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Iterative MultiFuzzy Set, Iterative MultiNeutrosophic Set, Iterative Multisoft set, and MultiPlithogenic Sets

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Abstract. Concepts such as Fuzzy Sets [1, 2], Neutrosophic Sets [3, 4], and Plithogenic Sets [5] have been extensively studied as effective frameworks for handling uncertainty, with applications spanning various fields. As extensions of Fuzzy Sets and Neutrosophic Sets, MultiFuzzy Sets [6, 7] and MultiNeutrosophic Sets [8, 9] have been introduced in the literature. In this paper, we further extend these frameworks by introducing and defining new structures: Iterative MultiFuzzy Sets, Iterative MultiNeutrosophic Sets, Iterative MultiSoft Sets, Iterative MultiRough Sets, and MultiPlithogenic Sets.

Keywords: Plithogenic Set, MultiPlithogenic Set, Iterative MultiFuzzy Sets, Iterative MultiNeutrosophic Sets, Iterative MultiSoft Sets

1. Preliminaries and Definitions

This section introduces the fundamental concepts and definitions necessary for the discussions in this paper. Throughout this paper, we consider finite and undirected sets. For readers seeking a more comprehensive understanding of set theory, we recommend referring to [10].

1.1. *Multiset*

A multiset is a generalized set where elements can appear multiple times, defined by a base set and a multiplicity function (cf. [11–13]).

Definition 1.1 (Multiset)**.** (cf. [13, 14]) Let X be a nonempty set and let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denote the set of nonnegative integers. A *multiset* M on X is defined as a pair (A, m) where:

(1) A is a subset of X (called the *support* or *carrier set* of M);

(2) $m : A \to \mathbb{N}_0$ is a function, called the *multiplicity function*, which assigns to each element $a \in A$ its multiplicity $m(a)$.

For any $x \in X \setminus A$, we set $m(x) = 0$. We often denote the multiset by

$$\{a^{m(a)} : a \in A\},$$

or, when the support is understood, simply by A. This definition permits us to represent a multiset either as a pair (A, m) or as a collection in which each element a is repeated $m(a)$ times.

Example 1.2 (A Simple Multiset)**.** Let $X = \{a, b\}$. Consider the multiset M with support $A = \{a, b\}$ and multiplicity function m defined by

$$m(a) = 2 \quad \text{and} \quad m(b) = 1.$$

Then M may be represented by

$$\{a, a, b\}.$$

Alternatively, introducing indices to distinguish repeated elements, we may write

$$\{a^{(1)}, a^{(2)}, b^{(1)}\}.$$

1.2. *MultiFuzzy Set*

A Fuzzy Set assigns each element a single membership degree in $[0, 1]$ [2, 15, 16], whereas a *MultiFuzzy Set* assigns multiple membership values derived from different lattices [17–19]. As a related concept, the notion of *HyperFuzzy Sets* has also been studied [20, 21].

Definition 1.3 (Fuzzy Set)**.** [2] Let X be a nonempty set. A *fuzzy set* A on X is characterized by its membership function

$$\mu_A : X \to [0, 1].$$

That is, the fuzzy set A is defined as

$$A = \{(x, \mu_A(x)) \mid x \in X\},$$

where $\mu_A(x)$ represents the degree to which the element $x \in X$ belongs to the set A.

Definition 1.4 (Multi-Fuzzy Set)**.** [17] Let X be a nonempty set and let $\{L_i\}_{i\in\mathbb{N}}$ be a family of complete lattices. A *multi-fuzzy set* A in X is an object of the form

$$A = \{(x, \mu_1(x), \mu_2(x), \mu_3(x), \dots) \mid x \in X\},$$

where, for each $i \in \mathbb{N}$, the function

$$\mu_i : X \to L_i$$

is called the i-th membership function of A.

If there exists a positive integer k such that only the first k membership functions are considered (i.e., the sequence has finite length k), then A is said to be a multi-fuzzy set of *dimension* k. In this case, A may be identified with a mapping

$$\mu_A : X \to \prod_{i=1}^{k} L_i,$$

and often one takes $L_i = [0, 1]$ for all $i = 1, 2, \ldots, k$.

Example 1.5 (A Multi-Fuzzy Set of Dimension 2)**.** Let $X = \{x_1, x_2\}$ and take $L_1 = L_2 = [0, 1]$. Define the membership functions

$$\mu_1(x_1) = 0.8, \quad \mu_2(x_1) = 0.2, \quad \mu_1(x_2) = 0.4, \quad \mu_2(x_2) = 0.5.$$

Then the multi-fuzzy set A is given by

$$A = \{(x_1,\, 0.8,\, 0.2),\ (x_2,\, 0.4,\, 0.5)\}.$$

This is a standard multi-fuzzy set of dimension 2.

1.3. *MultiNeutrosophic Set*

A Neutrosophic Set assigns to each element three independent membership degrees—truth (T), indeterminacy (I), and falsity (F)—each taking values in $[0, 1]$, thereby enabling flexible modeling of uncertainty [22–25]. The notion of *MultiNeutrosophic Sets* has been investigated in a growing body of work [26–29]. Owing to their scientific importance, a wide range of related applications has also been explored in recent studies [30–33]. In addition, related concepts such as the *HyperNeutrosophic Set* have been introduced and studied [34, 35].

Definition 1.6 (Neutrosophic Set)**.** [3, 36] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Example 1.7 (Neutrosophic Set in Real Life: Medical Diagnosis)**.** Consider a doctor diagnosing a disease in a patient. Let X be the set of all possible diagnoses, and let A be a neutrosophic set representing the likelihood of a particular disease D being present in a patient.

For a given diagnosis $x \in X$, the doctor assigns three independent membership degrees:

- $T_A(x)$ (Truth degree): The degree of confidence that the patient has disease D based on lab results.

- $I_A(x)$ (Indeterminacy degree): The uncertainty due to insufficient or conflicting data.
- $F_A(x)$ (Falsity degree): The degree of confidence that the patient does not have disease D based on medical history.

For example, let $x = D$ represent a flu diagnosis, and assume:

$$T_A(D) = 0.7, \quad I_A(D) = 0.2, \quad F_A(D) = 0.1.$$

This means that the doctor believes with 70% confidence that the patient has the flu, there is 20% uncertainty, and a 10% chance that the patient does not have it.

Definition 1.8 (MultiNeutrosophic Set)**.** (cf. [26]) Let $\mathcal{U}$ be a universe of discourse, and let M be a subset of $\mathcal{U}$. A *MultiNeutrosophic Set (MNS) M* is defined as:

$$M = \{(x, \langle T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s\rangle) \mid x \in \mathcal{U}\},$$

where:

- $p, r, s \geq 0$ with $p + r + s = n \geq 2$,
- At least one of p, r, s satisfies ≥ 2 to ensure the multiplicity of truth (T), indeterminacy (I), or falsehood (F),
- $T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s \subseteq [0, 1]$,
- The following condition is satisfied:

$$0 \leq \sum_{j=1}^{p} \inf T_j + \sum_{k=1}^{r} \inf I_k + \sum_{l=1}^{s} \inf F_l \leq \sum_{j=1}^{p} \sup T_j + \sum_{k=1}^{r} \sup I_k + \sum_{l=1}^{s} \sup F_l \leq n.$$

Example 1.9 (MultiNeutrosophic Set in Real Life: Stock Market Analysis)**.** In financial decision-making, investors assess stocks based on multiple uncertain parameters. Let X be the set of all stocks under consideration, and let M be a multi-neutrosophic set representing the evaluation of a stock S based on various expert analyses.

For each stock $x \in X$, multiple experts provide their independent evaluations:

- $T_1(x), T_2(x), \ldots, T_p(x)$ represent the degrees of belief from different financial analysts that the stock will increase in value.
- $I_1(x), I_2(x), \ldots, I_r(x)$ represent the uncertainty due to market fluctuations and unpredictable external factors.
- $F_1(x), F_2(x), \ldots, F_s(x)$ represent the degrees of belief that the stock will decrease in value.

For instance, let $x = S$ be a particular stock, and assume:

$$T_1(S) = 0.6, \quad T_2(S) = 0.7, \quad I_1(S) = 0.3, \quad I_2(S) = 0.4, \quad F_1(S) = 0.1, \quad F_2(S) = 0.2.$$

Here, two experts predict the stock price increase with confidence levels of 60% and 70%, while the uncertainty levels are 30% and 40%, and the chances of price decline are 10% and 20%.

This model allows investors to make decisions considering multiple perspectives and uncertainties.

1.4. *Multisoft set*

A Soft Set is a parameterized family of subsets used to handle uncertainty, introduced by Molodtsov in 1999 for decision-making problems [37,38]. A Multisoft Set generalizes Soft Sets by handling multiple parameters, allowing more flexible and complex uncertainty modeling in decision-making and classification problems [39–41]. Related concepts such as the *HyperSoft Set* [42–44] and the *SuperHyperSoft Set* [45,46] have also been studied.

Definition 1.10 (Soft Set [38])**.** Let U be a universe set and E be a set of parameters. Let $A \subseteq E$ and denote by $\mathcal{P}(U)$ the power set of U. A pair (F, A) is called a *soft set* over U if

$$F : A \to \mathcal{P}(U).$$

For each parameter $\epsilon \in A$, the set $F(\epsilon)$ is called the ϵ-approximation of the soft set (F, A). In other words, a soft set over U is a parameterized family of subsets of U.

Example 1.11 (Soft Set in a Housing Decision Problem)**.** Let the universe be a set of houses

$$U = \{h_1, h_2, h_3, h_4, h_5\},$$

and let the set of parameters be

$$E = \{\text{Expensive}, \text{Modern}, \text{Spacious}\}.$$

Choose $A = E$. Define the soft set (F, A) by

$$F(\text{Expensive}) = \{h_1, h_2\}, \quad F(\text{Modern}) = \{h_2, h_3, h_4\}, \quad F(\text{Spacious}) = \{h_1, h_3, h_5\}.$$

This soft set represents the attractiveness of houses based on the given criteria.

Definition 1.12 (Multisoft Set)**.** [39–41] Let U be a universe set and E be a set of parameters. Let $A \subseteq E$ and let $\mathcal{P}(U)$ denote the power set of U. A pair (F, A) is called a *multisoft set* over U if

$$F : A \to \mathcal{P}(U).$$

That is, a multisoft set over U is a parameterized family of subsets of U. The term "multisoft set" is used to emphasize its role in handling multiple approximations or decision parameters, thereby generalizing the classical notion of a soft set.

Example 1.13 (Multi-Soft Set in a Car Selection Problem)**.** Let the universe be a set of cars

$$U = \{c_1, c_2, c_3, c_4, c_5\},$$

and let the set of parameters be

$$E = \{\text{Cheap}, \text{Fuel-Efficient}, \text{Reliable}\}.$$

Choose $A = E$. Define the multisoft set (F, A) by

$$F(\text{Cheap}) = \{c_1, c_3, c_5\}, \quad F(\text{Fuel-Efficient}) = \{c_1, c_2, c_4\}, \quad F(\text{Reliable}) = \{c_2, c_3, c_4, c_5\}.$$

This multisoft set is used to model multiple approximations for car selection based on cost, fuel efficiency, and reliability.

1.5. *Multirough Set*

A Rough Set approximates a subset using lower and upper bounds based on equivalence classes, capturing certainty and uncertainty in membership [47–49]. A Multirough Set is a generalization of a Rough Set [50]. The definitions are provided below.

Definition 1.14 (Rough Set Approximation)**.** [51] Let X be a non-empty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (or indiscernibility relation) on X. The equivalence relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for $x \in X$, where:

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ of U are defined as follows:

(1) *Lower Approximation* $\underline{U}$:

$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$

The lower approximation $\underline{U}$ includes all elements of X whose equivalence classes are entirely contained within U. These are the elements that *definitely* belong to U.

(2) *Upper Approximation* $\overline{U}$:

$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$

The upper approximation $\overline{U}$ contains all elements of X whose equivalence classes have a non-empty intersection with U. These are the elements that *possibly* belong to U.

The pair $(\underline{U}, \overline{U})$ forms the *rough set* representation of U, satisfying the relationship:

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

Example 1.15 (Rough Set in a Housing Market)**.** Consider a universe $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ of houses. Define an equivalence relation R by neighborhood so that:

$$[h_1]_R = \{h_1, h_2\}, \quad [h_3]_R = \{h_3, h_4\}, \quad [h_5]_R = \{h_5, h_6\}.$$

Let $X = \{h_1, h_3, h_5\}$ be the set of houses with gardens. Then:

$$\underline{X} = \{x \in U \mid [x]_R \subseteq X\} = \varnothing,$$

since in every equivalence class, at least one house is missing from X; and

$$\overline{X} = \{x \in U \mid [x]_R \cap X \neq \varnothing\} = U.$$

Thus, the rough set approximation of X is $(\varnothing, U)$, capturing complete uncertainty.

Definition 1.16. [50] Let U be a universal set, and let $R_1, R_2, \ldots, R_n$ be equivalence relations (indiscernibility relations) on U. For any subset $X \subseteq U$, the *Multirough Set* of X is defined by the collection of lower and upper approximations with respect to each equivalence relation R_i.

For each $i = 1, 2, \ldots, n$, we define:

- The *Lower Approximation* of X with respect to R_i:

$$\underline{X}_i = \{x \in U \mid [x]_{R_i} \subseteq X\},$$

  where $[x]_{R_i}$ denotes the equivalence class of x under R_i.

- The *Upper Approximation* of X with respect to R_i:

$$\overline{X}_i = \{x \in U \mid [x]_{R_i} \cap X \neq \emptyset\}.$$

The *Multirough Set* of X is then the collection:

$$\mathcal{MR}(X) = \left\{ \left(\underline{X}_i, \overline{X}_i\right) \mid i = 1, 2, \ldots, n \right\}.$$

Example 1.17 (Multirough Set in a Housing Market with Two Criteria)**.** Using the same universe $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$, consider two equivalence relations:

(1) R_1 (by neighborhood):

$$[h_1]_{R_1} = \{h_1, h_2\}, \quad [h_3]_{R_1} = \{h_3, h_4\}, \quad [h_5]_{R_1} = \{h_5, h_6\}.$$

(2) R_2 (by house type): Assume detached houses $D_1 = \{h_1, h_3, h_5\}$ and semi-detached houses $D_2 = \{h_2, h_4, h_6\}$.

Let $X = \{h_1, h_3, h_5\}$ (houses with gardens, coinciding with detached houses). Then for R_1:

$$\underline{X}_1 = \varnothing \quad \text{and} \quad \overline{X}_1 = U.$$

For R_2:

$$\underline{X}_2 = D_1 = \{h_1, h_3, h_5\} \quad \text{and} \quad \overline{X}_2 = D_1 = \{h_1, h_3, h_5\}.$$

The multirough set of X is

$$\mathcal{MR}(X) = \{(\underline{X}_1, \overline{X}_1), (\underline{X}_2, \overline{X}_2)\} = \{(\varnothing, U), (\{h_1, h_3, h_5\}, \{h_1, h_3, h_5\})\},$$

illustrating how different criteria yield distinct approximations.

1.6. *Plithogenic Set*

The Plithogenic Set is a mathematical framework designed to integrate multi-valued degrees of appurtenance and contradiction, making it particularly effective for addressing complex decision-making scenarios. Numerous studies have explored the properties and applications of Plithogenic Sets, as highlighted in works such as [52–55]. The formal definition is presented below.

Definition 1.18 (Plithogenic Set)**.** [56,57] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)*
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

(1) *Reflexivity of Contradiction Function*:

$$pCF(a, a) = 0$$

(2) *Symmetry of Contradiction Function*:

$$pCF(a, b) = pCF(b, a)$$

Example 1.19 (Plithogenic Set — Laptop Selection under Conflicting Priorities)**.** Let the universe of laptops be $P = \{\ell_1, \ell_2, \ell_3\}$ and consider the single attribute $v =$ "priority objective" with value set

$$Pv = \{\text{lightweight, performance, battery}\}.$$

Degree of Appurtenance Function (DAF). Let $pdf : P \times Pv \to [0,1]$ be given by

	lightweight	performance	battery
ℓ_1	0.85	0.55	0.50
ℓ_2	0.60	0.85	0.40
ℓ_3	0.45	0.65	0.80

(Each entry is the membership degree of the laptop to the corresponding value of v.)

It is important to note that the definition of the Degree of Appurtenance Function varies across different papers. Some studies define this concept using the power set, while others simplify it by avoiding the use of the power set [58]. The author has consistently defined the Classical Plithogenic Set without employing the power set.

Degree of Contradiction Function (DCF). Let $pCF : Pv \times Pv \to [0,1]$ be symmetric with $pCF(a,a) = 0$:

$$pCF = \begin{array}{c|ccc} & \text{lightweight} & \text{performance} & \text{battery} \\ \hline \text{lightweight} & 0 & 0.60 & 0.30 \\ \text{performance} & 0.60 & 0 & 0.50 \\ \text{battery} & 0.30 & 0.50 & 0 \end{array}$$

(Heavier performance parts contradict lightweight more strongly, and high performance tends to contradict long battery life.)

A simple plithogenic aggregation (relative to a dominant value). Suppose the decision-maker's dominant value is d = battery. Define compatibility weights $w(a \mid d) := 1 - pCF(a,d)$ for $a \in Pv$:

$$w(\text{lightweight} \mid d) = 1 - 0.30 = 0.70,$$

$$w(\text{performance} \mid d) = 1 - 0.50 = 0.50,$$

$$w(\text{battery} \mid d) = 1 - 0 = 1.00.$$

For $x \in P$, compute a (normalized) plithogenic score

$$S(x \mid d) \ := \ \frac{\displaystyle\sum_{a \in Pv} w(a \mid d)\, pdf(x,a)}{\displaystyle\sum_{a \in Pv} w(a \mid d)} \quad \in [0,1].$$

Explicit calculations. The denominator is $0.70 + 0.50 + 1.00 = 2.20$.

$$\begin{aligned} \text{For } \ell_1 : \quad & \sum w\, pdf = 0.70 \cdot 0.85 \ + \ 0.50 \cdot 0.55 \ + \ 1.00 \cdot 0.50 \\ & = 0.595 + 0.275 + 0.500 \ = \ 1.370, \qquad S(\ell_1 \mid d) = \frac{1.370}{2.20} \approx 0.62. \\ \text{For } \ell_2 : \quad & \sum w\, pdf = 0.70 \cdot 0.60 \ + \ 0.50 \cdot 0.85 \ + \ 1.00 \cdot 0.40 \\ & = 0.420 + 0.425 + 0.400 \ = \ 1.245, \qquad S(\ell_2 \mid d) = \frac{1.245}{2.20} \approx 0.57. \\ \text{For } \ell_3 : \quad & \sum w\, pdf = 0.70 \cdot 0.45 \ + \ 0.50 \cdot 0.65 \ + \ 1.00 \cdot 0.80 \\ & = 0.315 + 0.325 + 0.800 \ = \ 1.440, \qquad S(\ell_3 \mid d) = \frac{1.440}{2.20} \approx 0.65. \end{aligned}$$

Outcome. Ranking by $S(\cdot \mid \text{battery})$ gives ℓ_3 (≈ 0.65) $>$ ℓ_1 (≈ 0.62) $>$ ℓ_2 (≈ 0.57). This example shows how a Plithogenic Set uses the contradiction matrix pCF to attenuate/boost membership degrees relative to a chosen dominant value.

2. **Results of This Paper**

This section presents the results obtained in this paper.

2.1. *nth-Iterative Multiset*

We now generalize the notion of a multiset by allowing its elements to themselves be multisets, iteratively.

Definition 2.1 (nth-Iterative Multiset)**.** Let X be a nonempty set. We define the collection of nth-iterative multisets on X recursively as follows:

(1) For $n = 0$, define

$$\mathcal{M}^0(X) := X.$$

(2) For each integer $n \geq 1$, an *nth-iterative multiset* on X is a multiset whose elements are $(n-1)$th-iterative multisets. Formally, define

$$\mathcal{M}^n(X) := \Big\{ m \ : \ \mathcal{M}^{n-1}(X) \to \mathbb{N}_0 \ \Big| \ \mathrm{supp}(m) \text{ is finite} \Big\},$$

where the *support* of m is given by

$$\mathrm{supp}(m) := \{ \, x \in \mathcal{M}^{n-1}(X) \mid m(x) \neq 0 \, \}.$$

An element of $\mathcal{M}^n(X)$ is called an *nth-iterative multiset* on X. In particular, $\mathcal{M}^1(X)$ is the set of all usual multisets on X.

Example 2.2 (1st-Iterative Multiset)**.** Let $X = \{a, b\}$. Then a 1st-iterative multiset on X is just a multiset on X. For instance, define $m : X \to \mathbb{N}_0$ by

$$m(a) = 2, \quad m(b) = 1.$$

Then $m \in \mathcal{M}^1(X)$ represents the multiset

$$\{a, a, b\}.$$

Example 2.3 (2nd-Iterative Multiset)**.** Let $X = \{a, b\}$. First, consider two distinct multisets in $\mathcal{M}^1(X)$:

- M_1 given by $m_1(a) = 2$ and $m_1(b) = 1$, so that $M_1 = \{a, a, b\}$;
- M_2 given by $m_2(a) = 1$ and $m_2(b) = 1$, so that $M_2 = \{a, b\}$.

Now, a 2nd-iterative multiset on X is a multiset whose elements are multisets from $\mathcal{M}^1(X)$. For example, define

$$m : \mathcal{M}^1(X) \to \mathbb{N}_0 \quad \text{with} \quad m(M_1) = 3, \quad m(M_2) = 2,$$

and $m(M) = 0$ for all other $M \in \mathcal{M}^1(X)$. Then m represents the 2nd-iterative multiset

$$\{M_1, M_1, M_1, M_2, M_2\},$$

which can be thought of as a multiset of multisets on X.

2.2. *Iterative Multi-Fuzzy Set*

We now generalize the concept of a multi-fuzzy set by allowing its membership values to themselves be multi-fuzzy sets. This leads to an iterative (or recursive) structure.

Definition 2.4 (Iterative Multi-Fuzzy Set)**.** Let X be a nonempty set and fix a positive integer k (the dimension). Define the *base level* as

$$\mathcal{F}_k^0 := \prod_{i=1}^{k} L_i,$$

where each L_i is a complete lattice (usually $L_i = [0, 1]$). For each integer $n \geq 1$, define the collection of nth-iterative multi-fuzzy sets on X recursively by

$$\mathcal{F}_k^n(X) := \Big\{ M \ : \ X \to \mathcal{F}_k^{n-1} \Big\}.$$

An element $M \in \mathcal{F}_k^n(X)$ is called an *nth-iterative multi-fuzzy set* on X of dimension k. In particular, when $n = 1$ we have

$$\mathcal{F}_k^1(X) = \Big\{ M : X \to \mathcal{F}_k^0 \Big\} = \Big\{ M : X \to \prod_{i=1}^{k} L_i \Big\},$$

which is exactly the collection of standard multi-fuzzy sets of dimension k.

Example 2.5 (1st-Iterative Multi-Fuzzy Set (Standard Multi-Fuzzy Set))**.** Let $X = \{x_1, x_2\}$ and $k = 1$ with $L_1 = [0, 1]$. Then a 1st-iterative multi-fuzzy set is a function

$$M : X \to [0, 1].$$

For instance, define

$$M(x_1) = 0.7, \quad M(x_2) = 0.3.$$

This corresponds to the fuzzy set $\{(x_1, 0.7), (x_2, 0.3)\}$.

Example 2.6 (2nd-Iterative Multi-Fuzzy Set)**.** Let $X = \{x_1, x_2\}$ and $k = 1$ with $L_1 = [0, 1]$. Then a 2nd-iterative multi-fuzzy set is a function

$$N : X \to \mathcal{F}_1^1(X),$$

so that for each $x \in X$, $N(x)$ is a fuzzy set on X. For example, define:

$$N(x_1) : X \to [0, 1], \quad N(x_1)(x_1) = 0.8, \quad N(x_1)(x_2) = 0.2,$$

$$N(x_2) : X \to [0, 1], \quad N(x_2)(x_1) = 0.4, \quad N(x_2)(x_2) = 0.6.$$

Thus, the 2nd-iterative multi-fuzzy set N assigns to x_1 the fuzzy set $\{(x_1, 0.8), (x_2, 0.2)\}$ and to x_2 the fuzzy set $\{(x_1, 0.4), (x_2, 0.6)\}$.

Theorem 2.7. *Let X be a nonempty set and let k be a fixed positive integer. Then every multi-fuzzy set A of dimension k on X is an element of $\mathcal{F}_k^1(X)$. In other words,*

$$\mathcal{F}_k^1(X) = \{\, A : X \to \prod_{i=1}^{k} L_i \,\}$$

coincides with the collection of all multi-fuzzy sets of dimension k on X.

Proof. By definition, a multi-fuzzy set A of dimension k is given by a mapping

$$\mu_A : X \to \prod_{i=1}^{k} L_i,$$

where for each $x \in X$, the tuple $\mu_A(x) = (\mu_1(x), \mu_2(x), \ldots, \mu_k(x))$ represents the membership degrees with respect to each lattice L_i. On the other hand, by the recursive definition of iterative multi-fuzzy sets, the collection $\mathcal{F}_k^1(X)$ is defined as

$$\mathcal{F}_k^1(X) = \{\, M : X \to \mathcal{F}_k^0 \,\},$$

with $\mathcal{F}_k^0 = \prod_{i=1}^{k} L_i$. Hence, every function

$$\mu_A : X \to \prod_{i=1}^{k} L_i$$

is an element of $\mathcal{F}_k^1(X)$. Therefore, the notion of a multi-fuzzy set of dimension k is recovered exactly as the first-iterative level of the iterative multi-fuzzy set hierarchy. □

Theorem 2.8 (Iterative Multi-Fuzzy Sets as Multi-Fuzzy Sets of Multi-Fuzzy Sets)**.** *Let X be a nonempty set and k a positive integer. Then for any $n \geq 1$, an nth-iterative multi-fuzzy set*

$$M \in \mathcal{F}_k^n(X) = \{\, F : X \to \mathcal{F}_k^{n-1}(X) \,\}$$

is a multi-fuzzy set whose membership values are $(n-1)$th-iterative multi-fuzzy sets. In particular, a 2nd-iterative multi-fuzzy set is a multi-fuzzy set of (standard) multi-fuzzy sets.

Proof. We prove by induction on n.

Base case $(n = 1)$: By definition,

$$\mathcal{F}_k^1(X) = \{\, M : X \to \mathcal{F}_k^0 \,\} = \{\, M : X \to \prod_{i=1}^{k} L_i \,\},$$

which is exactly the collection of standard multi-fuzzy sets of dimension k.

Inductive step: Assume that for some $n \geq 1$, every element of $\mathcal{F}_k^n(X)$ is a multi-fuzzy set whose membership values are $(n-1)$th-iterative multi-fuzzy sets. Then by definition,

$$\mathcal{F}_k^{n+1}(X) = \{\, M : X \to \mathcal{F}_k^n(X) \,\}.$$

Since by the induction hypothesis each element of $\mathcal{F}_k^n(X)$ is itself a multi-fuzzy set, it follows that every function $M : X \to \mathcal{F}_k^n(X)$ is a multi-fuzzy set whose membership values are multi-fuzzy sets (namely, the $(n-1)$th-iterative ones). This completes the induction. □

2.3. *Iterative Multi-Neutrosophic Set*

Iterative Multi-Neutrosophic Sets extend Multi-Neutrosophic Sets by recursively applying multi-valued truth, indeterminacy, and falsity functions, capturing hierarchical uncertainty across multiple levels.

Definition 2.9 (Base Level for Multi-Neutrosophic Sets)**.** Fix nonnegative integers p, r, s with $p + r + s \geq 2$. Define the *base neutrosophic space* by

$$\mathcal{N}^0_{p,r,s} := \Big(\prod_{j=1}^{p}[0,1]\Big) \times \Big(\prod_{k=1}^{r}[0,1]\Big) \times \Big(\prod_{l=1}^{s}[0,1]\Big).$$

An element of $\mathcal{N}^0_{p,r,s}$ is a tuple

$$\big(T_1, \ldots, T_p;\ I_1, \ldots, I_r;\ F_1, \ldots, F_s\big)$$

that represents the membership degrees in the truth, indeterminacy, and falsity components.

Definition 2.10 (Iterative Multi-Neutrosophic Set)**.** Let X be a nonempty set. For each integer $n \geq 1$, define the collection of *nth-iterative multi-neutrosophic sets* on X recursively as follows:

(1) For $n = 1$, set

$$\mathcal{N}^1_{p,r,s}(X) := \Big\{\, M : X \to \mathcal{N}^0_{p,r,s} \,\Big\}.$$

That is, a 1st-iterative multi-neutrosophic set is exactly a multi-neutrosophic set on X with p truth-membership functions, r indeterminacy-membership functions, and s falsity-membership functions.

(2) For each integer $n \geq 2$, define

$$\mathcal{N}^n_{p,r,s}(X) := \Big\{\, M : X \to \mathcal{N}^{n-1}_{p,r,s}(X) \,\Big\}.$$

An element of $\mathcal{N}^n_{p,r,s}(X)$ is called an *nth-iterative multi-neutrosophic set* on X (of type (p, r, s)).

Example 2.11 (1st-Iterative Multi-Neutrosophic Set)**.** Let $X = \{x_1, x_2\}$ and choose $p = 2$, $r = 1$, $s = 1$ so that each element is characterized by two truth-membership values, one indeterminacy value, and one falsity value. Define $M \in \mathcal{N}^1_{2,1,1}(X)$ by:

$$M(x_1) = (0.7,\ 0.6;\ 0.2;\ 0.1), \quad M(x_2) = (0.4,\ 0.5;\ 0.3;\ 0.2).$$

Then

$$M = \{(x_1, (0.7, 0.6; 0.2; 0.1)),\ (x_2, (0.4, 0.5; 0.3; 0.2))\}$$

is a multi-neutrosophic set on X (the standard case).

Example 2.12 (2nd-Iterative Multi-Neutrosophic Set)**.** Let $X = \{x_1, x_2\}$ and consider the same parameters $p = 2$, $r = 1$, $s = 1$. A 2nd-iterative multi-neutrosophic set $N \in \mathcal{N}^2_{2,1,1}(X)$ is a function from X to $\mathcal{N}^1_{2,1,1}(X)$. For example, define:

$$N(x_1) \in \mathcal{N}^1_{2,1,1}(X) \quad \text{by} \quad \begin{cases} N(x_1)(x_1) = (0.8,\ 0.7;\ 0.1;\ 0.0), \\ N(x_1)(x_2) = (0.5,\ 0.4;\ 0.3;\ 0.2), \end{cases}$$

and

$$N(x_2) \in \mathcal{N}^1_{2,1,1}(X) \quad \text{by} \quad \begin{cases} N(x_2)(x_1) = (0.6,\ 0.5;\ 0.2;\ 0.1), \\ N(x_2)(x_2) = (0.3,\ 0.2;\ 0.4;\ 0.3). \end{cases}$$

Thus, N assigns to each $x \in X$ a multi-neutrosophic set on X, and hence N is a 2nd-iterative multi-neutrosophic set.

Theorem 2.13. *Let X be a nonempty set and let p, r, s be fixed nonnegative integers with $p + r + s \geq 2$. Then:*

(a) The collection $\mathcal{N}^1_{p,r,s}(X)$ of 1st-iterative multi-neutrosophic sets coincides with the collection of all multi-neutrosophic sets on X with p truth-, r indeterminacy-, and s falsity-membership functions.

(b) If we set $r = s = 0$ and $p = k$ (with $k \geq 1$), then $\mathcal{N}^1_{k,0,0}(X)$ is isomorphic to the collection of multi-fuzzy sets of dimension k on X. Consequently, the iterative multi-neutrosophic set framework generalizes both multi-neutrosophic sets and iterative multi-fuzzy sets.

Proof. (a) By definition, a 1st-iterative multi-neutrosophic set on X is a function

$$M : X \to \mathcal{N}^0_{p,r,s},$$

where $\mathcal{N}^0_{p,r,s} = \left(\prod_{j=1}^{p}[0,1]\right) \times \left(\prod_{k=1}^{r}[0,1]\right) \times \left(\prod_{l=1}^{s}[0,1]\right)$. This is exactly the structure of a multi-neutrosophic set as defined earlier.

(b) Suppose now that $r = s = 0$ and $p = k$. Then

$$\mathcal{N}^0_{k,0,0} = \prod_{j=1}^{k}[0,1],$$

and a 1st-iterative multi-neutrosophic set is a function

$$M : X \to \prod_{j=1}^{k}[0,1].$$

This is precisely the definition of a multi-fuzzy set of dimension k on X (see, e.g., the definition of multi-fuzzy sets). Hence, the framework of iterative multi-neutrosophic sets subsumes the classical notion of multi-fuzzy sets. □

Theorem 2.14 (Iterative Multi-Neutrosophic Sets as Multi-Neutrosophic Sets of Multi-Neutrosophic Sets)**.** *Let X be a nonempty set and let p, r, s be fixed nonnegative integers with $p + r + s \geq 2$. Then for any $n \geq 1$, an nth-iterative multi-neutrosophic set*

$$M \in \mathcal{N}^{n}_{p,r,s}(X) = \{ F : X \to \mathcal{N}^{n-1}_{p,r,s}(X) \}$$

is a multi-neutrosophic set whose membership values are $(n-1)$th-iterative multi-neutrosophic sets. In particular, a 2nd-iterative multi-neutrosophic set is a multi-neutrosophic set of (standard) multi-neutrosophic sets.

Proof. The proof is analogous to that for iterative multi-fuzzy sets.

Base case ($n = 1$): By definition,

$$\mathcal{N}^{1}_{p,r,s}(X) = \{ M : X \to \mathcal{N}^{0}_{p,r,s} \},$$

where

$$\mathcal{N}^{0}_{p,r,s} = \Big(\prod_{j=1}^{p}[0,1]\Big) \times \Big(\prod_{k=1}^{r}[0,1]\Big) \times \Big(\prod_{l=1}^{s}[0,1]\Big).$$

Thus, $\mathcal{N}^{1}_{p,r,s}(X)$ is exactly the collection of standard multi-neutrosophic sets.

Inductive step: Assume that for some $n \geq 1$, every element of $\mathcal{N}^{n}_{p,r,s}(X)$ is a multi-neutrosophic set whose membership values are $(n-1)$th-iterative multi-neutrosophic sets. Then by definition,

$$\mathcal{N}^{n+1}_{p,r,s}(X) = \{ M : X \to \mathcal{N}^{n}_{p,r,s}(X) \}.$$

Since each element of $\mathcal{N}^{n}_{p,r,s}(X)$ is a multi-neutrosophic set by the induction hypothesis, any mapping $M : X \to \mathcal{N}^{n}_{p,r,s}(X)$ is a multi-neutrosophic set whose membership values are multi-neutrosophic sets. This completes the induction. □

2.4. *Iterative Multi-Soft Set*

We now generalize the concept of a multisoft set by defining an *iterative multi-soft set* recursively. The idea is to allow the approximations themselves to be multisoft sets, thereby creating a hierarchy or nested structure of approximations.

Definition 2.15 (Iterative Multi-Soft Set)**.** Let U be a nonempty universe and E be a set of parameters. Define the *base level* as

$$\mathcal{S}^0(U) := \mathcal{P}(U),$$

the power set of U. Then, for a fixed $n \in \mathbb{N}$ and for any subset $A \subseteq E$, an *nth-iterative multi-soft set* over U is defined recursively as follows:

(1) For $n = 1$, define

$$\mathcal{S}^1(U) := \{ F : A \to \mathcal{S}^0(U) \},$$

so that a 1st-iterative multi-soft set is simply a soft set (or multisoft set) over U.

(2) For $n \geq 2$, define

$$\mathcal{S}^n(U) := \{ F : A \to \mathcal{S}^{n-1}(U) \}.$$

That is, an nth-iterative multi-soft set is a mapping from a set of parameters A into the collection of $(n-1)$th-iterative multi-soft sets over U.

An element $F \in \mathcal{S}^n(U)$ is called an *nth-iterative multi-soft set* over U with parameter set A.

Example 2.16 (1st-Iterative and 2nd-Iterative Multi-Soft Sets)**.** Let $U = \{h_1, h_2, h_3\}$ be a set of houses, and let $E = \{\text{expensive}, \text{modern}, \text{green surroundings}\}$ be a set of parameters.

1st-Iterative Level: A soft set (or multisoft set) $F \in \mathcal{S}^1(U)$ is defined as

$$F : E' \to \mathcal{P}(U),$$

where $E' \subseteq E$. For instance, let $E' = \{\text{expensive}, \text{modern}\}$ and define

$$F(\text{expensive}) = \{h_1, h_2\}, \quad F(\text{modern}) = \{h_2, h_3\}.$$

Then (F, E') is a 1st-iterative multi-soft set over U.

2nd-Iterative Level: A 2nd-iterative multi-soft set $G \in \mathcal{S}^2(U)$ is a mapping

$$G : A \to \mathcal{S}^1(U),$$

for some $A \subseteq E$. For example, let $A = \{\alpha_1, \alpha_2\}$ be a new set of meta-parameters, and define G as follows:

$$G(\alpha_1) = F_1, \quad G(\alpha_2) = F_2,$$

where $F_1, F_2 \in \mathcal{S}^1(U)$ are defined by:

$$F_1(\text{expensive}) = \{h_1\}, \quad F_1(\text{modern}) = \{h_1, h_2\},$$

$$F_2(\text{expensive}) = \{h_2, h_3\}, \quad F_2(\text{modern}) = \{h_3\}.$$

Then G is a 2nd-iterative multi-soft set over U. In this structure, each meta-parameter in A is associated with a soft set (i.e., a 1st-iterative multi-soft set) over U.

Theorem 2.17. *Let U be a universe set and E be a set of parameters. Then every multisoft set (F, A), where $F : A \to \mathcal{P}(U)$, is a 1st-iterative multi-soft set over U; that is,*

$$\mathcal{S}^1(U) = \{ F : A \to \mathcal{P}(U) \mid A \subseteq E \},$$

and hence the concept of an iterative multi-soft set generalizes the classical notion of a multisoft set.

Proof. By definition, a multisoft set over U is a mapping

$$F : A \to \mathcal{P}(U),$$

which is exactly the definition of a 1st-iterative multi-soft set:

$$\mathcal{S}^1(U) := \{ F : A \to \mathcal{S}^0(U) \},$$

where $\mathcal{S}^0(U) = \mathcal{P}(U)$. Thus, every multisoft set is an element of $\mathcal{S}^1(U)$. Consequently, the iterative framework

$$\mathcal{S}^n(U) = \{ F : A \to \mathcal{S}^{n-1}(U) \}$$

for $n \geq 2$ extends the notion of a multisoft set to higher levels, thereby generalizing it. □

2.5. *Iterative Multi-Rough Set*

Iterative Multi-Rough Sets generalize Multi-Rough Sets by recursively applying rough approximations, capturing multi-level uncertainty through nested equivalence relations.

Definition 2.18 (Iterative Multi-Rough Set)**.** Let U be a nonempty set and let $\{R_1^0, R_2^0, \ldots, R_{n_0}^0\}$ be a family of equivalence relations on U. For any subset $X \subseteq U$, define the *first-order multi-rough set* of X as

$$\mathcal{MR}^1(X) = \left\{ \left(\underline{X}_i^{(0)}, \overline{X}_i^{(0)} \right) \;\middle|\; i = 1, 2, \ldots, n_0 \right\},$$

where for each i,

$$\underline{X}_i^{(0)} = \{ x \in U \mid [x]_{R_i^0} \subseteq X \}, \quad \overline{X}_i^{(0)} = \{ x \in U \mid [x]_{R_i^0} \cap X \neq \emptyset \}.$$

Assume now that for each integer $k \geq 1$ we are given a family of equivalence relations

$$\{R_1^k, R_2^k, \ldots, R_{n_k}^k\}$$

on the set

$$U_k := \mathcal{MR}^k(X).$$

Then the $(k+1)$th *iterative multi-rough set* of X is defined by

$$\mathcal{MR}^{k+1}(X) = \left\{ \left(\underline{Y}_j^{(k)}, \overline{Y}_j^{(k)} \right) \;\middle|\; Y \in U_k,\ j = 1, 2, \ldots, n_k \right\},$$

where for each j and for a given $Y \in U_k$,

$$\underline{Y}_j^{(k)} = \{ Z \in U_k \mid [Z]_{R_j^k} \subseteq Y \}, \quad \overline{Y}_j^{(k)} = \{ Z \in U_k \mid [Z]_{R_j^k} \cap Y \neq \emptyset \}.$$

An element of $\mathcal{MR}^{k+1}(X)$ is called a $(k+1)$th-order iterative multi-rough set of X.

Example 2.19 (First- and Second-Order Multi-Rough Sets)**.** Let $U = \{1, 2, 3, 4\}$ and consider the equivalence relation R^0 on U that partitions U into the blocks $\{1, 2\}$ and $\{3, 4\}$. Let $X = \{1, 3\}$.

First-Order: For the single equivalence relation R^0 (i.e., $n_0 = 1$), compute:

$$\underline{X}^{(0)} = \{\, x \in U \mid [x]_{R^0} \subseteq X \,\} = \varnothing,$$

since neither $\{1, 2\} \subseteq \{1, 3\}$ nor $\{3, 4\} \subseteq \{1, 3\}$. Also,

$$\overline{X}^{(0)} = \{\, x \in U \mid [x]_{R^0} \cap X \neq \emptyset \,\} = \{1, 2, 3, 4\},$$

since both equivalence classes have a non-empty intersection with X (i.e., $1 \in \{1, 2\}$ and $3 \in \{3, 4\}$). Hence,

$$\mathcal{MR}^1(X) = \Big\{ \big(\varnothing,\ \{1, 2, 3, 4\}\big) \Big\}.$$

Second-Order: Now, let $U_1 = \mathcal{MR}^1(X) = \{\, Y \,\}$ where $Y = (\varnothing, \{1, 2, 3, 4\})$. Define an equivalence relation R^1 on U_1 as the universal relation (i.e., $[Y]_{R^1} = U_1$ for the unique element Y). Then, for this R^1,

$$\underline{Y}^{(1)} = \{\, Z \in U_1 \mid [Z]_{R^1} \subseteq Y \,\}.$$

Since $[Y]_{R^1} = U_1 = \{Y\}$ and $\{Y\} \subseteq Y$ is false (as Y is a pair, not a set containing Y), we have $\underline{Y}^{(1)} = \varnothing$. However, the upper approximation is

$$\overline{Y}^{(1)} = \{\, Z \in U_1 \mid [Z]_{R^1} \cap Y \neq \emptyset \,\} = U_1 = \{Y\}.$$

Thus,

$$\mathcal{MR}^2(X) = \Big\{ \big(\varnothing,\ \{Y\}\big) \Big\}.$$

This simple example illustrates the recursive construction of iterative multi-rough sets.

Theorem 2.20. *Let U be a universal set, $X \subseteq U$, and let $\{R^0_1, R^0_2, \ldots, R^0_{n_0}\}$ be a family of equivalence relations on U. Then the first-order iterative multi-rough set $\mathcal{MR}^1(X)$ coincides with the classical multirough set of X. That is,*

$$\mathcal{MR}^1(X) = \{\, (\underline{X}_i, \overline{X}_i) \mid i = 1, 2, \ldots, n_0 \,\},$$

where

$$\underline{X}_i = \{\, x \in U \mid [x]_{R^0_i} \subseteq X \,\} \quad \textit{and} \quad \overline{X}_i = \{\, x \in U \mid [x]_{R^0_i} \cap X \neq \emptyset \,\}.$$

Proof. By definition, the first-order iterative multi-rough set is given by

$$\mathcal{MR}^1(X) = \Big\{ \Big(\underline{X}^{(0)}_i,\ \overline{X}^{(0)}_i \Big) \ \Big|\ i = 1, 2, \ldots, n_0 \Big\},$$

where

$$\underline{X}^{(0)}_i = \{\, x \in U \mid [x]_{R^0_i} \subseteq X \,\} \quad \text{and} \quad \overline{X}^{(0)}_i = \{\, x \in U \mid [x]_{R^0_i} \cap X \neq \emptyset \,\}.$$

This is precisely the definition of the classical multirough set of X as provided earlier. Hence, $\mathcal{MR}^1(X)$ generalizes (in fact, equals) the classical multirough set. □

2.6. *MultiPlithogenic Set*

MultiPlithogenic Sets are mathematical structures that extend Plithogenic Sets by incorporating multiple attributes, handling uncertainty, and contradictions in decision-making.

Definition 2.21 (MultiPlithogenic Set)**.** Let S be a universal set and $P \subseteq S$ be a subset. Let v be an attribute with a set of possible values P_v. Let $\{L_i\}_{i=1}^k$ and $\{L'_j\}_{j=1}^l$ be families of complete lattices (typically, $L_i, L'_j = [0,1]$ or products thereof). A *MultiPlithogenic Set* (MPS) on P is defined as the 5-tuple

$$\text{MPS} = \Big(P,\ v,\ P_v,\ \{pdf_i\}_{i=1}^k,\ \{pCF_j\}_{j=1}^l\Big),$$

where:

- For each $i = 1, \ldots, k$, the function

  $$pdf_i : P \times P_v \to L_i$$

  is called a *Degree of Appurtenance Function (DAF)*. For every $x \in P$ and $\alpha \in P_v$, the value $pdf_i(x, \alpha)$ represents one component of the membership degree of x with respect to the attribute value α.
- For each $j = 1, \ldots, l$, the function

  $$pCF_j : P_v \times P_v \to L'_j$$

  is called a *Degree of Contradiction Function (DCF)*. These functions satisfy, for all $\alpha, \beta \in P_v$:

  $$pCF_j(\alpha, \alpha) = 0 \quad \text{(reflexivity)} \quad \text{and} \quad pCF_j(\alpha, \beta) = pCF_j(\beta, \alpha) \quad \text{(symmetry)}.$$

Example 2.22 (Color-Based Decision Making)**.** Let $S = \mathbb{R}$ and $P = [0, 10]$. Suppose the attribute v is "color" with possible values

$$P_v = \{\text{red}, \text{green}, \text{blue}\}.$$

Define two Degree of Appurtenance Functions:

$$pdf_1(x, \alpha) = \begin{cases} \frac{x}{10} & \text{if } \alpha = \text{red}, \\ 1 - \frac{x}{10} & \text{if } \alpha = \text{green}, \\ 0.5 & \text{if } \alpha = \text{blue}, \end{cases}$$

and

$$pdf_2(x,\alpha) = \begin{cases} 0.5 & \text{if } \alpha = \text{red}, \\ \frac{x}{10} & \text{if } \alpha = \text{green}, \\ 1 - \frac{x}{10} & \text{if } \alpha = \text{blue}. \end{cases}$$

Also, define a single Degree of Contradiction Function:

$$pCF_1 : P_v \times P_v \to [0,1], \quad pCF_1(\alpha,\beta) = \begin{cases} 0 & \text{if } \alpha = \beta, \\ 0.3 & \text{if } \alpha \neq \beta. \end{cases}$$

Then the 5-tuple

$$\text{MPS} = \Big([0,10],\ v,\ \{\text{red}, \text{green}, \text{blue}\},\ \{pdf_1,\ pdf_2\},\ \{pCF_1\}\Big)$$

defines a MultiPlithogenic Set that assigns to each $x \in [0,10]$ a pair of membership degrees (via pdf_1 and pdf_2) for each color, along with a contradiction measure between any two color values.

Theorem 2.23. *Let S be a universal set and $P \subseteq S$. Consider the following specializations of a MultiPlithogenic Set:*

(a) If $k = 1$ and $l = 1$, then the MultiPlithogenic Set

$$(P,\ v,\ P_v,\ \{pdf_1\},\ \{pCF_1\})$$

reduces to the classical Plithogenic Set.

(b) If the attribute v is omitted (or considered trivial) and $pdf_1 : P \to [0,1]^k$ (with $k \geq 1$), then the structure is equivalent to a multi-fuzzy set of dimension k on P.

(c) If we choose $k = 3$ and define

$$pdf_1(x,\alpha) = T(x,\alpha), \quad pdf_2(x,\alpha) = I(x,\alpha), \quad pdf_3(x,\alpha) = F(x,\alpha),$$

with an appropriate definition of pCF_1 (or by ignoring it), then the MultiPlithogenic Set specializes to a multi-neutrosophic set.

Thus, the MultiPlithogenic Set framework unifies and generalizes Plithogenic Sets, Multi-Neutrosophic Sets, and Multi-Fuzzy Sets.

Proof. (a) In the classical Plithogenic Set (see, e.g., [57]), one considers a single Degree of Appurtenance Function $pdf : P \times P_v \to [0,1]^s$ and a single Degree of Contradiction Function $pCF : P_v \times P_v \to [0,1]^t$. By setting $k = 1$ and $l = 1$ in the definition of a MultiPlithogenic Set, we obtain

$$(P,\ v,\ P_v,\ \{pdf_1\},\ \{pCF_1\}),$$

which is identical to the classical definition.

(b) A multi-fuzzy set of dimension k on P is defined as a mapping

$$\mu : P \to \prod_{i=1}^{k}[0,1],$$

assigning to each element $x \in P$ a k-tuple of membership degrees. By omitting the attribute v and setting P_v to a singleton (or ignoring it), and by taking the single DAF $pdf_1 : P \to [0,1]^k$, the MultiPlithogenic Set reduces to the multi-fuzzy set structure.

(c) In a multi-neutrosophic set, each element $x \in P$ is characterized by three membership degrees: truth $T(x)$, indeterminacy $I(x)$, and falsity $F(x)$. By setting $k = 3$ and defining

$$pdf_1(x,\alpha) = T(x,\alpha), \quad pdf_2(x,\alpha) = I(x,\alpha), \quad pdf_3(x,\alpha) = F(x,\alpha),$$

for each $\alpha \in P_v$, the MultiPlithogenic Set becomes a structure in which each element is assigned a triple of membership values. With a suitable (or even trivial) definition of the contradiction function pCF_1, this framework exactly recovers the notion of a multi-neutrosophic set.

In each case, the MultiPlithogenic Set is seen to specialize to the corresponding classical set by appropriate choices of the parameters k, l and the functions pdf_i and pCF_j. Thus, the general framework indeed unifies and generalizes the three types of sets. □

3. Iterative MultiPlithogenic Set: Definition and Reductions

An Iterative MultiPlithogenic Set recursively assigns plithogenic-valued memberships, layering appurtenance and contradiction across levels for hierarchical, multi-attribute, robust uncertainty modeling.

Notation 1. *Fix a nonempty universe P (of objects to be evaluated), a single attribute v with a finite value set P_v, and two finite index sets:*

$$\mathcal{I} = \{1,\dots,k\} \quad \textit{for DAF components}, \qquad \mathcal{J} = \{1,\dots,\ell\} \quad \textit{for DCF components.}$$

Let $\{L_i\}_{i\in\mathcal{I}}$ and $\{L'_j\}_{j\in\mathcal{J}}$ be complete lattices (typically $L_i = L'_j = [0,1]$).

Definition 3.1 (Iterative MultiPlithogenic codomains)**.** Define the level-0 plithogenic fiber as

$$\mathfrak{L}^{(0)} := \prod_{i\in\mathcal{I}} L_i.$$

Recursively, for $n \geq 1$ set

$$\mathfrak{L}^{(n)}(P,P_v) := \Big\{ \Phi : P \times P_v \longrightarrow \mathfrak{L}^{(n-1)}(P,P_v) \Big\}.$$

(Thus an element of $\mathfrak{L}^{(n)}$ is a *plithogenic-valued* mapping on $P \times P_v$ whose values are level-$(n-1)$ objects.)

Definition 3.2 (Iterative MultiPlithogenic Set (order n)). Let $n \in \mathbb{N}$, $n \geq 1$. An *Iterative MultiPlithogenic Set of order* n on P is a tuple

$$\mathrm{IMPS}^{(n)} := \Big(P,\ v,\ P_v,\ \{\,\mathrm{PDF}_i^{(n)}\,\}_{i\in\mathcal{I}},\ \{\,\mathrm{pCF}_j\,\}_{j\in\mathcal{J}}\Big),$$

where

- for each $i \in \mathcal{I}$, the i-th *iterative Degree of Appurtenance Function* (DAF) is

$$\mathrm{PDF}_i^{(n)} :\ P \times P_v \ \longrightarrow\ \mathfrak{L}^{(n-1)}(P, P_v);$$

- for each $j \in \mathcal{J}$, the *Degree of Contradiction Function* (DCF)

$$\mathrm{pCF}_j :\ P_v \times P_v \ \longrightarrow\ L'_j$$

  is symmetric and reflexive: $\mathrm{pCF}_j(\alpha, \alpha) = 0, \quad \mathrm{pCF}_j(\alpha, \beta) = \mathrm{pCF}_j(\beta, \alpha)$ for all $\alpha, \beta \in P_v$.

When $n = 1$ we get the classical (non-iterative) case $\mathrm{PDF}_i^{(1)} : P \times P_v \to \mathfrak{L}^{(0)} = \prod_{i\in\mathcal{I}} L_i$, i.e., ordinary MultiPlithogenic DAFs.

Remark 3.3 (Semantics). For $n = 1$, each $\mathrm{PDF}_i^{(1)}(x, \alpha) \in \prod_{r\in\mathcal{I}} L_r$ is a k-tuple of base degrees. For $n \geq 2$, the value $\mathrm{PDF}_i^{(n)}(x, \alpha)$ is itself a (level-$(n-1)$) plithogenic-valued mapping on $P \times P_v$, enabling hierarchical (*self-similar*) refinement of membership information across several layers.

Example 3.4 (IMPS of order 2: Bike-share station selection with time-of-day contexts). Let the universe of candidate stations be $P = \{s_1, s_2, s_3\}$. Consider the attribute $v =$ "user priority" with value set

$$P_v = \{\text{commute } (C),\ \text{leisure } (L),\ \text{safety } (S)\}.$$

Contradiction (symmetric, reflexive) is specified by

$$\mathrm{pCF} = \begin{array}{c|ccc} & C & L & S \\ \hline C & 0 & 0.3 & 0.5 \\ L & 0.3 & 0 & 0.2 \\ S & 0.5 & 0.2 & 0 \end{array}.$$

We build an *Iterative MultiPlithogenic Set of order* 2, where the iterative DAF $\mathrm{PDF}^{(2)} : P \times P_v \to [0,1]^{\Xi}$ returns a vector over the context set $\Xi = \{\text{peak}, \text{off}\}$. Let the context-weights be $\lambda_{\text{peak}} = 0.7$ and $\lambda_{\text{off}} = 0.3$. The level-2 DAFs are:

	C	L	S
s_1	(0.80, 0.60)	(0.50, 0.70)	(0.60, 0.70)
s_2	(0.70, 0.50)	(0.60, 0.60)	(0.80, 0.90)
s_3	(0.60, 0.60)	(0.70, 0.80)	(0.50, 0.60)

(entries are (peak, off)).

Inner (context) aggregation. For $x \in P$ and $a \in P_v$ set

$$m_{x,a} := 0.7\,\mathrm{PDF}^{(2)}(x,a)_{\mathrm{peak}} + 0.3\,\mathrm{PDF}^{(2)}(x,a)_{\mathrm{off}}$$

. Explicitly:

For $s_1:\ C: 0.7\cdot 0.80 + 0.3\cdot 0.60 = 0.74,\ L: 0.7\cdot 0.50 + 0.3\cdot 0.70 = 0.56,\ S: 0.7\cdot 0.60 + 0.3\cdot 0.70 = 0.63;$

For $s_2:\ C: 0.64,\ L: 0.60,\ S: 0.83;$ For $s_3:\ C: 0.60,\ L: 0.73,\ S: 0.53.$

Plithogenic aggregation (dominant value $d = S$). Set weights $w(a \mid d) := 1 - \mathrm{pCF}(a,d)$: $w(C \mid S) = 1 - 0.5 = 0.5$, $w(L \mid S) = 1 - 0.2 = 0.8$, $w(S \mid S) = 1$; denominator $\sum_a w(a \mid S) = 2.3$. Define the score

$$S_d(x) := \frac{\sum_{a\in P_v} w(a \mid d)\, m_{x,a}}{\sum_{a\in P_v} w(a \mid d)}.$$

Numerics.

$$S_S(s_1) = \frac{0.5\cdot 0.74 + 0.8\cdot 0.56 + 1.0\cdot 0.63}{2.3} = \frac{0.37 + 0.448 + 0.63}{2.3} \approx 0.630,$$
$$S_S(s_2) = \frac{0.5\cdot 0.64 + 0.8\cdot 0.60 + 1.0\cdot 0.83}{2.3} = \frac{0.32 + 0.48 + 0.83}{2.3} \approx 0.709,$$
$$S_S(s_3) = \frac{0.5\cdot 0.60 + 0.8\cdot 0.73 + 1.0\cdot 0.53}{2.3} = \frac{0.30 + 0.584 + 0.53}{2.3} \approx 0.615.$$

s_2 is preferred under safety-dominant policy: $s_2\,(0.709) > s_1\,(0.630) \gtrsim s_3\,(0.615)$.

Example 3.5 (IMPS of order 3: Wind-farm site planning with season and day-night layers)**.** Let $P = \{\mathrm{A}, \mathrm{B}\}$ be candidate sites and $v =$ "planning objective" with

$$P_v = \{\text{capacity (Cap)},\ \text{cost (Cost)},\ \text{wildlife (Bird)}\}.$$

Contradiction (symmetric, reflexive) is

$$\mathrm{pCF} = \begin{array}{c|ccc} & \text{Cap} & \text{Cost} & \text{Bird} \\ \hline \text{Cap} & 0 & 0.4 & 0.6 \\ \text{Cost} & 0.4 & 0 & 0.2 \\ \text{Bird} & 0.6 & 0.2 & 0 \end{array}.$$

We construct an *order*-3 *IMPS* whose iterative DAF $\mathrm{PDF}^{(3)} : P \times P_v \to [0,1]^{S\times T}$ returns a 2×2 array over seasons $S = \{\mathrm{Winter}, \mathrm{Summer}\}$ and times $T = \{\mathrm{Day}, \mathrm{Night}\}$. Context weights: $(\theta_{\mathrm{Day}}, \theta_{\mathrm{Night}}) = (0.6, 0.4)$ and $(\sigma_{\mathrm{Winter}}, \sigma_{\mathrm{Summer}}) = (0.5, 0.5)$. Level-3 DAFs (entries are $(\mathrm{Day}, \mathrm{Night})$):

	Cap	Cost	Bird
A : Winter	(0.75, 0.70)	(0.50, 0.55)	(0.80, 0.85)
A : Summer	(0.65, 0.60)	(0.45, 0.50)	(0.70, 0.75)
B : Winter	(0.85, 0.80)	(0.60, 0.65)	(0.60, 0.65)
B : Summer	(0.75, 0.70)	(0.55, 0.60)	(0.55, 0.60)

Inner (time-of-day) aggregation. For each site/objective/season,

$$m^{(T)} := 0.6\,(\text{Day}) + 0.4\,(\text{Night}).$$

For site A:

$$\text{Cap}: \;\text{Winter } 0.6{\cdot}0.75 + 0.4{\cdot}0.70 = 0.73,\; \text{Summer } 0.63;$$

$$\text{Cost}: \;\text{Winter } 0.52,\; \text{Summer } 0.47;$$

$$\text{Bird}: \;\text{Winter } 0.82,\; \text{Summer } 0.72.$$

For site B:

$$\text{Cap}: \;(0.83, 0.73), \quad \text{Cost}: \;(0.62, 0.57), \quad \text{Bird}: \;(0.62, 0.57).$$

Season aggregation. For each site/objective,

$$m^{(S)} := 0.5\,(\text{Winter}) + 0.5\,(\text{Summer}).$$

Thus

$$\text{A: Cap} = 0.5(0.73 + 0.63) = 0.68,\; \text{Cost} = 0.495,\; \text{Bird} = 0.77;$$

$$\text{B: Cap} = 0.78,\; \text{Cost} = 0.595,\; \text{Bird} = 0.595.$$

Plithogenic aggregation (dominant value $d = \text{Bird}$). Weights $w(a \mid d) = 1 - \text{pCF}(a, d)$ give

$$w(\text{Cap} \mid \text{Bird}) = 0.4, \quad w(\text{Cost} \mid \text{Bird}) = 0.8, \quad w(\text{Bird} \mid \text{Bird}) = 1.0, \quad \sum w = 2.2.$$

Define $S_d(x) := \dfrac{\sum_{a \in P_v} w(a \mid d)\, m^{(S)}_{x,a}}{2.2}$. *Numerics.*

$$S_{\text{Bird}}(\text{A}) = \frac{0.4 \cdot 0.68 + 0.8 \cdot 0.495 + 1.0 \cdot 0.77}{2.2} = \frac{0.272 + 0.396 + 0.770}{2.2} = \frac{1.438}{2.2} \approx 0.655,$$

$$S_{\text{Bird}}(\text{B}) = \frac{0.4 \cdot 0.78 + 0.8 \cdot 0.595 + 1.0 \cdot 0.595}{2.2} = \frac{0.312 + 0.476 + 0.595}{2.2} = \frac{1.383}{2.2} \approx 0.629.$$

With wildlife protection dominant, site A (≈ 0.655) slightly outperforms site B (≈ 0.629). Changing the dominant value (e.g., to capacity) would alter the contradiction weights and may reverse the ranking.

Theorem 3.6 (IMPS generalizes MultiPlithogenic Sets)**.** *Let $n = 1$. Then the class of order-1 IMPS coincides with the class of MultiPlithogenic Sets. In particular, the correspondence*

$$\Big(P, v, P_v, \{\text{PDF}^{(1)}_i\}_{i \in \mathcal{I}}, \{\text{pCF}_j\}_{j \in \mathcal{J}}\Big) \;\longleftrightarrow\; \Big(P, v, P_v, \{\text{pdf}_i\}_{i \in \mathcal{I}}, \{\text{pCF}_j\}_{j \in \mathcal{J}}\Big),$$

with $\text{pdf}_i := \text{PDF}^{(1)}_i : P \times P_v \to \prod_{r \in \mathcal{I}} L_r$, is an identity of data.

Proof. By Definition 3.1, $\mathfrak{L}^{(0)} = \prod_{r \in \mathcal{I}} L_r$. For $n = 1$, Definition 3.2 prescribes $\text{PDF}^{(1)}_i : P \times P_v \to \mathfrak{L}^{(0)}$ for each i, which are exactly the DAF components of a (non-iterative) MultiPlithogenic Set, with the same DCFs. No further structure is added or removed; hence the two notions coincide. □

Theorem 3.7 (IMPS generalizes Iterative MultiFuzzy Sets). *Fix $k = |\mathcal{I}| \geq 1$, let all $L_i = [0,1]$, take $\ell = 0$ (no DCFs), and let $P_v = \{\alpha_0\}$ be a singleton. For each $n \geq 1$ there is a natural bijection*

$$\Big\{\text{order-}n \text{ IMPS on } P \text{ with } P_v = \{\alpha_0\}\Big\} \;\cong\; \mathcal{F}_k^n(P),$$

where $\mathcal{F}_k^n(P) = \{\, M : P \to \mathcal{F}_k^{n-1}(P) \,\}$ with $\mathcal{F}_k^0 = \prod_{i=1}^k [0,1]$ is the space of order-n Iterative MultiFuzzy Sets.

Proof. We proceed by induction on n.
Base $n = 1$. An order-1 IMPS as specialized in the statement consists of $\mathrm{PDF}_i^{(1)} : P \times \{\alpha_0\} \to \prod_{r=1}^k [0,1]$. Identify $P \times \{\alpha_0\}$ with P and define

$$M(x) \;:=\; \Big(\mathrm{PDF}_1^{(1)}(x,\alpha_0), \ldots, \mathrm{PDF}_k^{(1)}(x,\alpha_0)\Big) \in \prod_{i=1}^k [0,1] = \mathcal{F}_k^0,$$

which yields a bijection between specialized order-1 IMPS and maps $M : P \to \mathcal{F}_k^0 = \mathcal{F}_k^1(P)$.
Step $n \to n+1$. An order-$(n+1)$ IMPS has $\mathrm{PDF}_i^{(n+1)} : P \times \{\alpha_0\} \to \mathfrak{L}^{(n)}(P, \{\alpha_0\})$. By Definition 3.1, $\mathfrak{L}^{(n)}(P, \{\alpha_0\})$ is the set of maps $P \to \mathfrak{L}^{(n-1)}(P, \{\alpha_0\})$, which by the induction hypothesis is naturally identified with $\mathcal{F}_k^n(P)$. Hence for each $x \in P$ we may define

$$M(x) \;:=\; \Big(\mathrm{PDF}_1^{(n+1)}(x,\alpha_0), \ldots, \mathrm{PDF}_k^{(n+1)}(x,\alpha_0)\Big) \;\in\; \mathcal{F}_k^n(P),$$

obtaining a bijection $P \to \mathcal{F}_k^n(P)$, i.e. $M \in \mathcal{F}_k^{n+1}(P)$. The construction is reversible at every stage; thus we have a natural bijection for all n. □

Theorem 3.8 (IMPS generalizes Iterative MultiNeutrosophic Sets). *Fix nonnegative integers p, r, s with $p + r + s \geq 2$ and set $k = p + r + s$, $L_i = [0,1]$. Let $P_v = \{\alpha_0\}$ and choose $\ell = 0$. For each $n \geq 1$ there is a natural bijection*

$$\Big\{\text{order-}n \text{ IMPS on } P \text{ with } P_v = \{\alpha_0\} \text{ and } k = p + r + s\Big\} \;\cong\; \mathcal{N}_{p,r,s}^n(P),$$

where $\mathcal{N}_{p,r,s}^n(P) = \{\, M : P \to \mathcal{N}_{p,r,s}^{n-1}(P) \,\}$ and $\mathcal{N}_{p,r,s}^0 = \big([0,1]^p \times [0,1]^r \times [0,1]^s\big)$ is the order-n Iterative MultiNeutrosophic space.

Proof. Order the index set $\mathcal{I}$ so that the first p coordinates correspond to truth, the next r to indeterminacy, and the last s to falsity. The proof mirrors Theorem 3.7. For $n = 1$, the specialized IMPS data are $\mathrm{PDF}_i^{(1)} : P \to \prod_{t=1}^{p+r+s} [0,1]$. Grouping the coordinates into $(T_1, \ldots, T_p; I_1, \ldots, I_r; F_1, \ldots, F_s)$ yields a map $M : P \to [0,1]^p \times [0,1]^r \times [0,1]^s = \mathcal{N}_{p,r,s}^0$, i.e. an element of $\mathcal{N}_{p,r,s}^1(P)$. For $n \to n+1$, the codomain $\mathfrak{L}^{(n)}(P, \{\alpha_0\})$ identifies with $\mathcal{N}_{p,r,s}^n(P)$ by the induction hypothesis, hence $x \mapsto \big(\mathrm{PDF}_i^{(n+1)}(x,\alpha_0)\big)_{i \in \mathcal{I}}$ is a map $P \to \mathcal{N}_{p,r,s}^n(P)$, i.e. an element of $\mathcal{N}_{p,r,s}^{n+1}(P)$. All steps are bijective and natural, completing the proof. □

4. Conclusion and Future Works

In this paper, we extended the frameworks by introducing and defining new structures: Iterative MultiFuzzy Sets, Iterative MultiNeutrosophic Sets, Iterative MultiSoft Sets, Iterative MultiRough Sets, and MultiPlithogenic Sets. In the future, we aim to investigate further extensions utilizing Graphs [59], HyperGraphs [60–62], and SuperHyperGraphs [63, 64].

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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