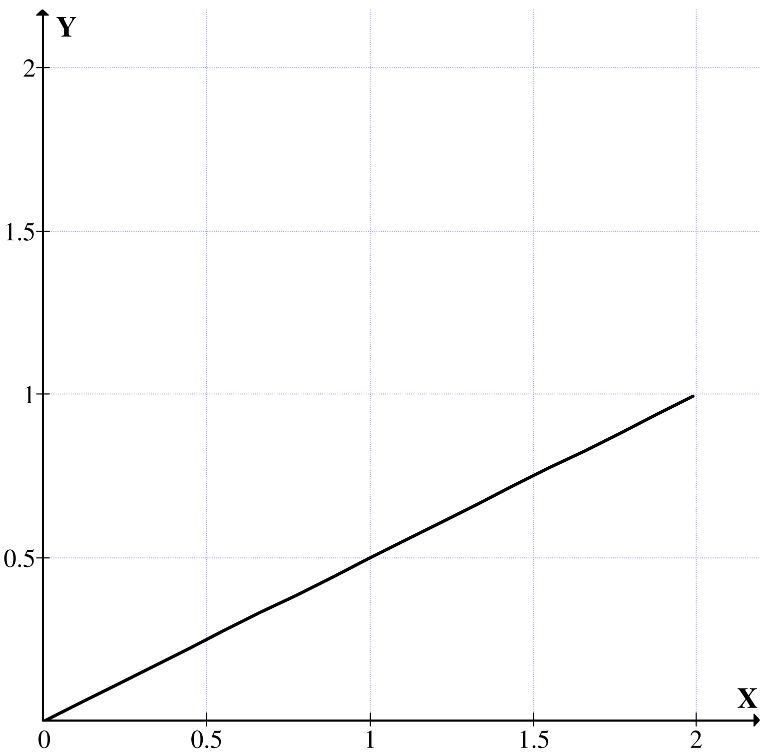

Figure 2.16

Example 2.9: Let us consider the MOD real plane $R_n(3)$.

Let y = 0.5x be the MOD function in $R_n(3)$.

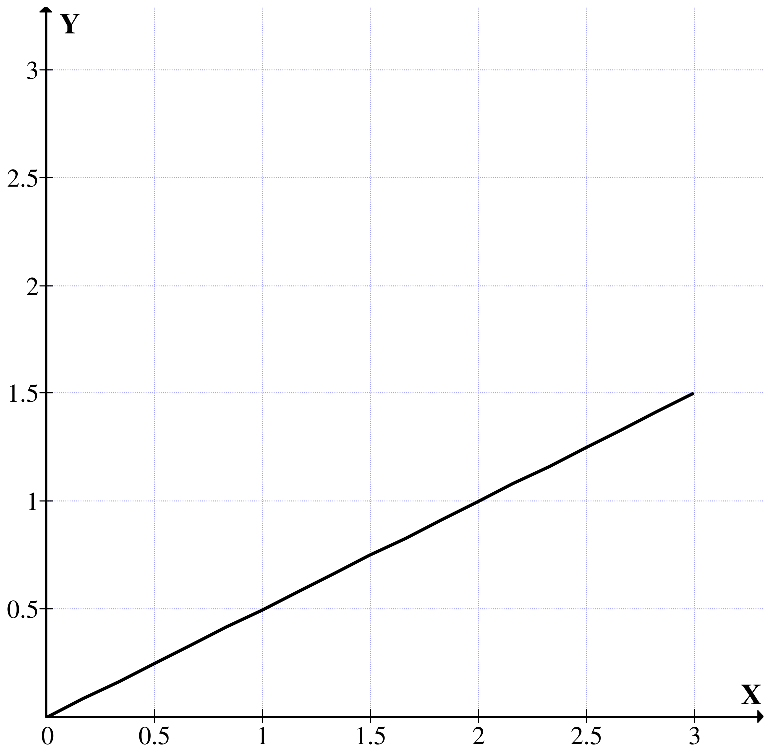

Figure 2.17

Consider the function y = 0.5x. When x = 1 then y = 0.5 and when x = 2, y = 1.

When x = 2.9 then y = 0.5 × 2.9 = 1.45.

So the greatest value is bound by 1.5.

Example 2.10: Let us consider the MOD real plane $R_n(4)$ built using [0, 4).

Consider the function y = 0.5x. The graph of y = 0.5x is as follows:

Figure 2.18

Example 2.11: Let us consider the MOD real plane $R_n(20)$ using the interval [0, 20).

Let y = 0.5x, the MOD function; the graph of this in the real MOD plane $R_n(20)$ is given in Figure 2.19.

In all cases we saw in a MOD real plane $R_n(m)$ we see the function y = 0.5x for the highest value of x say (m–1) .99 … .9; the y value is only $\left(\frac{m}{2}-1\right)$ .99 … 9.

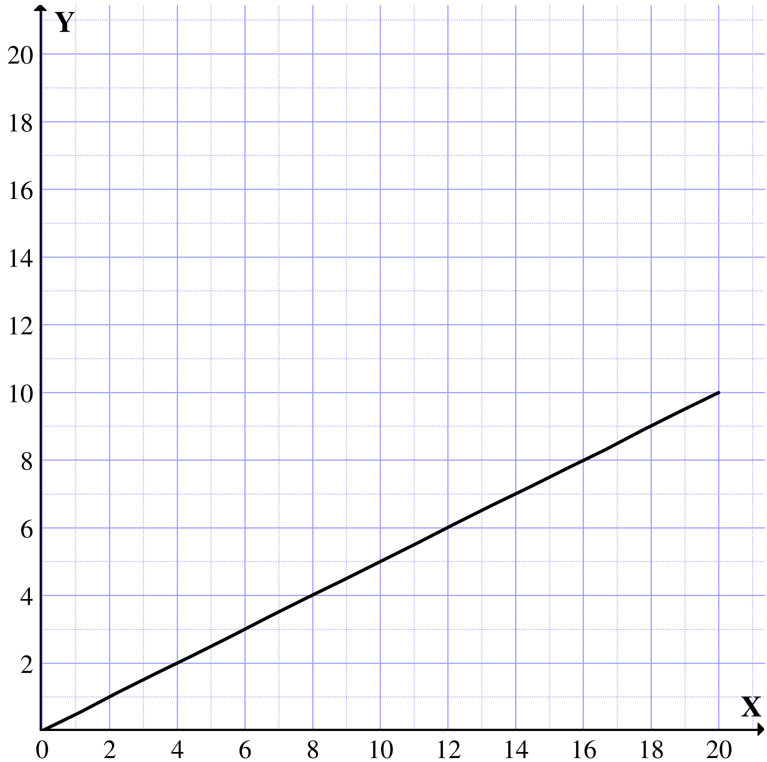

Figure 2.19

When we compare this with the real plane the function $y = 0.5\,x$ is given Figure 3.20.

Thus we see they extend to infinity.

Next we analyse the function $y = x$.

This function behaves in a nice way or it is alike in both the real plane as in well as the MOD real plane.

Next we consider the function $y = 2x$ and see how it behaves in each of the MOD planes and the real plane.

In fact in the real plane it has the unique representation.

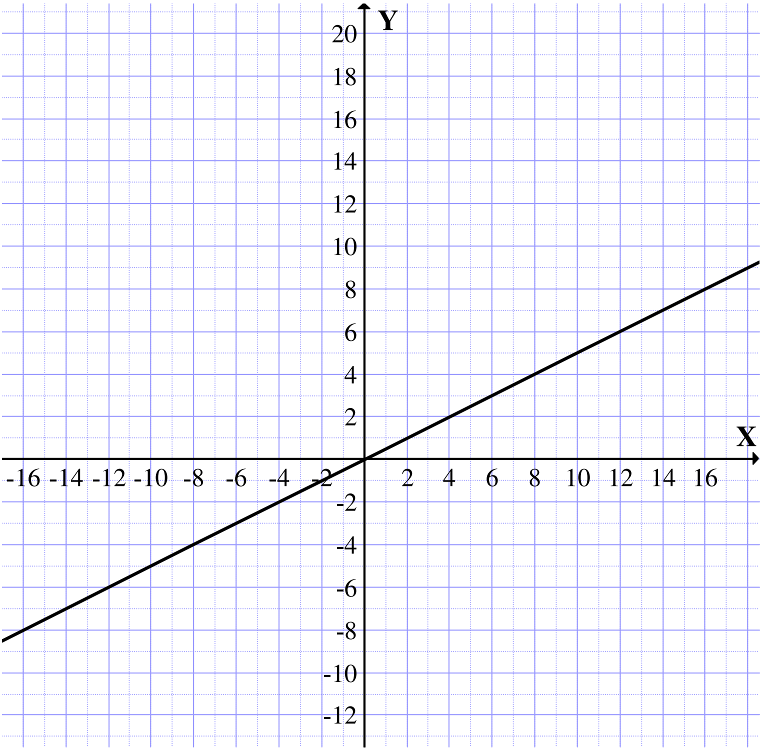

Figure 2.20

However in the MOD real plane they have a different representation represented by two lines.

Clearly y = 2x is not defined in the MOD plane $R_n(2)$.

Just to have some way of analyzing the function y = 2x in MOD real planes.

The graph of y = 2x in the real plane is given in Figure 3.21.

We give the graph of the function y = 2x only in two MOD planes $R_n(6)$ and $R_n(7)$ over the intervals [0, 6) and [0, 7) respectively; the graphs of the MOD function y = 2x is given in Figure 3.22 and Figure 2.23 respectively.

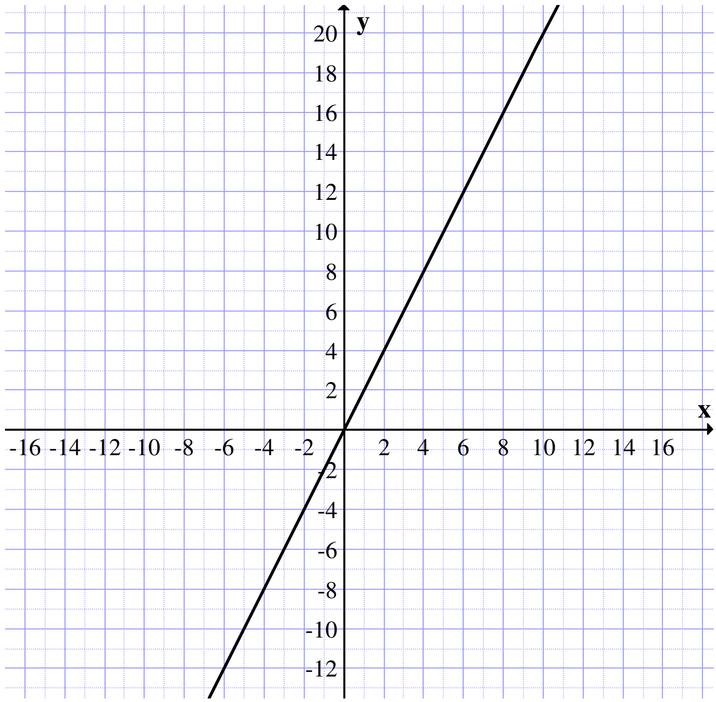

Figure 2.21

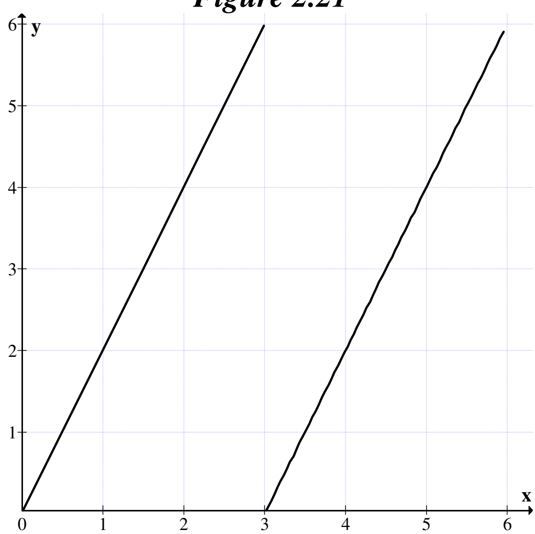

Figure 2.22

$R_n(6)$ MOD plane on the interval [0, 6).

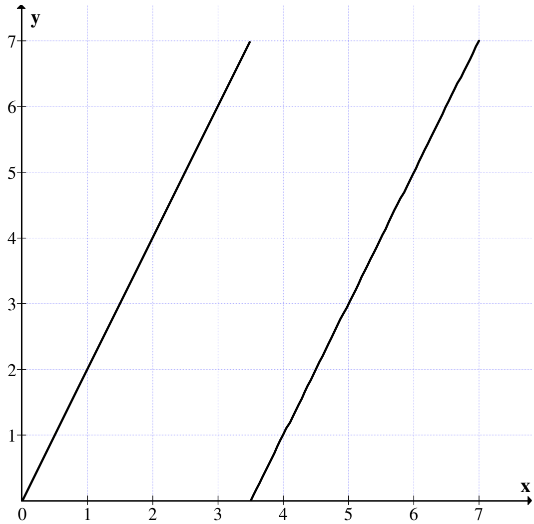

Figure 2.23

$R_n(7)$ the MOD real plane on the interval [0, 7).

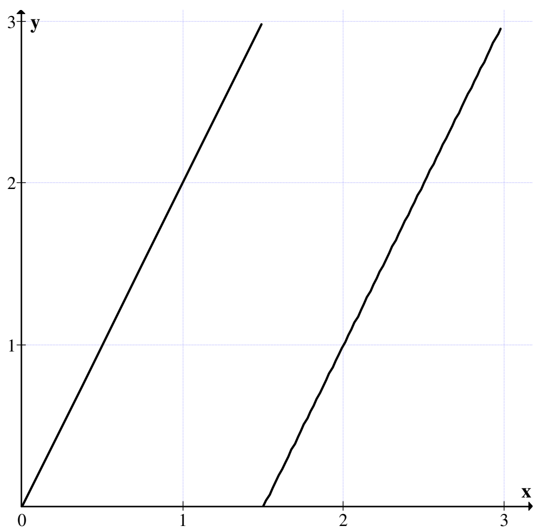

Figure 2.24

Clearly the function y = 2x can be defined only in the MOD real plane $R_n(m)$; $m \geq 3$.

We just see how y = 2x has its graph in the MOD real plane $R_n(3)$ as given in Figure 2.24.

Thus we see y = 2x is a two line graph represented in that form.

We see this is true in all MOD real planes $R_n(m)$.

Next we study the function y = 3x in the real plane R and the MOD real plane $R_n(m)$ on the interval [0, m); clearly y = 3x is not defined in the MOD plane $R_n(3)$.

Example 2.12: Let $R_m(5)$ be the MOD real plane.

We see how the graph of the function y = 3x looks.

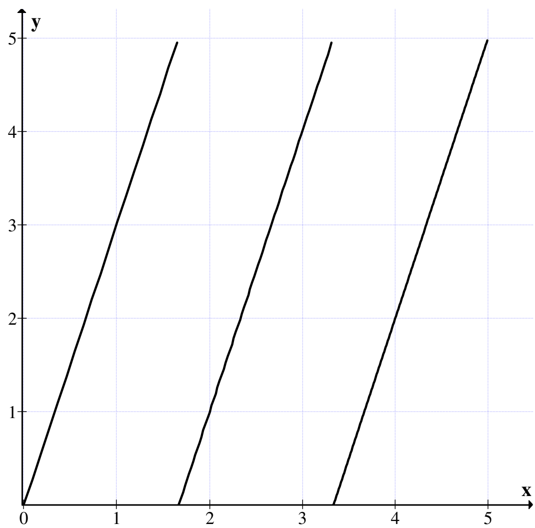

Figure 2.25

We see we have three disconnected line graphs represents the function y = 3x in the MOD plane $R_n(5)$.

Consider the MOD real plane $R_n(6)$ in the following example where 3/6.

We will see how these graphs look like.

Example 2.13: Consider the MOD real plane $R_n(6)$

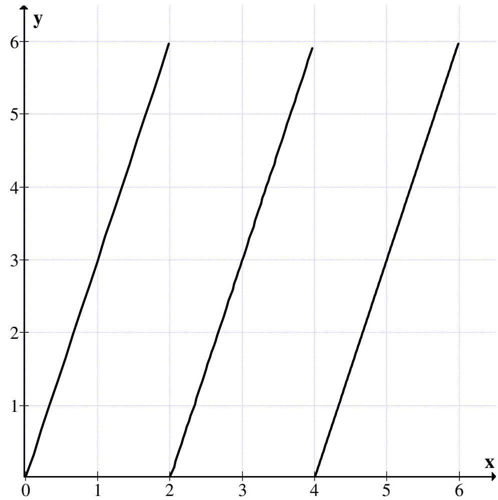

Figure 2.26

In another example we will consider $R_n(8)$; (3, 8) =1.

We see when 3/6 the graphs of the function y = 3x behave in a chaotic way. Thus we first see how the function y = 3x behaves on the MOD real plane $R_n(8)$.

This graph is given in Figure 2.27.

(3, 8) = 1 we get the about graph for y = 3x in the MOD plane $R_n(8)$ using [0, 8).

Example 2.14: Consider the MOD plane $R_n(9)$.

We see this has 3 lines which are disjoint but are parallel to each other. This graph is given in Figure 2.28.

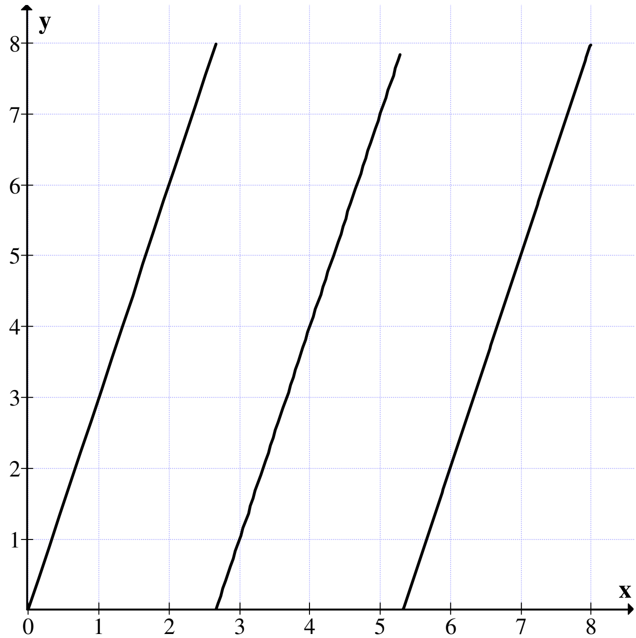

Figure 2.27

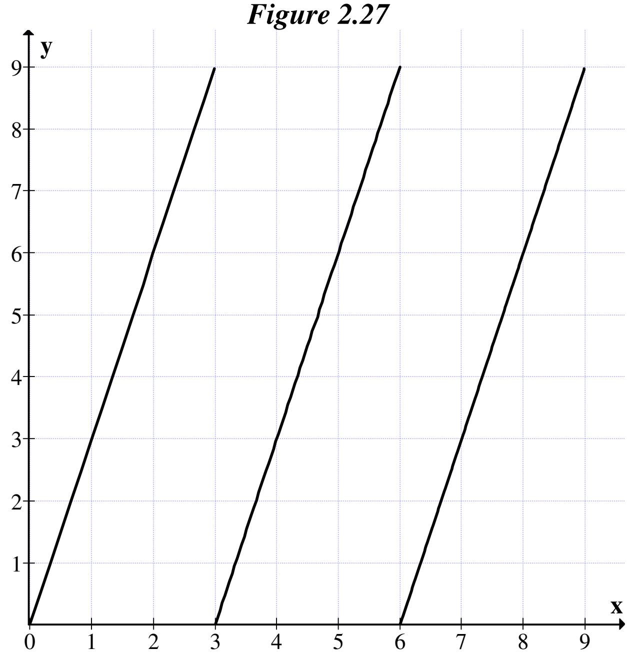

Figure 2.28

We see this has 3 lines which are disjoint but are parallel to each other.

Example 2.15: Consider the MOD real plane $R_n(11)$. Consider the graph of the MOD function y = 3x.

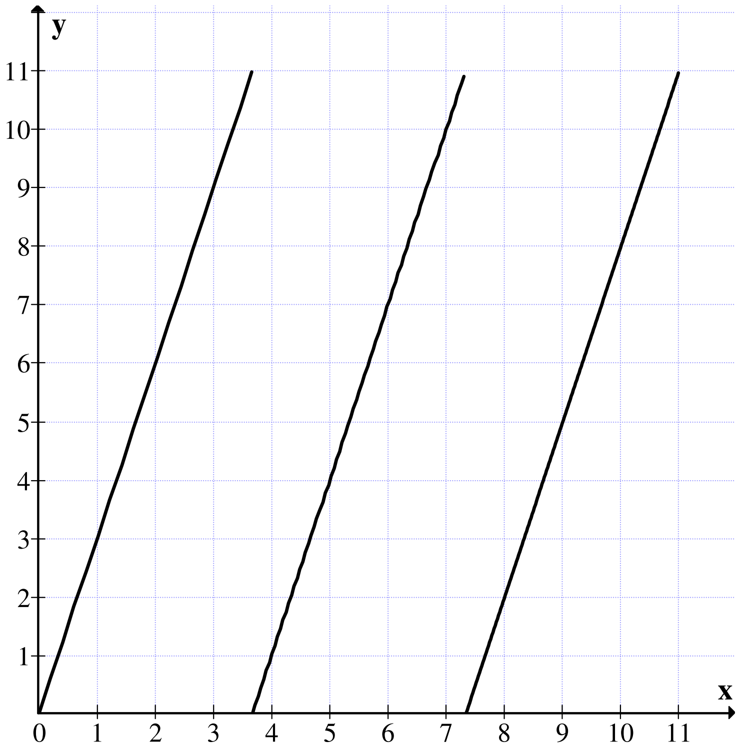

Figure 2.29

Here also we get 3 parallel lines associated with the graph for the MOD function y = 3x.

Next we consider the function y = 0.4x in the MOD real planes $R_n(1)$, $R_n(2)$, $R_n(3)$, $R_n(4)$ and $R_n(8)$.

Consider the MOD fuzzy real plane.

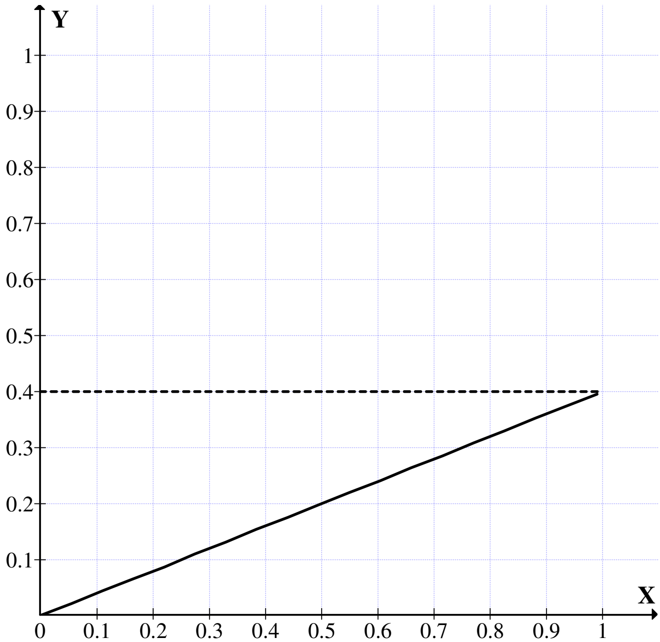

Figure 2.30

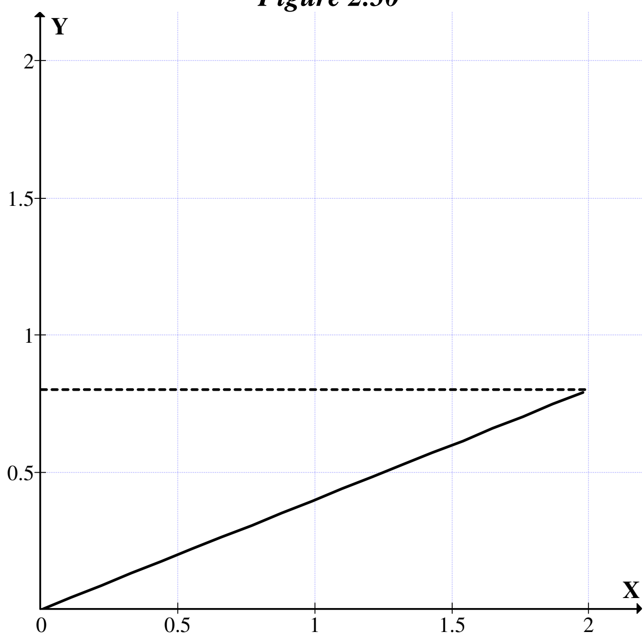

Figure 2.31

Consider the MOD real plane $R_n(2)$ given in Figure 2.31.

Consider the MOD real plane $R_n(3)$ given in Figure 2.32.

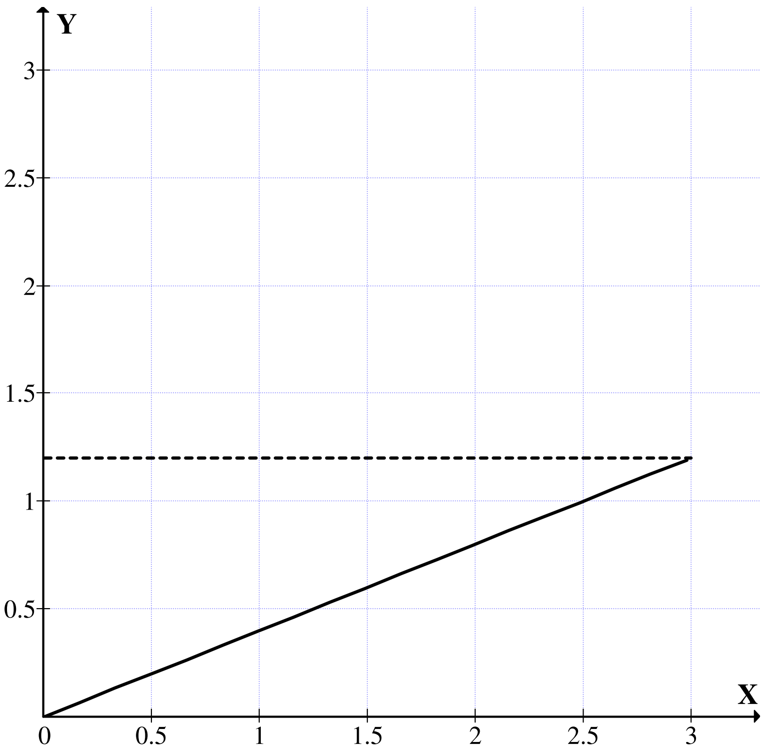

Figure 2.32

We see the graph of y = 0.4x is a single line. The maximum value is bounded by 1.2.

Consider the MOD real plane $R_n(4)$ given in Figure 2.33.

The graph is a straight line and the maximum value is bounded by 1.6.

Consider the MOD real plane $R_n(8)$ given in Figure 2.34.

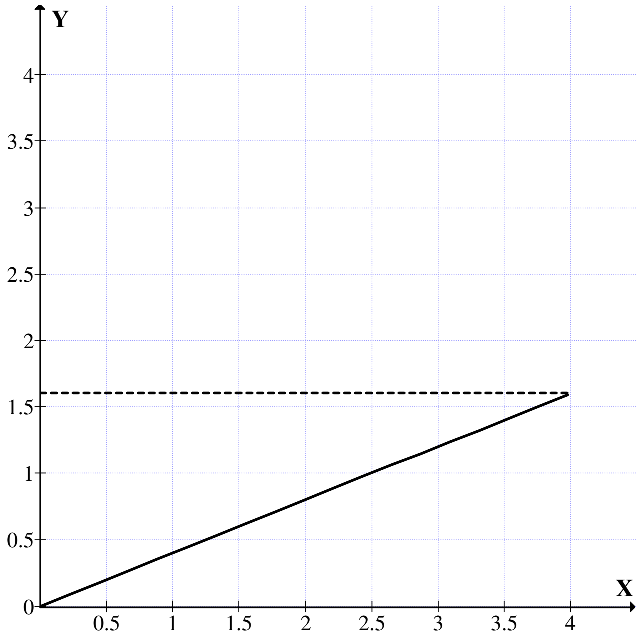

Figure 2.33

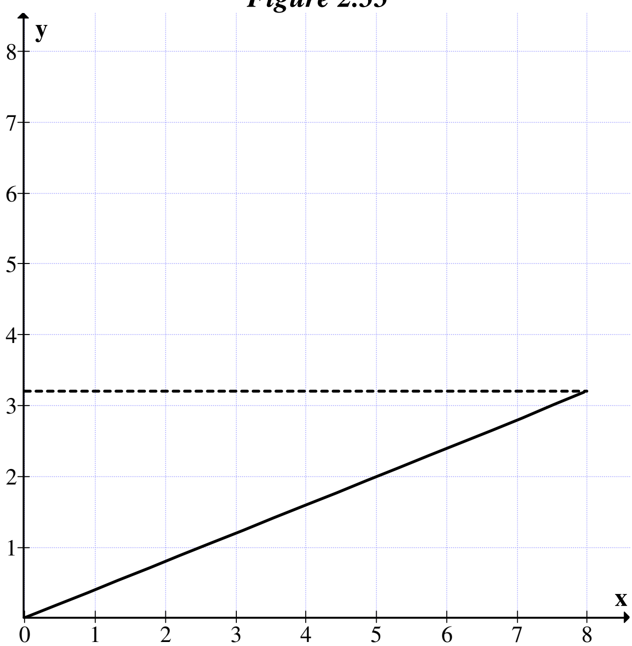

Figure 2.34

This is also a straight line reaching a maximum value for this is bounded by 3.2.

Let y = 4x.

We can have this function to be defined in the MOD real planes $R_n(5)$, $R_n(6)$ and so on.

Now we will study the function y = 4x in the MOD real plane $R_n(5)$.

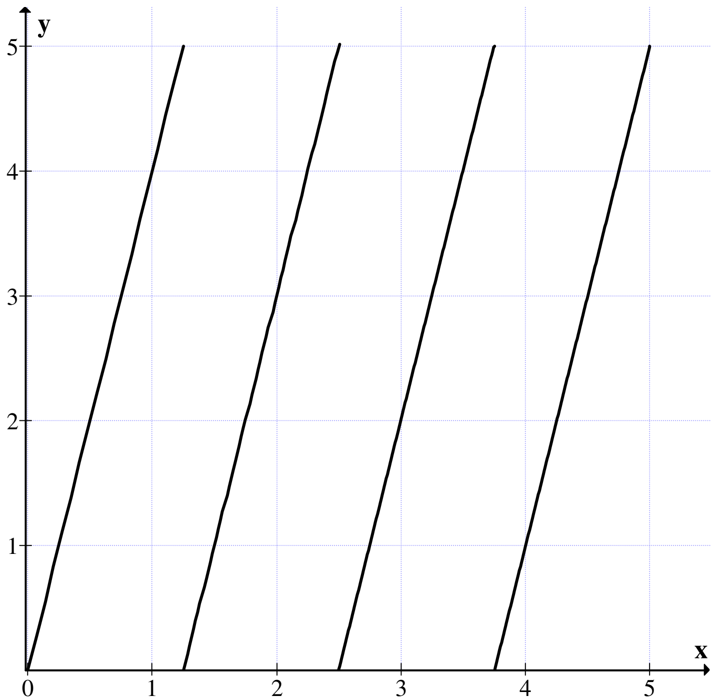

Figure 2.35

The graph y = 4x in the MOD real plane y = 4x gives 4 parallel lines as indicated in the graph.

Consider y = 4x in the MOD real plane $R_n(8)$ given in Figure 2.36.

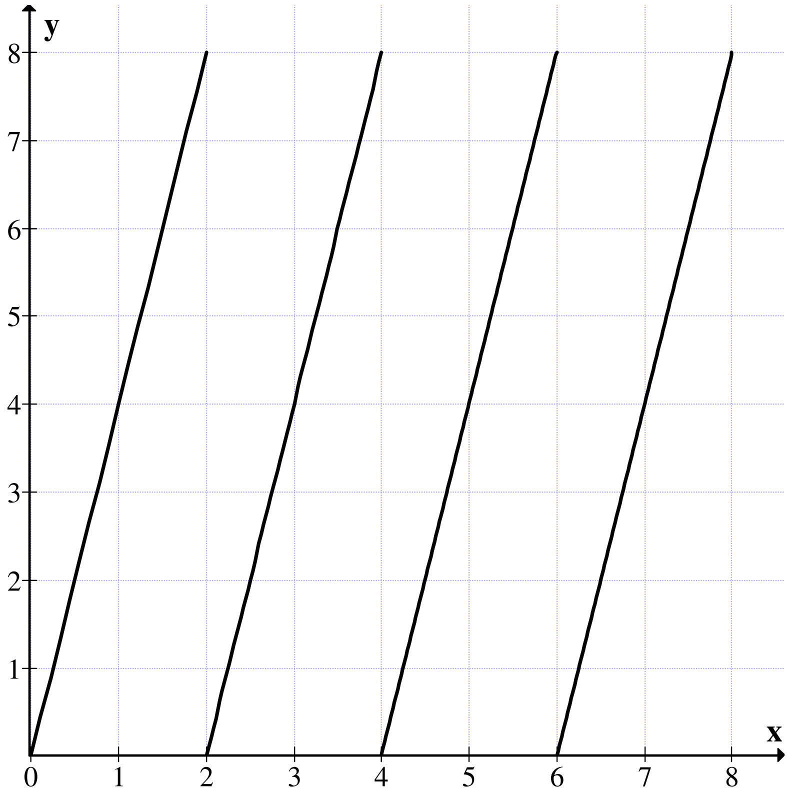

Figure 2.36

In this case also we get four parallel lines for the graph y = 4x in the MOD real plane.

We study the function y = 4x in the MOD plane $R_n(16)$ which is given in Figure 2.37.

This function y = 4x which is a single line graph in the real plane has four parallel line representation in the MOD real plane $R_n(16)$.

Next we study y = 5x.

This function can be defined only in $R_n(m)$, m ≥ 6.

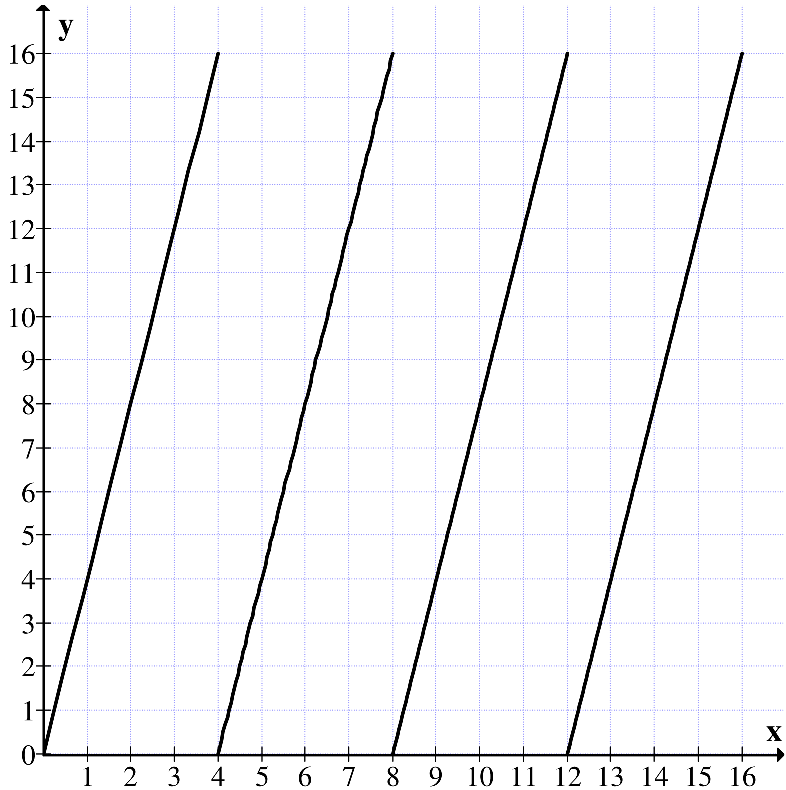

Figure 2.37

For if $m \leq 5$ this function does not exist for those MOD real planes.

We study the function $y = 5x$ in the MOD plane $R_n(6)$ and the graph is given in Figure 2.38.

We get five parallel lines for the graph of the function $y = 5x$ in the MOD real plane $R_n(6)$.

Now we study the function $y = 5x$ in the MOD real plane $R_n(10)$, the graph is given in Figure 2.39.

Thus we get in this case also 5 parallel lines as shown in the figure.

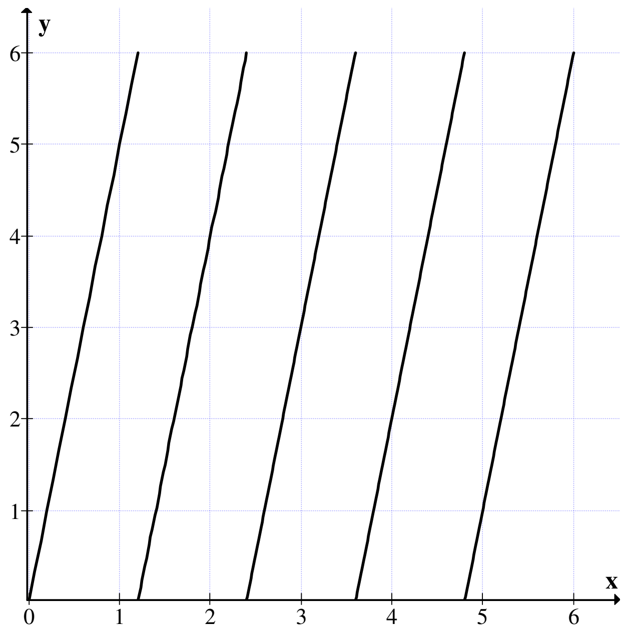

Figure 2.38

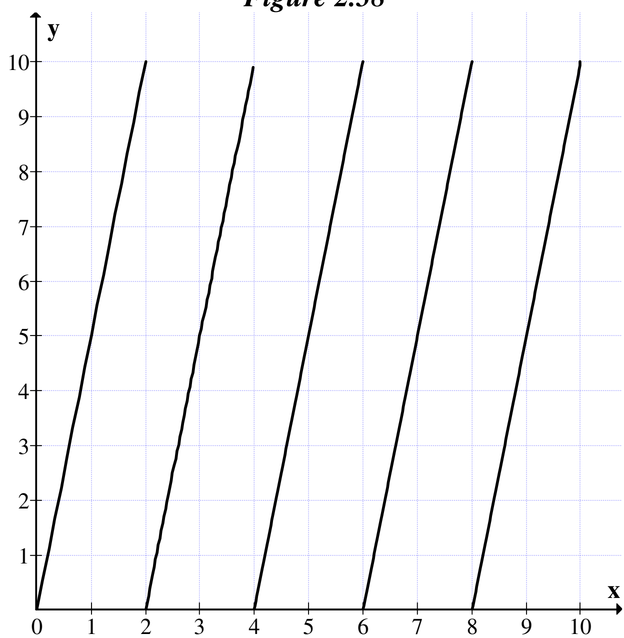

Figure 2.39

In view of all these examples we concentrate on the study of these functions of the form $y = ts$, t a positive integer can be defined only over the MOD real plane $R_n(m)$, but $m > t$.

We propose the following problems.

Conjecture 2.1: If $R_n(m)$ is a MOD real plane, Let $y = tx$ (t an integer) be the function $m > t$.

Can we say we have t number of parallel lines as the graph of $y = tx$ in the MOD real plane $R_n(m)$?

Note: Clearly if $m \leq t$ then the very function remains undefined.

Let us now consider the functions of the form $y = tx + s$.

We see in the first case $y = tx + s$ can be defined over the MOD real plane $R_n(m)$ if and only if $m > s$ and t.

We will describe them by some examples for appropriate values of m, s and t.

Example 2.16: Let $R_n(9)$ be the MOD real plane. Consider $y = 5x + 2$ the function. We find the representation of this function in the MOD real plane $R_n(9)[x]$. The associated graph is given in Figure 2.40.

We get for this function $y = 5x + 2$ also in the MOD real plane $R_n(9)$ six parallel disjoint lines with different starting points.

Let us consider the MOD function $y = 5x + 4$ in the MOD real plane $R_n(9)$. The associated graph is given in Figure 2.41.

We get 6 distinct lines for the function $y= 5x + 4$.

Let us consider the function $y = 5x + 4$ on the MOD real plane $R_n(11)$. The associated graph is given in Figure 2.42.

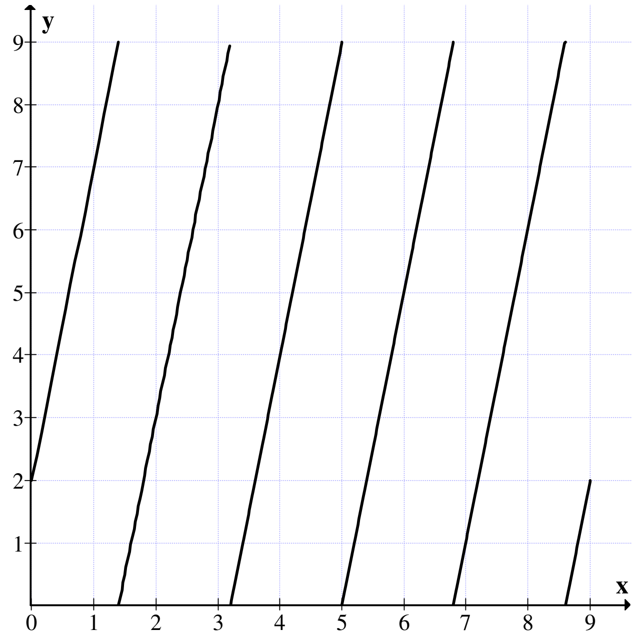

Figure 2.40

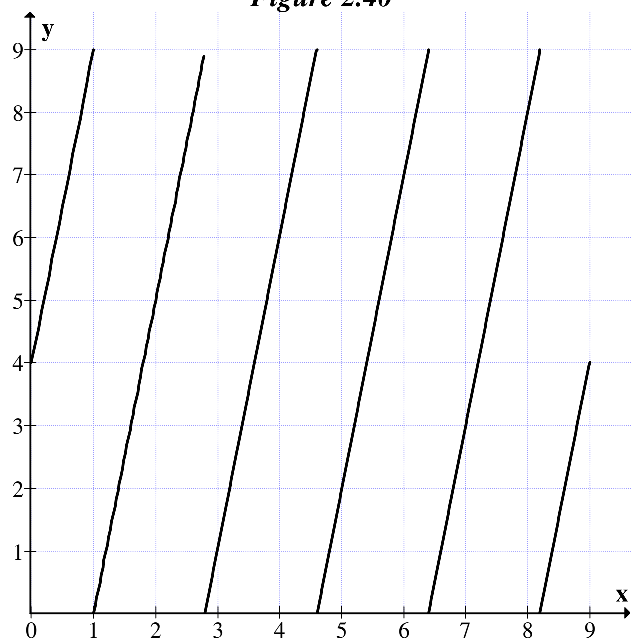

Figure 2.41

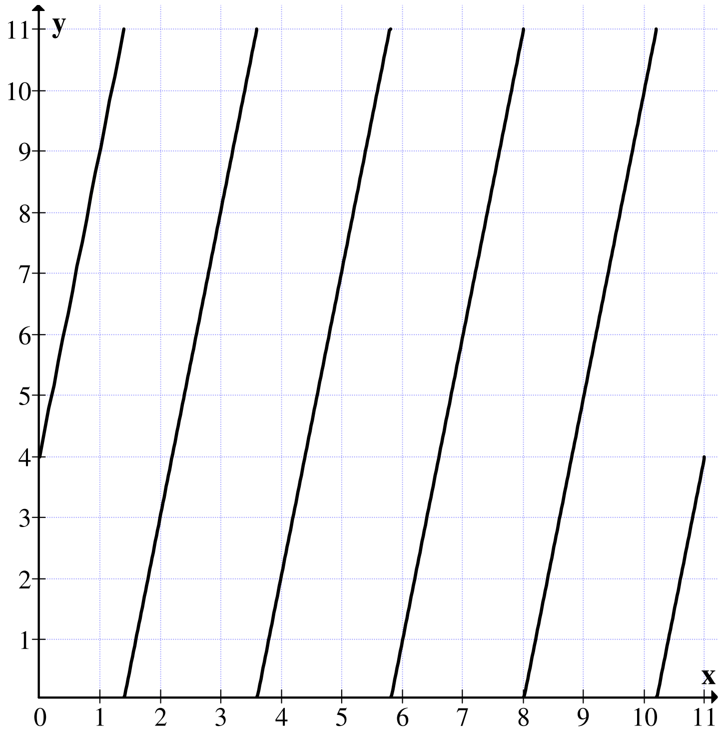

Figure 2.42

Here also we get only six lines given by the figure.

Thus we propose one more conjecture.

Conjecture 2.2: Let $R_n(m)$ be a MOD real plane, $y = tx + s$ be a function defined on $R_n(m)$; $m > t$ and s.

Can we say the graph of $y = tx + s$ will have t or t+1 lines in the MOD real plane $R_n(m)$? Will they be parallel lines?

If they are parallel lines we have to study whether they correspond to the same MOD slope, such study is both innovative and interesting.

Let us now find the graph of the function $y = x^2$ in the MOD plane $R_n(m)$; $m > 1$.

Let us consider the function in the MOD real plane $R_n(4)$.

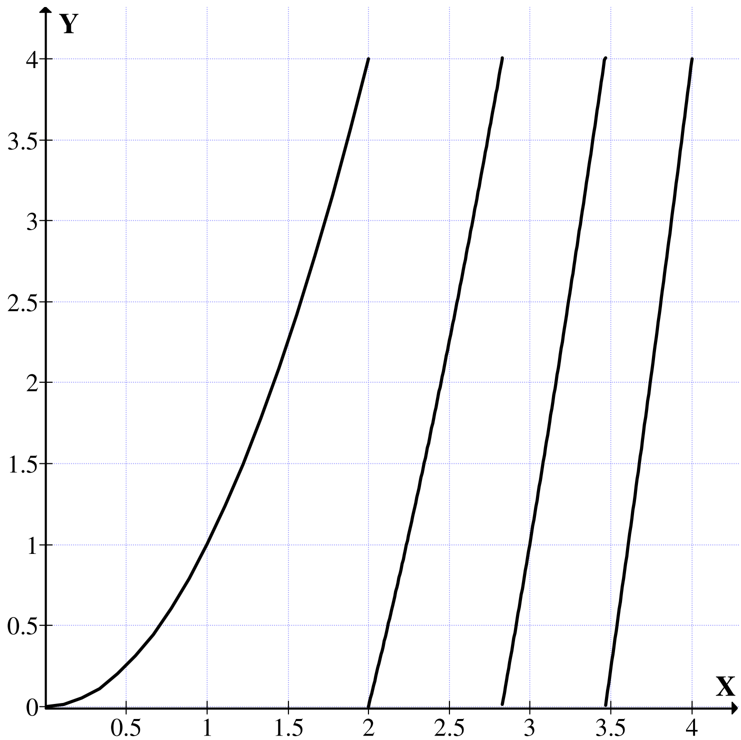

Figure 2.43

We get two parallel lines and the function $y = x^2$ is not continuous in the MOD real plane $R_n(4)$.

Let us consider $y = x^2$ in the MOD real plane $R_n(2)$. The associated graph is given in Figure 2.44.

We can only say the function $y = x^2$ becomes zero in the interval x =1.41 to x = 1.42 but we cannot find the point as $(1.41)^2 = 1.9881$ and

$$\begin{aligned} 1.42 \times 1.42 &= 2.0164 \\ 1.412 \times 1.412 &= 1.993744 \\ 1.413 \times 1.413 &= 1.996569 \end{aligned}$$

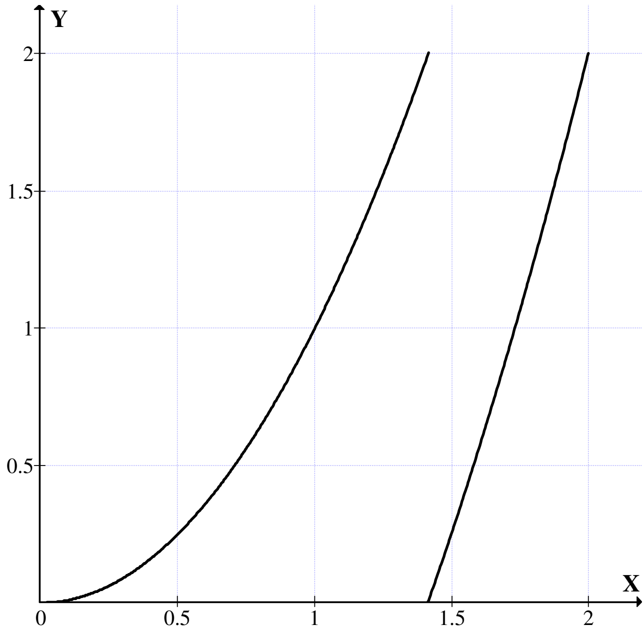

Figure 2.44

1.414×1.414	=	1.999396
1.4142×1.4142	=	1.99996164
1.41421×1.41421	=	1.999989924
1.414213×1.414213	=	1.999998409
$1.4142134 \times 1.4142134$	=	1.999999541
$1.4142136 \times 1.4142136$	=	2.000000106
$1.41421354 \times 1.41421354$	=	1.999999937
$1.41421358 \times 1.41421358$	=	2.00000005
$1.41421356 \times 1.41421356$	=	1.999999993
$1.414213562 \times 1.414213562$	=	1.999999999
$1.414213563 \times 1.414213563$	=	2.000000002.

So approximately the zero lies in the interval

[1.414213562, 1.414213563]

How ever $1.4142135625 \times 1.4142135625 = 2$.

Thus when x = 1.4142135625 we get $y = x^2 = 0$.
We get for $y = x^2$ two parallel lines.

Thus the function is not a continuous function. Now we study the functions of the form $y = tx^2$ in $R_n(m)$; $m > t$.

Let us consider $y = 3t^2$ in the real MOD plane be and $R_n(4)$.

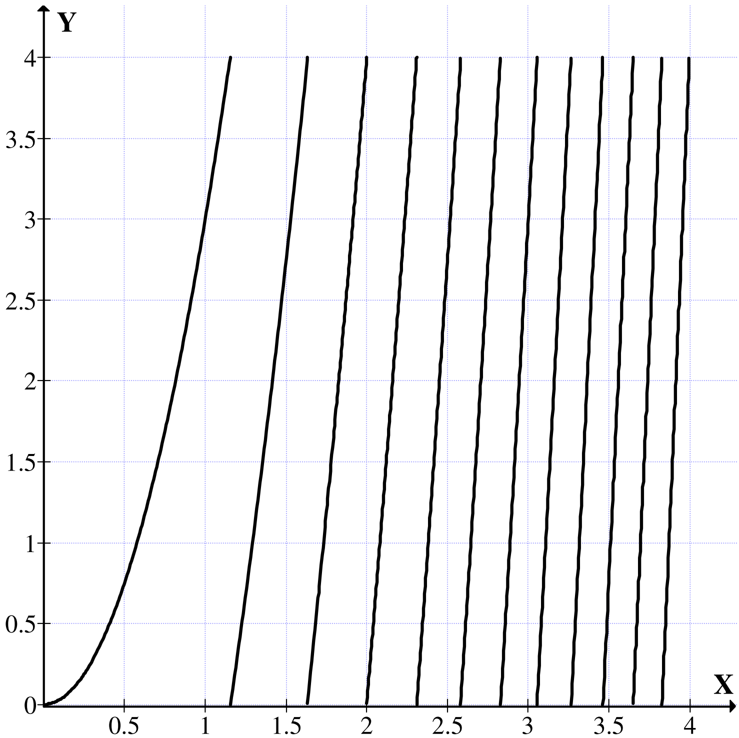

Figure 2.45

Let $x = 1.2$, $y = 3\ (1.2)$

$$\begin{aligned} x \times y &= 3 \times (1.44) \\ &= 5.32 \pmod 4 \\ &= 1.32. \end{aligned}$$

If $x = 1.1$ then $y = 3\ (1.1)^2$

$$x \times y = 3 \times 1.21$$

= 3.63.

So zero lies between x = 1.1 and x =1.2.

To be more closer zero lies
between x = 1.1 and x = 1.16
or between x = 1.1 and x = 1.155
or between x = 1.144 and 1.155
or between x = 1.153 and 1.155
or between x = 1.154 and 1.155
or between x = 1.1544 and x = 1.1550.

That is zero lies between 1.1545 and 1.155.

So zero lies between 1.1545 and 1.1548.

Infact between 1.154700537 and 1.15470054.

Approximately x = 1.1547005384 is such that $y = 3x^2 = 0$. Another zero lies between x = 2.58 and x = 2.6.

Infact zero lies between x = 2.58 and x = 2.59.
Infact zero lies between x = 2.58 and x = 2.583.
Infact zero lies between x = 2.581 and x = 2.582.
Infact between x = 2.5816 and 2.582.

Infact zero lies between x = 2.5819 and 2.582.
Infact zero lies in between x = 2.58198 and 2.582.
Infact zero lies between x =2.581988 and 2.58199.
Infact zero lies between x = 2.5819888965 and 2.581988899.

Infact zero lies between x = 2.581988897 and 2.581988898.

Infact at x = 2.58199888975 $y = 3x^2 = 0 \bmod (4)$.

We are yet to study completely the properties of $y = 3x^2$ in $R_n(4)$ and in $R_n(m)$; $m > 4$.

It is important to note in the MOD real plane $y = x^2$ has zeros other than $x = 0$.

Infact the $y = 3x^2$ in the real plane attains $y = 0$ if and only if $x = 0$.

But $y = 3x^2$ in $R_n(4)$ has zeros when $x = 0$, $x = 2$, $x = 2.58199888975$, and $x = 1.1547005384$ are all such that $y = 3x^2 = 0 \text{ mod } 4$.

Thus we have seen four zeros. Now a zero lies between $x = 3.26598632$ and $x = 3.265986324$.

Clearly $x = 3.2659863235$ is such that $y = 3x^2 \equiv 0 \pmod 4$. Thus $y = 3x^2$ has 4 zeros or more.

We see $y = 3x^2$ is a continuous function in the real plane where as $y = 3x^2$ is a non continuous function in the MOD plane $R_n(m)$; $m > 4$ with for several values x; $y = 0$.

Such study is innovative and interesting.

We will see more examples of them.

Let $y = 4x^2$ be the function defined on the MOD real plane $R_n(5)$ associated with the interval $[0, 5)$.

Interested reader can find the zeros of $y = 4x^2$ that is those points for which $y = 0$ for $x \neq 0$.

Now we leave the following open conjecture.

Conjecture 2.3: Let $R_n(m)$ be the MOD plane. Let $y = tx^2$ where $m > t$.

(i) Find those x for which $y = 0$.
(ii) Prove the graph is not continuous.
(iii) How many line graphs we have for $y = tx^2$ on the MOD real plane $R_n(m)$?

(iv) Are they parallel ?
(v) Find the slopes of these lines.
(vi) Are these lines with the same slope?

We will give some examples of finding the slope of the lines of functions defined in the MOD real plane $R_n(4)$.

Example 2.17: Let y = 3x be the MOD function in the MOD plane $R_n(4)$. The associated graph is given in Figure 2.46.

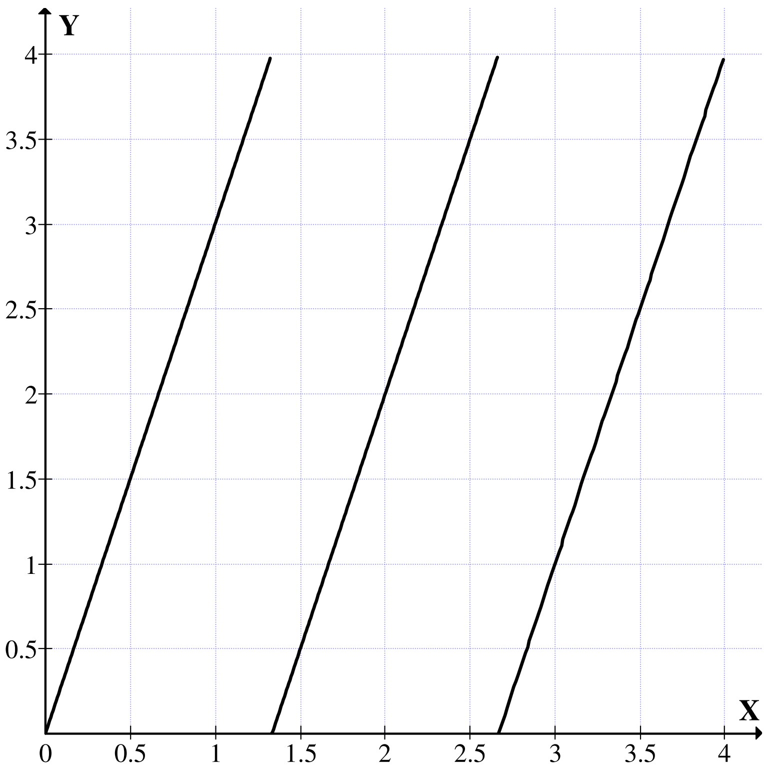

Figure 2.46

We see there are three disjoint lines and the angle which it makes with the MOD axis X_n is tan 70.

Example 2.18: Let us consider the MOD real plane $R_n(6)$.

Let y = 4x be the function to find the graph and slope of the graph relative to $R_n(6)$.

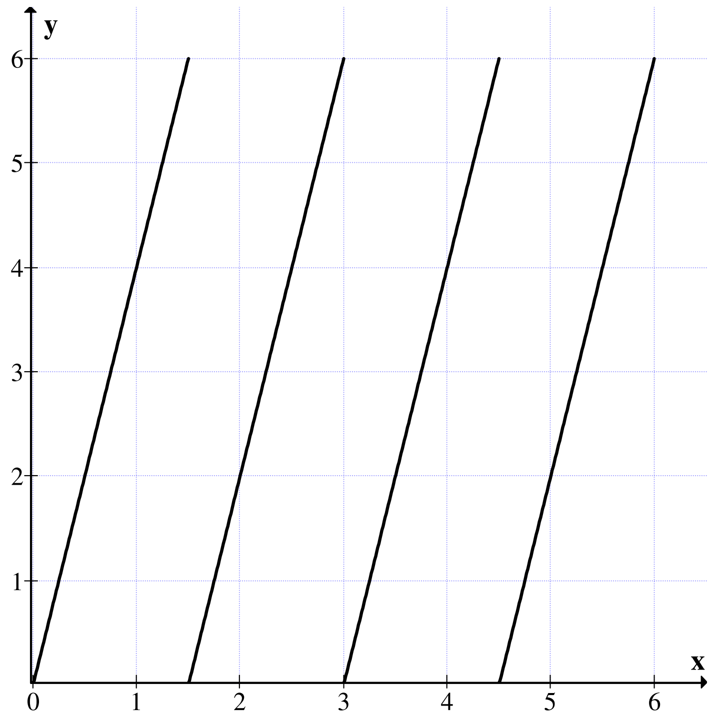

Figure 2.47

The angle made by that graph is 75°.

We see there are four parallel lines and we see the slope of them are approximately 4.

Example 2.19: Let $R_n(6)$ be the MOD real plane. y = 5x be the function. We find the graph y = 5x on $R_n(6)$. The associated graph is given in Figure 2.48.

We see there are five parallel lines all of them have the same slope tan 80°.

Now we find graphs of the form y = tx + s in the MOD real plane $R_n(m)$, m > t and s.

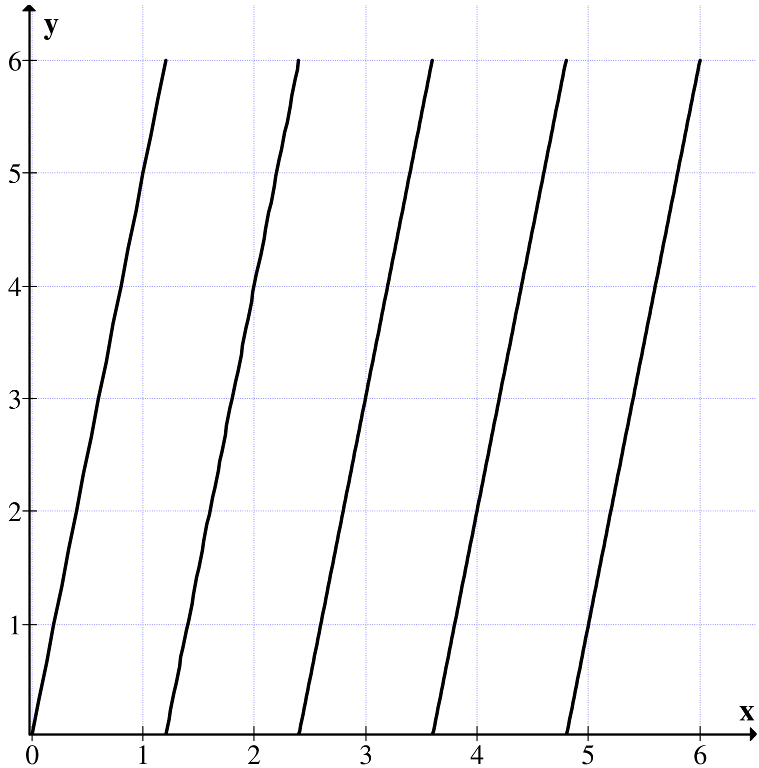

Figure 2.48

Example 2.20: Let $R_n(5)$ be the MOD real plane. Consider the function y = 4x + 2 on $R_n(5)$.

The associated graph is given in Figure 2.49.

Example 2.21: Let us consider the MOD real plane $R_n(6)$ and the function (or equation) y = 4x + 2 in the MOD real plane.

The graph of it is given in Figure 2.50.

All the four lines have the same slope.

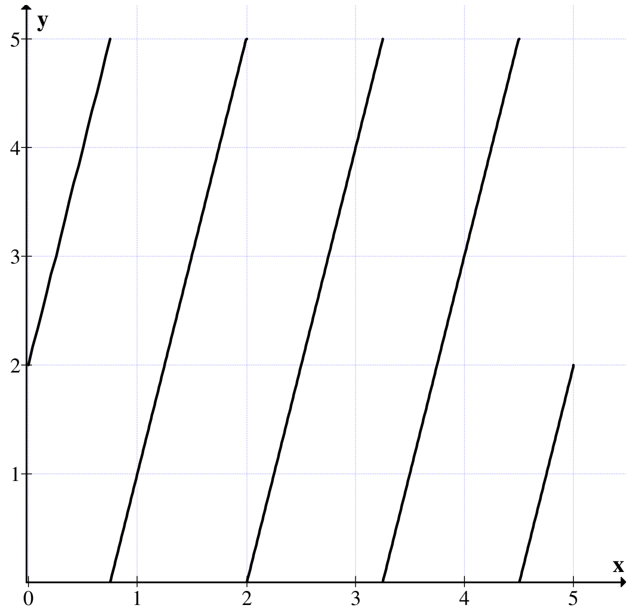

Figure 2.49

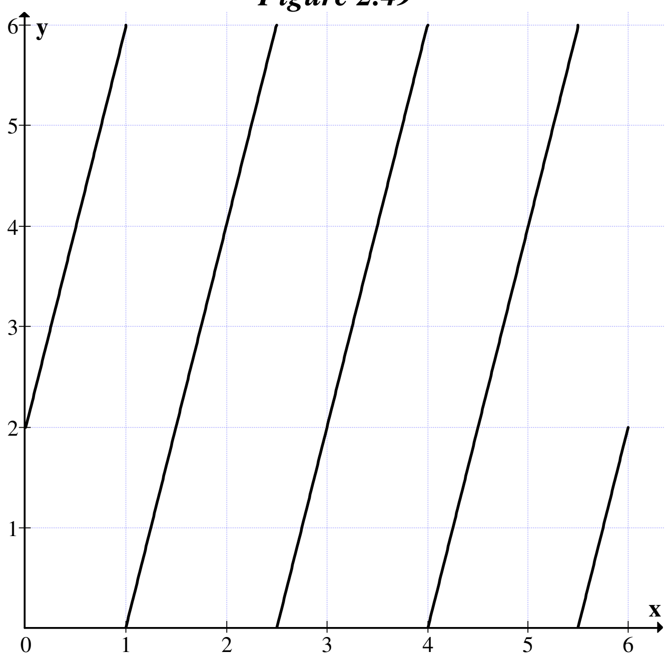

Figure 2.50

Next we consider the equations of the form $y = tx$ in the MOD real plane $R_n(m)$, $m > t$. We consider $y = x^2$ in $R_n(2)$.

The graph is as follows:

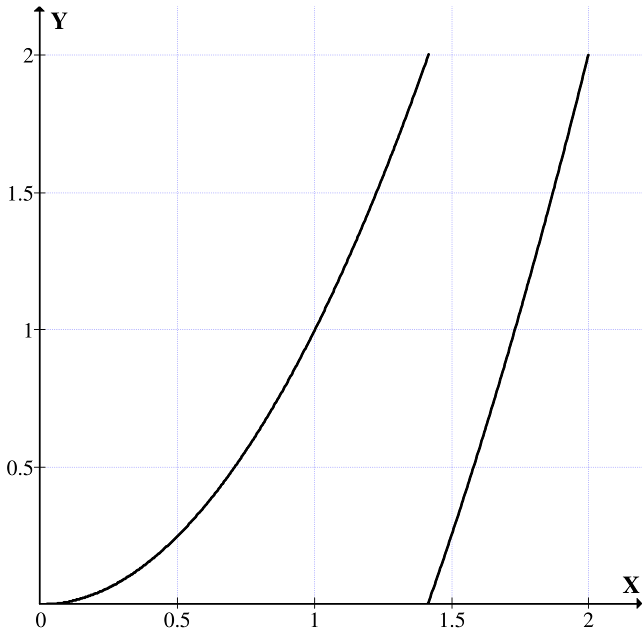

Figure 2.51

$y = 0$ for some value in between $x = 1.4$ and 1.5. We see $y = x^2$ is not a continuous line.

Let us consider $y = x^3$ in $R_n(2)$. The associated graph is given in Figure 2.52.

We see for x in between (1.25, 1.26); y takes the value zero. So this function is also not continuous in the MOD real plane $R_n(2)$.

Let $y = x^3$ a function in the MOD real plane $R_n(3)$. The graph is given in Figure 2.53.

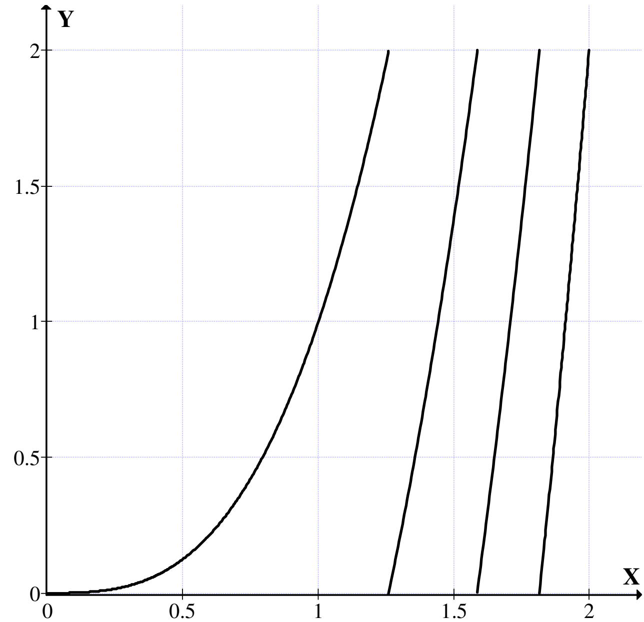

Figure 2.52

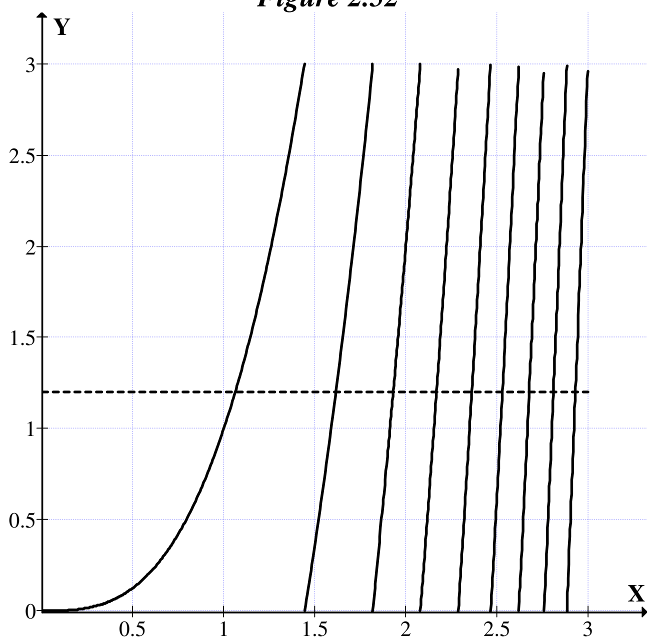

Figure 2.53

Zero lies between (1.4422, 1.4423).

We see this yields many lines which are not parallel. Let us consider $y = x^3$ on the MOD plane $R_n(4)$. The graph is as follows:

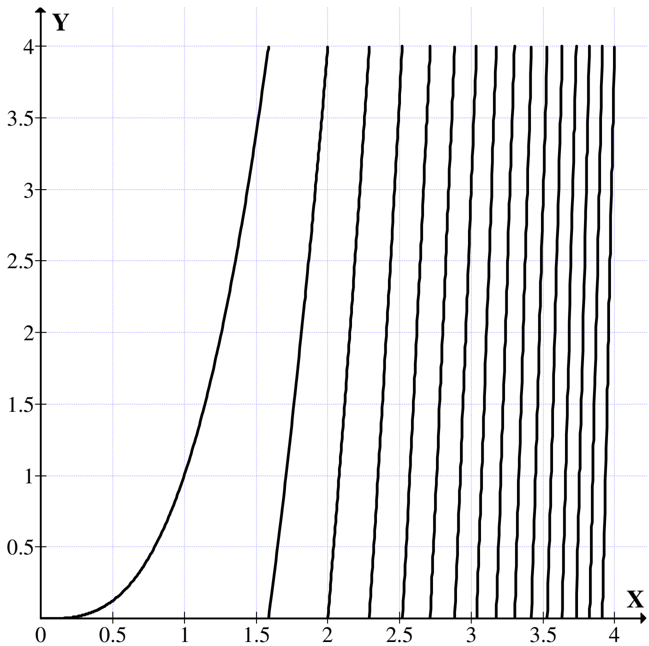

Figure 2.54

If x = 0 then y = 0,
If x = 1 then y = 1,
If x = 1.2 then y = 1.728,
If x = 1.4 then y = 2.744,
If x = 1.5 then y = 3.375,
If x = 1.6 then y = 4.096,
If x = 3 then y = 3,

If x = 1.7 then y = 0.913
If x = 2 then y = 0
If x = 2.2 then y = 2.648
If x = 2.3 then y = 0.167
If x = 2.5 then y = 3.625
If x = 2.6 then y = 1.576
If x = 2.7 then y = 3.683
If x = 2.8 then y = 3.952
If x = 2.9 then y = 0.398
If x = 3.2 then y = 0.768
If x = 3.3 then y = 3.937
If x = 3.31 then y = 0.264691
If x = 3.4 then y = 3.304

If $x = 3.5$ then $y = 2.875$
If $x = 3.42$ then $y = 0.001688$
If $x = 3.64$ then $y = 0.2285$
If $x = 3.7$ then $y = 2.653$
If $x = 3.8$ then $y = 0.872$
If $x = 3.9$ then $y = 3.319$
If $x = 3.92$ then $y = 0.236288$

If $x = 0$ then $y = 0$,
If $x = 2$ then $y = 0$,

If $x \in (1.5, 1.6)$ then $y = 0$
For $x \in (2.2, 2.6)$, $y = 0$
For $x \in (2.5, 2.6)$, $y = 0$
For $x \in (2.7, 2.74)$, $y = 0$
For $x \in (2.8, 2.9)$, $y = 0$
For $x \in (3.1, 3.2)$, $y = 0$
and so on.

$y = x^3$ has over 12 zeros in $R_n(4)$.

Zero lies between 1.585 to 1.6 and at $x = 2$, $y = 0$ and so on. Consider $y = x^3$ at the MOD real plane $R_n(6)$ given in Figure 2.55.

In between 1.8 and 1.9 we have a zero and so on. We see whether they are curves or straight lines.

It is left as an open conjecture for researchers to trace the curves $y = tx^p$ where $t > 1$ in the MOD real plane $R_n(m)$; $m > t$. ($p > 2$).

We see this is a difficult problem and needs a complete analysis. Study of the curve $y = tx^p + s$ $p > 2$, $t, s < m$ over the MOD real plane $R_n(m)$.

Is the function continuous in $R_n(m)$ is an open conjecture.

Next we see we can in the first place define functions $f(x) = x^2 + x + 2$ or any polynomials. To this end we make the following definition.

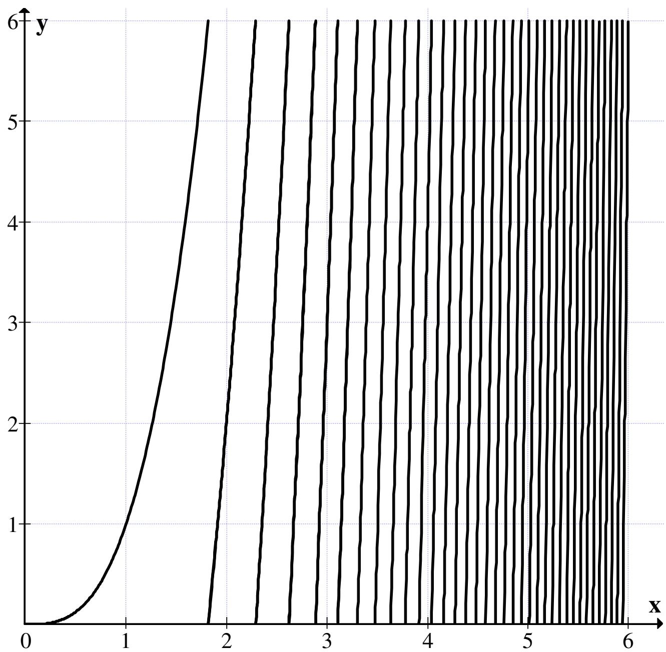

Figure 2.55

DEFINITION 2.1: *Let $R_n(m)$ be the MOD plane.*

Let $[0, m)[x] = I[x]$ where $I = \{a \mid a \in [0, m)\}$ be the pseudo ring with 1, $(m > 1)$.

Any polynomial in $I[x]$ can be defined on the MOD plane in $R_n(m)$; $(m > 1)$.

So to each MOD plane $R_n(m)$ we have the polynomial pseudo ring $I[x]$ where $I = [0,m)$ and

$I[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, m) \right\}$ associated with it any polynomial $p(x)$ in $I[x]$ can be treated as a MOD function in $I[x]$ or as a MOD polynomial in $I[x]$.

A natural question will be, can we define differentiation and integration of the MOD polynomials in $I[x]$ which is associated with the MOD plane $R_n(m)$.

We know the MOD polynomials in general are not continuous on the interval [0, m) or in the MOD real plane $R_n(m)$.

We assume all these functions are continuous on the small intervals $[a, b) \subseteq [0,m)$.

We call these sub intervals as MOD intervals and further define these polynomials to be MOD continuous on the MOD real plane.

Thus all MOD functions are MOD continuous on the MOD intervals in the MOD plane.

We will illustrate this situation by some examples.

Example 2.22: Let $R_n(5)$ be the MOD plane on the MOD interval [0,5).

Let $p(x) = 3x^2 + 2 \in [0, 5)[x]$, we see p(x) is a MOD continuous function on MOD subintervals of [0,5).

So if we define MOD differentiation of p(x) as

$$\frac{dp(x)}{dx} = \frac{d}{dx}(3x^2 + 2) = x.$$

We know $p(x) = 3x^2 + 2$ the MOD function is not continuous on the whole interval [0, 5) only on subintervals of [0, 5) that is why MOD function and they are MOD differentiable. Consider the function $q(x) = 4x^5 + 3x^2 + 3 \in [0,5)[x]$ be MOD polynomial in I[x].

$$(I = [0,5)).\ \frac{d(q(x))}{dx} = 0 + 6x = x.$$

We make the following observations. That is the difference between usual differentiation of polynomials and MOD differentiation of MOD polynomials.

(i) MOD polynomials are continuous only in MOD subintervals of [0, m).

(ii) If p(x) is in I[x] (I = ([0,m))) is a nth degree polynomial in I[x] then the derivative of p(x) need not be a polynomial of degree n – 1.

This is very clear from the case of $q(x) = 4x^5 + 3x^2 + 3$.

Further this q(x), the MOD polynomial is only MOD continuous for q (1) = 0, q(0) = 3 and so on.

$$\frac{dq(x)}{dx} = x.$$

Consider $r(x) = 2x^7 + 2x^5 + 3x + 3 \in I[x]$ (I = [0,5)) be the MOD polynomial the MOD derivative of r(x) is $\frac{dr(x)}{dx} = 4x^6 + 3$.

We can as in case of usual polynomial find derivatives of higher orders of a MOD polynomials.

Let us consider $r(x) = 3x^8 + 4x^5 + 3x + 2$ a MOD polynomial in the MOD interval pseudo ring I(x) (where I = [0, 5)).

$$\frac{d(r(x))}{dx} = 4x^7 + 3 = r'(x)$$

$$\frac{d^2(r(x))}{dx^2} = 3x^6$$

$$\frac{d^3(r(x))}{dx^3} = 3x^5 \text{ and } \frac{d^4(r(x))}{dx^4} = 0.$$

Thus the MOD differentiation behaves in a different way.

However for $r(x) \in R[x]$ then r(x) can be derived 8 times but r(x) as a MOD polynomial can be MOD derived only four times and

$$\frac{d^4(r(x))}{dx^4} = 0.$$

This is yet another difference between the usual differentiation and MOD differentiation of polynomials.

The study in this direction is innovative and interesting.

Likewise we can define the MOD integration of polynomials. This is also very different from usual integration.

Now finally we introduce the simple concept of MOD standardization of numbers or MODnization of numbers.

For every positive integer m to get the MOD real plane we have to define MODnization or MOD standardization of numbers to m which is as follows:

If p is a real number positive or negative, if $p < m$ we take p as it is. If $p = m$ then MODnize it as 0. If $p > m$ and positive divide p by m the remaining value is taken as the MODnized number with respect to m or MOD standardization of p relative (with respect) to m. If p is negative divide p by m the remainder say is t and t is negative then we take $m - t$ as the MODnized number.

We will illustrate this situation for the readers.

Let $m = 5$; the MODnization of the real plane to get the MOD real plane using 5 is as follows.

Let $p \in$ reals, if $p < 5$ take it as it is. If $p = 5$ or a multiple of p then MODnize it as zero.

If $p > 5$ then find $\frac{p}{5} = t + \frac{s}{5}$ then p is MODnized as s if $p > 0$ if $p < 0$ then as $5 - s$.

Thus for instance if $p = 251.7$ to MODnize it in the MOD real plane $R_n(5)$; $\frac{p}{5} = \frac{251.7}{5} = 1.7 \in [0, 5)$.

Let $p = -152.89 \in R$. $\frac{p}{5} = -2.89$ so we have -2.89 to be $2.11 \in [0,5)$.

Let $p = 3.2125$ then $p \in [0,5)$. If $p = 25$ then $p = 0$.

This is the way we MOD standardize the numbers.

MOD standardized numbers will be known as MOD numbers. Thus [0, m) can be also renamed as MOD interval.

Let us consider the MOD standardization of numbers to the MOD plane $R_n(12)$.

Let $p = 3.7052 \in R$ then $p \in [0,12)$.

Let $p = -3.7052 \in R$ then MOD standardized value is $8.2948 \in [0,12)$.

If $p = 15.138 \in R$ then the MOD standardized value is 3.138 in [0,12).

If $p = -19.15 \in R$ the MOD standardized value is 4.85 in [0,12).

This is the way real numbers are MOD standardized to a MOD real plane.

We like wise can MOD standardize any function in the real numbers.

For instance if $y = 7x + 8$ defined in the real plane then the MOD standardized function in the MOD real plane $R_n(6)$ is $y = x + 2$. All functions $y = f(x)$ defined in R, can be MOD standardized in the MOD plane $R_n(m)$.

Also all polynomials in R[x], R-reals can be made into a MOD standardized polynomials in the MOD real plane $R_n(m)$.

We will illustrate this situation by some simple examples.

Let $p(x) = 21x^7 + 14x^2 - 5x + 3 \in R[x]$.

The MOD standardized polynomial in $R_n(7)$ is as follows, $p_n(x) = 0x^7 + 0x^2 + 2x + 3$.

So the first important observation is if degree of p(x) is s then the MOD standardized polynomial in $R_n(m)[x]$ need not be of degree s it can be of degree < s or a constant polynomial or even 0.

For if $p(x) = 14x^8 - 7x^4 + 21x^2 - 35x + 7 \in R[x]$ then the MOD standardized polynomial $p_n(x)$ in $R_n(7)[x]$ is 0.

The same p(x) in $R_n(5)$ is $p_n(x) = 4x^8 + 3x^4 + x^2 + 2$.

In view of all these we have the following theorems.

THEOREM 2.2: *Let $p \in R$ be a real number. $p_n = 0$ can occur after MOD standardization of p in [0, m) if and only if m/p or m = p where p_n is the MODnized number of p.*

The proof is direct hence left as an exercise to the reader.

THEOREM 2.3: *Let y = f(x) be a function defined in R. The MOD standardized function y_n of y can be a zero function or a constant function or a function other than zero or a constant.*

Proof is direct and hence left as an exercise to the reader.

We have give examples of all these situations.

However we will give some more examples for the easy understanding of these new concepts by the reader.

Example 2.23: Let $y = 8x^2 + 6$ be a function defined in the real plane R[x].

Consider the MODnization of y to y_n in the MOD real plane $R_n(4)$; then $y_n = 2$ is a constant function in the MOD real plane.

Example 2.24: Let $y^2 = 8x^3 + 4x^2 + 2$ be the function defined in the real plane. y_n^2 defined in the MOD real plane $R_n(2)$ after MODnization is zero.

Example 2.25: Let $y = 9x^4 + 5x^2 + 23$ be the function defined in the real plane R[x].

Let y_n be the MODnization of y in the plane $R_n(3)$.

$$y_n = 2x^2 + 2.$$

Clearly when MODnization takes place, the polynomial or the function is totally changed.

In view of all these we have the following theorem.

THEOREM 2.4: *Let $p(x) \in R[x]$ be the polynomial in the polynomial ring.*

Let $R_n(m)$ be the MOD real plane. Every polynomial p(x) in R[x] can be MODnized into a polynomial $p_n(x)$ whose coefficients are from $R_n(m)$.

This mapping is not one to one but only many to one. So such mappings are possible. Thus we see R[x] to $R_n(m)[x]$ is given by $p_n(x)$ for every p(x). Several p(x) can be mapped on to $p_n(x)$. It can be a infinite number of p(x) mapped on to a single $p_n(x)$.

We will just illustrate this situation by an example.

Let $p(x) = 5x + 15 \in R[x]$.

In the MOD real plane $R_n(5)$ we see p (x) = 0.

If $p(x) = 15x^3 + 10x^2 + 5 \in R[x]$ then also $p_n(x) = 0$.

If $p(x) = 20x^7 + 5 \in R[x]$ then also $p_n(x) = 0$.

Thus $p(x) = 5(q(x)) \in R[x]$ is such that $p_n(x) = 0$ in the MOD real plane $R_n(5)$.

Thus we see we have infinite number of polynomials in R[x] mapped on to the MOD polynomial in the MOD real plane.

We call $R_n(m)[x]$ as the MOD real polynomial plane.

We see we have infinite number of MOD real polynomial planes. Clearly these MOD real polynomial plane is only a pseudo ring which has zero divisors.

However R[x] is a commutative ring with no zero divisors.

Thus an integral domain is mapped on to the pseudo MOD ring which has zero divisors.

We see R is a field but $R_n(m)$ ($m \geq 2$) is only a pseudo MOD ring which has zero divisors.

Study in all these directions are carried out in the forth coming books.

Now we suggest the following problems.

Problems

(1) Find any other interesting property associated with the MOD real plane.

(2) Define the function $y = 7x$ in the real MOD plane $R_n(9)$.

(i) How many lines are associated with $y = 7x$ on R_n (9)

(ii) Find all x in [0, 9) such that $y = 7x = 0 \pmod 9$.

(iii) What are the slopes of those lines ?

(iv) Does it have any relation with 7?

(3) Show by some techniques it is impossible to define distance between any pair of points in the MOD real plane $R_n(m)$ using the interval [0, m).

(4) Can co ordinate geometry be imitated or defined on the MOD real planes?

(5) Can you prove / disprove that MOD real plane is not a Euclidean plane?

(6) Prove or disprove the MOD real plane can never have the coordinate geometry properties associated with it.

(7) Can a circle $x^2 + y^2 = 2$ be represented as a circle in the MOD real plane $R_n(5)$?

(8) Can a parabola $y^2 = 4x$ in the real plane be represented as a parabola in the MOD real plane $R_n(6)$?

(9) Let $\frac{x^2}{4}+\frac{y^2}{9} = 1$ be the ellipse in the real plane.
What is the corresponding curve in the MOD real plane $R_n(11)$?

(10) Show the curve $\frac{x^2}{4}+\frac{y^2}{9} = 1$ in the real plane R cannot be defined in the MOD real plane $R_n(4)$. Is it defined?

(11) Show $\frac{x^2}{4}+\frac{y^2}{9} = 1$ in the real plane R cannot be defined on the real MOD plane $R_n(36)$.

(12) Find the corresponding equation of $\frac{x^2}{4}+\frac{y^2}{9} = 1$ in the MOD real plane $R_n(19)$.

(13) Find the corresponding equation of $\frac{x^2}{4}+\frac{y^2}{9} = 1$ in $R_n(p)$, p a prime $p > 9$.

(14) Prove or disprove one cannot define the function $3x^2 + 4y^2 = 4$ in the MOD real plane $R_n(2)$.

(15) What is MOD standardization?

(16) MOD standardize the function $y^2 = 38x$ in the MOD real plane $R_n(7)$.

(17) Prove or disprove the MOD standardization of the function $y = 8x^3 + 12x + 16$ is a zero function in $R_n(2)$ and $R_n(4)$.

(18) Prove a nth degree polynomial in R[x] after MOD standardization need not be in general a n^{th} degree polynomial in $R_n(m)$.

(19) Let $p(x) = 10x^9 + 5x^6 + 4x^3 + 3x + 7 \in R[x]$; what is the degree of the polynomial after standardization in

(i) $[0,2)[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0,2)\right\}$.

(ii) $[0,5)[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0,5)\right\}$.

(iii) $[0,10)[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0,10)\right\}$.

(20) Obtain some special and interesting features about MODnization of numbers.

(21) Obtain some special features enjoyed by MOD real planes $R_n(m)$ (m even).

(22) Obtain some special features enjoyed by MOD real planes $R_n(m)$ (m a prime).

(23) Obtain some special features enjoyed by pseudo MOD real rings of $R_n(m)$.

(24) Obtain some special features enjoyed by pseudo MOD polynomials in $R_n(m)[x]$.

(25) Find the properties enjoyed by differentiation of MOD polynomials in $R_n(m)[x]$.

(26) Find the special properties enjoyed by integration of MOD polynomials in $R_n(m)[x]$.

(27) Let $p(x) = 8x^9 + 5x^6 + 2x + 7x + 1 \in R_n(9)[x]$.

Find all higher derivatives of p(x). Integrate p(x) in $R_n(9)[x]$.

(28) Let $R_n(7)[x]$ be the MOD real polynomial pseudo ring.

(i) Show $p(x) \in R_n(7)[x]$ is such that $p'(x) = 0$.
(ii) Characterize those $p(x) \in R_n(7)[x]$ which has only 3 derivatives where deg $p(x) = 12$.
(iii) Let $p(x) \in R_n(7)[x]$ show p(x) is not always integrable.
(iv) Find methods to solve $p(x) \in R_n(7)[x]$.

(29) Let $R_n(12)[x]$ be the MOD real polynomial. Study questions (i) to (iv) of problem (28) for this $R_n(12)[x]$.

(30) Let $R_n(15)[x]$ be the MOD real polynomial. Study questions (i) to (iv) of problem (28) for this $R_n(15)[x]$.

(31) Let $y = 8x^3 + 7 \in R_n(9)$ be the MOD real polynomial. Draw the graph of $y = 8x^3 + 7$.

(32) Let $y = 4x^2 + 6 \in R_n(8)$ be the function in MOD real plane. Draw the graph and find the slope of y.

(33) Let $y= 5x \in R_n(7)$ be the function in the MOD real plane.

(i) Is $y = 5x$ continuous in the MOD real plane $R_n(7)$?
(ii) Is $y = 5x$ have only 5 parallel lines that represents the MOD curve?
(iii) Can $y = 5x$ have all the curves with the same slope?

(34) Is R[x] to $R_n(10)[x]$ a one to one map?

(35) Can R [x] to $R_n(10)[x]$ be a many to one map?

(36) Can R [x] to $R_n(10)[x]$ be a one to many map?

(37) Let p [x] $\in$ $R_n(13)[x]$ find the derivative of p(x) and integral of p(x).

(38) Let $p(x) = 8x^3 + 12x^2 + x + 8 \in R_n(8)$.
(i) Find $p'(x)$.
(ii) Find $\int p(x)dx$.

(39) Let $p(x) = 7x^8 + 4x^3 + 5x + 1 \in R_n(8)[x]$. Find $p'(x)$ and $\int p(x)dx$.

(40) Let $p(x) = 12x^{36} + 5x^{27} + 3 \in R_n(9)[x]$. Find $p'(x)$ and $\int p(x)\, dx$.

(41) Let us consider $y = 8x^2 + 4$ defined in the MOD real plane $R_n(18)$.
(i) Find the graph of $y = 8x^2 + 4$ in the MOD real plane.
(ii) Is the function continuous in the real MOD plane?
(iii) Find the slopes of the curve.
(iv) Find all zeros of $y = 8x^2 + 4$.

(42) Let y = 8x + 1 be a function defined on the MOD real plane $R_n(5)$. Study questions (i) to (iv) of problem 41 for this function $y^2 = 8x + 1$.

(43) Let y = 4x be a function in the MOD real plane $R_n(9)$.
(i) Is the function a parabola in the MOD real plane?
(ii) Does y = 4x a continuous function in the MOD real plane $R_n(9)$?
(iii) Find the slope of the curve y = 4x in $R_n(9)$.
(iv) Study questions (i) to (iv) of problem 41 for this function y = 4x in $R_n(9)$.

(44) Let $\frac{x^2}{9} + \frac{y^2}{4} = 1$ be the function defined in the MOD real plane $R_n(13)$.

(i) Study the nature of the curve.
(ii) What is the equation of the transformed curve?
(iii) Obtain the number of zeros of the curve.
(iv) Is it a line or a curve?

(45) Let $y = x^2$ be the function defined in the MOD real plane $R_n(15)$.
 (i) Study questions (iii) and (iv) of problem 44 for this function.
 (ii) Find all the zeros of the function $y = x^2$ in $R_n(15)$.

(46) Obtain the zeros of the function $y = x^2$ in $R_n(12)$.

(47) Study the function $y = 8x^3 + 3$ in the MOD real plane $R_n(10)$.

(48) Does there exists a function in the MOD plane $R_n(7)$ which is continuous on the semi open interval [0, 7)?

(49) Does there exist a function in the MOD real plane $R_n(12)$ which is continuous in the semi open interval [0, 12) ?

(50) Prove or disprove distance concept cannot defined on the MOD real plane?

(51) Can we define an ellipse in a MOD real plane?

(52) Can the notion of rectangular hyperbola exist in a MOD real plane?

Chapter Three

MOD FUZZY REAL PLANE

In chapter II we have defined the notion of MOD real plane $R_n(m)$, $m \geq 1$. We have mentioned that if $m = 1$ we call the MOD real plane as the MOD fuzzy real plane and denote it by $R_n(1)$.

We will study the properties exhibited by the MOD fuzzy real plane $R_n(1)$. We will also call this as fuzzy MOD real plane.

We study how functions behave in this plane. First of all we have no power to define $y = tx$, $t \geq 1$ for such functions do not take its values from the interval $[0, 1)$. The linear equation or function $y = tx + s$ is defined in the MOD fuzzy plane only if $0 < t, s < 1$. We can have $s = 0$ is possible with $0 < t < 1$.

Here we proceed onto describe them by examples.

Example 3.1: Let $R_m(1)$ be the MOD fuzzy real plane. Consider $y = 0.2x$. The graph of $y = 0.2x$ in $R_m(1)$ is given in Figure 3.1.

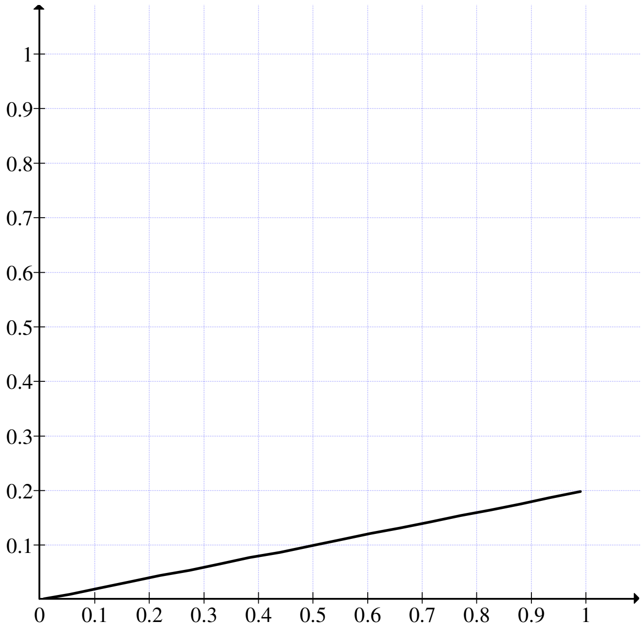

Figure 3.1

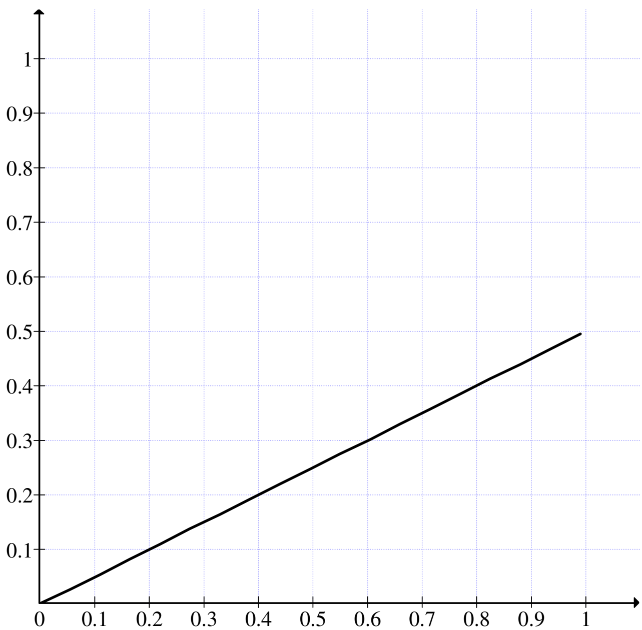

Figure 3.2

This graph is a continuous curve. But bounded on y as $\cong 0.2$.

Example 3.2: Let us consider the function y = 0.5x in the MOD fuzzy real plane $R_n(1)$. The graph of it is given Figure 3.2

We see in the first place we can map the entire real plane on the MOD fuzzy real plane $R_n(1)$. Any integer positive or negative is mapped to zero.

Any (t, s) = (19.00321, 25) is mapped on to (0.00321, 0) and (l, m) = (–8.4521, 0.321) is mapped on to (0.5479, 0.321).

Thus the real plane can be mapped to the MOD fuzzy plane $R_n(1)$. This mapping is infinitely many to one element.

The total real plane is mapped to the MOD fuzzy plane.

All integer pairs both positive or both negative and one positive and other negative are mapped onto (0, 0).

Only positive decimal pairs to itself and all negative decimal pairs (x, y) to (1–x, 1–y) and mixed decimal pairs are mapped such that the integer part is marked to zero and the decimal part marked to itself if positive otherwise to 1– that decimal value.

Some more illustration are done for the better grasp of the readers.

Let P (0.984, –0.031) $\in$ R.

This point R is mapped onto the MOD fuzzy real plane as $P_n(0.984, 0.969) \in R_n(1)$.

Likewise if P (–3.81, –19.06) $\in$ R.

This point P is mapped onto the MOD fuzzy real plane as $P_n(0.19, 0.94) \in R_n(1)$.

Let P (4.316, –8.19) $\in$ R, P is mapped onto
(0.316, 0.81) $\in$ $R_n(1)$.

$P_n(–9.182, 6.01) \in$ R, P is mapped onto
(0.818, 0.01) $\in$ $R_n(1)$.

P(8, 27) $\in$ R is mapped to (0, 0) $\in$ $R_n(1)$.

P(–9, 81) $\in$ R is mapped to (0, 0) $\in$ $R_n(1)$.

This is the way transformation from the real plane R to the MOD fuzzy plane $R_n(1)$ is carried out.

The transformation is unique and is called as the MOD fuzzy transformation of R to $R_n(1)$ denoted by η_1. For instance $\eta_1((0.78, –4.69)) = (0.78, 0.31)$.

$\eta_1 : R \to R_n(1)$ is defined in a unique way as follows:

$$\eta_1((x, y)) = \begin{cases} (0,0) \text{ if } x, y \in Z \\ \quad \text{decimal pair if } x, y \in R^+ \setminus Z \\ 1 - \text{decimal pair } x, y \in R^- \setminus Z \end{cases}$$

The other combinations can be easily defined.

Now we have to study what sort of MOD calculus is possible in this case.

Throughout this book η_1 will denote the MOD fuzzy transformation of R to $R_n(1)$.

Now we recall [6-10]. The concept of MOD fuzzy polynomial can be defined using the real polynomials.

Let R[x] be the real polynomial and $R_n(1)[x]$ be the MOD fuzzy polynomials.

Let $p(x) \in R[x]$; we map using $\eta_1 : R[x] \to R_n(1)[x]$.

$$\text{Let } p(x) = \sum_{i=0}^{\infty} a_i x^i \mid a_i \in R.$$

$$\eta_1(a_i) = 0 \text{ if } a_i \in Z$$

$$\eta(a_i) = \text{decimal if } a_i \in R^+ \setminus Z^+$$

$$\eta(a_i) = 1 - \text{decimal part if } a_i \in R^- \setminus Z^-$$

$$\eta(a_i) = 0 \text{ if } a_i \in Z^-.$$

Thus we can have for every collection of real polynomials a MOD fuzzy polynomials.

The problems with this situation is $\frac{dp_n(x)}{dx} = 0$ for some $p_n(x) \in R_n(1)[x]$.

We will illustrate these situation by some examples.

$$p_n(x) = 0.7x^{16} + 0.6x^{10} + 0.1x^2 + 0.75$$

$$\frac{dp_n(x)}{dx} = 0.7 \times 16x^{15} + 10 \times 0.6 \times x^9 + 2 \times 0.1 \times x + 0$$

$$= 11.2x^{15} + 6x^9 + 0.2x$$

$$= 0.2x^{15} + 0.2x.$$

$$\frac{d^2p_n(x)}{dx^2} = 3.x^{14} + 0.2$$

$$= 0.2 \text{ so}$$

$$\frac{d^3 p_n(x)}{dx^3} = 0.$$

Thus it is unusual the third derivative of a sixteenth degree polynomial is 0.

Let $p_n(x) = 0.34x^5 + 0.01x^3 + 0.07x^2 + 0.9$ is in $R_n(1)[x]$.

$$\frac{dp_n(x)}{dx} = 0.34 \times 5x^4 + 0.01 \times 3x^2 + 0.07 \times 2x + 0$$

$$= 1.70x^4 + 0.03x^2 + 0.14x$$

$$\frac{d^2 p_n(x)}{dx^2} = 0.7 \times 3 \times x^3 + 0.03 \times 2 \times x + 0.14$$

$$= 2.1x^3 + 0.06x + 0.14$$

$$\frac{d^3 p_n(x)}{dx^3} = 0.1 \times 3 \times x^2 + 0.06$$

$$= 0.3 \times x^2 + 0.06$$

$$\frac{d^4 p_n(x)}{dx^4} = 0.3 \times 2 \times x + 0$$

$$= 0.6\,x$$

$$\frac{d^5 p_n(x)}{dx^5} = 0.6.$$

For this function $p_n(x)$ which is the five degree polynomial, in this case the 5^{th} derivative is a constant.

It is important to keep on record that even while deriving implicitly the MOD transformation function η_1 is used.

We have avoided the explicit appearance of η_1 in every derivative.

$$p_n(x) = 0.3x^5 + 0.8\times x^4 + 0.9$$

$$\frac{dp_n(x)}{dx} = \eta_1 (0.3 \times 5)x^4 + \eta_1 (0.8 \times 4)\, x^3 + 0$$

$$= \eta_1 (1.5)x^4 + \eta_1 (3.2)x^3$$

$$= 0.5x^4 + 0.2x^3$$

$$\frac{d^2p_n(x)}{dx^2} = \eta_1 (0.5 \times 4)x^3 + \eta_1 (0.2 \times 3)x^2$$

$$= \eta_1 (2.0)x^3 + \eta_1 (0.6)x^2$$

$$= 0 + 0.6x^2.$$

$$\frac{d^3p_n(x)}{dx^3} = \eta_1 (0.6 \times 2)x = \eta_1 (1.2)x = 0.2x$$

$$\frac{d^4p_n(x)}{dx^4} = \eta_1 (0.2) = 0.2.$$

Thus is the way the role of η_1 takes place in the derivatives.

So in the case of MOD fuzzy polynomials the derivatives are not natural.

We have the MOD fuzzy transformation function plays a major role in this case without the use of this MOD fuzzy transformation η_1 we cannot differentiate the functions.

Now can we integrate the MOD fuzzy polynomials?

The answer is no for we know if

$$p_n(x) \in R_n(1)[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i\text{'s are positive decimals in } R_n(1)\right\}$$

Now suppose $p_n(x) = 0.8x^9 + 0.9x^8 + 0.2x^3 + 0.7$.

Then

$\int p_n(x)\, dx = \int 0.8x^9\, dx + \int 0.9x^8\, dx + \int 0.2\, x^3\, dx + \int 0.7dx$

$$= \frac{0.8x^{10}}{10} + \frac{0.9x^9}{9} + \frac{0.2x^4}{4} + 0.7x + c$$

For $\frac{1}{10}$ or $\frac{1}{9}$ or $\frac{1}{4}$

is undefined in $R_n(1)$ so $\int p_n(x)\, dx$ is undefined whenever $p_n(x) \in R_n(1)[x]$.

Thus we cannot integrate any $p_n(x) \in R_n(1)[x]$ of degree greater than or equal to one. So how to over come this problem or can we define polynomials differently.

The authors suggest two ways of defining polynomials and also we have to define different types of integrations differently.

We in the first place define $R^d[x] = \left\{\sum_{i \in R} a_i x^i \,\middle|\, a_i \in R\right\}$

$$p^d(x) = 9.3101x^{4.3132} + 8.14x^{3.001} + 0.984x^{0.92007} - 9.99079$$

to be the decimal polynomial ring. We can find product, sum and work with. We can also differentiate and integrate it.

Let $p^d(x)$ be given as above;

$$\frac{dp^d(x)}{dx} = 4.3132 \times 9.3101x^{3.3132} + 8.14 \times 3.001x^{2.001} + 0.984 \times 0.92007x^{-0.07993} - 0$$

Now we can also integrate

$\int pd(x)dx$

$$= \frac{9.3101x^{5.3132}}{5.3132} + \frac{8.14x^{4.001}}{4.001} + \frac{0.984x^{1.92007}}{1.92007} - 9.99079x + C$$

This is the way integration is carried out in the decimal ring $R^d[x]$.

Now while defining the MOD polynomials we can define the MOD transformation in two ways.

$$\eta_1 : R^d[x] \to R_n^d(1)[x]$$

$$\text{by } \eta_1(p(x) = \sum_{i \in R} a_i x^i) = \sum_{i \in R} \eta_1(a_i) x^i .$$

This is similar to or same as η_1 for no operation is performed on the variable x.

Another transformation is defined as follows:

$$\eta_1^* : R^d[x] \to R_n^*(1)[x] = \{\sum a_i\, x^i \mid i \in [0, 1), a_i \in [0, 1)\}$$

$$\eta_1^* \,(p^d(x) : \sum_{i \in R} a_i x^i) = \sum \eta_1(a_i) x^{\eta_1(i)} .$$

$\eta_1 : R \to R_n(1)$, the usual MOD transformation; η_1^* is defined as the MOD fuzzy decimal transformation.

We will illustrate both the situations by some examples.

Let $p^d(x) = 6.731x^{8.501} + 3.2x^{1.42} + 6.5 \in R^d[x]$.

$\eta_1 (p^d(x)) = 0.731x^{8.501} + 0.2x^{1.42} + 0.5 \in R_n^d(1)[x] = q(x)$.

$$\frac{dq(x)}{dx} = 0.731 \times \eta_1 (8.501)x^{7.501} + 0.2\eta_1(1.42)\, x^{0.42}$$

$$= (0.731 \times 0.501)x^{7.501} + 0.2 \times 0.42x^{0.42}.$$

This is the way derivatives are found in $R_n^d(1)[x]$.

Now if $p^d(x) \in R^d[x]$, then

$\eta_1^* (p^d(x)) = 0.731x^{0.501} + 0.2x^{0.42} + 0.5 = q^*(x)$.

Now we see this $q^*(x) \in R_n^*(1)[x]$ cannot be easily differentiated for

$$\frac{dq^*(x)}{dx} = 0.731 \times 0.501x^{0.501-1} + 0.2 \times 0.42x^{0.42-1}$$

$= q^*(x)$ (therefore $0.499 = 0.501$ and $-0.58 = 0.42$).

So the power will remain the same; so in case of MOD fuzzy decimal polynomials in $R_n^*(1)[x]$ we follow a different type of differentiation and integration which is as follows:

This differentiation will be called as MOD differentiation.

We see if $p(x) = x^{0.312}$ we decide to MOD differentiate it in the following way.

$$\text{We } \frac{dp(x)}{dx(0.1)} = \frac{dx^{0.312}}{dx(0.1)} = 0.312x^{0.312-0.1}.$$

We say the function is MOD differentiated using 0.1.

Let $p^d(x) = 0.312x^{0.51} + 0.9x^{0.02} + 0.47x^{0.1} + 0.3 \in R_n^*(1)[x]$.

$$\frac{dp^d(x)}{dx(0.1)} = (0.312 \times 0.51) \times x^{0.51-0.1} + (0.9 \times 0.02) \times x^{0.02-0.1} + (0.47 \times 0.1)x^{0.1-0.1} + 0$$

$$= (0.312 \times 0.51)x^{0.41} + (0.9 \times 0.02)x^{0.902} + (0.47 \times 0.1)$$

$$= (0.312 \times 0.51)x^{0.41} + (0.9 \times 0.02)x^{0.902} + 0.047.$$

The advantage of MOD decimal differentiation is that in usual differentiation power of x is reduced by 1 at each stage but in case of MOD decimal representation the MOD power of x can be reduced by any $a \in [0, 1)$.

This is the advantage for we can differentiate a function $p^*(x) \in R_n^*(1)[x]$ in infinite number of ways for every $a \in (0, 1)$ gives a different value for $\frac{dp^*(x)}{dx(a)}$.

We will illustrate this situation by an example or two.

Let $p^*(x) = 0.58x^{0.792} + 0.063x^{0.631} + 0.62x^{0.5} + 0.7x^{0.24} + 0.5x^{0.08} + 0.71 \in R_n^*(1)[x]$.

We MOD decimal differentiate $p^*(x)$ with respect to 0.2

$$\frac{dp^*(x)}{dx(0.2)} = (0.58 \times 0.792)x^{0.592} + (00063 \times 0.631)x^{0.431} + (0.62)(0.5)x^{0.3} + (0.7 \times 0.24)x^{0.04} + (0.5 \times 0.08)x^{0.88} + 0 \quad \text{--- I}$$

We now differentiate with respect to 0.03;

$$\frac{dp^*(x)}{dx(0.03)} = (0.58 \times 0.792)x^{0.762} + (0.063 \times 0.631)x^{0.601} + (0.62 \times 0.5)x^{0.47} + 0.7 \times 0.28x^{0.21} + (0.5 \times 0.8)x^{0.05} + 0 \quad \text{--- II}$$

Clearly I and II are distinct the only common feature about them is that both $\frac{dp^*(x)}{dx(0.2)}$ and $\frac{dp^*(x)}{dx(0.03)}$ are such that every respective term has the same coefficient but the MOD degree of the polynomial is different in both cases for each of the coefficients.

Recall when the degree of x lies in the interval (0, 1) we call it as MOD degree of the polynomials.

The highest MOD degree and lowest MOD degree is similar to usual polynomials.

Now we MOD decimal differentiate p*(x) with respect to 0.5.

$$\frac{dp^*(x)}{dx(0.5)} = (0.58 \times 0.792)x^{0.292} + (0.063 \times 0.631)x^{0.131} + (0.62 \times 0.5) + 0.7 \times 0.24x^{0.74} + (0.5 \times 0.08) \times x^{0.58} \quad \text{--- III}$$

This is also different from I and II.

We see the degree of the MOD decimal polynomial in all the three MOD differentiate are distinct.

This is an interesting and an important feature about the MOD fuzzy differentiation of the MOD fuzzy polynomials.

If one is interested in finding integration then only MOD decimal integration on $R_n^*(1)[x]$ is valid all other differentiation fails to yield any form of meaningful results.

So we can integrate in similar way as we have differentiated using MOD decimal differentiation techniques.

Let $p^*(x) = 0.92x^{0.81} + 0.65x^{0.63} + 0.139x^{0.21} + 0.03x^{0.7} + 0.14 \in R_n^*(1)[x]$.

We integrate with respect to 0.1.

$$\int p^*(x)dx(0.1) = \frac{0.92x^{0.81+0.1}}{0.81+0.1} + \frac{0.65x^{0.63+0.1}}{0.63+0.1} + \frac{0.139x^{0.21+0.1}}{0.21+0.1} + \frac{0.03x^{0.7+0.1}}{0.7+0.1} + \frac{0.14x^{0.1}}{0.1} + C \quad \text{--- I}$$

We integrate with MOD decimal with respect to 0.2.

$$\int p^*(x)dx(0.2) = \frac{0.92x^{0.81+0.2}}{0.81+0.2} + \frac{0.65x^{0.63+0.2}}{0.63+0.2} + \frac{0.139x^{0.21+0.2}}{0.21+0.2} + \frac{0.03x^{0.7+0.2}}{0.7+0.2} + \frac{0.14x^{0.2}}{0.2} + C$$

$$= \frac{0.92x^{0.01}}{0.01} + \frac{0.65x^{0.83}}{0.83} + \frac{0.139x^{0.41}}{0.41} + \frac{0.3x^{0.9}}{0.9}$$

$$+ \frac{0.14x^{0.2}}{0.2} + C \qquad \text{--- II}$$

Clearly I and II are different and we see we can integrate the MOD decimal polynomial in infinite number of ways but they have no meaning as $\frac{1}{0.01}$ or $\frac{1}{0.83}$ etc; have no meaning in [0, 1) as no n $\in$ [0, 1) has inverse.

Now we will integrate with respect to the MOD decimal 0.9 $\in$ (0, 1).

$$\int p^*(x)dx(0.9) = \frac{0.92x^{0.81+0.9}}{0.81+0.9} + \frac{0.65x^{0.63+0.9}}{0.63+0.9} + \frac{0.139x^{0.21+0.9}}{0.21+0.9}$$

$$+ \frac{0.03x^{0.7+0.9}}{0.7+0.9} + \frac{0.14x^{0.9}}{0.9} + C$$

$$= \frac{0.92x^{0.71}}{0.71} + \frac{0.65x^{0.53}}{0.53} + \frac{0.139x^{0.11}}{0.11} + \frac{0.03x^{0.6}}{0.6}$$

$$+ \frac{0.14x^{0.9}}{0.9} + C \qquad \text{--- III is in } R_n^*(1)[x].$$

I, II and III are distinct and infact we can get infinite number of MOD decimal integral values of p*(x) $\in$ $R_n^*(1)[x]$.

So it is an open conjecture to define integrals in fuzzy MOD planes.

The next challenging problem is to find methods of solving equations in $R_n^*(1)[x]$.

If p*(x) = 0 how to solve for the roots in $R_n^*(1)[x]$.

Equally difficult is the solving of equations $p^d(x) = 0$ in $R_n^d(1)[x]$.

We will first illustrate this situation by some examples.

$$\text{Let } p^*(x) = (x^{0.3} + 0.4)(x^{0.1} + 0.5)$$

$$= x^{0.4} + 0.4x^{0.1} + 0.5x^{0.3} + 0.2.$$

The roots are $x^{0.3} = 0.6$ and $x^{0.1} = 0.5$.

$$\text{For } p^*(0.6) = (0.6 + 0.4)(0.6 + 0.5) = 0.$$

Likewise $p^*(0.5) = (0.5 + 0.4)(0.5 + 0.5) = 0$.

This is the way roots can be determined.

It is left as an open conjecture to solve the MOD decimal polynomials in $R_n^*(1)[x]$.

Let us consider $p^*(x) = (x^{0.5} + 0.2)^2 \in R_n^*(1)[x]$.

$$p^*(x) = 0.4x^{0.5} + 0.04$$

$$\frac{dp^*(x)}{dx(0.3)} = \frac{d(0.4x^{0.5} + 0.04)}{dx(0.3)}$$

$$= 0.4 \times 0.5x^{0.5-0.3}$$

$$= 0.2x^{0.2} = q^*(x).$$

$x^{0.5} = 0.8$ is a root but
$q^*(0.8) \neq 0$.

Thus if $p^*(x)$ has a multiple root to be $a \in (0, 1)$ then a is not a root of $\frac{dp^*(x)}{dx(a)}$, $a \in (0, 1)$.

So the classical property of calculus of differential is not in general true in case MOD decimal differentiation. Several such properties are left as open problems for the interested reader.

$R_n^*(1)[x]$ is only a pseudo MOD ring of decimal polynomials.

For if $q^*(x) = 0.73x^{0.25} + 0.921x^{0.7} + 0.21x^{0.2} + 0.1$

and $p^*(x) = 0.53x^{0.25} + 0.13x^{0.7} + 0.35x^{0.1} + 0.9$ are in $R_n^*(1)[x]$; then we see

$$\begin{aligned} p^*(x) + q^*(x) &= (0.73x^{0.25} + 0.921x^{0.7} + 0.21x^{0.2} + 0.1) + \\ &\quad (0.53x^{0.25} + 0.13x^{0.7} + 0.35x^{0.1} + 0.9) \\ &= (0.73x^{0.25} + 0.53x^{0.25}) + (0.921x^{0.7} + \\ &\quad 0.13x^{0.7}) + 0.21x^{0.2} + 0.35x^{0.1} + \\ &\quad (0.1 + 0.9) \\ &= 0.26x^{0.25} + 0.051x^{0.7} + 0.21x^{0.2} + \\ &\quad 0.35x^{0.1} + 0 \in R_n^*(1)[x]. \end{aligned}$$

We see with respect to addition closure axiom is true. Infact $p^*(x) + q^*(x) = q^*(x) + p^*(x)$.

That is addition is commutative.

Further for $p^*(x), q^*(x), r^*(x) \in R_n^*(1)[x]$

We see $(p^*(x) + q^*(x)) + r^*(x)$

$= p^*(x) + (q^*(x) + r^*(x)).$

That is the property of addition of MOD decimal polynomials is associative.

Finally $0 \in [0, 1)$ acts as the additive identity of the MOD decimal polynomial.

Further to each $p^*(x) \in R_n^*(1)[x]$

we see there exists a unique $s^*(x) \in R_n^*(1)[x]$ such that

$$p^*(x) + s^*(x) = 0.$$

That is $s^*(x)$ is the unique MOD decimal polynomial inverse of $p^*(x)$ and vice versa.

Thus $(R_n^*(1)[x], +)$ is an infinite MOD decimal polynomial group of infinite order.

Now we see how the operation $\times$ acts on $R_n^*(1)[x]$.

Before we define $\times$ we just show for any $p^*(x)$ the additive inverse of it in $R_n^*(1)[x]$.

Let $p^*(x) = 0.378x^{0.15} + 0.9213x^{0.9} + 0.113x^{0.01} + 0.017x^{0.5} + 0.1113 \in R_n^*(1)[x]$.

The additive inverse of $p^*(x)$ is $-p^*(x) = 0.622x^{0.15} + 0.0787x^{0.9} + 0.887x^{0.01} + 0.983x^{0.5} + 0.8887 \in R_n^*(1)[x]$ such that $p^*(x) + (-p^*(x)) = 0$.

Let $p^*(x) = 0.2x^{0.8} + 0.9x^{0.3} + 0.11x^{0.4} + 0.3$

and $q^*(x) = 0.9x^{0.6} + 0.3x^{0.5} + 0.1x^{0.7} + 0.9$ be in $R_n^*(1)[x]$. To define product for these two MOD decimal polynomials.

$$p^*(x) \times q^*(x) = (0.2x^{0.8} + 0.9x^{0.3} + 0.11x^{0.4} + 0.3) \times (0.9x^{0.6} + 0.3x^{0.5} + 0.1x^{0.7} + 0.9)$$

$$= 0.18x^{0.4} + 0.81x^{0.9} + 0.099 + 0.27x^{0.6} + 0.06x^{0.3} + 0.7x^{0.8} + 0.033x^{0.9} + 0.09x^{0.5} + 0.02x^{0.5} + 0.09 + 0.011x^{0.1} + 0.03x^{0.7} + 0.18x^{0.8} + 0.81x^{0.3} + 0.099x^{0.4} + 0.27$$

$$= 0.843x^{0.9} + 0.88x^{0.8} + 0.03x^{0.7} + 0.27x^{0.6} + 0.11x^{0.5} + 0.099x^{0.4} + 0.87x^{0.3} + 0.011x^{0.1} + 0.459 \in R_n^*(1)[x].$$

It is easily verified $p^*(x) \times q^*(x) = q^*(x) \times p^*(x)$ for all $p^*(x), q^*(x)$ in $R_n^*(1)[x]$.

We next show whether $p^*(x) \times (q^*(x) \times r^*(x))$ is equal to $(p^*(x) \times q^*(x)) \times r^*(x)$.

Let $p^*(x) = 0.7x^{0.6} + 0.8x^{0.4} + 0.5$

$q^*(x) = 0.8x^{0.9} + 0.7$ and $r^*(x) = 0.2x^{0.2} + 0.3x^{0.1} + 0.1 \in R_n^*(1)[x]$.

We find $[p^*(x) \times q^*(x)] \times r^*(x)]$

$$= [(0.7x^{0.6} + 0.8x^{0.4} + 0.5) \times (0.8x^{0.9} + 0.7) \times (0.2x^{0.2} + 0.3x^{0.1} + 0.1)$$

$$= (0.56x^{0.5} + 0.64x^{0.3} + 0.4x^{0.9} + 0.49x^{0.6} + 0.56x^{0.4} + 0.35) \times (0.2x^{0.2} + 0.3x^{0.1} + 0.1)$$

$$= (0.112x^{0.7} + 0.128x^{0.5} + 0.08x^{0.1} + 0.098x^{0.8} + 0.112x^{0.6} + 0.07x^{0.2} + 0.168x^{0.6} + 0.192x^{0.4} + 0.12 + 0.147x^{0.7} + 0.168x^{0.5} + 0.105x^{0.1} + 0.056x^{0.5} + 0.064x^{0.3} + 0.04x^{0.9} + 0.049x^{0.6} + 0.056x^{0.4} + 0.035)$$

$$= 0.04x^{0.9} + 0.098x^{0.8} + 0.259x^{0.7} + 0.319x^{0.6} + 0.352x^{0.5} + 0.248x^{0.4} + 0.105x^{0.3} + 0.07x^{0.2} + 0.185x^{0.1} + 0.155 \quad \ldots \quad \text{I}$$

Consider $p^*(x) \times [q^*(x) \times r^*(x)]$

$p^*(x) \times [(0.8x^{0.9} + 0.7) \times (0.2x^{0.2} + 0.3x^{0.1} + 0.1)]$

$$= [0.7x^{0.6} + 0.8x^{0.4} + 0.5] [0.16x^{0.1} + 0.14x^{0.2} + 0.24 + 0.21x^{0.1} + 0.08x^{0.9} + 0.07]$$

$$= 0.112x^{0.7} + 0.128x^{0.5} + 0.080x^{0.1} + 0.098x^{0.8} + 0.112x^{0.6} + 0.07x^{0.2} + 0.120 + 0.168x^{0.6} + 0.192x^{0.4} + 0.105x^{0.1} + 0.147x^{0.7} + 0.168x^{0.5} + 0.056x^{0.5} + 0.064x^{0.3} + 0.040x^{0.9} + 0.049x^{0.6} + 0.056x^{0.4} + 0.035$$

$$= 0.040x^{0.9} + 0.098x^{0.8} + 0.259x^{0.7} + 0.319x^{0.6} + 0.352x^{0.5} + 0.248x^{0.4} + 0.105x^{0.3} + 0.07x^{0.2} + 0.185x^{0.1} + 0.155 \quad \ldots \quad \text{II}$$

I and II are not distinct so the operation is associative.

Let $p^*(x) = 0.9x^{0.7}$,

$q^*(x) = 0.6x^{0.9}$ and

$r^*(x) = 0.12x^{0.3} \in R_n^*(1)[x]$.

$p^*(x) \times [q^*(x) \times r^*(x)]$

$= p^*(x)\ [0.072x^{0.2}]$

$= 0.9x^{0.7} \times 0.072x^{0.2}$

$= 0.0648x^{0.9} \quad \ldots \quad \text{I}$

$[p^*(x) \times q^*(x)] \times r^*(x)$

$$= (0.9x^{0.7} \times 0.6x^{0.9}) \times 0.12x^{0.3}$$

$$= 0.54x^{0.6} \times 0.12x^{0.3}$$

$$= 0.0648x^{0.9} \qquad \ldots \quad \text{II}$$

I and II are equal.

$a \times (b \times c) = a \times (b \times c)$ for a, b, c $\in [0, 1)$.

So $\times$ operation on $R_n^*(1)[x]$ is always associative.

Infact $p^*(x) \times [q^*(x) + r^*(x)] \neq p^*(x) \times q^*(x) + p^*(x) \times r^*(x)$.

Take $p^*(x) = 0.5x^{0.8}$

$q^*(x) = 0.7x^{0.4}$ and $r^*(x) = 0.3x^{0.4}$ be the MOD decimal polynomials in $R_n^*(1)[x]$.

Consider $p^*(x) \times [q^*(x) + r^*(x)]$

$$= p^*(x) \times [0.7x^{0.4} + 0.3x^{0.4}]$$
$$= p^*(x) \times 0 = 0. \qquad \ldots \quad \text{I}$$

Consider

$p^*(x) \times q^*(x) + p^*(x) \times r^*(x)$

$$= 0.5x^{0.8} \times 0.7x^{0.4} + 0.5x^{0.8} \times 0.3x^{0.4}$$
$$= 0.35x^{0.2} + 0.15x^{0.2}$$
$$= 0.5x^{0.2} \qquad \ldots \quad \text{II}$$

I and II are distinct so the distributive law is not in general.

Let us consider $R_n^d(1)[x]$.

Is $R_n^d(1)[x]$ a ring or only a pseudo ring?

$$R_n^d(1)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 1) \right\}.$$

Let $p^d(x) = 0.3x^{9.2} + 0.4x^{2.1} + 0.8$

and $q^d(x) = 0.7x^{6.8} + 0.7x^{1.5} + 0.3 \in R_n^d(1)[x]$.

$$p^d(x) + q^d(x) = 0.3x^{9.2} + 0.7x^{6.8} + 0.4x^{2.1} + 0.7x^{1.5} + 0.1 \in R_n^d(1)[x].$$

We see '+' is a commutative operation with 0 as the additive identity.

For every polynomial $p^d(x) \in R_n^d(1)[x]$ we have a unique $q^d(x) \in R_n^d(1)[x]$ such that $p^d(x) + q^d(x) = (0)$.

Let $p^d(x) = 0.71x^{2.3} + 0.912x^{0.8} + 0.016x^{2.1} + 0.71 \in R_n^d(1)[x]$.

$q^d(x) = 0.29x^{2.3} + 0.088x^{0.8} + 0.984x^{2.1} + 0.29 \in R_n^d(1)[x]$ is such that $p^d(x) + q^d(x) = 0$. Clearly $q^d(x) = -p^d(x)$.

Thus $R_n^d(1)[x]$ under + is a group of infinite order which is abelian.

Let $p^d(x) = 0.7x^9 + 0.81x^2 + 0.3$ and

$q^d(x) = 0.1x^8 + 0.5x + 0.4 \in R_n^d(1)[x]$ we see

$$p^d(x) \times q^d(x) = (0.7x^9 + 0.81x^2 + 0.3) \times (0.1x^8 + 0.5x + 0.4)$$

$= 0.07x^{17} + 0.081x^{10} + 0.03x^8 + 0.35x^{10} + 0.405x^3 + 0.15x + 0.28x^9 + 0.324x^2 + 0.12 \in R_n^d(1)[x]$, this is the way $\times$ operation is performed on $R_n^d(1)[x]$

We see for $p^d(x) = 0.3x^8 + 0.9x + 0.1$

$q^d(x) = 0.4x^7 + 0.8$ and $r^d(x) = 0.1x^6 \in R_n^d(1)[x]$.

We find $[p^d(x) \times q^d(x)] \times r^d(x)$

Now $[p^d(x) \times q^d(x)]\ r^d(x) =$

$$[(0.3x^8 + 0.9x + 0.1) \times [0.4x^7 + 0.8)]\ r^d(x)$$

$$= [0.12x^{15} + 0.36x^8 + 0.04x^7 + 0.24x^8 + 0.72x + 0.08] \times 0.1x^6$$

$$= 0.012x^{21} + 0.036x^{14} + 0.004x^{13} + 0.024x^{14} + 0.008x^6 + 0.072x^7 \quad \ldots \quad \text{I}$$

Consider $p^d(x) \times (q^d(x) \times r^d(x))$

$$= p^d(x) \times [0.4x^7 + 0.8 \times 0.1x^6]$$

$$= p^d(x) \times [0.04x^{13} + 0.08x^6]$$

$$= (0.3x^8 + 0.9x + 0.1)\ (0.04x^{13} + 0.08x^6)$$

$$= 0.012x^{21} + 0.036x^{14} + 0.004x^{13} + 0.024x^{14} + 0.072x^7 + 0.008x^6 \quad \ldots \quad \text{II}$$

I and II are identical infact the operation $\times$ is associative on $R_n^d(1)[x]$.

Now let us consider

$p^d(x) = 0.3x^7 + 0.8$,

$q^d(x) = 0.5x^5 + 0.2$ and

$r^d(x) = 0.7 + 0.2x^8 + 0.8x^5 \in R_n^d(1)[x]$.

$p^d(x) \times (q^d(x) + r^d(x))$

$$= p^d(x) \times [0.5x^5 + 0.2 + 0.2x^8 + 0.8x^5 + 0.7]$$

$$= (0.3x^7 + 0.8)\ [0.3x^5 + 0.9 + 0.2x^8]$$

$$= 0.09x^{12} + 0.27x^7 + 0.06x^{15} + 0.24x^5 + 0.72 + 0.16x^8 \quad \ldots \quad \text{I}$$

$p^d(x)\ q^d(x) + p^d(x)\ r^d(x)$

$$= (0.3x^7 + 0.8) \times (0.5x^5 + 0.2) + (0.3x^7 + 0.8) \times (0.7 + 0.2x^8 + 0.8x^5)$$

$$= 0.15x^{12} + 0.40x^5 + 0.16 + 0.06x^7 + 0.21x^7 + 0.56 + 0.06x^{15} + 0.16x^8 + 0.24x^{12} + 0.64x^5 \quad \ldots \quad \text{II}$$

Clearly I and II are distinct so we see ($R_n^d(1)[x]$, +, ×) is not a ring only a pseudo MOD polynomial ring as the distributive law is not true.

Thus both $R_n^d(1)[x]$ and $R_n^*(1)[x]$ are only pseudo rings.

Infact study of properties on MOD fuzzy plane would be a substantial work for any researcher.

Thus both $R_n^d(1)[x]$ and $R_n^*(1)[x]$ enjoy different properties relative to differentiation and integration of polynomials.

For in $R_n^*(1)[x]$ usual integration and differentiation cannot be defined; infact we have infinite number of differentials and integrals for a given MOD decimal polynomials in $R_n^*(1)[x]$ never exists.

Infact for every a ∈ (0, 1) we have a MOD derivative and a MOD integral not possible.

We call this structure as MOD for two reasons one they are made up of small values and secondly they do not have the usual integrals or derivatives of the functions.

Defining equations of circles, ellipse, hyperbola, parabola or for that matter any conic on the MOD fuzzy plane happens to be a challenging problem for any researcher.

Infact continuous curves in the real plane happens to be discontinuous parallel lines in the MOD fuzzy plane. Infact they have zeros in certain cases. Sometimes a continuous line and so on.

We suggest some problems.

Problem

1. Obtain any special and interesting feature about MOD fuzzy plane.

2. What is the striking property enjoyed by the MOD fuzzy plane?

3. Is the MOD fuzzy plane orderable?

4. Find the curve $y = 0.5x^2$ in the MOD fuzzy plane.
   Is the function continuous in $R_n(1)$?

5. Draw the function $y = 0.7x + 0.4$ in the MOD fuzzy plane $R_n(1)$.

6. Draw the function $y = 0.8x^2 + 0.6x + 0.1$ on the MOD fuzzy plane Rn(1).

   (i) Is this function continuous or discontinuous?
   (ii) Does the function have zeros?
   (iii) Is this function derivable?

7. Plot the function $y^2 = 0.4x$ on the MOD fuzzy plane $R_n(1)$.

8. What are the advantages of using functions in the fuzzy MOD plane?

9. Is it possible to define the distance concept on $R_n(1)$?

10. Plot the function $0.7x^2 + 0.7y^2 = 0.2$ on the MOD fuzzy plane $R_n(1)$.
    (i) Will this be a circle in $R_n(1)$? Justify your claim.
    (ii) Is this function a line?

11. Let $0.5x^2 + 0.25x^2 = 0.3$ be a function in the MOD fuzzy plane $R_n(1)$.
    (i) Can we say the resultant is a curve?
    (ii) Will the graph be a straight line or a curve?
    (iii) Is this function a continuous curve or a line?

12. Can $R_n(1)$, the fuzzy MOD plane be considered as the MOD form of the coordinate plane?

13. Can we have the notion of conics in the MOD fuzzy plane $R_n(1)$?

14. Is it possible to have more than one MOD fuzzy transformation η_1 from R to $R_n(1)$?

15. Let R[x] be the real polynomials η : R[x] to $R_n^d(1)[x]$ be a map.
    How many such MOD fuzzy transformations exist?

16. Let $R^d[x]$ be the real polynomials $\eta_1 : R^d[x] \to R_n^*(1)[x]$ be a MOD fuzzy real transformation.
    How many such MOD real transformations can be defined?

17. Let
    $p(x) = 0.3x^{2.3} + 0.7x^{5.7} + 0.2x^{12.3} + 0.5 \in R_n^d(1)[x]$.
    (i) Find $\frac{dp(x)}{dx}$ and $\int p(x)\, dx$.
    (ii) Is the integration the reverse process of differentiation or vice versa?
    (iii) Does integrals in $R^d(1)[x]$ exist?

18. Let $p(x) = 0.9x^{0.8} + 0.7x^{0.5} + 0.61x^{0.2} + 0.71 \in R_n^*(1)[x]$.
   (i) Find $\frac{dp(x)}{dx(0.2)}$.
   (ii) Find $\int p(x)\,dx$ (0.2), does it exist?
   (iii) Is $\frac{d\int p(x)dx(0.2)}{dx(0.2)} = p(x)$ possible?

19. Let
   $p(x) = 0.27x^{0.71} + 0.512x^{0.3} + 0.3107x^{0.2} + 0.7 \in R_n^d(1)[x]$.
   Study questions (i) to (iii) of problem 18 for this p(x).

20. Let $p(x) = 0.312x^{0.9} + 0.881x^{0.7} + 0.031x^{0.003} + 0.009 \in R_n^d(1)[x]$. Take $\alpha = 0.007$.
   Study questions (i) to (iii) of problem 18 for this p(x) with $\alpha = 0.007 \in (0, 1)$.

21. Let $R_n^d(1)[x]$ be the MOD decimal fuzzy polynomial plane.
   (i) Prove on $R_n^d(1)[x]$ we can define infinite number of derivatives and no integrals.
   (ii) Let $p(x) = 0.72x^{0.72} + 0.49x^{0.49} + 0.332x^{0.332} + 0.7 \in R_n^d(1)[x]$; does there exist $\alpha \in (0, 1)$ such that $\frac{dp(x)}{dx(\alpha)} = 0$ and $\int p(x)dx\ (\alpha) \neq 0$?
   (iii) Find $\int p(x)\,dx$ (0.32), does it exist?
   (iv) Find $\frac{dp(x)}{dx(0.32)}$.

22. Let $R_n^*(1)[x]$ be the MOD fuzzy decimal polynomials.
   (i) Study question (i) to (iv) problem 21 for p(x) with respect $\alpha = 0.98$.
   (ii) Compare the results.

23. Obtain some special features enjoyed by $R_n^d(1)[x]$.

24. Obtain the special features associated with $R_n^*(1)[x]$.

25. Is $R_n^d(1)[x]$ a group with respect to +?

26. Is $R_n^d(1)[x]$ a semigroup with respect to ×?

27. Is $R_n(1)$ a ring?

28. Does the MOD fuzzy plane $R_n(1)$ contain zero divisors?

29. Can the MOD fuzzy plane $R_n(1)$ contain idempotents?

30. Can the MOD fuzzy plane $R_n(1)$ contain S-zero divisors?

31. Is the MOD fuzzy plane $R_n(1)$ be a semigroup with zero divisors?

32. Compare the MOD fuzzy plane $R_n(1)$ with $R_n^d(1)[x]$ and $R_n^*(1)[x]$.

33. Can $R_n^*(1)[x]$ contain zero divisors?

34. Let $R_n^*(1)[x]$ be the MOD fuzzy plane.
    Let $y = 0.3x^2 + 0.4x + 1 \in R_n^*(1)[x]$.
    Draw the graph of y on $R_n^*(1)$

35. Obtain some special features enjoyed by $R^d[x]$.

36. Compare $R_n^d(1)[x]$ with $R_n^*(1)[x]$.

37. What can be the practical uses of integration and differentiation using $\alpha \in (0, 1)$?

38. Can we define for every $\alpha \in R^+ \setminus \{0\}$ the new type of differentiation on R[x]?
    What is the problem faced while defining it?

39. Draw the graph of the function $y = 0.3x^2 + 0.2$ in the MOD fuzzy plane $R_n(1)$?

40. Obtain a method to solve polynomial equations in $R_n^d(1)[x]$.

41. Show if α is a multiple root of $p(x) \in R_n^d(1)[x]$ then α, in general is not a root of $p'(x)$.

42. Show that there exists a lot of difference between MOD fuzzy differentiation and usual differentiation.

43. Show that there exists difference between usual integration and MOD fuzzy integration.

44. What are the advantages of using derivatives with every $\alpha \in (0, 1)$ in $p(x) \in R_n^*(1)[x]$?

45. What is the justification in using the derivatives of p(x) the power of x decreases by 1 and powers of x increases by 1 in integrals?

46. What is the draw back in replacing 1 by any real?

47. Is every function which is continuous in the real plane R continuous in the MOD fuzzy plane $R_n(1)$?

48. Can we give representation for trigonometric functions in the MOD fuzzy plane? Justify your claim.

49. Obtain any interesting features enjoyed by the MOD fuzzy plane $R_n(1)$.

50. What are the special properties associated with $R^d[x]$?

51. What are the special features enjoyed by the MOD fuzzy decimal polynomials $R_n^*(1)[x]$?

52. If $p(x) \in R_n^*(1)[x]$ does there exist distinct $\alpha, \beta \in (0, 1)$

$$(\alpha \neq \beta) \text{ but } \frac{dp'(x)}{dx(\alpha)} = \frac{dp'(x)}{dx(\beta)} ?$$

53. Can we say if $\alpha_i, \alpha_j \in (0, 1)$ are distinct then $\int p(x)\, dx\, (\alpha_i) \neq \int p(x)\, dx\, (\alpha_j)$ for every $p(x) \in R_n^*(1)[x]$ $\alpha_i \neq \alpha_j$?

54. Is it possible to define step functions in the MOD fuzzy plane $R_n(1)$?

55. Can we say for every $p(x) \in R_n^*(1)[x]$ there exists a $q(x) \in R_n^*(1)[x]$ such that

$$\frac{dp(x)}{dx(\alpha_i)} = \frac{dq(x)}{dx(\alpha_j)}$$

$\alpha_i \neq \alpha_j$, $\alpha_i, \alpha_j \in (0, 1)$?

56. What is the graph of $y = 0.001x^4$ in the MOD fuzzy plane $R_n(1)$?

57. What is the graph of $y = 0.01x^4$ in the MOD fuzzy plane $R_n(1)$?

    Compare the graphs in problems 56 and 57.

58. Find the graph of $y = 0.1x^4$ in the MOD fuzzy plane.

59. Does the graph $y = 0.5x^5$ defined in $R_n(1)$ have 5 zeros? Justify.

60. $y = 0.5x^2$, $y = 0.5x^3$, $y = 0.5x$ and $y = 0.5x^4$ be four distinct functions on the MOD fuzzy plane.
    Compare the MOD fuzzy graphs of the four functions?

61. Let $f(x) = 0.9x^{0.8} + 0.3x^{0.6} + 0.21x^{0.5} + 0.01x^{0.3} + 0.92 \in R_n^*(1)[x]$.

    (i) Find $\dfrac{df(x)}{dx(0.8)}$ and $\int f(x)\, dx\, (0.8)$.

(ii) Find $\frac{df(x)}{dx(0.6)}$ and ∫ f(x) dx (0.6).

(iii) Find $\frac{df(x)}{dx(0.5)}$ and ∫ f(x) dx (0.5).

(iv) Find $\frac{df(x)}{dx(0.3)}$ and ∫ f(x) dx (0.3).

(v) Find $\frac{df(x)}{dx(0.120)}$ and ∫ f(x) dx (0.120).

Show none of the integrals are defined.

62. Is it possible to define the notion of circle in a MOD fuzzy plane?

63. Can distance concept be adopted on a MOD fuzzy plane?

64. Is a MOD fuzzy plane Euclidean?

65. Can a metric be defined over the MOD fuzzy plane?

66. How to solve equations in $R_n^*(1)[x]$?

67. If $p(x) = 0.33x^{0.2} + 0.75x^{0.1} + 0.2 \in R_n^*(1)[x]$.
    (i) Is p(x) solvable?
    (ii) Can p(x) have more than two roots?
    (iii) Does the roots of p(x) belong to [0, 1)?

68. Can we say 0.5^{th} degree polynomials can have five roots?

69. Can a MOD quadratic polynomial be defined in $R_n^*(1)[x]$?

70. If $p(x) = 0.7x^{0.6} + 0.83x^{0.3} + 0.2 \in R_n^*(1)[x]$ can we say p(x) is a special quadratic in 0.3 that is $0.7x^{(0.3)^2} + 0.83x^{0.3} + 0.2$?

Chapter Four

MOD NEUTROSOPHIC PLANES AND MOD FUZZY NEUTROSOPHIC PLANES

In this chapter we introduce the notion of MOD neutrosophic planes and MOD fuzzy neutrosophic planes. This is very important as the concept of indeterminacy plays a vital role in all types of problems.

Recall I is called the indeterminacy defined and developed by Florentin Smarandache in [3, 4] and $I^2 = I$ which will denote the neutrosophic or the indeterminacy concept.

For more about this concept refer [3, 4].

We denote the MOD neutrosophic plane by

$$R^{I}_{m}(m) = \{a + bI \mid a, b \in [0, m)\}$$

$= \{(a, b) \mid$ a is the real part and b is the neutrosophic part$\}$, $m \geq 1$.

The MOD neutrosophic plane is represented as

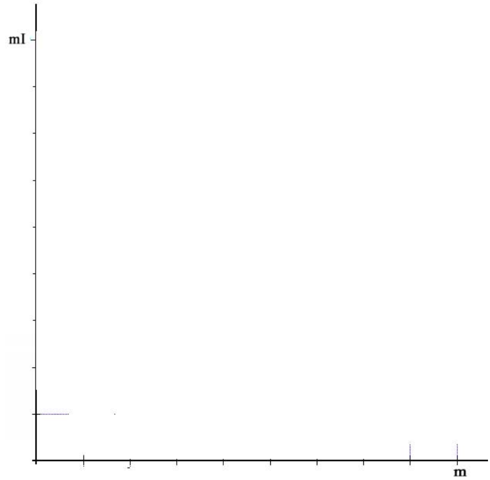

Figure 4.1

When m = 1 that is $R_n^I(1)$ is defined as the MOD neutrosophic fuzzy plane.

We will give examples of them and work with them.

Example 4.1: Let $R_n^I(5)$ be the MOD neutrosophic plane associated with the MOD interval [0, 5).

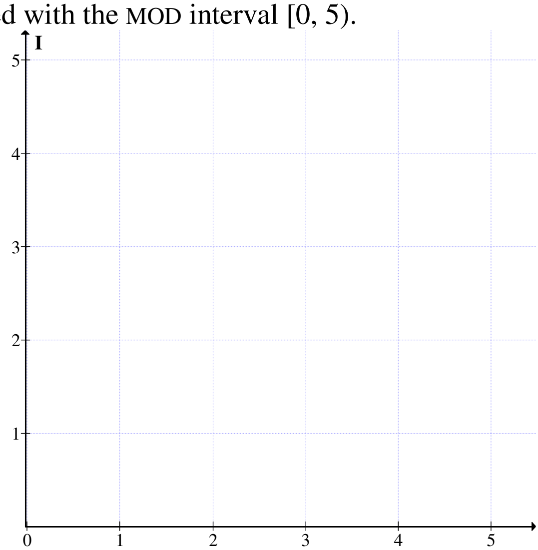

Figure 4.2

We see how representations of the MOD neutrosophic numbers are made in the MOD neutrosophic plane.

Recall $\langle R \cup I \rangle = \{a + bI \mid a, b \in R, I^2 = I\}$ is the real neutrosophic plane. We can get the map of $\langle R \cup I \rangle$ into $R_n^I(m)$ in the following way.

We know $R_n^I(m)$ is built using the intervals [0, m) and [0, mI).

We show how the MOD neutrosophic transformation works.

$$\eta_I : \langle R \cup I \rangle \to R_n^I(m)$$

$$\eta_I(I) = I, \ \eta_I(tI) = tI \text{ if } t < m$$

and

$$\eta_I(tI) = rI \text{ where } \frac{r}{m} + s = \frac{t}{m} \text{ if } t > m.$$

$$\eta_I(tI) = 0 \text{ if } t = m.$$

$$\eta(t) = t \text{ if } t < m;$$

$$= r \text{ if } t > m \text{ and } \frac{t}{m} = s + \frac{r}{m}$$

$$= 0 \text{ if } t = m.$$

This is the way by the MOD neutrosophic transformation the neutrosophic real plane is mapped on to the MOD neutrosophic plane $R_n^I(m)$.

We will illustrate this by some examples.

Let $8 + 25.2I \in \langle R \cup I \rangle$.

η_I (8 + 25.2I) = 3 + 0.2I $\in R_n^I(5)$.

Let –7 + 14.8I ∈ $\langle R \cup I\rangle$.
η_I (–7 + 14.8I) = (3 + 4.8I) $\in R_n^I(5)$.

Let –15 + 25I ∈ $\langle R \cup I\rangle$;

η_I (–15 + 25I) = 0 $\in R_n^I(5)$.

η_I (–72 – 49I) = (3 + I) $\in R_n^I(5)$.

η_I (24 – 71I) = (4 + 4I) $\in R_n^I(5)$.

This is the way MOD neutrosophic transformation from $\langle R \cup I\rangle$ to $R_n^I(5)$ is made.

Let $R_n^I(12)$ be the MOD neutrosophic plane.

η_I (24 + 40I) = 4I $\in R_n^I(12)$.

η_I (27 + 48I) = 3 $\in R_n^I(12)$.

η_I (10 + 4I) = 10 + 4I $\in R_n^I(12)$

η_I (–2I) = 10I $\in R_n^I(12)$

η_I (–4) = 8 $\in R_n^I(12)$

η_I (–5 – 20I) = (7 + 4I) $\in R_n^I(12)$ and so on.

Infact we have only a unique MOD neutrosophic transformation from $\langle R \cup I\rangle$ to $R_n^I(m)$.

Let us now study the behaviour of functions on the neutrosophic plane.

Let y = (8 + 5I)x be a function defined on the real neutrosophic plane.

Now this function is defined on $R_n^I(6)[x]$ as $y_n = (2 + 5I)x$.

Thus if $\langle R \cup I\rangle[x]$ is the real neutrosophic polynomial collection then we can map every p(x) to

$$p_n(x) \in R_n^I(m)[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R_n^I(m)\right\}.$$

We will illustrate this situation by some simple examples.

Let $p(x) = 5.3x^9 + 3.21x^7 + 40x^2 + 3 \in \langle R \cup I\rangle[x]$

$\eta_I : \langle R \cup I\rangle[x] \to R_n^I(4)[x]$.

$\eta_I(p(x)) = \eta_I(5.34x^9 + 3.21x^7 + 40x^2 + 3)$

$= 1.34x^9 + 3.21x^7 + 3 \in R_n^I(4)$.

Thus we can say several p(x) is mapped on to one.

$$p_n(x) \in R_n^I(4).$$

Let $p(x) = 246x^{21} - 481x^{18} + 46x^{10} - 5x^8 + 6x^2 + 5x - 17 \in \langle R \cup I\rangle[x]$.

$\eta_I(p(x)) = 2x^{21} + 3x^{18} + 2x^{10} + 3x^8 + 2x^2 + x + 3 \in R_n^I(4)$.

This is the way MOD neutrosophic transformations are carried out.

One of the natural questions can be differential of $p_n(x) \in R_n^I(4)[x]$.

The answer is yes.

Let $p_n(x) = (1 + 3I)x^8 + (3I + 2.2)x^5 + 1.6Ix^3 + 2.15Ix + 3.001 \in R_n^I(4)$.

$$\frac{dp_n(x)}{dx} = 8(1+3I)x^7 + 5(3I + 2.2)x^4 + 3(1.6I)x^2 + 2.15I + 0$$

$$= 0 + (3I + 2.2)x^4 + 0.8Ix^2 + 2.15I \in R_n^I(4).$$

Now we can also have polynomials of the form

$$p(x) = (140 + 25I)x^5 + (-30 + 7I)x^4 + (13 - 43I)x^3 + 7Ix^2 + 2x + (17 - 26I) \in \langle R \cup I \rangle [x].$$

$$\eta_1(p(x)) = Ix^5 + (2 + 3I)x^4 + (1 + 2I)x^3 + 3Ix^2 + 2x + (1 + 2I) \in R_n^I(4).$$

This is the way the MOD transformation of polynomials are carried out.

Let $R_n^I(7)[x]$ be the MOD neutrosophic polynomial.

$$\text{Let } p(x) = (40 + I)x^{18} + (43 - 14I)x^{15} + (25 + 7I)x^6 + (-23 + 12I)x^5 + (-3 - 14I)x^3 + (-2 + 10I)x + 3 - 15I \in \langle R \cup I \rangle[x].$$

Let $\eta_I : \langle R \cup I \rangle[x] \to R_n^I(7)[x]$ be the MOD neutrosophic transformation.

$$\eta_I(p(x)) = (5 + I)x^{18} + 1x^{15} + 4x^6 + (5 + 5I)x^5 + (4)x^3 + (5 + 3I)x + (3 + 6I) \in R_n^I(7)[x].$$

We can thus see this transformation is such that infinite number of neutrosophic real numbers are mapped onto a single MOD neutrosophic element in $R_n^I(m)[x]$.

Let p(x) = $3.17x^{10} + (I + 1.89)x^8 + (2I + 0.3)x^5 + (4.37 + 3.1107I)x^3 + (0.79 + 0.82I)x + (1.3071 + I) \in \langle R \cup I\rangle[x]$.

We define $\eta_I : \langle R \cup I\rangle[x] \to R_n^I(6)[x]$ as $\eta_I(p(x)) = p(x)$ as every coefficient is less than 6.

We find the derivative of p(x).

$$\text{Now } \frac{dp(x)}{dx} = 10 \times 3.17x^9 + 8 \times (I + 1.89)x^7 + 5(2I + 0.3)\,x^4 + 3(4.37 + 3.1107I)x^2 + (0.79 + 0.82I)$$

$$= 1.7x^9 + (2I + 3.12)x^7 + (4I + 1.5)x^4 + (1.11 + 3.3321I)\,x^2 + (0.79 + 0.82I) \in R_n^I(6).$$

It is important to note that if p(x) is a polynomial in the MOD neutrosophic plane of degree say s (s > 1), then the first derivative need not be of degree (s – 1); it can be zero or a lesser power than s–1.

All these will be illustrated by some examples.

Let $p(x) = 2x^{10} + 3x^5 + 4.7301 \in R_n^I(5)[x]$.

Clearly degree of p(x) is 10 but the first derivative of p(x) is

$$\frac{dp(x)}{dx} = 20x^9 + 15x^4 + 0 = 0.$$

Thus the first derivative of the MOD polynomial $p(x) \in R_n^I(5)[x]$ is zero.

Let $p(x) = 2.5x^{40} + 3x^{10} + 2x^5 + 3.711 \in R_n^I(5)[x]$.

We now find the derivative of p(x),

$$\frac{dp(x)}{dx} = 2.5 \times 40x^{39} + 30x^9 + 10x^4 + 0$$

$$= 0.$$

$$\text{Let } p(x) = 4x^5 + 2.31x^2 + 4.5 \in R_n^I(5)[x].$$

$$\frac{dp(x)}{dx} = 20x^4 + 4.62x + 0$$

$$= 4.62x \in R_n^I(5)[x].$$

Thus p(x) is a degree 5 and the first derivative is of degree one.

So the fifth degree polynomial in $R_n^I(5)[x]$ has its first derivative to be of degree one.

Let $p(x) \in R_n^I(4)[x]$ we can find $p'(x)$ which can be zero or a constant or of degree much less than degree $p(x) - 1$.

$p(x) = 0.3x^6 + 2.7x^5 + 1.3x^4 + 0.7x + 1.2 \in R_n^I(4)[x]$.

$$\frac{dp(x)}{dx} = 1.8x^5 + 1.5x^4 + 1.2x^3 + 0.7 \in R_n^I(4)[x].$$

Degree of p(x) is 6 and its first derivative of p(x) which is $p'(x)$ is of degree 5.

Let $p(x) = 2x^5 + 4x + 0.3 \in R_n^I(4)[x]$;

$\frac{dp(x)}{dx} = 2x^4$. Thus a fifth degree polynomial is a constant.

So polynomials in $R_n^I(m)[x]$ and in $\langle R \cup I\rangle[x]$ behave differently for a sth degree polynomial in $\langle R \cup I\rangle[x]$ has its derivative to be of degree (s–1).

But a sth degree polynomial in $R_n^I(m)[x]$ may have its first derivative to be zero or constant or of degree s–k ($k \neq 1$) is possible.

Next we have the main result which is based on the roots of the polynomial in $R_n^I(m)[x]$.

It is pertinent to keep on record that in general it is very difficult to solve polynomial equations in $R_n^I(m)[x]$ ($m > 1$).

Thus it is left as an open conjecture to solve the following problems.

(i) Can we say a sth degree polynomial in $R_n^I(m)[x]$ has s and only s roots?

(ii) Can we have a sth degree polynomial in $R_n^I(m)[x]$ to have more than s roots?

(iii) Can sth degree polynomial in $R_n^I(m)[x]$ has less than s roots?

(iv) If α is a multiple root of $s(x) \in R_n^I(m)[x]$; can we say α is a root of $\frac{ds(x)}{dx}$?

All the four problems are left as open conjectures.

Let $s(x) = (x + 2.1)\ (x +1)^2\ (x + 3) \in R_n^I(4)[x]$.

The roots in the usual sense are 3, 3 1 and $1.9 \in [0, 4)$.

$$s(x) = (x + 2.1)\ (x^2 + 2x + 1)\ (x+3)$$

$$= (x^3 + 2x^2 + x + 2.1x^2 + 2.1 + 0.2x)\ (x + 3)$$

$$= (x^4 + 2x^3 + x^2 + 2.1x^3 + 2.1x + 0.2x^2 + 3x^3 + 2x^2 + 3x + 2.3x^2 + 2.3 + 0.6x)$$

$$= x^4 + (2 + 3 + 2.1)x^3 + (1 + 0.2 + 2 + 2.3)x^2 + (2.1 + 3 + 0.6)x + 2.3$$

$$s(x) = x^4 + 3.1x^3 + 1.5x^2 + 1.7x + 2.3.$$

$$s(3) = 3^4 + 3.1 \times 27 + 1.5 \times 9 + 1.7 \times 3 + 2.3$$

$$= 81 + 1.3 + 1.1 + 1.5 + 83.7$$

$$= 1 + 1.3 + 1.1 + 1.5 + 2.3$$

$$= 3.2 \neq 0.$$

But if

$$s(x) = (x + 2.1)\ (x+1)^2\ (x+3)$$

$$s(3) = (3 + 2.1)\ (3 + 1)^2\ (3 + 3)$$

$$= 0.$$

Thus one may be surprised to see 3 is a root of

$$s(x) = (x + 2.1)\ (x + 1)^2\ (x+3).$$

But when the same s(x) is expanded $s(3) \neq 0$.

So we now claim this is due to the fact in general in $R_n^I(m)$ or $R_n^I(m)[x]$ the distributive law in not true.

That is why studying the properties of MOD neutrosophic polynomials in $R_n^I(m)[x]$ is a difficult as well as it is a new study in case of polynomials.

That is why we have proposed those open problems.

Now so we can think of integrating the MOD neutrosophic polynomials in $R_n^I(m)[x]$ to see the problems we face in this adventure.

Let us to be more precise take

$$p(x) = 4.2x^5 + 3.2x^4 + 2.5x + 3 \in R_n^I(6)[x].$$

$$\int p(x)\,dx = \int 4.2x^5 dx + \int 3.2x^4\,dx + \int 2.5x dx + \int 3dx$$

$$= \text{undefined} + \frac{3.5x^5}{5} + \frac{2.5x^2}{2} + 3x + c.$$

Clearly $\frac{2.5x^2}{2}$ is not defined as 2 does not have inverse in [0, 6).

Thus in general integration is not defined for every $p(x) \in R_n^I(m)[x]$.

So integration in case of MOD neutrosophic polynomials in general is not an easy task.

Now can we define on the MOD neutrosophic plane the notion of conics?

This also remains as an open problem.

Now we study the algebraic structure enjoyed by $R_n^I(m)$.

We will first give some examples before we proceed onto prove a few theorems.

Example 4.2: Let $R_n^I(7) = \{a + bI \mid a, b \in [0, 7), I^2 = I\}$ be the MOD neutrosophic plane.

Let $(6.3 + 3I)$ and $3.7 + 4.1I \in R_n^I(7)$.

$$(6.3 + 3I) + (3.7 + 4.1I)$$

$$= 10 + 7.1I$$

$$= 3 + 0.1I \in R_n^I(7).$$

It is easily verified $0 = 0 + 0I \in R_n^I(7)$ acts as the identity, for $0 + x = x + 0$ for all $x \in R_n^I(7)$.

Let $x = 0.7 + 3.4I \in R_n^I(7)$ we see

$y = 6.3 + 3.6I \in R_n^I(7)$ is such that $x + y = 0$.

Thus to every $x \in R_n^I(7)$ we have a unique $y \in R_n^I(7)$ such that $x + y = 0$ and y is called the additive inverse of x and vice versa.

Thus $(R_n^I(7), +)$ is an abelian group with respect to addition known as the MOD neutrosophic plane group.

DEFINITION 4.1: *Let $R_n^I(m)$ be the MOD neutrosophic plane. $R_n^I(m)$ under + (addition) is a group defined as the MOD neutrosophic plane group.*

Example 4.3: Let $(R_n^I(12); +)$ be a MOD neutrosophic group. Clearly $R_n^I(12)$ is of infinite order and is commutative.

Example 4.4: Let $\{R_n^I(19), +\}$ be a MOD neutrosophic group of infinite order.

Next we proceed onto define the notion of product '×' in $R_n^I(m)$.

DEFINITION 4.2: *Let $R_n^I(m)$ be the MOD neutrosophic plane relative to the MOD interval [0, m).*

Define × in $R_n^I(m)$ as for x = a + bI and y = c + dI ∈ $R_n^I(m)$;

$$x \times y = (a + bI) \times (c + dI)$$

$$= ac + bcI + adI + bdI$$

$$= ac + (bc + ad + bd)I \ (mod\ n)$$

is in $R_n^I(m)$.

Thus $\{R_n^I(m), \times\}$ is a semigroup which is commutative and is of infinite order.

$\{R_n^I(m), \times\}$ is defined as the MOD neutrosophic plane semigroup.

We will give some examples of this in the following.

Example 4.5: Let S = $\{R_n^I(10), \times\}$ be the MOD plane neutrosophic semigroup.

Let x = 4 + 8I and y = 5 + 5I ∈ S;

$$x \times y = (4 + 8I) \times (5 + 5I)$$

$$= 0.$$

Thus S has zero divisors.

Let $x = 5I \in R_n^I(10)$. $x^2 = x$. Thus the semigroup S has idempotents. The MOD neutrosophic semigroup has both zero divisors and idempotents.

Example 4.6: Let $\{R_n^I(17), \times\}$ be the MOD neutrosophic semigroup. Let x = 8.5I and y = 2 + 2I be in $R_n^I(17)$.

$$x \times y = 0.$$

Example 4.7: Let $\{R_n^I(20), \times\} = S$ be the MOD neutrosophic semigroup. S has zero divisors, units and idempotents. x = 3 and y = 7 ∈ S is such that x × y = 1 (mod 20).

$$x = 9 \in S \text{ such that } x^2 = 1 \text{ (mod 20)}.$$

Example 4.8: Let $\{R_n^I(25), \times\} = S$ be the MOD neutrosophic semigroup.

We see P = {a + bI | a, b ∈ {0, 5, 10,15, 20} ⊆ [0, 25)} is a subsemigroup of finite order.

Interested reader can find subsemigroups.

Example 4.9: Let M =$\{R_n^I(127), \times\}$ be the MOD neutrosophic plane semigroup. M has subsemigroups of finite order.

We see $R_n^I(m)$ is a group under '+' and semigroup under ×, however $\{R_n^I(m), +, \times\}$ is not a ring. The main problem being the distributive law is not true.

For a × (b + c) ≠ a × b + a × c for all a, b, c ∈ $R_n^I(m)$.

Example 4.10: Let $R_n^I(6)$ be the MOD neutrosophic plane. $R_n^I(6)$ is a MOD neutrosophic plane group and $R_n^I(6)$ is a MOD neutrosophic plane semigroup.

Let x = 3.7 + 4.2I,

y = 2.4 + 16I and z = 3.6 + 4.4I $\in R_n^I(6)$.

Consider

$$
\begin{aligned}
x \times (y + z) &= (3.7 + 4.2I) \times [(2.4 + 1.6I) + (3.6 + 4.4I)] \\
&= (3.7 + 4.2I)\,(0) \\
&= 0 \qquad \ldots \quad \text{I}
\end{aligned}
$$

Now

$$
\begin{aligned}
x \times y + x \times z &= (3.7 + 4.2I) \times (2.4 + 1.6I) + (3.7 + 4.2I) \times (3.6 + 4.4I) \\
&= (2.88 + 4.08I + 5.92I + 0.76I) + (1.32 + 3.12I + 4.28I + 2.48I) \\
&= 2.88 + 4.76I + 1.32 + 3.88I \\
&= 4.2 + 2.64I \qquad \ldots \quad \text{II}
\end{aligned}
$$

Clearly I and II are distinct hence the claim.

That is the distributive law in general is not true in $R_n^I(m)$.

DEFINITION 4.3: *Let {$R_n^I(m)$, +, ×} be defined as the pseudo MOD neutrosophic plane ring which is commutative and is of infinite order.*

We give some more examples of this situation.

Example 4.11: Let $\{R_n^I(7), +, \times\} = R$ be the MOD neutrosophic plane pseudo ring. R is commutative and is of infinite order.

Take x = 3.2 + 5.4I

y = 5.9 + 2.1I and z = 6 + 4I ∈ $R_n^I(7)$.

$$
\begin{aligned}
x \times (y + z) &= (3.2 + 5.4I) \times (5.9 + 2.1I + 6 + 4I) \\
&= (3.2 + 5.4I) \times (1.9 + 6.1I) \\
&= (6.08 + 3.26I + 5.52I + 4.94I) \\
&= 6.08 + 6.72I \qquad \ldots \quad I
\end{aligned}
$$

Now we find

$$x \times y + x \times z = 3.2 + 5.4I \times 5.9 + 2.1I + 3.2 + 5.4I \times 6 + 4I$$

$$= (4.88 + 3.86I + 6.72I + 4.34I) + 5.2 + 4.4I + 5.8I + 0.6I$$

$$= 2.08 + 4.72I \qquad \ldots \quad II$$

Clearly I and II are distinct, hence the distributive law in general is not true in $R_n^I(7)$.

Now we see that is why we had problems in solving equations.

Recall $R_n^I(m)[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R_n^I(m)\right\}$ is the MOD neutrosophic polynomials.

The natural interest would be the highest structure enjoyed by this MOD neutrosophic polynomials $R_n^I(m)[x]$.

Example 4.12: Let $R_n^I(9)[x]$ be the MOD neutrosophic polynomials. $R_n^I(9)[x]$ under addition is a group of infinite order which is abelian. For take

$p(x) = 0.8x^7 + 5x^6 + 7.2x^4 + 3.12$ and

$q(x) = 3.4x^7 + 2.7x^4 + + 1.3x^3 + 5.7$ be in $R_n^I(9)[x]$.

We find

$$p(x) + q(x) = 0.8x^7 + 5x^6 + 7.2x^4 + 3.12 + 3.4x^7 + 2.7x^4 + 1.3x^3 + 5.7$$

$$= 4.2x^7 + 5x^6 + 0.9x^4 + 1.3x^3 + 8.82 \in R_n^I(9)[x].$$

This is the way the operation of addition is performed.

Clearly '+' operation is commutative for

$p(x) + q(x) = q(x) + p(x)$ can be easily verified.

Now $0 \in R_n^I(9)\,[x]$ is such that $p(x) + 0 = 0 + p(x) = p(x)$.

We see for every p(x) in $R_n^I(9)[x]$ there is a unique $q(x) \in R_n^I(9)[x]$ such that $p(x) + q(x) = 0$.

For let $p(x) = 0.321x^9 + 6.312x^8 + 0.929x^6 + 6.2x^5 + 4.3x^2 + 5.09x + 0.9991 \in R_n^I(9)[x]$.

Now $q(x) = 8.679x^9 + 2.688x^8 + 8.071x^6 + 2.8x^5 + 4.7x^2 + 3.91x + 8.0009$ in $R_n^I(9)[x]$ is such that $p(x) + q(x) = 0$.

q(x) is the unique additive inverse of p(x) and verse versa.

Thus $R_n^I(9)[x]$ is an abelian group under +.

Example 4.13: Let $(R_n^I(5)[x], +)$ be the MOD polynomial neutrosophic group.

For every

$p(x) = 0.43x^{10} + 3.1x^9 + 2.5x^2 + 3.3x + 0.914 \in R_n^I(5)[x]$;

the inverse of p(x) is denoted by

$-p(x) = 4.57x^{10} + 1.9x^9 + 2.5x^2 + 1.7x + 4.086 \in R_n^I(5)[x]$

is such that p(x) + (–p(x)) = 0.

DEFINITION 4.4: *$\{R_n^I(m)[x], +\}$ is defined to be the MOD polynomial group of infinite order which is commutative.*

Example 4.14: Let $R_n^I(12)[x]$ be the MOD neutrosophic polynomial set. Define the operation × on $R_n^I(12)[x]$ as follows.

Let $p(x) = 6.16x^{10} + 8.21x^5 + 7.01x^2 + 5.1$ and

$q(x) = 3.1x^8 + 6.4x^2 + 7 \in R_n^I(12)[x]$.

$$\begin{aligned} p(x) \times q(x) &= (6.16x^{10} + 8.21x^5 + 7.01x^2 + 5.1) \times (3.1x^8 + 6.4x^2 + 7) \\ &= 7.096x^{18} + 1.451x^{13} + 9.731x^{10} + 3.81x^8 + 3.424x^{12} + 4.544x^7 + 8.864x^4 + 8.64x^2 + 1.07x^2 + 7.12x^{10} + 11.7 + 9.47x^5 \\ &= 7.096x^{18} + 1.451x^{13} + 4.851x^{10} + 3.424x^{12} + 3.81x^8 + 4.544x^7 + 8.864x^4 + 9.77x^2 + 9.47x^5 + 11.7 \in R_n^I(12)[x]. \end{aligned}$$

This is the way product is obtained.

It is easily verified that p(x) × q(x) = q(x) × p(x).

Further $R_n^I(m)[x]$ for any m is a commutative semigroup of infinite order.

This semigroup is defined as the MOD neutrosophic polynomial semigroup.

Next we proceed onto give examples of pseudo MOD neutrosophic rings of polynomials.

Example 4.15: Let $R_n^I(10)[x]$ be the MOD neutrosophic polynomial. $(R_n^I(10)[x], +)$ is a group and $(R_n^I(10)[x], \times)$ is a semigroup.

$$\text{Let } p(x) = 0.3x^8 + 6.1x^4 + 0.8,$$

$$q(x) = 0.4x^2 + 0.9x + 1$$

$$\text{and } r(x) = 5x^6 + 0.3x^3 + 0.1 \in R_n^I(10)[x].$$

Consider $p(x) \times (q(x) + r(x)) =$

$$= (0.3x^8 + 6.1x^4 + 0.8) \times [0.4x^2 + 0.9x + 1 + 5x^6 + 0.3x^3 + 0.1]$$

$$= 0.12x^{10} + 2.44x^6 + 0.32x^2 + 0.27x^9 + 5.49x^5 + 0.72x + 0.3x^8 + 6.1x^4 + 0.8 + 1.5x^{14} + 0.5x^{10} + 4x^6 + 0.09x^{11} + 1.83x^7 + 0.24x^3 + 0.03x^8 + 0.61x^4 + 0.08$$

$$= 1.5x^{14} + 0.09x^{11} + 0.62x^{10} + 0.27x^9 + 0.33x^8 + 0.72x + 1.83x^7 + 6.44x^6 + 5.49x^5 + 0.24x^3 + 0.32x^2 + 6.71x^4 + 0.88 \quad \ldots \text{ I}$$

Consider $p(x) \times q(x) + p(x) \times r(x)$

$$= (0.3x^8 + 6.1x^4 + 0.8)(0.4x^2 + 0.9x + 1) + (0.3x^8 + 6.1x^4 + 0.8)(5x^6 + 0.3x^3 + 0.1)$$

$$= 0.12x^{10} + 2.44x^6 + 0.32x^2 + 0.27x^9 + 5.49x^5 + 0.72x + 0.3x^8 + 6.1x^4 + 0.8 + 1.5x^{14} + 0.5x^{10} + 4x^6 + 0.03x^8 + 0.61x^4 + 0.08 + 0.09x^{11} + 1.83x^7 + 0.24x^3$$

$$= 1.5x^{14} + 0.09x^{11} + 0.62x^{10} + 0.27x^9 + 0.33x^8 + 1.83x^7 + 6.44x^6 + 5.49x^5 + \ldots + 0.88 \qquad \ldots \text{II}$$

It is easily verified I and II are not distinct for this set of MOD polynomials.

Let us consider $p(x) = 0.3x^4 + 8$,

$q(x) = 4x^3 + 8x + 0.1$ and

$r(x) = 6x^3 + 0.4 \in R_n^I(10)[x]$.

$p(x) \times [q(x) + r(x)]$

$$= (0.3x^4 + 8) \times [4x^3 + 8x + 0.1 + 6x^3 + 0.4]$$

$$= (0.3x^4 + 8)(8x + 0.5)$$

$$= 2.4x^5 + 4x + 0.15x^4 + 4 \qquad \ldots \text{I}$$

$p(x) \times q(x) + p(x) \times r(x)$

$$= (0.3x^4 + 8)(4x^3 + 8x + 0.1) + (0.3x^4 + 8)(6x^3 + 0.4)$$

$$= 1.2x^7 + 2x^3 + 2.4x^4 + 4 + 0.03x^4 + 0.8 + 1.8x^7 + 4x^3 + 0.12x^4 + 3.2$$

$$= 3x^7 + 2.55x^4 + 6x^3 + 8 \qquad \ldots \text{II}$$

Clearly I and II are distinct so for this triple p(x), q(x), r(x) $\in R_n^I(10)[x]$;

$p(x) \times (q(x) + r(x)) \neq p(x) \times q(x) + p(x) \times r(x)$.

Thus $(R_n^I(m)[x], +, \times)$ is only a MOD neutrosophic polynomial pseudo ring and is not a ring for the distributive law in general is not true.

We proceed to define the MOD pseudo neutrosophic polynomial ring.

DEFINITION 4.5: *Let $R_n^I(m)[x]$ be the MOD neutrosophic polynomial, $(R_n^I(m)[x], +)$ is an abelian group under +.*

$(R_n^I(m)[x], \times)$ is a commutative semigroup. Clearly for $p(x), q(x), r(x) \in R_n^I(m)[x]$ we have $p(x) \times [q(x) + r(x)] \neq p(x) \times q(x) + p(x) \times q(x)$ in general for every triple. Hence $S = (R_n^I(m)[x], +, \times)$ is defined as the MOD neutrosophic polynomial pseudo ring.

We see S is of infinite order and S contains idempotents and zero divisors and is commutative.

We will give one to two examples of them.

Example 4.16: Let $\{R_n^I(7)[x], +, \times\}$ be the MOD pseudo neutrosophic polynomial ring.

Example 4.17: Let $\{R_n^I(12)[x], +, \times\}$ be the MOD pseudo neutrosophic polynomial ring.

Now we see solving equations in $R_n^I(m)[x]$ is a very difficult task.

For even if we consider

$$\begin{aligned} s(x) &= (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n) \neq x^n - \\ &\quad (\alpha_1 + \ldots + \alpha_n)\, x^{n-1} + \ldots \pm \alpha_1 \ldots \alpha_n (\text{mod } m) \\ &= s_1(x). \end{aligned}$$

Thus this leads to a major problem for $s(\alpha_i) = 0$ but $s_1(\alpha_i) \neq 0$.

So working with pseudo rings we encounter with lots of problems.

However in several places the integrals in $R_n^I(m)[x]$ is not defined.

Differentiation does not in general follow the classical properties in $R_n^I(m)[x]$.

Thus lots of research in MOD neutrosophic polynomials pseudo rings remains open.

Next we proceed onto define the new concept of MOD fuzzy neutrosophic plane.

In the MOD neutrosophic plane if we put m = 1 we get the MOD neutrosophic fuzzy plane.

We will denote the MOD fuzzy neutrosophic plane by $R_n^I(1) = \{a + bI \mid a, b \in [0, 1), I^2 = I\}$.

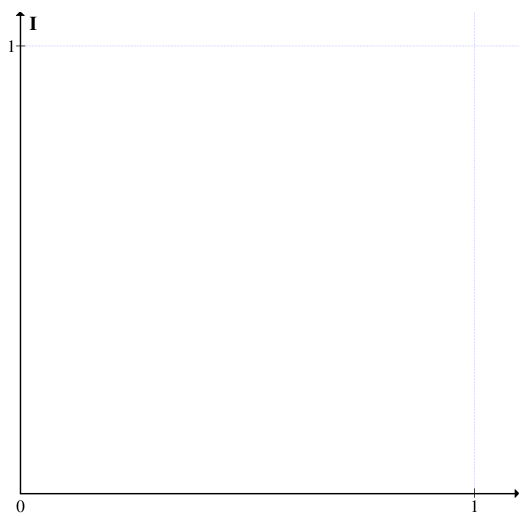

Figure 4.3

Now we see $R_n^I(1)$ is the MOD fuzzy neutrosophic plane. All properties of $R_n^I(m)$ can be imitated for $R_n^I(1)$, however the MOD fuzzy neutrosophic polynomials do not help in integration or differentiation if m = 1.

For if $p(x) = 0.3x^5 + 0.2x^3 \in R_n^I(1)[x]$;

$$\frac{dp(x)}{dx} = 5 \times 0.3x^4 + 0.6x^2$$

$$= 0.5x^4 + 0.6x^2.$$

But if $p(x) = 0.9x^{10} + 0.5x^2$ is in $R_n^I(1)[x]$ then $\frac{dp(x)}{dx} = 0$.

So we see p(x) is degree 10 but its derivative is zero.

However $\int p(x)\, dx = \int 0.3x^5\, dx + \int 0.2x^3\, dx$ is not defined.

Let x = 0.3 + 0.7I and y = 0.9 + 0.4I $\in R_n^I(1)$.

Now x + y = 0.3 + 0.7I + 0.9 + 0.4I

= 0.2 + 0.1I $\in R_n^I(1)$.

It is easily verified $0 \in R_n^I(1)$ is such that x + 0 = 0 + x = x for all $x \in R_n^I(1)$.

We see for every $x \in R_n^I(1)$ there exists a unique y such that x + y = 0.

For x = 0.9 + 0.2I in $R_n^I(1)$ we have y = 0.1 + 0.8I such that x + y = 0.

$\{R_n^I(1), +\}$ is a MOD fuzzy neutrosophic abelian group of infinite order under +.

Consider $x = 0.8 + 0.6I$ and

$y = 0.5 + 0.8I \in R_n^I(1)$.

$$\begin{aligned} x \times y &= (0.8 + 0.6I) \times (0.5 + 0.8I) \\ &= 0.4 + 0.3I + 0.64I + 0.48I \\ &= 0.4 + 0.42I \in R_n^I(1). \end{aligned}$$

This is the way product is defined and $\{R_n^I(1), \times\}$ is defined as the MOD fuzzy neutrosophic semigroup of infinite order.

Let $x = 0.4$ and $y = 0.8I \in R_n^I(1)$;

$x \times y = 0.24\, I \in R_n^I(1)$.

Now this MOD fuzzy neutrosophic semigroup has subsemigroups and ideals.

$P = \langle 0.2 + 0.2I\rangle$ is a fuzzy neutrosophic MOD subsemigroup of $\{R_n^I(1), \times\}$.

Let $M = \{aI \mid a \in [0, 1)\} \subseteq \{R_n^I(1), \times\}$ be an ideal; called the fuzzy MOD neutrosophic ideal of $\{R_n^I(1), \times\}$.

Thus the MOD fuzzy neutrosophic semigroup has both ideals and subsemigroups.

Let $x = 0.3 + 0.8I \in R_n^I(1)$

then $x^2 = (0.3 + 0.8I) \times (0.3 + 0.8I)$

$= 0.09 + 0.24I + 0.24I + 0.64I$

$= 0.09 + 0.12I \in R_n^I(1)$.

$\{R_n^I(1), +, \times\}$ is only a pseudo MOD fuzzy neutrosophic ring for distributive law is not true.

Let $x = 0.7 + 0.4I$

$y = 0.5 + 0.2I$ and

$z = 0.5 + 0.8I \in R_n^I(1)$.

$x \times (y + z) = 0.7 + 0.4I \times (0.5 + 0.2I + 0.5 + 0.8I)$
$= 0.7 + 0.4I \times 0 = 0$ … I

$xy + xz = 0.7 + 0.4I \times 0.5 + 0.2I + 0.7 + 0.4I \times 0.5 + 0.8I$

$= 0.35 + 0.2I + 0.14I + 0.08I + 0.35 + 0.2I + 0.56I + 0.32I$

$= 0.7 + 0.5I$ … II

Clearly I and II are different that is why we call $\{R_n^I(1), +, \times\}$ as the MOD fuzzy neutrosophic pseudo ring.

Clearly this pseudo ring has subrings and ideals.

Consider $P = \{aI \mid a \in [0, 1)\} \subseteq R_n^I(1), +, \times\}$. P is a pseudo ideal of $R_n^I(1)$ of infinite order.

$I = \{a \mid a \in [0, n)\}$ is only a pseudo subring of $R_n^I(1)$ and is not an ideal of $R_n^I(1)$.

We see MOD fuzzy neutrosophic planes behave like MOD neutrosophic planes except some special features like

$$x \times y = (a + bI)(c + d)$$

$$= 0 \text{ is not possible with } a, b, c, d \in (0, 1)$$

Study of these concepts is left as an exercise to the reader.

Next we proceed onto study the notion of MOD fuzzy neutrosophic polynomials of two types.

$$\text{Let } R_n^I(1)[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R_n^I(1)\right\}.$$

$$\text{Let } p(x) = 0.3x^7 + 0.7x^2 + 0.1$$

$$\text{and } q(x) = 0.7x^3 + 0.5x^2 + 0.8 \in R_n^I(1)[x];$$

$$p(x) + q(x) = 0.3x^7 + 0.7x + 0.1 + 0.7x^3 + 0.5x^2 + 0.8$$

$$= 0.3x^7 + 0.7x^3 + 0.2x^2 + 0.9.$$

It is easily verified $p(x) + q(x) \in R_n^I(1)[x]$.

Further $p(x) + q(x) = q(x) + p(x)$ and $0 \in R_n^I(1)[x]$ is such that $0 + p(x) = p(x) + 0 = p(x)$.

Now we show for every $p(x) \in R_n^I(1)[x]$ there exists a unique $q(x) \in R_n^I(1)[x]$ such that $p(x) + q(x) = q(x) + p(x) = 0$.

We call q(x) the inverse of p(x) with respect to addition and vice versa.

Thus { $R_n^I(1)[x]$, +} is an additive abelian group of infinite order.

Let p(x) = $0.9x^{21} + 0.331x^{20} + 0.108x^8 + 0.912x^4 + 0.312 \in R_n^I(1)[x]$.

We see

$q(x) = 0.1x^{21} + 0.669x^{20} + 0.892x^8 + 0.088x^4 + 0.688 \in R_n^I(1)[x]$ is such that p(x) + q(x) = 0.

Consider p(x), q(x) $\in R_n^I(1)[x]$ we can find product

p(x) × q(x) and p(x) × q(x) $\in R_n^I(1)[x]$.

Let $p(x) = 0.31x^8 + 0.24x^3 + 0.197$ and

$q(x) = 0.9x^3 + 0.6x^2 + 0.5$ be in $R_n^I(1)[x]$.

$$p(x) \times q(x) = (0.31x^8 + 0.24x^3 + 0.197) \times (0.9x^3 + 0.6x^2 + 0.5)$$

$$= 0.279x^{11} + 0.216x^6 + 0.1773x^3 + 0.186x^{10} + 0.144x^5 + 0.1182x^2 + 0.155x^8 + 0.120x^3 + 0.985 \in R_n^I(1)[x].$$

All these happens when the coefficients are real.

Let $p(x) = (0.3 + 0.7I)x^{10} + (0.8 + 0.4I)x^3 + (0.7 + 0.5I)$

and $q(x) = (0.5 + 0.2I)x^3 + (0.4 + 0.9I) \in R_n^I(1)[x]$.

$$p(x) + q(x) = (0.3 + 0.7I)x^{10} + (0.8 + 0.4I)x^3 + (0.7 + 0.5I) + (0.5 + 0.2I)x^3 + (0.4 + 0.9I)$$

$= \quad (0.3 + 0.7I)x^{10} + (0.3 + 0.6I)x^6 + (0.1 + 0.4I)$ is in $R_n^I(1)[x]$.

$p(x) \times q(x) = \quad [(0.3 + 0.7I)x^{10} + (0.8 + 0.4I)x^3 + (0.7 + 0.5I)] \times (0.5 + 0.2I)x^3 + (0.4 + 0.9I)$

$= \quad (0.3 + 0.7I)(0.5 + 0.2I)x^{13} + (0.8 + 0.4I)(0.5 + 0.2I)x^6 + (0.7 + 0.5I)(0.5 + 0.2I)x^3 + (0.3 + 0.7I) \times (0.4 + 0.9I)x^{10} + (0.8 + 0.4I)(0.4 + 0.9I)x^3 + (0.7 + 0.5I)(0.4 + 0.9I)$.

$= \quad (0.15 + 0.35I + 0.06I + 0.14I)x^{13} + (0.32 + 0.2I + 0.08I + 0.16I)x^6 + (0.35 + 0.25I + 0.14I + 0.1I)x^3 + (0.12 + 0.28I + 0.63I + 0.27I)x^{10} + (0.32 + 0.16I + 0.72I + 0.36I)x^3 + (0.28 + 0.2I + 0.63I + 0.45I)$

$= \quad (0.15 + 0.55I)x^{13} + (0.32 + 0.44I)x^6 + (0.35 + 0.49I)x^3 + (0.12 + 0.18I)x^{10} + (0.32 + 0.24I)x^3 + (0.28 + 0.28I)$

$= \quad 0.15 + 0.55I)x^{13} + (0.32 + 0.44I)x^6 + (0.12 + 0.18I)x^{10} + (0.67 + 0.73I)x^3 + (0.28 + 0.28I)$ is in $R_n^I(1)[x]$.

$\{R_n^I(1)[x], \times\}$ is a semigroup of infinite order which is commutative.

We see $\{R_n^I(1)[x], +, \times\}$ is not a MOD fuzzy neutrosophic polynomial ring only a pseudo ring; for

$$p(x) \times (q(x) + r(x)) \neq p(x) \times q(x) + p(x) \times r(x)$$
for $p(x), q(x), r(x) \in R_n^I(1)[x]$.

Let $p(x) = \quad 0.3x^{20} + 0.7x^{10} + 0.3$,

$q(x) = \quad 0.5x^2 + 0.7x + 0.12$ and

$$r(x) = \quad 0.9x^6 + 0.5x^2 + 0.88 \in R_n^I(1)[x].$$

Consider $p(x) \times (q(x) + r(x))$

$= p(x) \times [0.5x^2 + 0.7x + 0.12 + 0.9x^6 + 0.5x^2 + 0.88]$

$= (0.3x^{20} + 0.7x^{10} + 0.30) \times [0.9x^6 + 0.7x]$

$= 0.27x^{26} + 0.63x^{16} + 0.27x^6 + 0.21x^{21} + 0.49x^{11} + 0.21x$ … I

We find $p(x) \times q(x) + p(x) \times r(x)$

$= 0.3x^{20} + 0.7x^{10} + 0.3 \times 0.5x^2 + 0.7x + 0.12 + 0.3x^{20} + 0.7x^{10} + 0.3 \times 0.9x^6 + 0.5x^2 + 0.88$

$= 0.15x^{22} + 0.35x^{12} + 0.15x^2 + 0.21x^{21} + 0.49x^{11} + 0.21x + 0.036x^{20} + 0.084x^{10} + 0.036 + 0.27x^{26} + 0.63x^{16} + 0.27x^6 + 0.15x^{22} + 0.35x^{12} + 0.15x^2 + 0.264x^{20} + 00.616x^{10} + 0.264$

$= 0.27x^{26} + 0.3x^{22} + 0.21x^{21} + 0.3x^{20} + 0.63x^{16} + 0.7x^{12} + 0.49x^{11} + 0.7x^{10} + 0.27x^6 + 0.3x^2 + 0.21x + 0.3$ … II

We see I and II are distinct hence in general

$p(x) \times (q(x) + r(x)) \neq p(x) \times q(x) + p(x) \times r(x);$

for $p(x), q(x), r(x) \in R_n^I(1)[x]$.

Thus $\{R_n^I(1)[x], +, \times\}$ is a MOD fuzzy neutrosophic polynomial pseudo ring of infinite order.

We say we cannot perform usual integration on p(x) $\in R_n^I(1)[x]$.

However differentiation can be performed for some fuzzy neutrosophic polynomials, it may be zero.

Let p(x) = $0.7x^{12} + 0.5x^{10} + 0.2x^6 + 0.8x^4 + 0.17x + 0.5$ be in $R_n^I(1)[x]$.

We find the derivative of p(x).

$$\frac{dp(x)}{dx} = 0.4x^{11} + 0 + 0.2x^5 + 0.2x^3 + 0.17 + 0 \text{ is in } R_n^I(1)[x].$$

The second derivative of p(x) is

$$\frac{d^2p(x)}{dx^2} = 0.4x^{10} + 0 + 0.6x^2 + 0 \text{ is in } R_n^I(1)[x].$$

The third derivative of p(x) is

$$\frac{d^3p(x)}{dx^3} = 0.2x.$$

Thus a twelfth degree MOD fuzzy polynomial has its third derivative to be a polynomial of degree one and the forth derivative is a constant viz $\frac{d^4p(x)}{dx^4} = 0.2$ and the fifth derivative of p(x); $\frac{d^5p(x)}{dx^5} = 0$. Thus a 12th degree polynomial in $R_n^I(1)[x]$ has its fifth degree to be zero.

Hence these derivatives of MOD fuzzy polynomials do not behave like the usual derivative polynomials in R[x].

Finally we cannot integrate any of these polynomials in $R_n^I(1)[x]$.

To over come this situation we define the new notion of MOD fuzzy neutrosophic decimal polynomials.

$$R_n^{Id}(1)[x] = \{\sum a_i x^i \mid a_i \in R_n^I(1) \text{ and } i \in [0, 1)\}.$$

$R_n^{Id}(1)[x]$ is defined as the decimal MOD fuzzy neutrosophic polynomials or MOD fuzzy neutrosophic decimal polynomials.

Let $p(x) = 0.3x^{0.12} + 0.2x^{0.17} + (0.1 + 0.3I)x^{0.01} + (0.34 + 0.21I)x^{0.015} + (0.45 + 0.32) \in R_n^{Id}(1)[x]$.

We see the powers of x are in [0, 1) and the coefficients of x are in $R_n^I(1)$.

We can add two polynomials in $R_n^{Id}(1)[x]$, as follows.

If $p(x) = 0.34x^{0.3} + 0.21x^{0.1} + 0.7$

and $q(x) = 0.4x^{0.3} + 0.84x^{0.1} + 0.5x^{0.004} + 0.8$ are in $R_n^{Id}(1)[x]$,

then $p(x) + q(x)$

$$= 0.34x^{0.3} + 0.21x^{0.1} + 0.7 + 0.4x^{0.3} + 0.84x^{0.1} + 0.5x^{0.004} + 0.8$$

$$= 0.74x^{0.6} + 0.05x^{0.1} + 0.5 + 0.5x^{0.004} \in R_n^{Id}(1)[x].$$

This is the way + operation is performed on $R_n^{Id}(1)[x]$. Further 0 acts as the additive identity and for every $p(x) \in R_n^{Id}(1)[x]$ we have a q(x) in $R_n^{Id}(1)[x]$ such that

$$p(x) + q(x) = 0.$$

Thus ($R_n^{Id}(1)$ [x], +) is an additive abelian group of infinite order.

Now we can find product of two polynomials p(x), q(x) in $R_n^{Id}(1)$[x].

Let $p(x) = 0.3x^{0.7} + 0.8x^{0.5} + 0.6x^{0.2} + 0.1$

and $q(x) = 0.9x^{0.5} + 0.6x^{0.2} + 0.7$ be in $R_n^{Id}(1)$[x].

$$\begin{aligned} p(x) \times q(x) &= (0.3x^{0.7}+ 0.8x^{0.5} + 0.6x^{0.2} + 0.1) \times \\ &\qquad (0.9x^{0.5} + 0.6x^{0.2} + 0.7) \\ &= 0.27x^{0.2} + 0.72 + 0.54x^{0.7} + 0.09x^{0.5} + 0.18x^{0.9} + 0.48x^{0.7} + 0.36x^{0.2} + 0.06x^{0.2} + \\ &\qquad 0.21x^{0.7} + 0.56x^{0.5} + 0.42x^{0.2} + 0.07 \\ &= 0.18x^{0.9} + 0.23x^{0.7} +0.65x^{0.5} + 0.11x^{0.2} + 0.79 \in R_n^{Id}(1)[x]. \end{aligned}$$

Now ($R_n^{Id}(1)$[x], ×) is defined as the MOD fuzzy decimal polynomial neutrosophic semigroup of infinite order.

However ($R_n^{Id}(1)$[x], +, ×) is not the MOD fuzzy decimal polynomial neutrosophic ring as

$p(x) \times (q(x) + r(x)) \neq p(x) \times q(x) + p(x) \times r(x)$ in general for $p(x), q(x), r(x) \in R_n^{Id}(1)[x]$.

$$\begin{aligned} \text{Consider } p(x) &= 0.3x^{0.21} + 0.4x^{0.5} + 0.5, \\ q(x) &= 0.8x^{0.4} + 0.2x^{0.2} + 0.4 \\ \text{and } r(x) &= 0.2x^{0.4} + 0.8x^{0.8} + 0.7 \in R_n^{Id}(1)[x]. \end{aligned}$$

$p(x) \times (q(x) + r(x))$

$= p(x) \times [0.8x^{0.4} + 0.2x^{0.2} + 0.4 + 0.2x^{0.4} + 0.8x^{0.8} + 0.7]$

$= 0.3x^{0.21} + 00.4x^{0.5} + 0.5 \times 0.1$

$= 0.03x^{0.21} + 0.04x^{0.5} + 0.05$ … I

Consider $p(x) \times q(x) + p(x) \times r(x)$

$= (0.3x^{0.21} + 0.4x^{0.5} + 0.5) \times (0.8x^{0.4} + 0.2x^{0.2} + 0.4) + (0.3x^{0.21} + 0.4x^{0.5} + 0.5) \times (0.2x^{0.4} + 0.8x^{0.8} + 0.7)$

$= 0.24x^{0.61} + 0.32x^{0.9} + 0.4x^{0.4} + 0.06x^{0.41} + 0.08x^{0.1} + 0.1x^{0.2} + 0.12x^{0.21} + 0.16x^{0.5} + 0.2 + 0.06x^{0.61} + 0.08x^{0.9} + 0.1x^{0.4} + 0.24x^{0.01} + 0.32x^{0.3} + 0.4x^{0.8} + 0.21x^{0.21} + 0.28x^{0.5} + 0.35$ … II

Clearly I and II are distinct hence the definition of pseudo ring.

We can differentiate and integrate in a very special way. We can not use usual differentiation or integration instead we use special integration or differentiation using $\alpha \in (0,1)$.

We will just illustrate this by an example or two.

Let $p(x) = (0.3 + 0.5I)x^{0.7} + (0.8 + 0.2I)x^{0.5} + (0.7 + 0.1I) \in R_n^{Id}(1)[x]$.

We take $\alpha = 0.3$ we find

$$\frac{dp(x)}{dx(0.3)} = (0.3 + 0.5I)\, 0.7 \times x^{0.4} + (0.8 + 0.2I) \times (0.5x^{0.2}) + 0$$

$$= (0.21 + 0.35I)x^{0.4} + (0.4 + 0.1I)\, x^{0.2}.$$

Similarly

$$\int p(x)\,dx = \frac{(0.3+0.5I)x^{0}}{0} + \frac{(0.8+0.2I)x^{0.8}}{0.5+0.3}.$$

We see a section of the integration is undefined in such cases we say the decimal integration does not exist for $\alpha = 0.3$.

Suppose we decimal integrate using $\alpha = 0.1$ then

$\int p(x)\,dx\ (0.1)$

$$= \frac{(0.3+0.5I)x^{0.8}}{0.8} + \frac{(0.8+0.2I)x^{0.6}}{0.6} + \frac{(0.7+0.1I)x^{0.1}}{0.1}$$

We see for the same polynomial the decimal integral is not defined for $\alpha = 0.1 \in (0, 1)$.

Let $p(x) = (0.2 + 0.5I)x^{0.8} + 0.72x^{0.4} + 0.31Ix^{0.2} + (0.03 + 0.4I) \in R_n^{Id}(1)[x]$.

We find the decimal derivative and the decimal integral of p(x) relative to $\alpha = 0.6 \in (0.1)$.

$$\frac{dp(x)}{dx(0.6)} = (0.2 + 0.5I)0.8x^{0.2} + 0.72 \times 0.4x^{0.8} + 0.31I \times 0.2x^{0.6}$$

$$= (0.16 + 0.4I)x^{0.2} + 0.288x^{0.8} + 0.062Ix^{0.6} \in R_n^{Id}(1)[x].$$

$$\int p(x)\,dx(0.6) = \frac{(0.2+0.5I)x^{0.4}}{0.4} + \frac{0.72x^{0}}{0} + \frac{0.31x^{0.8}}{0.6+0.2} +$$

$$\frac{(0.03+0.4I)x^{0.6}}{0.6} + C \text{ is defined.}$$

This is the way decimal integration and decimal differentiation are carried out. Study in this direction is interesting.

We propose the following problems for the reader.

Problems :

1. Derive some special features enjoyed by $R_n^I(m)$ the MOD neutrosophic plane.

2. Prove there exists infinite number of distinct MOD neutrosophic planes.

3. Prove solving equations in $R_n^I(m)[x]$ is not an easy task.

4. Study the properties enjoyed by $R_n^I(5)$.

5. What is the algebraic structure enjoyed by $R_n^I(8)$?

6. Study $R_n^I(m)$ when m is a prime.

7. In $R_n^I(m)$ when m is not a prime; study the special feature enjoyed by a non prime.

8. Let $p(x) = 9x^{20} + 5.37x^{15} + 10.315x^{10} + 15.7x^5 \in R_n^I(20)[x]$.

   Find the derivative of p(x). Does $\frac{d^{20}p(x)}{dx^{20}}$ exist?

9. Can we integrate $p(x) \in R_n^I(20)[x]$ in problem 8.

10. Find some special features enjoyed by the MOD neutrosophic semigroups ($R_n^I(m)$, ×).

    (i) Find ideals in $R_n^I(m)$.

    (ii) Can $R_n^I(m)$ have ideals of finite order?

    (iii) Can $R_n^I(m)$ have subsemigroups of finite order?

    (iv) Can we have subsemigroups of infinite order which are not ideals of $R_n^I(m)$?

11. Let S = {$R_n^I(20)$, ×} be the MOD neutrosophic semigroup.

    (i) Study questions (i) to (iv) of problem 10 for this S.

    (ii) Can S have S-zero divisors?

    (iii) Prove S has zero divisors.

    (iv) Can S have S-idempotents?

    (v) Prove S has idempotents.

    (vi) Can S have idempotents which are not S-idempotents?

12. Let S_1 = {$R_n^I(19)$, ×} be the MOD neutrosophic semigroup.

    Study questions (i) to (vi) of problem 11 for this S_1.

13. Let S_2 = {$R_n^I(25)$, ×} be the MOD neutrosophic semigroup.

Study questions (i) to (vi) of problem 11 for this S_2.

14. Obtain any other special feature enjoyed by the MOD fuzzy neutrosophic group ($R_n^I(m)$, +).

15. Let M = {$R_n^I(m)$, +, ×} be the MOD neutrosophic pseudo ring.

    (i) Show M has zero divisors.

    (ii) Prove M has subrings of finite order.

    (iii) Can M have ideals of finite order?

    (iv) Prove M has subrings which are not ideals.

    (v) Prove M has infinite order pseudo subrings which are not ideals.

    (vi) Can M have S-idempotents?

    (vii) Can M have S-ideals?

    (viii) Is M a S-pseudo ring?

    (ix) Can M have S-zero divisors?

    (x) Obtain any other special feature enjoyed by M.

16. Let N = {$R_n^I(29)$, +, ×} be the MOD neutrosophic pseudo ring.

    Study questions (i) to (x) of problem 15 for this N.

17. Let T = {$R_n^I(30)$, +, ×} be the MOD neutrosophic pseudo ring.

    Study questions (i) to (x) of problem 15 for this T.

18. Let $M_1 = \{R_n^I(2), +, \times\}$ be the MOD neutrosophic pseudo ring.

    Study questions (i) to (x) of problem 15 for this M_1.

19. Let $P = \{R_n^I(20), +, \times\}$ be the MOD neutrosophic pseudo ring.

    Study questions (i) to (x) of problem 15 for this P.

20. Obtain any other special feature enjoyed by the MOD neutrosophic pseudo ring $\{R_n^I(m), +, \times\}$.

21. Let $R_n^I(m)[x]$ be MOD neutrosophic polynomials.

    (i) Show all polynomials in $R_n^I(m)[x]$ are not differentiable.

    (ii) Show for several polynomials in $R_n^I(m)[x]$ the integrals does not exist.

    (iii) If degree of $p(x) \in R_n^I(m)[x]$ is t and t/m will the
    (a) derivative exist?
    (b) integral exist?

22. Let $R_n^I(7)[x]$ be the MOD neutrosophic polynomial pseudo ring.

    Study questions (i) to (x) of problem 15 for this $R_n^I(7)[x]$.

23. Let $R_n^I(24)[x]$ be the MOD neutrosophic polynomial pseudo ring.

Study questions (i) to (x) of problem 15 for this $R_n^I(24)[x]$.

24. Let M = $R_n^I(15)[x]$ be the MOD neutrosophic polynomial.

    Study questions (i) to (iii) of problem 21 for this M.

25. Let $p(x) = 9x^{20} + 4x^{12} + 3.5x^9 + 6.3x^7 + 2.71x^2 + 0.19 \in R_n^I(10)[x]$.

    (i) Find $\frac{dp(x)}{dx}$.

    (ii) Is p(x) an integrable function?

    (iii) How many higher derivatives of p(x) exist?

    (iv) Is every polynomial in $R_n^I(10)[x]$ integrable?

    (v) Can every derivative of p(x) in $R_n^I(10)[x]$ exists?

26. Give some special properties enjoyed by $R_n^I(m)[x]$.

27. M = ($R_n^I(m)[x]$, ×) be a MOD neutrosophic polynomial semigroup.

    (i) Can M have zero divisors?

    (ii) Can M have S-ideals?

    (iii) Is M a S-semigroup?

    (iv) Can M have zero divisors?

    (v) Can M have zero divisors which are not S-zero divisors?

(vi) Can M have subsemigroups of infinite order which are not ideals?

(vii) Can M have ideals of finite order?

(viii) Is deg (p(x)) × deg q(x) = deg (p(x) × q(x)) in general for p(x), q(x) in M?

28. Let N = ($R_n^I(12)[x]$, ×) be a MOD neutrosophic polynomial semigroup.

    Study questions (i) to (viii) of problem 27 for this N.

29. Let P = ($R_n^I(17)[x]$, ×) be a MOD neutrosophic polynomial semigroup.

    Study questions (i) to (viii) of problem 27 for this P.

30. Let M = ($R_n^I(m)[x]$, ×) be a MOD neutrosophic polynomial semigroup.

    (i) Can M have subgroups of finite order?

    (ii) Let p(x) and q(x) ∈ M. p(x) is of degree t and q(x) of degree s.

    Is deg (p(x) + q(x)) = deg p(x) + deg q(x)?

31. Let M_1 = ($R_n^I(20)[x]$, ×) be a MOD neutrosophic polynomial semigroup.

    Study questions (i) to (ii) of problem 30 for this M_1.

32. Let M_2 = ($R_n^I(47)[x]$, ×) be a MOD neutrosophic polynomial semigroup.

Study questions (i) to (ii) of problem 30 for this M_2.

33. What is the algebraic structure enjoyed by $\{R_n^I(m)[x], +, \times\}$?

34. Prove $\{R_n^I(m)[x], +, \times\}$ cannot be given the ring structure.

    (i) Find a triple in $R_n^I(m)[x]$ such that they are not distributive.

    (ii) Does there exist a triple in $R_n^I(m)[x]$ which is distributive?

35. Let $S = \{R_n^I(27)[x], +, \times\}$ be the MOD neutrosophic polynomial pseudo ring.

    Study questions (i) to (ii) of problem 34 for this S.

36. Study questions (i) to (ii) of problem 30 for $S_1 = \{R_n^I(29)[x], +, \times\}$ the MOD neutrosophic polynomial pseudo ring.

37. Let $R = \{R_n^I(m)[x], +, \times\}$ the MOD neutrosophic polynomial pseudo ring.

    (i) Can R have zero divisors?
    (ii) Can R have S-zero divisors?
    (iii) Can R be a S-pseudo ring?
    (iv) Can R have finite pseudo ideals?
    (v) Can R have finite subrings?
    (vi) Can R have S-units?
    (vii) Can R have finite pseudo subrings?
    (viii) Will the collection of all distributive elements in R from any proper algebraic structure?
    (ix) Can R have S-idempotents?

(x) Can we say integration and differentiation of all $p(x) \in R$ is possible?

38. Let $R_1 = \{R_n^I(216)[x]; +, \times\}$ be the MOD neutrosophic polynomial pseudo ring.

    Study questions (i) to (x) of problem 37 for this R_1.

39. Let $R_2 = \{R_n^I(29)[x]; +, \times\}$ be the MOD neutrosophic polynomial pseudo ring.

    Study questions (i) to (x) of problem 37 for this R_2.

40. Study the special features associated with MOD fuzzy neutrosophic polynomials $R_n^I(1)[x]$.

41. Study the special features associated with $R_n^I(1)$ the MOD fuzzy neutrosophic plane.

42. Let $\{R_n^I(1), +\}$ be the MOD fuzzy neutrosophic group.

    (i) Find subgroups of finite order.
    (ii) Find subgroups of infinite order.
    (iii) Give any special feature enjoyed by $R_n^I(1)$.

43. Let $S = \{R_n^I(1), \times\}$ be the MOD fuzzy neutrosophic semigroup.

    (i) Find ideals of S.
    (ii) Can S be a S-semigroup?
    (iii) Find S-ideals of S.
    (iv) Can S have S-zero divisors?
    (v) Can S have S-units?
    (vi) Can S have S-idempotents?
    (vii) Can the concept of units ever possible in S?
    (viii) Show ideal in S are of infinite order.
    (ix) Can S have finite order subsemigroup?

(x) Obtain any other interesting property associated with S.

44. Let P = { $R_n^I(1)$, +, ×} be the MOD fuzzy neutrosophic pseudo ring.

    (i) Study all special features enjoyed by P.
    (ii) Can P have zero divisors?
    (iii) Is P a S-pseudo ring?
    (iv) Can P have S-zero divisors?
    (v) Can P have units?
    (vi) Can P have finite subrings?
    (vii) Can P have ideals of finite order?
    (viii) Can P have S-ideals?
    (ix) Mention any other striking feature about P.

45. Let S = { $R_n^I(1)[x]$, +} be the MOD fuzzy neutrosophic polynomial group.

    Study questions (i) to (iii) of problem 42 for this S.

46. Let S = { $R_n^I(1)[x]$, ×} be the MOD fuzzy neutrosophic polynomial semigroup.

    Study questions (i) to (x) of problem 43 for this S.

47. Let B ={ $R_n^I(1)[x]$, +, ×} be the MOD fuzzy neutrosophic pseudo ring.

    Study questions (i) to (ix) of problem 44 for this B.

48. Study $R_n^{Id}(1)[x]$, the MOD neutrosophic fuzzy decimal polynomials.

49. Study for S = { $R_n^{Id}(1)[x]$, +} the MOD fuzzy neutrosophic decimal polynomial group questions (i) to (iii) of problem 42 for this S.

50. Let B = $\{R_n^{Id}(1)[x], \times\}$ be the MOD fuzzy neutrosophic decimal polynomial semigroup.

    Study questions (i) to (x) of problem 43 for this B.

51. Let M = $\{R_n^{Id}(1)[x], +, \times\}$ be the MOD fuzzy neutrosophic decimal polynomial pseudo ring.

    Study questions (i) to (ix) of problem 44 for this M.

52. Show in case of $R_n^{Id}(1)[x]$ for any $p(x) \in R_n^{Id}(1)[x]$ we can have infinite number of

    (i) decimal differentiation.
    (ii) decimal integration.
    (iii) If $\alpha = \beta$, $\alpha, \beta \in (0, 1)$ can we have $\frac{dp(x)}{dx(\alpha)} = \frac{dp(x)}{dx(\beta)}$ for some $p(x) \in R_n^{Id}(1)[x]$?
    (iv) Show for $\alpha \neq \beta$, $\alpha, \beta \in (0, 1)$ can we have a $p(x) \in R_n^{Id}(1)[x]$ such that $\int p(x)\,dx(\alpha) = \int p(x)\,dx(\beta)$?
    (v) Can we have a $p(x) \in R_n^{Id}(1)[x]$ for which $\frac{dp(x)}{dx(\alpha)} = 0$ for a $\alpha \in (0, 1)$?
    (vi) Can we have a $p(x) \in R_n^{Id}(1)[x]$ for which $\int p(x)\,dx(\alpha)$ is undefined for a $\alpha \in (0, 1)$?

53. What are advantages of using decimal differentiation and decimal integration of MOD fuzzy neutrosophic decimal polynomials?

54. Obtain any other interesting feature about $R_n^{Id}(1)[x]$.

Chapter Five

THE MOD COMPLEX PLANES

In this chapter we proceed onto define, develop and describe the concept of MOD complex planes.

Let C be the usual complex plane we can for every positive integer n define a MOD complex plane. Thus we have infinite number of MOD complex planes.

DEFINITION 5.1: *Let $C_n(m) = \{a + bi_F \mid a, b \in [0, m), i_F^2 = m-1\}$. $C_n(m)$ is a semi open plane defined as the MOD complex plane.*

We diagrammatically describe in Figure 5.1.

This plane is defined as the MOD complex plane.

We will describe the above by the following examples.

Example 5.1: Let $C_n(10) = \{a + bi_F \mid a, b \in [0, 10), i_F^2 = 9\}$ be the MOD complex plane associated with the MOD interval [0, 10).

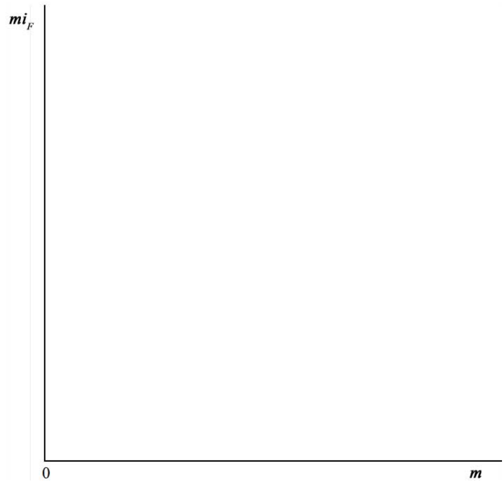

Figure 5.1

Example 5.2: Let $C_n(7) = \{a + bi_F \mid a, b \in Z_7$ and $i_F^2 = 6\}$ be the MOD complex plane associated with the MOD interval [0, 7).

Example 5.3: Let $C_n(3) = \{a + bi_F \mid a, b \in [0, 3)$ and $i_F^2 = 2\}$ be the MOD complex plane associated with the MOD interval [0, 3).

Example 5.4: Let $C_n(2) = \{a + bi_F \mid a, b \in [0, 2)$ and $i_F^2 = 1\}$ be the MOD complex plane associated with the MOD interval [0, 2).

$C_n(2)$ is the smallest MOD complex plane.

However we cannot say $C_n(m)$ for a particular m is the largest MOD complex plane as one can go for m+2, m+3, …, m+r; r a big integer.

How to get at these MOD complex plane from the complex plane C?

We know for every $a + bi \in C$ is mapped by the complex MOD transformation;

$\eta_c : C \rightarrow C_n(m)$; defined as

η_c (a + bi) = a + bi$_F$ if $0 \leq a, b < m$ and $i_F^2 = m-1$.

η_c (a + bi) = c + di$_F$ if both a, b > m is given by

$$\frac{a}{m} = x + \frac{c}{m} \text{ and d is given by } \frac{b}{m} = y + \frac{d}{m}.$$

η_c (a + bi) = 0 if a = b = m or a = mx and b = my.

η_c (a + bi) = a + ci$_F$ if $0 < a < m$ and c is given by

$$\frac{b}{m} = x + \frac{c}{m}.$$

η_c (a + bi) = d + bi$_F$ if $0 < b \leq m$ and

$$\frac{a}{m} = y + \frac{d}{m}.$$

η_c (a + bi) = m – a + (m – b)i$_F$ if a and b are negative but less than m.

η_c (a + bi) = m–c + (m–d)i$_F$ if both a and b are –ve and greater than m; c and d defined as earlier.

We will first illustrate these situations before we proceed to define properties associated with $C_n(m)$.

Example 5.5: Let $C_n(6)$ be the MOD complex plane. We define

$\eta_c : C \to C_n(6)$. For $25 + 46.3i \in C$.

$\eta_c (25 + 46.3i) = (1 + 4.3i_F)$ where $i_F^2 = 5$.

For $3.1 + 2.7i \in C$; $\eta_c (3.1 + 2.7i) = 3.1 + 2.7i_F$.

For $4.205 + 12.03i \in C$; $\eta_c (4.205 +12.03i) = 4.205 + 0.03i_F$

For $12 + 18i \in C$; $\eta_c (12 + 18i) = 0$.

For $24 + 0.089i \in C$; $\eta_c (24 + 0.089i) = 0.089i_F$.

For $-2 + 6.07i \in C$; $\eta_c (-2 + 6.07i) = 4 + 0.07i_F$.

$-46 - 27.3i \in C$; $\eta_c (-46 - 27.3i) = 2 + 2.7i_F \in C_n(6)$ and so on. This is the way the MOD complex transformation is carried out.

Example 5.6: Let $C_n(11) = \{a + bi_F \mid a, b \in [0, 11), i_F^2 = 10\}$ be the MOD complex plane.

$\eta_c : C \to C_n(11)$ is the MOD complex transformation and it is defined as follows:

$$\eta_c (a + bi) = a + bi_F \text{ if } a, b \in [0, 11).$$

$$\eta_c (19 + 0.2i) = 8 + 0.2i_F \in C_n(11)$$

$$\eta_c (240 + 361.3i) = (9 + 9.3i_F)$$

$$\eta_c (-421 + 3i) = (8 + 3i_F)$$

$$\eta_c (214 - 149.6i) = (5 + 4.4i_F).$$

This is the way the MOD complex transformation is done. Infact infinite number of points in C are mapped by η_c on to a single point.

Example 5.7: Let $\eta_c : C \to C_n(13)$.

$$\eta_c (27 + 0.3i) = (1 + 0.3i_F)$$

$$\eta_c (42 + 40.7i) = (3 + 1.7i_F) \text{ and so on.}$$

$$\eta_c (-52 - 20.8i) = (0 + 5.2i_F)$$

η_c (8 – 48.3i) = (8 + 3.7i_F) and

η_c ((6.1 + 5.2i) = 6.1 + 5.2i_F.

This is the way η_c the MOD complex transformation is defined from C to the MOD complex plane $C_n(13)$.

Now we proceed onto define the MOD complex plane polynomial or MOD complex polynomials with coefficients from the MOD complex plane $C_n(m)$.

$$C_n(m)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in C_n(m) = \{a + bi_F \mid a, b \in [0, m), \right.$$

$i_F^2 = m - 1\}$ is defined as the MOD complex polynomials.

Now we have infinite number of distinct MOD complex polynomials for varying m.

We will give examples of them.

Example 5.8: Let $C_n(5)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in C_n(5) = \{a + bi_F \mid a, \right.$ $b \in [0, 5),\; i_F^2 = 4\}$ be the MOD complex polynomial with coefficients from $C_n(5)$.

Example 5.9: Let $C_n(9)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in C_n(9) = \{a + bi_F \mid a, \right.$ $b \in [0, 9),\; i_F^2 = 8\}\}$ be the MOD complex polynomial.

Clearly $C_n(9)[x]$ is different from $C_n(5)[x]$ given in example 5.8.

Example 5.10: Let $C_n(12)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in C_n(12) = \{a + bi_F \mid a, b \in [0, 12),\ i_F^2 = 11\} \right\}$ be the MOD complex polynomial with coefficients from $C_n(12)$.

We can as in case of MOD complex transformation, $\eta_c : C \to C_n(m)$ in case of MOD complex polynomial also define the same MOD complex transformation.

$\eta_c : C[x] \to C_n(m)[x]$ as $\eta_c[x] = x$ and $\eta_c : C \to C_n(m)$ and η_c case of MOD complex transformation.

Thus we see the total complex polynomial C[x] is mapped on to the MOD complex polynomials.

We will illustrate this situation by some examples.

Example 5.11: Let C[x] be the complex polynomial ring and $C_n(20)[x]$ be the MOD complex polynomial.

We have a unique MOD transformation $\eta_c : C[x] \to C_n(20)[x]$ such that $\eta_c(x) = x$ and $\eta_c(C) = C_n(20)$.

Now we show how operations on $C_n(m)$ and $C_n(m)[x]$ are defined in the following.

Example 5.12: Let $C_n(15) = \{a + bi_F \mid a, b \in [0, 15),\ i_F^2 = 14\}$ be the MOD complex plane.

$$x = 5.3 + 12.5i_F \text{ and } y = 8.6 + 3i_F \in C_n(15),$$

$$\begin{aligned} x + y &= (5.3 + 12.5i_F) + (8.6 + 3i_F) \\ &= (13.9 + 0.5i_F) \in C_n(15). \end{aligned}$$

It is easily verified for every pair $x, y \in C_n(15)$

$$x + y \in C_n(15).$$

Infact $0 \in C_n(15)$ serves as the additive identity.

For $x + 0 = 0 + x = x$ for all $x \in C_n(15)$.

Further for every $x \in C_n(15)$ there exist a unique $y \in C_n(15)$ such that $x + y = 0$.

Let $x = 6.331 + 9.2305i_F \in C_n(15)$ we have

$y = 8.669 + 5.7695i_F \in C_n(15)$ such that $x + y = 0$.

Thus y is the inverse of x and x is the inverse of y.

Hence $(C_n(15), +)$ is a group of infinite order and is commutative.

Example 5.13: Let $C_n(10) = \{a + bi_F \mid a, b \in [0, 10), i_F^2 = 9\}$ be a group of infinite order under addition modulo 10.

Example 5.14: Let $C_n(7) = \{a + bi_F \mid a, b \in [0, 7), i_F^2 = 6\}$ be a group of infinite order under +.

Example 5.15: Let $C_n(20) = \{a + bi_F \mid a, b \in [0, 20), i_F^2 = 19\}$ be a group of infinite order under + modulo 20.

Now we can give some other operation on $C_n(m)$.

We define the product on $C_n(m)$.

Example 5.16: Let $C_n(12) = \{a + bi_F \mid a, b \in [0, 12), i_F^2 = 11\}$ be the MOD complex plane.

Let $x = 0.37 + 4.2i_F$ and $y = 5 + 9i_F \in C_n(12)$.

$x \times y \;=\; (0.37 + 4.2i_F) \times (5 + 9i_F)$

$$= 0.37 \times 5 + 4.2i_F \times 5 + 0.37 \times 9i_F + 4.2i_F \times 9i_F$$

$$= 1.85 + 9i_F + 3.33i_F + 1.8 \times 11$$

$$= 9.65 + 0.33i_F \in C_n(12).$$

We see for every $x, y \in C_n(12)$;

we have $x \times y \in C_n(12)$.

$x \times y = y \times x$ for all $x, y \in C_n(12)$.

We have $0 \in C_n(12)$ is such that $x \times 0 = 0$ for all $x \in C_n(12)$.

We may or may not have $C_n(12)$ to be a group. Clearly $\{C_n (12), \times\}$ is only a semigroup.

Further $\{C_n(12), \times\}$ has zero divisors, idempotents and units.

$C_n(12)$ is an infinite commutative semigroup under product.

Example 5.17: Let $\{C_n(10), \times\}$ be the semigroup.

Let $x = 5+ 5i_F$ and $y = 2 + 4i_F \in C_n(10)$ is such that

$$x \times y = 0.$$

$$x^2 = (5 + 5i_F) \times (5 + 5i_F)$$

$$= 5 + 5 \times i_F + 5i_F + 5 \times 9 = 0.$$

Let $x = 3.2 + 4.8i_F$ and $y = 9 +8i_F \in C_n(10)$.

$$x \times y = (3.2 + 4.8i_F) \times (9 + 8i_F)$$

$= 3.2 \times 9 + 4.8i_F \times 9 + 3.2 \times 8i_F + 4.8 \times 8 \times 9$

$= 8.8 + 3.2i_F + 5.6i_F + 8.4 \times 9$

$= 8.8i_F + 4.4 \in C_n(10).$

Example 5.18: Let $C_n(11)$ be the semigroup under $\times$.

$x = 1.9 + 5.2i_F$ and $y = 3 + 10i_F \in C_n(11)$.

$x \times y \quad = (1.9 + 5.2i_F) \times (3 + 10i_F)$

$= 5.7 + 4.6i_F + 8i_F + 3$

$= 8.7 + 1.6i_F \in C_n(11).$

Let $x = 0.37 + 0.5i_F$ and $y = 10 + 10i_F \in C_n(11)$;

$x \times y \quad = (0.37 + 0.5i_F) \times (10 + 10i_F)$

$= 3.7 + 5i_F + 3.7i_F + 5 \times 10$

$= 9.7 + 8.7i_F \in C_n(11).$

Example 5.19: Let $\{C_n(5), \times\}$ be the semigroup.

Let $x = 3.1 + 0.52i_F$ and $y = 0.8 + 4i_F \in C_n(5)$;

$x \times y \quad = (3.1 + 0.52i_F) \times (0.8 + 4i_F)$

$= 2.48 + 4.16i_F + 2.4i_F + 2.08 \times 4$

$= 0.8 + 1.56i_F \in C_n(5).$

Thus $C_n(5)$ is only a semigroup.

THEOREM 5.1: *Let $C_n(m)$ be the MOD complex plane. $\{C_n(m), +\}$ be a group.*

Then

(1) $\{C_n(m), +\}$ has subgroups of infinite order.

(2) If m is not a prime, $\{C_n(m), +\}$ has more number of subgroups.

The proof of the theorem is left as an exercise to the reader.

THEOREM 5.2: *Let $\{C_n(m), \times\} = S$ be the semigroup (m, a non prime).*

(i) S has subsemigroups of infinite order.
(ii) S has zero divisors.
(iii) S has idempotents.

The proof is left as an exercise to the reader.

Next we show $\{C_n(m), +, \times\}$, MOD complex plane is not a ring for distributive law is not true in general for elements in $C_n(m)$.

We will just show this fact by some examples.

Let $x = 0.31 + 0.2i_F$, $y = 6 + 7i_F$ and $z = 4 + 3i_F \in C_n(10)$.

Consider $x \times (y + z) = 0.31 + 0.2i_F \times (6 + 7i_F + 4 + 3i_F)$

$$= 0.31 + 0.2i_F \times 0$$

$$= 0 \qquad \ldots \text{ I}$$

We find

$x \times y + x \times z = 0.31 + 0.2i_F \times 6 + 7i_F + 0.31 + 0.2i_F \times 4 + 3i_F$

$$= 1.86 + 1.2i_F + 2.17i_F + 1.4 \times 9 + 1.24 + 0.8i_F + 0.93i_F + 0.6 \times 9$$

$$= (1.86 + 2.6 + 1.24 + 5.4) + (1.2 + 2.17 +$$

$0.8 + 0.93)i_F$

$$= \quad 1.1 + 5.10i_F \qquad \ldots \quad \text{II}$$

Clearly I and II are distinct.

Hence in general $x \times (y + z) \neq x \times y + x \times z$ for $x, y, z \in C_n(m)$.

Thus we define $\{C_n(m), +, \times\}$ be the MOD complex pseudo ring.

$\{C_n(m), +\}$ to be the MOD complex group and $\{C_n(m), \times\}$ to be the MOD complex semigroup.

We have several interesting properties associated with the pseudo ring of MOD complex plane.

The pseudo MOD complex ring has ideals, subsemirings, units, zero divisors and idempotents.

This will be illustrated by some examples.

Example 5.20: Let $\{C_n(6),+, \times\}$ be the MOD complex pseudo ring. $\{C_n(6), +, \times\}$ has subrings which are not ideals of both finite and infinite order.

$S = \{[0, 6) \subseteq C_n(6), +, \times\}$ is a pseudo subring of infinite order which is not an ideal of $\{C_n(6), +, \times\}$. $\{Z_6, +, \times\} \subseteq C_n(6)$ is a subring of finite order which is not an ideal.

$S_1 = \{\{0, 2, 4\}, +, \times\} \subseteq C_n(6)$ be a subring of finite order which is not an ideal.

Example 5.21: Let $S = \{C_n(10), +, \times\}$ be the MOD complex pseudo ring.

S has zero divisors, idempotents and units.

Example 5.22: Let S = $\{C_n(19), +, \times\}$ be the MOD complex pseudo ring.

R has zero divisors and units.

Now we proceed onto define MOD complex polynomial pseudo ring and describe their properties.

DEFINITION 5.2: *Let* $C_n(m)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in C_n(m) \right\}$ *be the MOD complex polynomial plane.*

We will illustrate this situation by some examples.

Example 5.23: Let $C_n(10)[x]$ be the MOD complex polynomial pseudo ring plane.

Example 5.24: Let $C_n(19)[x]$ be the MOD complex polynomial plane pseudo ring.

Example 5.25: Let $C_n(43)[x]$ be the MOD complex polynomial pseudo ring.

Example 5.26: Let $C_n(148)[x]$ be the MOD complex polynomial pseudo ring.

Let $(0.5 + 6.1i_F) + (7.2 + 0.5i_F)x^2 + (3.2 + 4i_F)x^3$

$$= p(x) \in C_n(8)[x]$$

where $C_n(8)[x]$ is the MOD complex polynomial plane.

Let $q(x) = (4 + 0.3i_F) + (3.8 + 0.7i_F)x + (6.8 + 0.7i_F)x^2 + (0.9 + 0.2i_F)x^4 \in C_n(8)[x]$

$p(x) + q(x)$

$$= (0.5 + 6.1i_F) + (7.2 + 0.5i_F)x^2 + (3.2 + 4i_F)x^3 + (4 + 0.3i_F) + (3.8 + 0.7i_F)x + (6.8 + 0.7i_F)x^2 + (0.9 + 0.2i_F)x^4$$

$$= (4.5 + 6.4i_F) + (3.8 + 0.7i_F)x + (6 + 1.2i_F)x^2 + (3.2 + 4i_F)x^3 + (0.9 + 0.2i_F)\, x^4.$$

This is the way the operation + is performed on $C_n(8)[x]$.

We see for every $q(x)$, $p(x) \in C_n(8)[x]$; $p(x) + q(x) \in C_n(8)[x]$, $0 \in C_n(8)[x]$ acts as the additive identity as $p(x) + 0 = 0 + p(x) = p(x) \in C_n(8)[x]$.

Now for every $p(x) \in C_n(8)[x]$ there exist a unique $q(x)$ in $C_n(8)[x]$ such that $p(x) + q(x) = 0$ and $q(x)$ is the called the additive inverse of $p(x)$ and vice versa.

Let $p(x) = (0.8 + 4.5i_F) + (1.8 + 3.75i_F)x + (4.2 + 3.5i_F)x^2 + (0.9 + 6.72i_F)x^3 \in C_n(8)[x]$.

We see

$q(x) = (7.2 + 3.5i_F) + (6.2 + 4.25i_F)x + (3.8 + 4.5i_F)x^2 + (7.1 + 1.28i_F)x^3 \in C_n(8)[x]$ is such that

$$q(x) + p(x) = p(x) + q(x) = 0.$$

Thus $\{C_n(8)[x], +\}$ is a group defined as the MOD complex polynomial group of infinite order which is abelian.

We will illustrate this situation by some examples.

Example 5.27: Let $\{C_n(6)[x], +\}$ be the MOD complex polynomial group.

Example 5.28: Let $G = \{C_n(37)[x], +\}$ be the MOD complex polynomial group.

Example 5.29: Let $G = \{C_n(64)[x], +\}$ be the MOD complex polynomial group.

Before we proceed to describe MOD semigroups we describe the MOD complex transformation from C[x] to $C_n(m)[x]$.

Let C[x] be the complex polynomials; $C_n(m)[x]$ be the MOD complex polynomial plane.

$\eta_c : C[x] \to C_n(m)[x]$ is defined as $\eta_c[x] = x$ and

$\eta_c : C \to C_n(m)$ is defined in the usual way that is η_c is the MOD complex transformation from C to $C_n(m)$.

Thus using this MOD complex transformation we can map the whole plane C[x] into the MOD complex polynomials plane $C_n(m)[x]$ for $m \in N$.

Infact just now we have proved $C_n(m)[x]$ to be a group under +.

Now we define the operation $\times$ on $C_n(m)[x]$.

For $p(x) = 3.2x^7 + 2.5x^3 + 4x + 1$ and

$q(x) = 0.5x^3 + 0.8x^2 + 2$ in $C_n(5)[x]$ we find

$p(x) \times q(x) = (3.2x^7 + 2.5x^3 + 4x + 1) \times (0.5x^3 + 0.8x^2 + 2)$

$$= 1.6x^{10} + 1.25x^6 + 2x^4 + 0.5x^3 + 2.56x^9 + 2x^5 + 3.2x^3 + 0.8x^2 + 1.4x^7 + 0 + 3x + 2 \in C_n(5)[x].$$

This is the way product operation is performed.

Let $p(x) = (3 + 2i_F)x^2 + (0.6 + 0.4i_F)x + (0.7 + 0.5i_F)$

and $q(x) = (0.2 + 0.4i_F)x^3 + (0.4 + 3i_F) \in C_n(5)[x]$.

We find $p(x) \times q(x) = ((3 + 2i_F)x^2 + (0.6 + 0.4i_F)x + (0.7 + 0.5i_F)) \times ((0.2 + 0.4i_F)x^3 + (0.4 + 3i_F))$

$= (0.12 + 0.08i_F + 0.24i_F + 0.16 \times 4)x^4 + (0.6 + 0.4i_F + 1.2i_F + 0.8 \times 4)x^5 + (0.14 + 0.1i_F + 0.28i_F + 0.2 \times 4)x^3 + (1.2 + 0.8i_F + 4i_F + 4)x^2 + (0.24 + 0.16i_F + 1.8i_F + 1.2 \times 4)x + (0.28 + 0.2i_F + 2.1i_F + 1.5 \times 4)$

$= (0.76 + 0.32i_F)x^4 + (3.8 + 1.6i_F)x^5 + (0.94 + 0.38i_F)x^3 + (0.2 + 4.8i_F)x^2 + (0.04 + 1.96i_F)x + (1.28 + 2.3i_F) \in C_n(5)[x]$.

This is the way product operation is performed on $C_n(5)[x]$.

$\{C_n(5)[x]\}$ is a semigroup of infinite order which is commutative.

Example 5.30: Let S = $\{C_n(8)[x], \times\}$ be the MOD complex polynomial semigroup. S has zero divisors and units.

Example 5.31: Let P = $\{C_n(12)[x], \times\}$ be the MOD complex polynomial semigroup. S has zero divisors and units.

Example 5.32: Let M = $\{C_n(13)[x], \times\}$ be the MOD complex polynomial semigroup.

$\{C_n(m)[x], \times\}$ = B is defined as the MOD complex polynomial semigroup.

B is of infinite order and is commutative.

Example 5.33: Let B = $\{C_n(12)[x], \times\}$ be the MOD complex polynomial semigroup.

$\{Z_{12}[x], \times\}$ is a subsemigroup of B and is of infinite order. $\{Z_{12}, \times\}$ is a finite subsemigroup of B.

$R_n(12)[x] \subseteq$ B is again only a subsemigroup and is not an ideal of B.

$\{[0, 12), \times\} \subseteq B$ is only a subsemigroup and is not an ideal of B.

$\{[0, 12)[x], \times\}$ is once again a subsemigroup of B and is not an ideal.

$p(x) = 4 + 4x^2 + 8x^3$ and $q(x) = 6 + 3x^2 \in B$ are such that $p(x) \times q(x) = 0$.

Infact B has infinite number of zero divisors.

Example 5.34: Let $\{C_n(29)[x], \times\} = S$ be the MOD complex polynomial semigroup. S has infinite number of zero divisors. S has subsemigroup of both finite and infinite order.

Now having seen MOD complex polynomial groups and semigroups.

We now proceed onto define MOD complex polynomial pseudo rings.

Let $S = \{C_n(m)[x], +, \times\}$ be the MOD complex polynomial under the binary operations + and $\times$.

Now S is define as the MOD complex polynomial pseudo ring as for $p(x), q(x), r(x) \in S$ we may not in general have

$$p(x) \times (q(x) + r(x)) = p(x) \times q(x) + p(x) \times r(x).$$

Because the triples need not always be distributive we define $\{C_n(m)[x], +, \times\}$ as only a pseudo ring.

We will proceed onto describe with examples the complex MOD polynomial pseudo rings.

Example 5.35: Let $\{C_n(15)\ [x], +, \times\}$ be the MOD complex polynomial pseudo ring.

Let $p(x) = 0.7x^8 + (2 + 4i_F)x^4 + 2i_F$,

$q(x) = 10.3x^5 + (10 + 0.8i_F)x^2 +(3.1 + 7.2i_F)$

and $r(x) = 4.7x^5 + (5 + 14.2i_F)x^2 + (11.9 + 7.8i_F) \in C_n(15)[x]$.

Consider $p(x) \times (q(x) + r(x))$

$$= (0.7x^8 + (2 + 4i_F)x^4 + 2i_F) \times (10.3x^5 + (10 + 0.8i_F)x^2 +(3.1 + 7.2i_F) + 4.7x^5 + (5 + 14.2i_F)x^2 + (11.9 + 7.8i_F))$$

$$= (0.7x^8 + (2 + 4i_F)x^4 + 2i_F) \times 0 \qquad \ldots \text{ I}$$

Now we find

$p(x) \times q(x) + p(x) + r(x)$

$$= [0.7x^8 + (2 + 4i_F)x^4 + 2i_F] \times (10.3x^5 + (10 + 0.8i_F)x^2 + (3.1 + 7.2i_F) + [0.7x^8 + (2 + 4i_F)x^4 + 2i_F] \times (4.7x^5 + (5+ 14.2i_F)x^2 + (11.9 + 7.8i_F))$$

$$= 7.21x^{13} + (7 + 0.56i_F)x^{10} + (2.17 + 5.04i_F) + (5.6 + 11.2i_F)x^9 + (14.8 + 11i_F)x^6 + (4.8 + 11.8i_F)x^4 + 5.6i_Fx^5 + (7.4 + 5i_F)x^2 +(6.2i_F + 6.6) + 3.29x^{13} +(5 + 9.94i_F)x^{10} +(8.33 + 5.46i_F)x^8 + 9.4i_Fx^5 +(10i_F + 7.6)x^2 +(8.8i_F + 9.6) + (9.4 + 3.8i_F)x^9 + (10.2 + 3.4i_F)x^6 + (3.2i_F +10.6) x^2$$

$$\neq 0 \qquad \ldots \text{ II}$$

Clearly I and II are distinct. Hence we see $\{C_n(m)[x], +, \times\}$ can only be a pseudo ring. It is not a ring.

Further $C_n(m)[x]$ has ideals subrings, zero divisors, units and idempotents.

Study in this direction is interesting as the non distributivity gives lots of challenges.

For instance we cannot write $(x + \alpha_1)(x + \alpha_2) \ldots (x + \alpha_n)$

$$= x^n (\alpha_1 + \ldots + \alpha_n)\ x^{n-1} + \sum_{i \neq j} x_i x_j\, x^{n-2} + \ldots + \alpha_1 \alpha_2 \ldots \alpha_n.$$

That is why α_i may make one side equal to zero but the other side of the equation is not zero in general this is mainly because the distributive law is not true.

Example 5.36: Let $\{C_n(6)[x], +, \times\}$ be the complex MOD polynomial pseudo ring.

Let $(x + 3.2)(x + 1.5)(x + 3) = p(x)$.

We see $p(3) = 0$,
$p(4.5) = 0$ and
$p(2.8) =0$.

$(x + 3.2)(x + 1.5)(x + 3) = (x^2 + 3.2x + 1.5x + 4.8)(x + 3)$

$= x^3 + 4.7x^2 + 4.8x + 3x^2 + 9.6x + 4.5x + 2.4$

$= x^3 +1.7x^2 + 0.9x + 2.4 = q(x).$

Put $x = 3$

$q(3) = 27 + 1.7 \times 9 + 0.9 \times 3 + 2.4$

$= 5.4 \neq 0.$

$q(2.8) = (2.8)^3 + 1.7(2.8)^2 + 0.9 \times 2.8 + 2.4$

$= 3.952 + 1.328 + 2.52 + 2.4$

$= 4.20 \neq 0.$

Thus $q(3) \neq 0$ but $p(3) = 0$ and so on.

It is an open problem how to over come this situation.

Study in this direction is being carried out.

Now regarding the derivatives and integrals we define a new type of MOD complex polynomials.

$$C_n^d(m)[x] = \left\{ \sum_i a_i x^i \,\middle|\, a_i \in C_n^d(m) \text{ and } i \in R^+ \text{ reals or zero} \right\}$$

In this case we call $C_n^d(m)[x]$ as the MOD complex decimal polynomials.

Let $p(x) = (3 + 4.1i_F)\, x^{2.1} + (1 + 0.2i_F)x^{0.1} + (0.1 + 0.5i_F)$ and $q(x) = (2 + 0.3i_F)\, x^{1.2} + (4 + 0.2i_F) \in C_n^d(5)[x]$; be the MOD complex decimal polynomial plane.

$$\begin{aligned} p(x) \times q(x) = \; & [(3 + 4.1i_F)x^{2.1} + (1 + 0.2i_F)x^{0.1} + \\ & (0.1 + 0.5i_F)] \times [(2 + 0.3i_F)x^{12} + (4 + 0.2i_F)] \end{aligned}$$

$$\begin{aligned} = \; & (1 + 3.2i_F + 0.9i_F + 1.23 \times 4)x^{3.3} + (2 + 0.4i_F + 0.6i_F + \\ & 0.6 \times 4)\, x^{1.3} + (0.2 + i_F + 0.03i_F + 0.15 \times 4)\, x^{1.2} + (12 \\ & + 16.4i_F + 0.6i_F + 0.82 \times 4)x^{2.1} + (4 + 0.8i_F + 0.2i_F + \\ & 0.04 \times 4)x^{0.1} + (0.4 + 2i_F + 0.02i_F + 0.4) \end{aligned}$$

$$\begin{aligned} = \; & (0.92 + 4.1i_F)x^{3.3} + (2.24 + i_F)x^{1.3} + (0.8 + 1.03i_F)x^{1.2} \\ & + (0.28 + 2i_F)\, x^{2.1} + (4.16 + i_F)x^{0.1} + (0.8 + 2.02i_F) \text{ is} \\ & \text{in } C_n^d(5)[x]. \end{aligned}$$

This is the way operations are performed on $C_n^d(5)[x]$.

Now we can both differentiate and integrate MOD decimal polynomials. Here also at times they will exist and at times they may not exist.

Example 5.37: Let $\{C_n^d(6)[x], +\}$ be the MOD decimal complex polynomial group.

Example 5.38: Let $\{C_n^d(7)[x], +\} = G$ be the MOD decimal complex polynomial group. G has subgroups of both finite and infinite order.

Example 5.39: Let $\{C_n^d(64)[x], +\} = G$ be the MOD decimal complex polynomial group. G has subgroups.

Now we define $G = \{C_n^d(m)[x], +\}$ to be the MOD complex decimal polynomial group.

Example 5.40: Let $\{C_n^d(10)[x], \times\} = S$ be the MOD decimal complex polynomial semigroup. S is of infinite order.

Example 5.41: Let $P = \{C_n^d(15)[x], \times\}$ be the MOD decimal complex polynomial semigroup. P has ideals and subsemigroups. P has units, zero divisors and idempotents.

Example 5.42: Let $X = \{C_n^d(13)[x], \times\}$ be the MOD complex polynomial semigroup.

$P = \{C_n^d(m)[x], \times\}$ is defined as the MOD complex decimal polynomial semigroup.

We have infinite number of such semigroups depending on $m \in Z^+ \setminus \{1\}$.

We will give some examples of them.

Example 5.43: Let $S = \{C_n^d(15)[x], \times\}$ be the MOD complex decimal polynomial semigroup. S has infinite number of zero divisors and few units.

Example 5.44: Let $M = \{C_n^d(13)[x], \times\}$ be the MOD complex polynomial semigroup. M has zero divisors, units and idempotents.

Example 5.45: Let S = { C_n^d (24)[x], ×} be the MOD complex polynomial semigroup. M has zero divisors and idempotents.

Example 5.46: Let T = { C_n^d (128)[x], ×} be the MOD complex decimal polynomial semigroup. T has zero divisors, units and idempotents.

Now having seen group and semigroup of MOD complex decimal polynomials, we proceed onto define MOD complex decimal polynomials pseudo ring.

{ C_n^d (m)[x], +, ×} is defined as the MOD complex decimal polynomial pseudo ring as the distributive laws are not true in case of MOD complex decimal polynomial triples.

That is in general
$p(x) \times (q(x) + r(x)) \neq p(x) \times q(x) + p(x) \times r(x)$

for p(x), q(x), r (x) ∈ { C_n^d (m) [x], +, ×}.

We will give examples of them.

Example 5.47: Let { C_n^d (4)[x], +, ×} = S be the MOD complex decimal polynomial pseudo ring.

Let $p(x) = (2 + i_F)x^8 + (3 + 0.5i_F)x^4 + (0.7 + 0.5i_F)$,

$q(x) = (3 +0.7i_F) + (0.6 + 2i_F)x^3$ and

$r(x) = (1 + 9.3i_F) + (3.4 + 2i_F)x^3 \in C_n^d(4)[x]$.

$p(x) \times [q(x) + r(x)]$

$$= (2 + i_F)x^8 + (3 + 0.5i_F)x^4 + (0.7 + 0.5i_F) [(3 + 0.7i_F) + (0.6 + 2i_F)x^3 + (1 + 9.7i_F) + (3.4 +2i_F)x^3]$$

$$= \quad p(x) \times 0$$

$$= \quad 0 \qquad \ldots \quad \text{I}$$

Now $p(x) \times q(x) + p(x) \times r(x)$

$$= [(2 + i_F)x^{8.1} + (3 + 0.5i_F)x^{4.2} + (0.7 + 0.5i_F)] \times (3 + 0.7i_F) + (0.6 + 2i_F)x^3 + [(2 + i_F)x^8 + (3 + 0.5i_F)x^4 + (0.7 + 0.5i_F)] \times (1 + 3.93i_F) + (3.4 + 2i_F)x^3$$

$$= [(6 + 3i_F + 1.4i_F + 0.7 \times 3)\, x^{8.1} + (9 + 1.5i_F + 2.1i_F + 0.35 \times 3)x^{4.2} + (2.1 + 1.5i_F + 0.49i_F + 0.35 \times 3) + (1.2 + 0.6i_F + 0 + 2 \times 3)x^{11.1} + (1.8 + 0.3i_F + 6i_F + 1 \times 3)x^{7.2} + (0.42 + 0.3i_F + 1.4i_F + 3)\, x^3 + (2 + i_F + 3.86i_F + 3.93 \times 3)x^{8.1} + (3 + 0.5i_F + 3.79i_F + 1.965 \times 3)x^{4.2} + (7 + 0.5i_F + 2.751i_F + 1.965 \times 3) + (2.8 + 0 + 3.4i_F + 2 \times 3)x^{11.1} + (3.2 + 1.7i_F + 2i_F + 3)x^{7.2} + (2.38 + 2i_F + 1.70i_F + 3)x^3$$

$$= (0.1 + 0.4i_F)x^{8.1} + (2.05 + 3.6i_F)x^{4.2} + (3.25 + 1.99i_F) + (3.2 + 1.7i_F)x^3 + (3.79 + 0.86i_F)x^{8.1} + (0.895 + 0.29i_F)x^{4.2} + (0.895 + 3.251i_F) + (0.8 + 3.4i_F)x^{11.1} + (2.2 + 3.7i_F)x^{7.2} + (1.38 + 3.7i_F)x^3$$

$$= (0.8 + 1.4i_F)\, x^3 + (3.89 + 1.26i_F)\, x^{8.1} + (2.945 + 3.89i_F)\, x^{4.2} + (0.145 + 1.241i_F) + 0 + (3 + 2i_F)x^{7.2} \qquad \ldots \quad \text{II}$$

Clearly I and II are distinct.

Hence $C_n^d(4)[x]$ is a MOD complex decimal polynomial pseudo ring.

We have shown how the distributive law is not true.

Example 5.48: Let B = { $C_n^d(128)[x]$, +, ×} be the MOD complex decimal polynomial pseudo ring.

Example 5.49: Let S = { $C_n^d(17)[x]$, +, ×} be the MOD complex decimal polynomial pseudo ring.

Example 5.50: Let P = { $C_n^d(18)[x]$, +, ×} be the MOD complex decimal polynomial pseudo ring.

Now having seen examples of MOD complex decimal polynomial rings we now proceed onto define different types of differentiation and integration on these polynomials.

Example 5.51: Let S = { $C_n^d(7)[x]$, +, ×} be the MOD complex decimal polynomial pseudo ring.

Let $p(x) = (2 + 3i_F)x^{3.2} + (0.71 + 0.53)x^{0.1} + (0.36 + 4.2i_F) \in S$.

$$\frac{dp(x)}{dx(0.5)} = (2 + 3i_F) \times 3.2x^{2.7} + \text{term not defined} + 0.$$

So we see this special type of derivatives are also not defined.

So we see the derivative with respect other values are not useful.

However the integration is defined in most cases with some simple modifications.

Let $p(x) = 0.31x^{2.1} + 2.432x^{0.1} + 4.2 \in C_n^d(5)[x]$.

$$\int p(x)\,dx\,(0.3) = \frac{0.31x^{2.4}}{2.4} + \frac{2.432x^{0.4}}{0.4} + 4.2x^{0.3} + C.$$

However we do not have meaning for this situation.

Thus we may have seemingly calculated the integral but is meaningless.

We just give in supportive of our argument that the division is meaningless in Z_n, [0, n), R_n([0, n)) and so on.

Consider Z_5 for instance we want to find $\frac{3}{2}$; for $3, 2 \in Z_5$.

$$\frac{3}{2} \neq 1.5 \text{ but } \frac{3\times 3}{2\times 3} = \frac{9}{1}$$

$$= 9 \equiv 4 \pmod 5).$$

So what sort of consistent division is to be used if we want to have multiplication to be the reverse process of division?

Let Z_7 be the ring of modulo integer $\frac{3}{4}$ = 0.75 in R; but $3, 4 \in Z_7$.

$$\frac{3}{4} = \frac{3\times 2}{4\times 2} = \frac{6}{1} = 6\,.$$

So with this hurdle even in modulo integers we leave this problem as an open problem.

Thus we feel it is not an easy task to integrate these $p(x) \in C_n(m)[x]$.

Let $p(x) = (3 + 0.2i_F)x^3 + (2.1 + 1.2i_F)x + (4 + 0.301i_F) \in C_n(5)\,[x]$

$$\int p(x)\,dx = \frac{(3+0.2i_F)x^4}{4} + \frac{(2.1+1.2i_F)x^2}{2} + (4 + 0.301i_F)x + k$$

the integral exist.

$\int p(x)\,dx$ (0.51)

$$= \frac{(3+0.2i_F)x^{4.51}}{4.51} + \frac{(2.1+1.2i_F)x^{2.51}}{2.51} + \frac{(4+0.301i_F)x^{1.51}}{1.51} + k$$

In R[x] and C[x] we see every polynomial can be both differentiated and integrated however in case $C_n^d[m][x]$ and $R_n^d[m][x]$ this is not possible.

More so this is not possible in case of $C_n(m)[x]$ and $R_n(m)[x]$.

Now we have studied such integration and differentiation of MOD complex polynomials and MOD complex decimal polynomials.

We now proceed on to suggest some problems.

Problems:

1. Find some special features enjoyed by MOD complex planes.

2. Show we have infinite number of MOD complex planes.

3. Prove we have for a given MOD complex plane $C_n(m)$ only one MOD complex transformation η_c.

4. Obtain interesting features enjoyed by η_c.

5. Define $\eta_c : C \to C_n(20)$ and show kernel $\eta_c \neq \{\phi\}$.

6. Prove $\{C_n(15), +\} = G$ is an abelian group of infinite order.

   (i) Find subgroups of G.

(ii) Can G have infinite order subgroups?
(iii) Can G have finite subgroups?

7. Let $\{C_n(29), +\}$ = S be the MOD complex plane group.

   Study questions (i) to (iii) of problem 6 for this S.

8. Let B = $\{C_n(24), +\}$ be the MOD complex group.

   Study questions (i) to (iii) of problem 6 for this B.

9. Let S = $\{C_n(12), \times\}$ be the MOD complex semigroup.

   (i) Find subsemigroups of S which are not ideals.
   (ii) Is S a S-semigroup?
   (iii) Find S-ideals if any in S.
   (iv) Can ideals of S be of infinite order?
   (v) Can S have S-units?
   (vi) Can S have idempotents which are not S-idempotents?
   (vii) Can S have zero divisors and S-zero divisors?

10. Let B = $\{C_n(19), \times\}$ be the MOD complex semigroup.

    Study questions (i) to (vii) of problem 9 for this B.

11. Let $S_1 = \{C_n(27), \times\}$ be the MOD complex semigroup.

    Study questions (i) to (vii) of problem 9 for this S_1.

12. Let M = $\{C_n(m), \times, +\}$ be the MOD complex pseudo ring.

    (i) Prove M is commutative.
    (ii) Can M have zero divisors?
    (iii) Can M have pseudo ideals?
    (iv) Can M have S-pseudo ideals?
    (v) Is every ideal of M is of infinite order?

(vi) Can M have subrings of finite order?
(vii) Is M a S-pseudo ring?
(viii) Can M have units?
(ix) Does every element in M invertible?
(x) Does M have only finite number of invertible elements?
(xi) Find any other special feature enjoyed by M.

(xii) Can M have idempotents which are not S-idempotents?

13. Let R = $\{C_n(29), \times, +\}$ be the MOD complex pseudo ring.

    Study questions (i) to (xii) of problem 12 for this R.

14. Let B = $\{C_n(28), \times, +\}$ be the MOD complex pseudo ring.

    Study questions (i) to (xii) of problem 12 for this B.

15. Let S = $\{C_n(64), \times, +\}$ be the MOD complex pseudo ring.

    Study questions (i) to (xii) of problem 12 for this S.

16. Let $C_n(m)[x]$ be the MOD complex plane.

    (i) Study all properties of $C_n(m)[x]$.
    (ii) Define η_c; $C[x] \to C_n(m)[x]$ so that η_c is a MOD complex transformation.

17. Let S $=\{C_n(m)[x], +\}$ be the MOD complex polynomial group.

    (i) Prove S is commutative.
    (ii) Prove $C_n(m) \subseteq S$.
    (iii) Can S have finite subgroups?
    (iv) Obtain any other special property enjoyed by S.

18. Let P = $\{C_n(24)[x], +\}$ be the MOD complex polynomial group.

    Study questions (i) to (iv) of problem 17 for this P.

19. Let B = $\{C_n(19)[x], +\}$ be the MOD complex polynomial group.

    Study questions (i) to (iv) of problem 17 for this B.

20. Let M = $\{C_n(625)[x], +\}$ be the MOD complex polynomial group.

    Study questions (i) to (iv) of problem 17 for this M.

21. Let S = $\{C_n(m)[x], \times\}$ be the MOD complex polynomial semigroup.

    (i) Is S a S-semigroup?
    (ii) Can S have S-ideals?
    (iii) Can S have S-units?
    (iv) Can S have units which are not S-units?
    (v) Can S have finite ideals?
    (vi) Can S have finite subsemigroups?
    (vii) Can S have idempotents?
    (viii) Can S have zero divisors?
    (ix) Obtain any other special property associated with S.

22. Let S = $\{C_n(12)[x], \times\}$ be the MOD complex polynomial semigroup.

    Study questions (i) to (ix) of problem 21 for this B.

23. Let S = $\{C_n(31)[x], \times\}$ be the MOD complex polynomial semigroup.

Study questions (i) to (ix) of problem 21 for this S.

24. Let T = $\{C_n(243)[x], \times\}$ be the MOD complex polynomial semigroup.

    Study questions (i) to (ix) of problem 21 for this T.

25. Let S = $\{C_n(m)[x], +, \times\}$ be the MOD complex polynomial ring.

    (i) Obtain some of the new properties associated with S.
    (ii) Can S be a S-pseudo ring?
    (iii) Can S have S-pseudo ideals?
    (iv) Can S have S-units?
    (v) Can S have zero divisors?
    (vi) Can S have infinite number of idempotents?
    (vii) Prove S has subrings of finite order.
    (viii) Can S have infinite number of units?
    (ix) Prove S has ideals of infinite order only.
    (x) Can S have infinite number of idempotents?
    (xi) Prove in S; $x \times (y + z) = x \times y + x \times z$ for some $x, y, z \in S$.

26. Let S_1 = $\{C_n(127)[x], +, \times\}$ be the MOD complex polynomial ring.

    Study questions (i) to (xi) of problem 25 for this S_1.

27. Let P = $\{C_n(48)[x], +, \times\}$ be the MOD complex polynomial ring.

    Study questions (i) to (xi) of problem 25 for this P.

28. Let M = $\{C_n(128)[x], +, \times\}$ be the MOD complex polynomial ring.

    Study questions (i) to (xi) of problem 25 for this M.

29. Obtain some ingenious methods of solving MOD complex polynomials in $C_n(20)[x]$.

30. Solve $p(x) = 0.39 + 4.28x^2 \in C_n(5)[x]$.

31. Solve $p(x) = 0.39 + 0.51x^2 + 6.7x \in C_n(7)[x]$.

32. Solve $p(x) = 5x^2 + 10x + 5 \in C_n(15)[x]$.

33. Prove if $p(x) \in C_n(81)[x]$ has a multiple root α then the derivative of p(x) may or may not have α to be a root of $p'(x)$ in general.

34. Prove for $(x + 0.3)^2\ (x + 6.5) \in C_n(10)[x]$,
$(x + 0.3)^2\ (x + 6.5) \neq (x^2 + 0.6x + 0.09) \times (x + 6.5)$.

35. Show because of the non distributivity of a triple in $C_n(m)[x]$;
α is a root of $p(x) = (x + \alpha_1) \ldots (x + \alpha_n)$ then α in general is not a root of $(x + \alpha_1) \ldots (x + \alpha_n)$ expanded by multiplying.

36. When can $p(x) \in C_n(m)[x]$ be integrated?

37. Show every $p(x) \in C_n(m)[x]$ need not have derivatives.

38. What are the problems associated with integral and differential of $p(x) \in C_n(m)[x]$?

39. Let $p(x) = 5x^2 + 0.3x + 5.3 \in C_n(6)[x]$.

    (i) Find $p'(x)$.
    (ii) Find $\int p(x)\, d(x)$.
    (iii) Find all the roots of p(x).
    (iv) Can p(x) have more then two roots?
    (v) $p_1(x) = 3.2x^6 + 4x^3 + 3x^2 + 0.37x + 0.7 \in C_n(6)[x]$.
    Study questions (i) to (iv) for this $p_1(x)$.
    (vi) How many times a nth degree polynomial in

$C_n(m)[x]$ be derived?

(vii) Show the difference between usual differentiation in $C[x]$ and in $C_n(m)[x]$.

40. Can we use any other modified form of differentiation and integration for $p(x) \in C_n(m)[x]$?

41. Let $p(x) = 14.3x^{30} + 3.3x^{15} + 7.5 \in C_n(30)[x]$.

    Find (i) $\frac{dp(x)}{dx}$.

    (ii) $\int p(x)\, dx$.

42. What are special features enjoyed by $C_n^d(m)[x]$?

43. Can $p(x) \in C_n^d(q)[x]$ be differentiable, where

    $p(x) = (3 + 5i_F)x^{9.1} + (2 + 0.3i_F)x^{2.5} + (0.31 + 0.94)x^{0.7} + (4.3 + 0.5i_F)$?

44. Find the integral of $p(x)$ given in problem 43.

45. Let $p(x) = 0.31x^{4.31} + (5.321 + 4i_F) + (3.2 + 0.6i_F)x^{3.12} + (0.33 + 6.71i_F)x^{2.1} \in C_n(7)[x]$.

    (i) Find $\frac{dp(x)}{dx(0.3)}$.

(ii) Find $\frac{dp(x)}{dx(6.3)}$; does the derivative exist?

(iii) Find $\int p(x)\,dx$ (0.3).

(iv) $\int p(x)\,dx$ (6.3).

46. Let $\{C_n(10)[x], +, \times\} = S$ be the MOD complex pseudo polynomial ring.

    (i) Prove S has ideal.
    (ii) Can S have S-ideals?
    (iii) Can S has S-zero divisors?
    (iv) Can S have S-units?
    (v) Can S have idempotents?
    (vi) Is S a S-pseudo ring?
    (vii) Can S have S-sub pseudo rings which are not S-ideals?
    (viii) Can S have finite S-ideals?
    (ix) Find all units of S.
    (x) Obtain any other special feature enjoyed by S.

47. Let $B = \{C_n(29)[x], +, \times\}$ be the MOD complex polynomial pseudo ring.

    (i) Study questions (i) to (x) of problem 46.
    (ii) Find 3 polynomials in B which does not have derivatives.
    (iii) Give 3 polynomials in B which has derivatives.
    (iv) Give 2 polynomials in B which is not integrable.
    (v) Give 2 polynomials in B which are integrable.

48. Let $P = \{C_n(24)[x], +, \times\}$ be the MOD complex polynomial pseudo ring.

    Study questions (i) to (v) of problem 47 for this P.

49. Let P = $\{C_n(4)[x], +, \times\}$ be the MOD complex polynomial pseudo ring.

    For $p(x) = (0.3 + 0.4i_F)x^8 + (0.7 + 2.1i_F)x^4 + (0.715 + 0.4i_F)x^{12} \in C_n(4)[x]$.

    (i) Find $\frac{dp(x)}{dx}$ and $\int p(x)\, dx$.

    (ii) Study questions (i) to (v) of problem 47 for this P.

50. Obtain the special features enjoyed by $C_n^d(m)[x]$.

51. Show all $p(x) \in C_n^d(m)[x]$ cannot be integrated.

52. Show B = $\{C_n^d(m)[x], \times, +\}$ is only a pseudo ring of MOD complex polynomials.

53. Let B = $\{C_n^d(24)[x], +, \times\}$ be the pseudo MOD decimal polynomial complex ring.

    (i) Show B has zero divisors.
    (ii) Can B have S-zero divisors?
    (iii) Find units B which are not S-units.
    (iv) Can B be a S-pseudo ring?
    (v) Can B have ideals of finite order?
    (vi) Does B have S-ideals?
    (vii) Find all finite subrings of B.
    (viii) Show there are subrings of infinite order in B which are not ideals.
    (ix) Find some special features associated with B.
    (x) Show B can have finite number of units.
    (xi) Show B cannot have infinite number of idempotents.

54. Let $C = \{C_n^d(128)[x], +, \times\}$ be the MOD complex decimal polynomial pseudo ring.

    Study questions (i) to (xi) of problem 53 for this C.

55. Let $p(x) = (0.25 + 100i_F)\, x^{13.9} + (4.3 + 25i_F)x^{3.9} + (2.7 + 3i_F) \in C$ (C given in problem 54).

    (i) Find $\frac{dp(x)}{dx(0.3)}$.

    (ii) How many times p(x) should be derived so that
    $$\frac{d^t p(x)}{dx(0.3)^t} = 0?$$

    (iii) Find $\int p(x)\, dx(0.3)$.

Chapter Six

MOD DUAL NUMBER PLANES

In this chapter we introduce the notion of MOD dual number plane. In [16] has introduced several properties about dual numbers.

Recall $R(g) = \{ a + bg / a, b$ are reals in R and g is such that $g^2 = 0 \}$ is defined as dual numbers.

Clearly R(g) is of infinite order we can give the dual numbers the plane representation. The associated graph is given in Figure 6.1.

This is defined as the real dual number plane. Several properties about these planes have been systematically carried out in [16].

Here we at the outset claim, we have infinite number of MOD dual number planes for a given 'g'.

Let $R_n(m)$ denote the usual MOD real plane associated with $[0, m)$.

$R_n(m)[g] = \{ a + bg \mid a, b \in [0, m)$ with $g^2 = 0\}$.

We define $R_n(m)[g]$ as the MOD dual number plane associated with $[0, m)$.

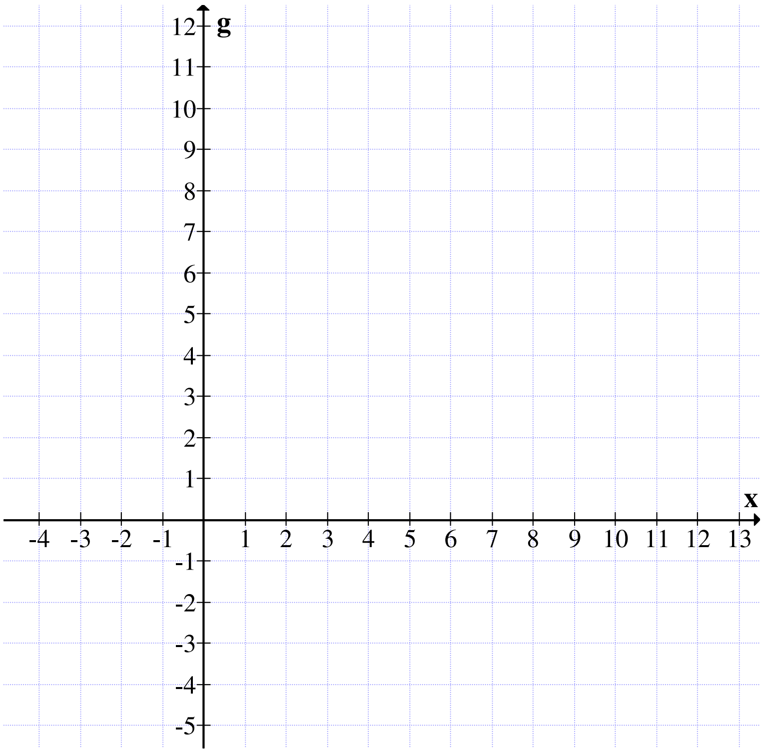

Figure 6.1

We will give examples of MOD dual number planes before we proceed on to define the concept of MOD dual number transformation.

Example 6.1: Let $R_n(3)$ [g] = { a + bg | a, b ∈ [0,3), $g^2 = 0$} be the MOD dual number plane associated with the MOD interval [0, 3). The associated graph is given in Figure 6.2.

0.5 + 1.5g in $R_n(3)$[g] is marked in the MOD dual number plane.

Example 6.2: Let $R^n(10)$[g] = { a + bg | a, b ∈ [0,10), $g^2 = 0$} be the MOD dual number plane. The associated graph is given in Figure 6.3.

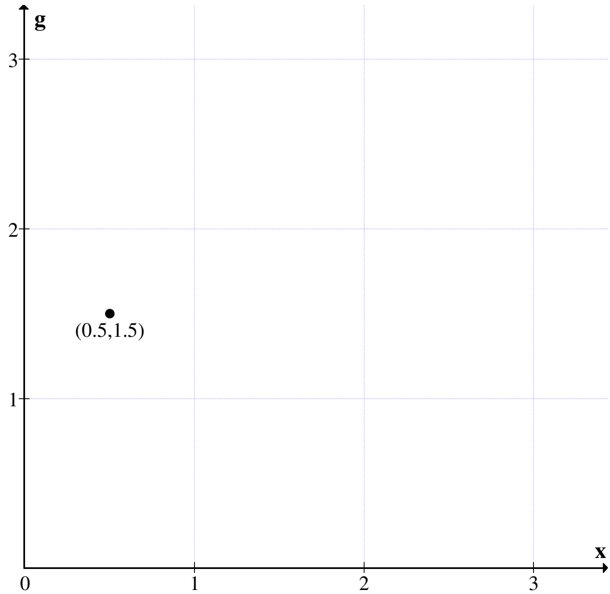

Figure 6.2

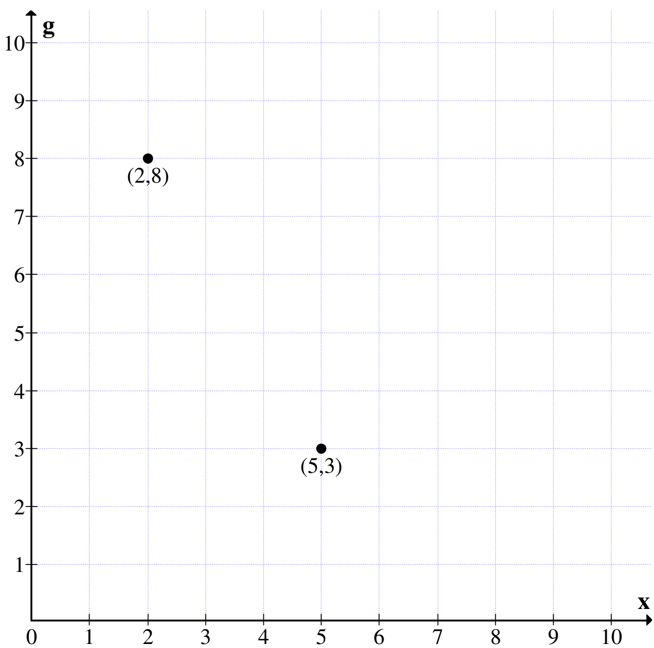

Figure 6.3

Let $5 + 3g \in R_n(10)$ [g]; it is represented in the MOD dual number plane as above.

Let $2 + 8g \in R_n(10)$ [g] it is represented as above.

This is the way MOD dual number are represented in the MOD dual number plane.

The elements $a + bg \in R_n(m)[g]$ are known as MOD dual numbers lying in the MOD dual number plane $R_n(m)[g]$.

Example 6.3: Let $R_n(7)$ [g] = $\{a + bg \mid a, b \in [0,7), g^2 = 0\}$ be the MOD dual number plane.

Any MOD dual numbers 3 = 4.1g and 2.5 + 2.6g $\in R_n(7)$ [g] are given the following representation in the MOD dual number plane.

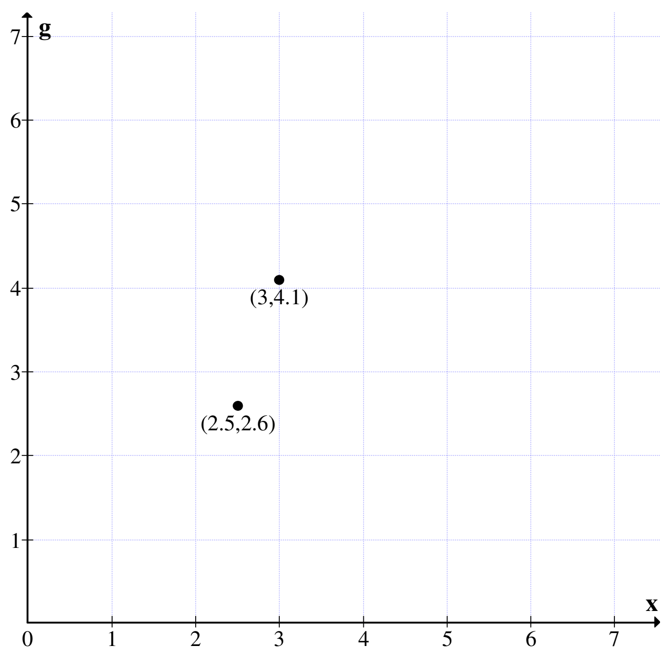

Figure 6.4

Example 6.4: Let $R_n(8)[g] = \{a + bg \mid a, b \in [0,8),\ g^2 = 0\}$ be the MOD dual number plane. Any MOD dual number in $R_n(8)[g]$ can be represented in the MOD dual number plane.

We have seen the concept of MOD dual number planes and the representation of MOD dual numbers in that plane.

Now we see for a given g with $g^2 = 0$ we can have one and only one real dual number plane but however MOD dual number planes are infinite in number for a given g with $g^2 = 0$.

For $R(g) = \{ a + bg \mid a, b \in R,\ g^2 = 0\}$ for the same g we have $R_n(m)[g] = \{ a + bg \mid a, b \in [0, m)\ \ g^2 = 0\}$ for varying finite integer $m \in Z^+ \setminus \{0\}$.

Thus we have infinite number of MOD dual number planes for a given g.

We can define $\eta_d : R(g) \to R_n(m)[g]$ by

$$\eta_d(a+bg) = \begin{cases} g \text{ if } a = 0 \text{ and } b = 1. \\ a + bg \text{ if both a and b are less than m.} \\ r + bg \text{ where } b<m \text{ and } \dfrac{a}{m} = d + \dfrac{r}{m} \\ a + sg \text{ when } a < m \text{ and } \dfrac{b}{m} = d + \dfrac{s}{m} \\ r + sg \text{ when } \dfrac{a}{m} = d + \dfrac{r}{m} \text{ and } \dfrac{b}{m} = e + \dfrac{s}{m} \\ = 0 \text{ if } a = mr,\ b = ms \\ = 0 + rg \text{ if } a = mt \text{ and } \dfrac{b}{m} = d + \dfrac{r}{m} \\ = s + 0 \text{ if } b = mt\ \dfrac{a}{m} = e + \dfrac{s}{m} \\ \text{when negative appropriate modification is made} \end{cases}$$

This map η_d is defined as the MOD dual number transformation.

We will illustrated this situation by some examples.

Example 6.5**:** Let $R_n(6)(g)$ be the MOD dual number plane. We will

define $\eta_d : R(g) \rightarrow R_n(6)(g)$

Let $24.3 + 8.7\,g \in R(g)$

$\eta_d\,(24.3 + 8.7g) = (0.3 + 2.7g) \in R_n(6)(g)$.

$$\begin{aligned}\eta_d\,(-40{:}4 + 17.5g) &= (6 - 4.4 + 5.5g)\\ &= (1.6 + 5.5g) \in R_n(6)[g].\end{aligned}$$

$\eta_d\,(13.93 + 18g) = (1.93 + 0) \in R_n(6)[g]$.

$\eta_d\,(56.31 - 14.7g) = 2.31 + 3.3g \in R_n(6)[g]$.

$\eta_d\,(12 + 24g) = 0$.

$\eta_d\,(48 - 15.3g) = 2.7g$ and so on.

This is the way operations are performed on $R_n(6)[g]$.

As in case of $R_n(m)$ or $C_n(m)$ or $R_n(m)[x]$ or $C_n(m)[x]$ we can show $R_n(m)[g]$ the MOD dual numbers plane is a group under addition +.

We will just illustrate this by an example or two.

Example 6.6: Let $R_n(5)(g) = \{a + bg \mid a, b \in [0, 5), g^2 = 0\}$ be the MOD dual number plane. $S = \{R_n(5)(g), +\}$ is a group.

For let $x = 3.001 + 0.2g$ and $y = 2.052 + 3.019g \in S$

$x + y = (3.001 + 0.2g) + (2.052 + 3.019g)$

$= 0.053 + 3.219g \in S.$

This is the way operation + is performed on S.

We see 0 acts as the additive identity of S.

For every $p \in R_n(5)(g)$ we have a unique $q \in R_n(5)(g)$ such that $p + q = 0$.

Let $p = 2.53 + 0.87g \in R_n(5)(g)$

then $q = 2.47 + 4.13g$ in $R_n(5)$ (g) is such that $p + q = 0$.

Thus we see $R_n(5)(g)$ is an abelian group under +.

$R_n(m)(g) = \{a + bg \mid a, b \in [0, m), g^2 = 0\}$ under + is a group defined as the MOD dual number group.

We will give some examples of them.

Example 6.7: Let $G = \{R_n(20)(g), +\}$ be the MOD dual number group of infinite order.

Example 6.8: Let $G = \{R_n(37)(g), +\}$ be the MOD dual number group of infinite order.

Example 6.9: Let $G = \{R_n(128)(g), +\}$ be the MOD dual number group.

Now we can define product on MOD dual number plane $R_n(m)(g)$.

$\{R_n(m)(g), \times\}$ is a semigroup which has infinite number of zero divisors, finite number of units and idempotents.

$S = \{R_n(m)g, \times, g^2 = 0\}$ is defined as the MOD dual number semigroup.

We will examples of them.

Example 6.10: Let G = $\{R_n(5)(g), \times\}$ be a MOD dual number semigroup.

Example 6.11: Let S = $\{R_n(15)(g), \times\}$ be the MOD dual number semigroup. S has zero divisors and units.

Example 6.12: Let T = $\{R_n(140)(g), \times\}$ be the MOD dual number semigroup.

Now having seen the definition and examples it is a matter of routine for the reader to study all the related properties.

Now $\{R_n(m)(g), +, \times\}$ is defined as the MOD dual number pseudo ring as in this case also the distributive law is not true in general.

We will give examples of them.

Example 6.13: Let S = $\{R_n(10)(g), +, \times\}$ be the MOD dual number pseudo ring. S has zero divisors, units and idempotents.

Example 6.14: Let S = $\{R_n(43)(g), +, \times\}$ be the MOD dual number pseudo ring. S has infinite number of zero divisors but only finite number of units and idempotents.

Example 6.15: Let S = $\{R_n(128)(g), +, \times\}$ be the MOD dual number pseudo ring. S has units, zero divisors and idempotents.

Now we proceed onto study the MOD dual number pseudo polynomial ring.

P =$\{R_n(m)(g)[x] = \{\sum a_i x^i \mid a_i \in R_n(m)(g)\}$ is defined as the MOD dual number polynomials.

Now it is a matter of routine to check $\{R_n(m)(g)[x], +\}$ is defined as the MOD dual number polynomial group.

We will first give one or two examples of them.

Example 6.16: Let S = {$R_n(9)(g)[x]$, +} be the MOD dual number polynomial group under +.

Example 6.17: Let S = {$R_n(12)(g)[x]$, +} be the MOD dual number polynomial group. This has subgroups of both finite and infinite order.

Example 6.18: Let S = {$R_n(37)(g)[x]$, +} be the MOD dual number polynomial group.

Now we can define the product operation on the MOD dual number polynomials {$R_n(m)(g)[x]$, ×} = S is defined as the MOD dual number polynomial semigroup.

The study of properties of S is a matter of routine and is left as an exercise to the reader.

Example 6.19: Let S = {$R_n(3)(g)[x]$, ×} be the MOD dual number polynomial semigroup which has infinite number of zero divisors.

Example 6.20: Let P = {$R_n(12)(g)[x]$, ×} be the MOD dual number polynomial semigroup.

P has idempotents and units only finite in number.

Example 6.21: Let B = {$R_n(243)(g)[x]$, ×} be the MOD dual number polynomial semigroup. B has infinite number of zero divisors.

Study of ideals, subsemigroups etc is considered as a matter of routine.

We now proceed onto study MOD dual number polynomial pseudo ring.

We define R ={$R_n(m)(g)[x]$, +, ×} to be the MOD dual number polynomial pseudo ring. R has units zero divisors, idempotents, subrings and ideals. This is left as a matter of routine.

However we will provide examples of them.

Example 6.22: Let M = $\{R_n(20)(g)[x], +, \times\}$ be the MOD dual number polynomial pseudo ring. M has infinite number of zero divisors. All ideas of M are of infinite order.

Example 6.23: Let B = $\{R_n(4)(g)[x], +, \times\}$ be the MOD dual number polynomial pseudo ring.

Let $p(x) = (3 + 0.2g)x^7 + (0.4 + 2g)x^3 + (0.8 + g)$ and

$q(x) = (0.4 + 2g)x^2 + (3.2 + 0.7g) \in B$.

$$p(x) \times q(x) = [(3 + 0.2g)x^7 + (0.4 + 2g)x^3 + (0.8 + g)] \times [(0.4 + 2g)x^2 + (3.2 + 0.7g)]$$

$$= (1.2 + 0.08g + 2g)x^9 + (0.16 + 0.8g + 0 + 0.8g)x^5 + (0.32 + 1.6g + 0.4g)x^2 + (1.6 + 2.1g + 0.64g)\, x^7 + (1.28 + 2.4g + 0.28g)\, x^3 + (2.56 + 0.56g + 3.2g)$$

$$= (1.2 + 2.08g)x^9 + (0.16 + 0.16g)x^5 + (0.32 + 2g)\, x^2 + (1.6 + 2.74g)x^7 + (1.28 + 2.68g)x^3 + (2.56 + 3.76g) \in B.$$

This is the way product is defined on these pseudo dual number polynomials.

We will now proceed onto show how integration and differentiation is possible on $p(x) \in R_n(m)[x]$ only if defined.

Let

$p(x) = (0.7 + 2g)x^3 + (4 + 0.8g)x^2 + (0.9 + 0.5g) \in R_n(6)(g)\,[x]$

$$\frac{dp(x)}{dx} = 3(0.7 + 2g)x^2 + 2(4 + 0.8g)x + 0$$

$$= (2.1 + 0)x^2 + (2 + 1.6g)\ x.$$

$$\int p(x)\ dx = \frac{(0.7+2g)x^4}{4} + \frac{(4+0.8g)x^3}{3} + (0.9 + 0.5g)x + c.$$

Clearly as $\frac{1}{4}$ and $\frac{1}{3}$ are not defined this integral is undefined.

This sort of study is considered as a matter of routine and is left as an exercise to the reader.

Next we proceed onto define MOD special dual like number planes.

Recall a + bg where a and b are reals with $g^2 = g$ is defined as the special dual like numbers.

We know $R(g) = \{a + bg \mid a, b \in R(\text{reals})\ g^2 = g\}$ is defined as the special dual like number collection.

For more about this please refer [17].

$R_n(m)(g) = \{a + bg \mid a, b \in [0, m), g^2 = g\}$ is defined as the MOD special dual like number plane.

We will illustrate this situation by some examples.

Example 6.24: Let $R_n(4)(g) = \{a + bg \mid a, b \in [0, 4), g^2 = g\}$ be the MOD special dual like number plane. The associated graph is given in Figure 6.5.

We call the elements of $R_n(4)(g)$ as MOD special dual like numbers.

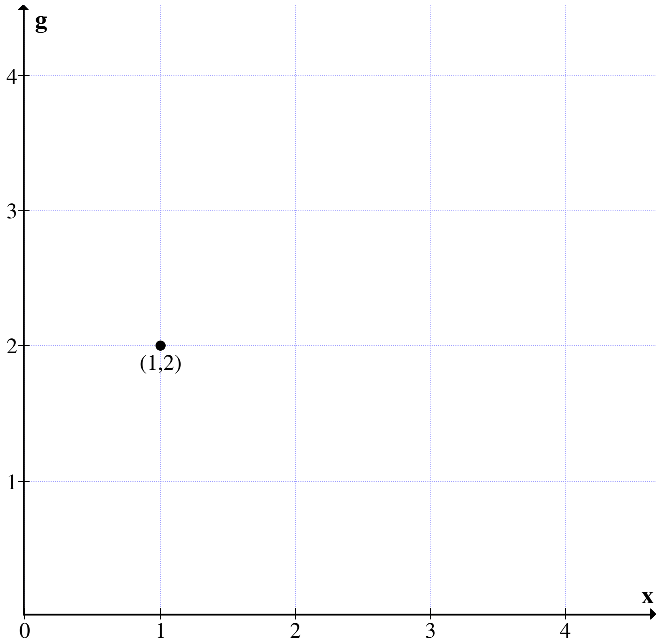

Figure 6.5

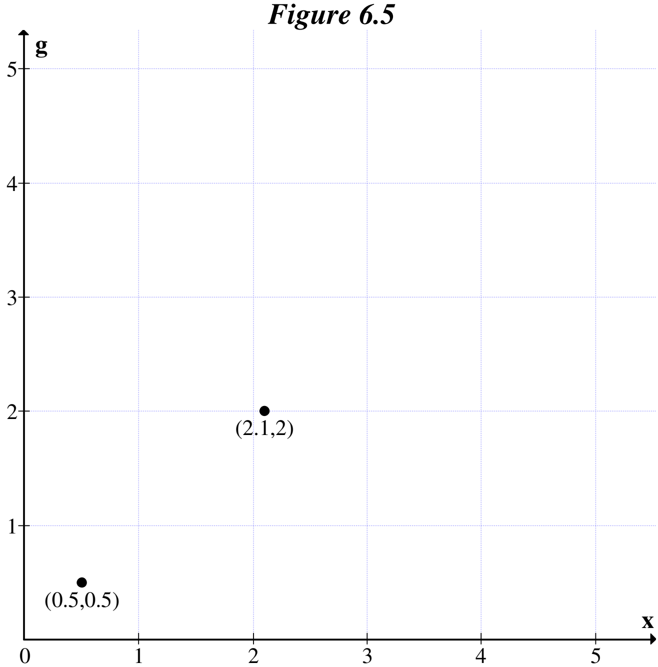

Figure 6.6

Example 6.25: Let S = $\{R_n(5)(g)\} = \{a + bg \mid a, b \in [0, 5)$ and $g^2 = g\}$ be the MOD special dual like number plane.

0.5 + 0.5g and 2.1 + 2g are known as MOD special dual like numbers of S. The associated graph is given in Figure 6.6.

We will show how product and sum are defined on MOD special dual like numbers.

Example 6.26: Let S = $\{R_n(10)(g)\} = \{a + bg \mid a, b \in [0, 10)$ and $g^2 = g\}$ be the MOD special dual like numbers.

Let x = 8.3 +0.74g and y = 5.72 +6.261g $\in$ S.

x + y = (8.3 + 0.74g) + (5.72 + 6.261g)

= (4.02 + 7.001g) $\in$ S.

It is easily verified (S, +) is a group under '+', 0 acts as the additive identity. Further for every x in S we have a unique y $\in$ S such that

x + y = y + x = 0 y is the inverse of x and vice versa.

Let x = 0.89 + 3.731g $\in$ S.

y = 9.11 + 6.269g $\in$ S is such that x + y = 0 and y is the inverse of x and x is the inverse of y.

Thus $\{R_n(m)(g) \mid g^2 = g, +\}$ is defined as the MOD special dual like number group.

We have subgroups of both finite and infinite order.

We will illustrate this situation by some examples.

Example 6.27: Let S = $\{R_n(23)(g) \mid g^2 = g, +\}$ be the MOD special dual like number group.

Example 6.28: Let $S = \{R_n(24)(g) \mid g^2 = g, +\}$ be the MOD special dual like number group.

Let us define product on $R_n(m)(g)$.

For $x = 10 + 0.8g$ and

$y = 4.6 + 2.8g \in R_n(24)(g)$;

we find $x \times y = (10 + 0.8g)\ (4.6 + 2.8g)$

$= 22+3.68g + 4g + 2.04g$

$= 22 + 9.72g \in R_n(24)(g)$.

We see $\{R_n(m)(g), \times\}$ is a semigroup defined as the MOD special dual like number semigroup.

We will give a few examples of them.

Example 6.29: Let $S = \{R_n(47)(g) \mid g^2 = g, \times\}$ be the MOD special dual like number semigroup.

Example 6.30: Let $S = \{R_n(48)(g) \mid g^2 = g, \times\}$ be the MOD special dual like number semigroup.

We see these semigroups have units, zero divisors, idempotents subsemigroups and ideals.

Study in this direction is a matter of routine hence left as an exercise to the reader.

Now we define $S = \{R_n(m)(g), +, \times\}$ to be the MOD special dual like number pseudo ring.

Cleary S is of infinite order we call it pseudo as
$a \times (b + c) = (a \times b + a \times c)$ for $a, b, c \in R$.

We will show this by examples.

Example 6.31: Let R = $\{R_n(6)(g), +, \times\}$ be the MOD special dual like number pseudo ring.

Let $x = 2.4 + 3.2g$ $y = 0.5 + 2.8g$ and

$z = (5.5 + 3.2g) \in R$.

$x \times (y + z) = (2.4 + 3.2g) \times [0.5 + 2.8g + 5.5 + 3.2g]$

$= 0$ --- I

Consider $x \times y + x \times z$

$= (2.4 + 3.2g) \times (0.5 + 2.8g) + (2.4 + 3.2g) \times (5.5 + 3.2g)$

$= (1.20 + 1.60g + 0.72g + 2.96g) + (1.2 + 5.6g + 4.24g + 1.68g)$

$= (1.2 + 5.28g) + (1.2 + 5.52g)$

$= 2.4 + 4.8g$ --- II

Clearly I and II are distinct that is why we call R to be the MOD special dual like number pseudo ring.

We will give an example or two of this.

Example 6.32: Let S = $\{R_n(17)(g) \mid g^2 = g, +, \times\}$ be the MOD special dual like number pseudo ring.

Example 6.33: Let M = $\{R_n(28)(g) \mid g^2 = g, +, \times\}$ be the MOD special dual like number pseudo ring.

M has units, zero divisors, idempotents, subrings and ideals.

This study is considered as a matter of routine and hence left as an exercise to the reader.

Next we proceed onto study the MOD special dual like number polynomials.

$T = \{R_n(m)(g)[x] = \{\sum a_i x^i \mid a_i \in R_n(m)(g); g^2 = g\}$ is defined as the MOD special dual like number polynomials.

We define + and × operation on T.

Infact T under + operation is a group and under × operation is a semigroup.

We will just give some examples for all these work is considered as a matter of routine.

Example 6.34: Let $S = \{R_n(6)(g)[x], +\}$ be the MOD special dual like number polynomial group.

Example 6.35: Let $S = \{R_n(11)(g)[x]\}$ be the MOD special dual like number polynomial group.

We define $\{R_n(m)(g) [x], +\} = R$ to be the MOD special dual like number polynomial group.

Now $\{R_n(m)(g)[x], \times\} = S$ is defined as the MOD special dual like number polynomial semigroup.

We give examples of them.

Example 6.36: Let $S = \{R_n(20)(g)[x] \mid g^2 = g, \times\}$ be the MOD special dual like number polynomial semigroup.

Let $p(x) = (6 + 0.8g)x^5 + (10 + 8.1g)x + (5 + 7g)$ and

$q(x) = (0.6 + 5g)x^2 + (0.7 + 0.2g) \in S$.

$p(x) \times q(x) = [(6 + 0.8g)x^5 + (10 + 8.1g)x + (5 + 7g)] \times [(0.6 + 5g)x^2 + (0.7 + 0.2g)]$

$= (3.6 + 0.48g + 10g + 4g)\, x^7 + (6 + 8.6g + 10g + 0.5g)x^6 + (3 + 4.2g + 5g + 15g)x^2 + (4.2 + 0.56g + 1.2g + 0.16g)\, x^5 + (7 + 5.67g + 2g + 16.2g)x + (3.5 + 4.9g + g + 1.4g)$

$= (3.6 + 14.48g)x^7 + (6 + 19.1g)x^6 + (3 + 4.2g)x^2 + (4.2 + 1.92g)x^5 + (7 + 3.87g)x + (3.5 + 6.3g) \in S.$

This is the way the product operation is performed on S.

Example 6.37: Let P = $\{R_n(43)(g)[x], \times\}$ be the MOD special dual like number semigroup.

Now having seen the form of MOD special dual like number group and semigroup we proceed onto define, describe and develop the notion of MOD special dual like number pseudo rings.

S = $\{R_n(m)(g)[x], +, \times\}$ is defined as the MOD special dual like number polynomial pseudo ring.

We call it pseudo ring as for p(x), q(x) and r(x) $\in$ S we see

$p(x) \times (q(x) + r(x)) \neq p(x) \times q(x) + p(x) \times r(x).$

We will give one or two examples before we proceed to define MOD special quasi dual numbers plane.

Example 6.38: Let S = $\{R_n(29)(g)[x], +, \times, g^2 = g\}$ be the MOD special dual like number pseudo polynomial ring.

S has zero divisors, units, idempotents ideals and subrings.

Study of this type is considered as a matter of routine and is left as an exercise to the reader.

Example 6.39: Let S = $\{R_n(48)(g)[x], +, \times, g^2 = g\}$ be the MOD special dual like number polynomial ring.

Now we can define differentiation integration on $p(x) \in S$.

Let $p(x) = (8 + 10g)x^7 + (4 + 0.5g)x^5 + (0.3 + 0.4g) \in S$.

$$\frac{dp(x)}{dx} = 7\,(8+10g)x^6 + 5\,(4+0.5g)x^4.$$

$$\int p(x)\,dx = \frac{(8+10g)x^8}{8} + \frac{(4+0.5g)x^6}{6} + (0.3 + 0.4g)x + C.$$

We can have derivatives or integrals to be defined or at times not defined in S. This is the marked difference between the usual polynomials in R[x] and MOD polynomials in $R_n(m)(g)[x]$.

Now we proceed onto define the MOD special quasi dual number plane.

Recall $R(g) = \{a + bg \mid a, b \in R \text{ (reals) } g^2 = -g\}$ is defined as the special quasi dual numbers.

We define

$R_n(m)(g) = \{a + bg \mid a, b \in [0, m) \text{ with } g^2 = (m-1)g\}$ to be the MOD special quasi dual numbers plane.

We will give examples of them and describe the plane. Further we can have the η_q to be MOD special quasi dual number transformation.

Example 6.40: Let $R_n(3)(g) = \{a + bg \mid a, b \in [0, 3), g^2 = 2g\}$ be the MOD special quasi dual number plane.

The associated graph is given in Figure 6.7.

$$2.1 + 0.5g = x \in R_n(3)(g).$$

a + bg are called MOD special quasi dual numbers.

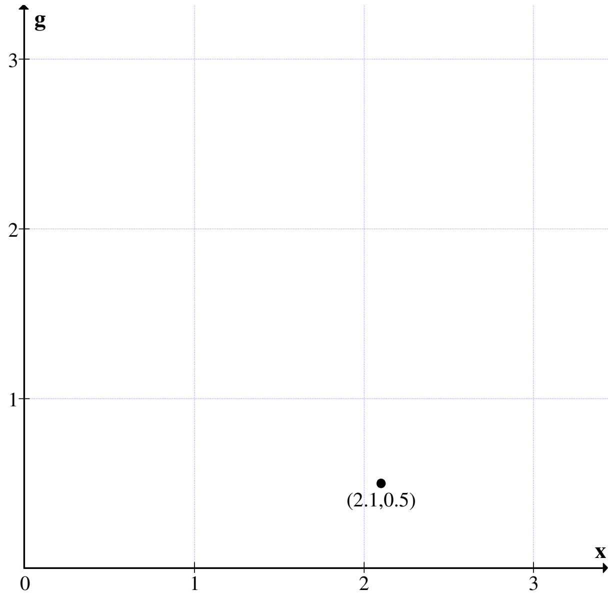

Figure 6.7

For η_q : R(g) → $R_n(m)g$ defined similar to η_d;
$\eta_q(g) = g$.

This is the way the mapping is done.

We can as in case of other MOD dual numbers define + and × on $\{R_n(m)g \mid g^2 = (m-1)g\}$

$\{R_n(m)(g), +, g^2 = (m-1)g\}$ is a group defined as the MOD special quasi dual number group.

We will give examples of them.

Example 6.41: Let S = $\{R_n(7)(g) \mid g^2 = 6g, +\}$ be the MOD special quasi dual number.

Example 6.42: Let M = $\{R_n(12)(g) \mid g^2 = 11g, +\}$ be a MOD special quasi dual number group.

Example 6.43: Let B = $\{R_n(256)(g) \mid g^2 = 255g, +\}$ be the MOD special quasi dual number group.

Now we can define product on $R_n(m)(g)$ with $g^2 = (m-1)g$ as follows:

Let us consider $R_n(8)$ g with $g^2 = 7g$.

Let $x = 3.1 + 4.5g$ and

$y = 0.8 + 5g \in R_n(8)(g)$.

$$x \times y = (3.1 + 4.5g) \times (0.8 + 5g)$$

$$= 2.48 + 4g + 7.5g + 6.5 \times 7g$$
$$= 2.48 + 3.5g + 5.5g$$

$$= 2.4g + g \in R_n(8)(g).$$

This is the way product is defined on $R_n(8)$ (g).

Cleary $R_n(m)(g)$ with $g^2 = (m-1)g$ is a semigroup under product.

Example 6.44: Let B = $\{R_n(9)(g) \mid g^2 = 8g, \times\}$ be the MOD special quasi dual number semigroup.

Example 6.45: Let B = $\{R_n(23)(g) \mid g^2 = 22g, \times\}$ be the MOD special quasi dual number semigroup.

All properties of this semigroup is considered as a matter routine and hence left as an exercise to the reader.

Now $\{R_n(m)(g)$ with $g^2 = (m-1)g, +, \times\}$ is defined as the MOD special quasi dual number pseudo ring.

We say it is only a pseudo ring as for any
$x, y, z \in R_n(m)(g)$;

$x \times (y + z) \neq x \times y + x + z$.

We will give one or two examples of them.

Example 6.46: Let $S = \{R_n(93)(g) \mid g^2 = 92g, +, \times\}$ be the MOD special quasi dual number pseudo ring.

S has units, zero divisors, idempotents, subrings and ideals.

This work is a matter of routine and is left as an exercise to the reader.

Example 6.47: Let $S = \{R_n(19)(g) \mid g^2 = 18g, +, \times\}$ be the MOD special quasi dual number pseudo ring.

Example 6.48: Let $B = \{R_n(128)(g) \mid g^2 = 127g, +, \times\}$ be the MOD special quasi dual number pseudo ring.

Now having seen examples them we proceed to define MOD special quasi dual number polynomials and the algebraic structures on them.

Let $R_n(m)(g)[x] = \{\sum a_i x^i \mid a_i \in R_n(m)g;\ g^2 = (m-1)g\}$ is defined as the MOD special quasi dual number polynomials.

Example 6.49: Let $S = \{R_n(17)(g)[x] \mid g^2 = 16g\}$ be the MOD special quasi dual number polynomials.

Example 6.50: Let $B = \{R_n(44)(g)[x] \mid g^2 = 43g\}$ be the MOD special quasi dual number polynomials.

We can define the operation $+$, $\times$ and $(+, \times\}$ on $R_n(m)(g)[x]$. This is also considered as a matter of routine.

Example 6.51: Let $B = \{R_n(10)(g)[x] \mid g^2 = 9g, +\}$ be the MOD special quasi dual number polynomial group.

Example 6.52: Let $B = \{R_n(43)(g)[x] \mid g^2 = 42g, +\}$ be the MOD special quasi dual number polynomial group.

Example 6.53: Let M = $\{R_n(24)(g)[x] \mid g^2 = 23g, +\}$ be the MOD special quasi dual number polynomial group.

Example 6.54: Let B = $\{R_n(19)(g)[x] \mid g^2 = 18g, \times\}$ be the MOD special quasi dual number polynomial semigroup.

B has zero divisors, units idempotents, ideals and subsemigroups.

Example 6.55: Let T = $\{R_n(25)(g)[x] \mid g^2 = 24g, \times\}$ be the MOD special quasi dual number polynomial semigroup.

S = $\{R_n(m)(g)[x], +, \times, g^2 = (m-1)g\}$ is defined as the MOD special quasi dual number polynomial pseudo ring.

It is a matter of routine to check in general

$p(x) \times q(x) + r(x)) \neq p(x) \times q(x) + p(x) \times r(x)$ for $p(x), q(x), r(x) \in S$.

We will give one or two examples of them.

Example 6.56: Let S = $\{R_n(19)(g)[x] \mid g^2 = 18g, +, \times\}$ be the MOD special quasi dual number polynomial pseudo ring.

S has units, zero divisors, idempotents, ideals and subrings.

Study in this direction is a matter of routine and left as an exercise to the reader.

We can integrate or differentiate any $p(x) \in S$ if and only if it is defined otherwise it may not be possible.

We will just give one or two illustrates is differentiation and integration.

Example 6.57: Let R = $\{R_n(7)(g)[x] \mid g^2 = 6g, +, \times\}$ be the MOD special quasi dual number polynomial pseudo ring.

Let $p(x) = (5 + 3g)x^8 + (0.8 + 2.5g)\ x^4 + (6.2+5.7g)$ belong to $R_n(7)(g)[x]$.

$$\frac{dp(x)}{dx} = 8(5 + 3g)x^7 + 4(0.8 + 2.5g)x^3$$

$$= (5 + 3g)x^7 + (3.2 + 3g)x^3.$$

For this p(x) the derivative exist.

Consider $\int p(x)dx = \frac{(5+3g)x^9}{9} + \frac{(0.8+2.5g)x^5}{5} +$

$$(6.2 + 5.7g)x + c$$

$$= \frac{(5+3g)x^9}{2} + \frac{(0.8+2.5g)x^5}{5} + (6.2 + 5.7g)\ x + c$$

$$= 4(5 + 3g)x^9 + 3(0.8 + 2.5g)\ x^5 + (6.2 + 5.7g)x + c$$

$$= (6 + 5g)x^9 + (2.4 + 0.5g)x^5 + (6.2 + 5.7g)x + c.$$

This is the way both integration and differentiation is carried out. The integration and differentiation may not exist for all $p(x) \in R$.

Let $p(x) = (5 + 5g)x^7 \in R$.

$$\frac{dp(x)}{dx} = 0.$$

However $\int p(x)\ dx$

$$= \int (5 + 5g)x^7 + (4.37 + 2.56g)$$

$$\frac{(5+5g)x^8}{8} + (4.37 + 256g)\ x = c$$

Thus the integral exists.

However $p(x) = (3 + 0.3g)x^6 + (0.8 + 4g)x^{13} \in R_n(7)(g)[x]$

$\int p(x)\, dx$ is not defined in this case.

Example 6.58: Let $R = \{R_n(19)(g)[x], g^2 = 18g, +, \times\}$ be the MOD special quasi dual number polynomial pseudo ring.

Let $p(x) = (18.1 + 3.5g)\, x^{18} + (10g + 3.7)x^{37} \in R$.

The integral of p(x) does not exist.

The derivative exist in this case.

Example 6.59: Let $R = \{R_n(4)(g)[x], g^2 = 3g, +, \times\}$ be the MOD special quasi dual number polynomial pseudo ring.

Let $p(x) = (2 + 2g)x^{20} + 2x^{15} + 2g$

and $q(x) = (2 + 2g) + 2x^{10} + 2gx^{15} \in R$.

Clearly $p(x) \times q(x) = 0$.

$$\int p(x)\, dx = \frac{(2+2g)x^{21}}{21} + \frac{2x^{16}}{16} + 2gx + c$$

is not defined.

$$\int q(x)\, dx = (2 + 2g)x + \frac{2\times x^{11}}{11} + \frac{2gx^{16}}{16} + c$$

this is also not defined as $16 \equiv 0 \pmod 4$.

$$\frac{dp(x)}{dx} = 20\,(2 + 2g)x^{19} + 15 \times 2x^{14}$$

$$= 2x^{14}.$$

$$\frac{d^2p(x)}{dx^2} = 2 \times 14x^{13} = 0.$$

Thus we see a 20^{th} degree MOD special quasi dual number polynomial in R is such that its second derivative itself is zero.

Thus we see in case of MOD special quasi dual numbers polynomials a nth degree polynomial in general need not have all the n derivatives to be non zero. In some cases even the first derivatives may be zero.

In some case the second derivatives may be zero and so on.

Thus all the classical properties of calculus may not in general be true in case of MOD special quasi dual polynomials.

Likewise the integrals may not be defined for every MOD special quasi dual number polynomials.

Example 6.60: Let S = $\{R_n(6)\ (g)\ [x],\ g^2 = 5g, +, \times\}$ be the MOD special quasi dual number polynomial pseudo ring. S has infinite number of zero divisors.

If $p(x) = x^{12} + x^6 + (3 + 4g)$ then $\frac{dp(x)}{dx} = 0$.

$$\int p(x)\ dx = \frac{x^{13}}{13} + \frac{x^7}{7} + (3 + 4g)x + c$$

$$= x^{13} + x^7 + (3+ 4g)x + c.$$

We see the basic or classical property integration is the reverse process of differentiation is not true.

$$\text{Further } \frac{d}{dx} \int p(x)\ dx = p\ (x) \text{ for this } p(x).$$

$$\frac{d}{dx}\int p(x)\,dx = \frac{d}{dx}\,(x^{13} + x^7 + (3 + 4g)x + c)$$

$$= 13x^{12} + 7x^6 + (3 + 4g)$$

$$= x^{12} + x^6 + (3 + 4g).$$

However consider

$p(x) = x^3 + x^2 + (4 + 3g) \in R$

$$\frac{dp(x)}{dx} = 3x^2 + 2x + 0$$

$$\int p(x)\,dx = \frac{x^4}{4} + \frac{x^3}{3} + (4 + 3g)x + c$$

Clearly as $\frac{1}{4}$ and $\frac{1}{3}$ are not defined in $R_n(6)(g)$.

We see $\int p(x)\,dx$ is not defined.

So $\frac{d}{dx}\int p(x)\,dx \neq p(x)$.

In view of all these we can state the following result.

THEOREM 6.1: *Let $\{R_n(m)(g)[x], +, \times\} = R$ be the MOD special quasi dual number polynomial pseudo ring.*

(i) *If m is not a prime and $p(x) = a_1x^{p_1} + a_2x^{p_2} + a_3x^{p_3}$ where $a_1, a_2, a_3 \in R_n(m)(g)$ and p_1, p_2 and p_3 divide m then $\int p(x)dx$ is undefined.*

(ii) *If* $p(x) = x^{tm} + x^{sm} + x^{rm} + \ldots + x^{m}$ *then* $\dfrac{dp(x)}{dx} = 0$.

The proof is direct and hence left as an exercise to the reader.

THEOREM 6.2: *Let* $R = \{R_n(m)(g)[x],\ g^2 = (m-1)g, +, \times\}$ *be the MOD special quasi dual number polynomial pseudo ring.*

If p(x) is a nth degree MOD special quasi dual number polynomial in R then in general p(x) need not have all the (n–1) derivatives to be non zero.

Proof is direct and hence left as an exercise to the reader.

However solving polynomials in R happens to be a challenging process as R happens to be a pseudo ring in the first place and secondly working in $R_n(m)(g)$ is a difficult task.

We suggest a few problems for the reader.

Problems:

1. Obtain some special features enjoyed by the MOD dual numbers plane $R_n(m)(g)$ where $g^2 = 0$.

2. Prove $\{R_n(9)(g),\ g^2 = 0, +\} = G$ be the MOD dual number group.
   (i) Can G have finite subgroups?
   (ii) Can G have subgroups of infinite order?

3. Let $\{R_n(43)(g);\ g^2 = 0, +\} = H$ be the MOD dual number group.
   Study questions (i) to (ii) of problem 2 for this H.

4. Enumerate all the special features enjoyed by MOD dual number groups.

5. Let S= $\{R_n(5)(g), g^2 = 0, \times\}$ be the MOD dual number semigroup.

   (i) Find S-subsemigroups if any of S.
   (ii) Is S a S-subsemigroup?
   (iii) Can S have S-ideals?
   (iv) Can ideals of S be of finite order?
   (v) Can S have S-units?
   (vi) Can S have S-idempotents?
   (vii) Can S have infinite number of idempotents?
   (viii) Can S have zero divisors?
   (ix) Find any other special feature enjoyed by S.
   (x) Can S have S-zero divisors?

6. Let $S_1 = \{R_n(24)g, g^2 = 0, \times\}$ be the MOD dual number semigroup.
   Study questions (i) to (x) of problem 5 for this S_1.

7. Let S = $\{R_n(43)g, g^2 = 0, \times\}$ be the MOD dual number semigroup.
   Study questions (i) to (x) of problem 5 for this S_1.

8. Let S = $\{R_n(27)g, g^2 = 0, +, \times\}$ be the MOD dual number pseudo ring.
   (i) Can S be a S-pseudo ring?
   (ii) Can S have ideals of finite order?
   (iii) Can S have S-ideals?
   (iv) Can S have S-subrings which are not S-ideals?
   (v) Can S have S-zero divisors?
   (vi) Can S have infinite number of units?
   (vii) Can S have infinite number of S-idempotents?
   (viii) Can S have S-zero divisors?
   (ix) Obtain some special features enjoyed by S.
   (x) Can S have S-units?
   (xi) Can S have idempotents which are not S-idempotents?

9. Let $S_1 = \{R_n(17)g, g^2 = 0, +, \times\}$ be the MOD dual number pseudo ring.
   Study questions (i) to (xi) of problem 8 for this S_1.

10. Obtain all special features associated with the MOD dual number polynomial pseudo rings.

11. Prove in case of such polynomials pseudo ring the usual properties of differentiation and integration are not in general true.

12. Let $R = \{R_n(12)(g)[x], g^2 = 0, +, \times\}$ be the MOD dual number pseudo polynomial ring.
    Study questions (i) to (xi) of problem 8 for this R.

13. Let $M = \{R_n(19)(g)[x], g^2 = 0, +, \times\}$ be the MOD dual number pseudo polynomial ring.
    Study questions (i) to (xi) of problem 8 for this M.

14. Let $W = \{R_n(12)(g)[x], g^2 = 0, +, \times\}$ be the MOD dual number pseudo polynomial ring.
    (i) Study questions (i) to (xi) of problem 8 for this W.
    (ii) Find 3 polynomials in W whose derivatives are zero.
    (iii) Find 3 polynomials in W whose integrals are zero.
    (iv) Does there exist a polynomial $p(x) \in W$ which is integrable but not differentiable and vice versa.

15. Let $R = \{R_n(m)(g), g^2 = g\}$ be the MOD dual like number plane.
    Study the special features enjoyed by M.

16. Let $R = \{R_n(24)(g)[x], g^2 = g, +\}$ be the MOD dual like number group.
    (i) What are the special features associated with P?
    (ii) Give the map
    $\eta : R(g) = \{a + bg \mid a, b \in \text{reals}, g^2 = g\} \to R_n(24)\, g$; which is a MOD dual like number transformation.

(iii) Find subgroups of both finite and infinite order in P.

17. Let R = $\{R_n(43)(g) \mid g^2 = g, +\}$ be the MOD dual like number group.

    Study questions (i) to (iii) of problem 16 for this P.

18. Let B = $\{R_n(28)(g) \mid g^2 = g, \times\}$ be the MOD dual like number semigroup.

    (i) Can B be a S-semigroup?
    (ii) Can B have subsemigroups which are S-subsemigroups?
    (iii) Can B have S-ideals?
    (iv) Can ideals in B be of finite order?
    (v) Can B have S-zero divisors?
    (vi) Can B have infinite number of S-units and units?
    (vii) Can B have S-idempotents?
    (viii) Find all special features enjoyed by B.

19. Let R = $\{R_n(43)(g) \mid g^2 = g, \times\}$ be the MOD dual like number semigroup.

    Study questions (i) to (viii) of problem 18 for this N.

20. What are the special features associated with MOD dual like number pseudo rings?

21. Let B = $\{R_n(12)(g) \mid g^2 = g, +, \times\}$ be the MOD dual like number pseudo ring.

    (i) Can B be a S-pseudo rings?
    (ii) Find all finite subrings of B.
    (iii) Can B be have S-zero divisors?
    (iv) Can B have idempotents which are not S-idempotents?
    (v) Can B have infinite number of units?

(vi) What are special features associated with B?
(vii)Can B have finite S-ideals?
(viii) Find S-subrings which are not ideals or S-ideals.

22. Let S = $\{R_n(43)(g) \mid g^2 = g, +, \times\}$ be the MOD special dual like number pseudo ring.

    Study questions (i) to (viii) of problem 21 for this S.

23. Let M = $\{R_n(m)(g)[x] \mid g^2 = g\}$ be the MOD special dual like number polynomials. Enumerate the special features enjoyed by M.

24. Let W = $\{R_n(24)(g)[x] \mid g^2 = g, +, \times\}$ be the MOD special dual like number polynomial pseudo ring.

    Study questions (i) to (viii) of problem 21 for this W.

25. Let B = $\{R_n(43)(g)[x] \mid g^2 = g, +, \times\}$ be the MOD special dual like number polynomial pseudo ring.

    Study questions (i) to (viii) of problem 21 for this B.

26. Let B = $\{R_n(m)(g) \mid g^2 = (m-1)g\}$ be the MOD special quasi dual number plane.

    Study the special features associated with B.

27. Let M = $\{R_n(8)(g) \mid g^2 = 7g, +\}$ be the MOD special quasi dual number group.

    (i) Find all subgroups of finite order M.

    (ii) Obtain all subgroups of infinite order in M.

    (iii) Study any other special features enjoyed by M.

28. Let B = $\{R_n(24)(g) \mid g^2 = 23g, +\}$ be the MOD special quasi dual number group.

Study questions (i) to (viii) of problem 21 for this B.

29. Let N = $\{R_n(m)(g) \mid g^2 = (m-1)g, \times\}$ be the MOD special dual number semigroup.

    Obtain all the special features enjoyed by N.

30. Let B = $\{R_n(6)(g) \mid g^2 = 5g, \times\}$ be the MOD special dual number semigroup.

    (i) Is B a S-semigroup?
    (ii) Can B have S-ideals?
    (iii) Can ideals of B have finite number of elements in them?
    (iv) Can B have infinite number of units?
    (v) Can B have units which are not S-units?
    (vi) Find all S-idempotents.
    (vii) Find all S-zero divisors of B.
    (viii) Can B have infinite number of zero divisors?
    (ix) Can B have S-subsemigroups?

31. Let W = $\{R_n(24)(g) \mid g^2 = 23g, \times\}$ be the MOD special quasi dual number semigroup.

    Study questions (i) to (ix) of problem 30 for this W.

32. Let S = $\{R_n(17)(g)[x] \mid g^2 = 16g, \times\}$ be the MOD special quasi dual number polynomial semigroup.

    Study questions (i) to (ix) of problem 30 for this S.

33. Let S_1 = $\{R_n(28)(g)[x] \mid g^2 = 27g, \times\}$ be the MOD special quasi dual number polynomial semigroup.

    Study questions (i) to (ix) of problem 30 for this S_1.

34. Let B = $\{R_n(m)(g) \mid g^2 = (m-1)g, +, \times\}$ be the MOD special quasi dual number pseudo ring.

    (i) Study all special features enjoyed by B.
    (ii) Is B a S-pseudo ring?
    (iii) Can B have S-ideals?
    (iv) Can B have ideals of finite order?
    (v) Can B have S-zero divisors?
    (vi) Can B have units?
    (vii) Can B have idempotents?
    (viii) Can B have units which are not S-units?
    (ix) Find all subrings of finite order.
    (x) Can B have a S-subrings which is not an S-ideals?

35. Let M = $\{R_n(24)(g) \mid g^2 = 23g, +, \times\}$ be the MOD special quasi dual number pseudo ring.

    Study questions (i) to (x) of problem 34 for this M.

36. Let N = $\{R_n(47)(g) \mid g^2 = 46g, +, \times\}$ be the MOD special quasi dual number pseudo ring.

    Study questions (i) to (x) of problem 34 for this N.

37. Let S = $\{R_n(m)(g)[x] \mid g^2 = (m-1)g, +, \times\}$ be the MOD special dual number pseudo ring.

    (i) Study questions (i) to (x) of problem 34 for this S.

    (ii) Give three MOD special quasi dual number polynomials such that their derivative is zero and integral does not exist.

    (iii) Give three MOD special quasi dual number polynomial such that their derivative exist and the integral does not exist.

38. Let M= $\{R_n(92)(g)[x] \mid g^2 = 91g, +, \times\}$ be the MOD special quasi dual number polynomial pseudo ring.

(i) Study questions (i) to (x) of problem 34 for this M.

(ii) Give three MOD special quasi dual number polynomials whose derivative is zero but the integral exist.

39. Enumerate the special features associated with MOD special quasi dual number polynomial pseudo rings.

40. Prove solving equations in $R_n(m)(g)[x]$ with $g^2=(m-1)g$ is a difficult problem. Prove this by example.

41. Let $p(x) = (3 + 4g)x^{10} + (0.2 + 3.2g)x^5 + (0.43 + 4.1g) \in R_n(5)(g)[x]$ where $g^2 = 4g$. Find

(i) $\dfrac{dp(x)}{dx}$.

(ii) $\int p(x)\,dx$.

(iii) Solve p(x) for roots.

(iv) Can p(x) generate an ideal in $R_n(5)(g)$?

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A new dimension is given to modulo theory by defining MOD planes. In this book, the authors consolidate the enitre four quadrant plane into a single quadrant plane defined as the MOD plane.

MOD plane can be transformed to infinite plane and vice versa. Several innovative results in this direction are obtained. This paradigm shift will certainly lead to new discoveries.

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