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Cognitive HyperGraphs and SuperHyperGraphs: A Novel Framework for Complex Relational Modeling

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Abstract. Graph theory explores the relationships between objects through mathematical structures composed of vertices (nodes) and edges (connections). A *hypergraph* generalizes the classical graph by introducing *hyperedges*, which can connect any number of vertices rather than just two, thus allowing the modeling of more complex multi-way relationships [1]. Building upon this, the concept of a *SuperHyperGraph* has been introduced as a further extension of hypergraphs and has recently become a subject of active research [2–4].

A *cognitive graph* is a structure designed to represent mental models of spatial environments, using nodes, edges, and labels to encode information such as location, direction, and navigational cues [5, 6]. Closely related concepts include *cognitive maps*, which are widely studied in fields such as artificial intelligence, social science, and computer science.

In this paper, we propose two new extended models: the *Cognitive HyperGraph* and the *Cognitive SuperHyperGraph*, which enhance the traditional cognitive graph framework using hypergraph and superhypergraph theory (cf. [7]). We hope these contributions will promote further development in cognitive modeling and its applications across disciplines such as AI, social sciences, and computational sciences.

Keywords: Cognitive Graph, HyperGraph, SuperHyperGraph, Graph Theory

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1. **Preliminaries**

This section introduces the fundamental concepts and definitions necessary for the discussions throughout this paper. Unless otherwise stated, all structures considered in this paper are assumed to be finite.

1.1. *SuperHyperGraph*

A *hypergraph* is a generalization of a classical graph in which *hyperedges* may connect any number of vertices rather than being restricted to two [8]. This framework enables the modeling of more intricate, multi-way relationships [1, 9–12]. A *SuperHyperGraph* is a more recent extension of the hypergraph concept that has attracted growing attention in the literature [2, 3, 13, 14]. It can be regarded as a recursive structure built upon hypergraphs, where the construction involves successive applications of the powerset operation, giving rise to the notion of the *n-th powerset* [15]. Due to its ability to capture hierarchical structures observed in real-world systems, the SuperHyperGraph has been the subject of extensive studies in various domains [16–20]. The formal definition is provided below.

Definition 1.1 (Powerset)**.** For a given set S, the *powerset* of S, written as $\mathcal{P}(S)$, is the family of all subsets of S:

$$\mathcal{P}(S) = \{ A \subseteq S \}.$$

By construction, both the empty set $\emptyset$ and the set S itself belong to $\mathcal{P}(S)$.

Definition 1.2 (n-th Powerset)**.** (cf. [21–23])

Let H be a set. The hierarchy of iterated powersets of H, denoted $\mathcal{P}_n(H)$, is defined inductively as

$$\mathcal{P}_1(H) := \mathcal{P}(H), \qquad \mathcal{P}_{n+1}(H) := \mathcal{P}\big(\mathcal{P}_n(H)\big), \quad n \geq 1.$$

Hence, for the first few cases one obtains

$$\mathcal{P}_2(H) = \mathcal{P}(\mathcal{P}(H)), \qquad \mathcal{P}_3(H) = \mathcal{P}(\mathcal{P}(\mathcal{P}(H))).$$

Similarly, the n-th *nonempty powerset*, denoted $\mathcal{P}^*_n(H)$, is defined recursively by

$$\mathcal{P}^*_1(H) := \mathcal{P}^*(H), \qquad \mathcal{P}^*_{n+1}(H) := \mathcal{P}^*\big(\mathcal{P}^*_n(H)\big), \quad n \geq 1,$$

where $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$.

Example 1.3 (Family and Subgroup Organization)**.** Let the base set be the members of a family

$$H = \{\text{Ayako}, \ \text{Taro}, \ \text{Kenji}\}.$$

First powerset:

$$\mathcal{P}_1(H) = \mathcal{P}(H) = \{\emptyset, \{\text{Ayako}\}, \{\text{Taro}\}, \{\text{Kenji}\}, \{\text{Ayako}, \text{Taro}\}, \{\text{Ayako}, \text{Kenji}\}, \{\text{Taro}, \text{Kenji}\}, H\}.$$

This represents all possible subgroups of family members.

Second powerset:

$$\mathcal{P}_2(H) = \mathcal{P}(\mathcal{P}(H)).$$

Here, each element of $\mathcal{P}_2(H)$ is a *collection of subgroups*. For instance,

$$\{ \{\text{Ayako}\}, \{\text{Taro}, \text{Kenji}\} \} \in \mathcal{P}_2(H),$$

which represents choosing one subgroup containing Ayako and another subgroup containing Taro and Kenji simultaneously. This captures higher-level groupings such as “alliances” within the family.

Example 1.4 (Course Curriculum Design). Let the base set be a small set of academic subjects

$$H = \{\text{Mathematics}, \text{ Physics}, \text{ Computer Science}\}.$$

First powerset:

$$\mathcal{P}_1(H) = \mathcal{P}(H)$$

lists all possible combinations of subjects (e.g. Mathematics only, Physics and Computer Science, etc.).

Second powerset:

$$\mathcal{P}_2(H) = \mathcal{P}(\mathcal{P}(H))$$

represents sets of such combinations. For example,

$$\{\,\{\text{Mathematics}, \text{Physics}\}, \ \{\text{Computer Science}\}\,\} \in \mathcal{P}_2(H),$$

which can be interpreted as a curriculum design that simultaneously considers a “science module” and a “computing module” as separate study tracks.

Third powerset:

$$\mathcal{P}_3(H) = \mathcal{P}(\mathcal{P}(\mathcal{P}(H))).$$

An element here might be a *set of curricula*, modeling multiple possible program structures chosen collectively, such as a university catalog containing different degree plans.

Definition 1.5 (Hypergraph [1,8]). A *hypergraph* $H = (V(H), E(H))$ is a pair where:

- $V(H)$: A non-empty set of vertices.
- $E(H)$: A set of hyperedges, each of which is a subset of $V(H)$.

This paper focuses exclusively on finite hypergraphs.

Definition 1.6 (n-SuperHyperGraph). (cf. [3, 24])

Let V_0 be a finite *base set* of vertices. For every integer $k \geq 0$, define the iterated powerset of V_0 inductively by

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big),$$

where $\mathcal{P}(\cdot)$ denotes the standard powerset operator.

An *n-SuperHyperGraph* is an ordered pair

$$\mathsf{SHG}^{(n)} = (V, E),$$

such that

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq \mathcal{P}(V).$$

The elements of V are called *n-supervertices*, and the elements of E are called *n-superedges*. In particular, each n-superedge is a subset of V, thereby generalizing the notion of hyperedges to the n-th powerset level.

Example 1.7 (Organizational Hierarchies)**.** Consider a company with a finite base set of employees

$$V_0 = \{\text{Ayano}, \text{Taro}, \text{Hiroko}, \text{Dave}\}.$$

At the first level, $\mathcal{P}(V_0)$ consists of all possible teams of employees. At the second level, $\mathcal{P}^2(V_0)$ contains collections of teams, which may represent *departments*. An n-SuperHyperGraph with $n = 2$ can thus model the structure of a company where

$$V \subseteq \mathcal{P}^2(V_0)$$

represents departments as 2-supervertices, and

$$E \subseteq \mathcal{P}(V)$$

represents inter-department collaborations as 2-superedges. This framework naturally encodes the hierarchical nature of modern organizations.

Example 1.8 (Social Media Communities)**.** Let the base set of vertices be

$$V_0 = \{\text{User}_1, \text{User}_2, \ldots, \text{User}_m\}.$$

At level one, $\mathcal{P}(V_0)$ corresponds to possible user groups. At level two, $\mathcal{P}^2(V_0)$ represents collections of such groups, such as *communities of communities*. In this setting, a 2-SuperHyperGraph

$$\mathsf{SHG}^{(2)} = (V, E)$$

can represent large-scale social media structures where 2-supervertices are meta-communities (clusters of groups), while 2-superedges capture relations such as shared interests or overlapping memberships between these meta-communities. This model allows the representation of complex multi-layered social dynamics.

1.2. *Cognitive Graph*

A cognitive graph models mental representations of spatial environments by using nodes, edges, and labels to encode locations and navigational knowledge [5, 6, 25]. A closely related concept is the cognitive map, which has been widely studied in cognitive science [26–28]. The formal definition is provided below.

Definition 1.9 (Cognitive Graph)**.** (cf. [5, 6]) Let S be a nonempty set of distinguished locations (e.g. landmarks or junctions). A *cognitive graph* is a quadruple

$$G = \bigl(V,\, E,\, \ell_V,\, \ell_E\bigr),$$

where

- $V \subseteq S$ is a finite set of *nodes* (significant places),
- $E \subseteq \bigl\{\{u,v\} \mid u,v \in V,\ u \neq v\bigr\}$ is a set of undirected *edges* (path segments),
- $\ell_V : V \to L_V$ is a *node–label function* assigning each $v \in V$ an identifying label (e.g. a name or local features),
- $\ell_E : E \to L_E$ is an *edge–label function* assigning each $\{u,v\} \in E$ a label encoding local metric information (e.g. distance, direction, or action sequence).

If ℓ_E is trivial (or omitted), G is called a *topological cognitive graph*, encoding only connectivity. Otherwise, G is a *labeled cognitive graph*, which incorporates local geometric or metric cues.

Example 1.10 (Urban Metro Network as a Cognitive Graph)**.** Let

$$S = \{\text{Central},\ \text{Museum},\ \text{University},\ \text{Stadium},$$
$$\text{Airport},\ \text{Harbor}\}$$

be the set of key transit stops in a city. We model a commuter's mental map of the metro as the cognitive graph

$$G = (V,\, E,\, \ell_V,\, \ell_E),$$

where:

- $V = S$ (all major stations are nodes).
- Edges represent direct metro connections:

$$E = \bigl\{\{\text{Central}, \text{Museum}\}, \{\text{Central}, \text{University}\}, \{\text{University}, \text{Stadium}\},$$
$$\{\text{Stadium}, \text{Airport}\}, \{\text{Central}, \text{Harbor}\}\bigr\}.$$

- The node–label function ℓ_V assigns each station its name and line color:

$$\ell_V(\text{Central}) = \bigl(\text{“Central”},\ \text{Red Line}\bigr),$$
$$\ell_V(\text{Museum}) = \bigl(\text{“Museum”},\ \text{Red Line}\bigr),\ \ldots$$

- The edge–label function ℓ_E assigns each connection its approximate travel time (in minutes):

$$\ell_E\bigl(\{\text{Central}, \text{Museum}\}\bigr) = 2,$$
$$\ell_E\bigl(\{\text{Central}, \text{University}\}\bigr) = 3,\ \ldots$$

Usage: A commuter uses this cognitive graph to plan routes by considering connectivity, line transfers, and estimated travel times between significant stops.

2. Result: Cognitive HyperGraph

In this section, we present the formal definition of the Cognitive HyperGraph(cf. [7]).

Definition 2.1 (Cognitive HyperGraph)**.** Let S be a nonempty set of distinguished locations (e.g. landmarks or junctions). A *Cognitive HyperGraph* is a quadruple

$$\mathcal{CH} = (V,\, E,\, \ell_V,\, \ell_E),$$

where

- $V \subseteq S$ is a finite set of *nodes* (significant places),
- $E \subseteq \mathcal{P}^*(V)$ is a set of nonempty *hyperedges*, each hyperedge $e \in E$ being a subset of V of arbitrary cardinality,
- $\ell_V : V \to L_V$ is a *node–label function* assigning each $v \in V$ a semantic label (e.g. name, type, or feature vector),
- $\ell_E : E \to L_E$ is a *hyperedge–label function* assigning each $e \in E$ a label encoding relational information (e.g. type of spatial relation, corridor, or region).

Example 2.2 (University Campus Cognitive HyperGraph)**.** Let

$$S = \{\text{Library, LectureHall, Cafeteria, StudentUnion, Gymnasium, AdministrationBuilding, Park}\}$$

be the set of significant campus locations. We construct a Cognitive HyperGraph $\mathcal{CH} = (V, E, \ell_V, \ell_E)$ as follows:

- $V = S$, so every landmark is a node.
- Define hyperedges grouping related locations:

$$\begin{aligned} e_1 &= \{\text{Library, LectureHall}\}, & \ell_E(e_1) &= \text{“Academic Zone”}, \\ e_2 &= \{\text{Cafeteria, StudentUnion}\}, & \ell_E(e_2) &= \text{“Social Zone”}, \\ e_3 &= \{\text{Gymnasium, Park}\}, & \ell_E(e_3) &= \text{“Recreational Zone”}, \\ e_4 &= \{\text{AdministrationBuilding}\}, & \ell_E(e_4) &= \text{“Admin Cluster”}. \end{aligned}$$

  Thus

$$E = \{\, e_1, e_2, e_3, e_4\} \subseteq \mathcal{P}^*(V).$$

- The node–label function ℓ_V assigns each $v \in V$ its name:

$$\ell_V(v) = v.$$

- The hyperedge–label function ℓ_E is given above, encoding each zone's type.

Interpretation: This Cognitive HyperGraph models how a student mentally groups campus buildings into functional zones—academic, social, recreational, and administrative—thereby supporting navigation and planning across the campus.

Theorem 2.3 (Cognitive Graphs as Special Cases). *Every Cognitive Graph $G = (V, E_2, \ell_V, \ell_E)$, where $E_2 \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$, can be viewed as a Cognitive HyperGraph by treating its edges as 2-element hyperedges.*

Proof. Let $G = (V, E_2, \ell_V, \ell_E)$ be a Cognitive Graph. Define

$$E = E_2 \subseteq \mathcal{P}^*(V),$$

since each $\{u, v\} \in E_2$ is a nonempty subset of V. Then $\mathcal{CH} = (V, E, \ell_V, \ell_E)$ satisfies all conditions of a Cognitive HyperGraph. Hence G is embedded as the special case in which every hyperedge has cardinality two. □

Theorem 2.4 (Cognitive HyperGraphs Are Hypergraphs). *If $\mathcal{CH} = (V, E, \ell_V, \ell_E)$ is a Cognitive HyperGraph, then the pair (V, E) forms a (finite) hypergraph.*

Proof. By definition, V is a finite set and $E \subseteq \mathcal{P}^*(V)$ is a collection of nonempty subsets of V. This exactly matches the definition of a finite hypergraph $H = (V, E)$. Therefore, (V, E) is a hypergraph. □

Theorem 2.5 (Intersection of Cognitive HyperGraphs). *Let $\mathcal{CH}_1 = (V, E_1, \ell_V, \ell_E)$ and $\mathcal{CH}_2 = (V, E_2, \ell_V, \ell_E)$ be two Cognitive HyperGraphs on the same vertex set V with identical label functions. Then*

$$E = E_1 \cap E_2$$

together with ℓ_V and $\ell_E\big|_E$ defines a Cognitive HyperGraph $\mathcal{CH} = (V, E, \ell_V, \ell_E)$.

Proof. Since $E_1, E_2 \subseteq \mathcal{P}^*(V)$, their intersection E is also a collection of nonempty subsets of V. Labels restrict consistently because for every $e \in E$, $\ell_E(e)$ is the same in both $\mathcal{CH}_1$ and $\mathcal{CH}_2$. Thus (V, E) with those label functions satisfies the definition of a Cognitive HyperGraph. □

Theorem 2.6 (Union of Cognitive HyperGraphs). *Let $\mathcal{CH}_1 = (V, E_1, \ell_V, \ell_E)$ and $\mathcal{CH}_2 = (V, E_2, \ell_V, \ell_E)$ be as above. Then*

$$E = E_1 \cup E_2$$

with ℓ_V and $\ell_E\big|_E$ also defines a Cognitive HyperGraph $\mathcal{CH} = (V, E, \ell_V, \ell_E)$.

Proof. The union $E_1 \cup E_2$ remains a collection of nonempty subsets of V. Labels agree on overlaps and are well-defined on new hyperedges since ℓ_E was already specified on both E_1 and E_2. Hence (V, E) is a valid Cognitive HyperGraph. □

Theorem 2.7 (2-Section is a Cognitive Graph). *Let $\mathcal{CH} = (V, E, \ell_V, \ell_E)$ be a Cognitive HyperGraph. Its* 2-section $G = (V, E_2, \ell_V, \ell'_E)$, *where*

$$E_2 = \{\{u, v\} \mid \exists\, e \in E,\ \{u, v\} \subseteq e\},$$

and $\ell'_E(\{u, v\}) = \{\ell_E(e) \mid e \in E,\ \{u, v\} \subseteq e\}$, *is a Cognitive Graph.*

Proof. By construction, $E_2 \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$. Each edge $\{u, v\}$ inherits labels from all hyperedges containing it; we collect these in a set $\ell'_E(\{u, v\})$. Since ℓ_V remains unchanged on vertices, $G = (V, E_2, \ell_V, \ell'_E)$ satisfies the definition of a Cognitive Graph. □

Theorem 2.8 (Dual Cognitive HyperGraph). *Let $\mathcal{CH} = (V, E, \ell_V, \ell_E)$ be a Cognitive HyperGraph. Define its* dual *by*

$$V^* = E, \quad E^* = \{\, \{\, e \in E : v \in e\} \mid v \in V \} \ \subseteq \mathcal{P}^*(E),$$

with label functions $\ell^*_V(e) = \ell_E(e)$ and $\ell^*_E(\{e \mid v \in e\}) = \ell_V(v)$. Then $\mathcal{CH}^* = (V^*, E^*, \ell^*_V, \ell^*_E)$ is a Cognitive HyperGraph.

Proof. By definition, $V^* = E \subseteq \mathcal{P}^*(V)$ and each element of E^* is a nonempty subset of E. The dual labels are well-defined by swapping the original node and hyperedge labels. Therefore $\mathcal{CH}^*$ meets all requirements of a Cognitive HyperGraph. □

3. Result: Cognitive SuperHyperGraph

In this section, we present the formal definition of the Cognitive SuperHyperGraph (cf. [7]).

Definition 3.1 (Cognitive n-SuperHyperGraph). Let S be a nonempty base set and let $n \geq 0$ be an integer. Define the iterated powersets by

$$P^0(S) = S, \qquad P^{k+1}(S) = \mathcal{P}\big(P^k(S)\big) \quad (k \geq 0).$$

A *Cognitive n-SuperHyperGraph* is a quadruple

$$\mathcal{C}\mathrm{SH}^{(n)} = (V,\, E,\, \ell_V,\, \ell_E),$$

where

- $V \subseteq P^n(S)$ is a finite set of *n-supervertices*,
- $E \subseteq \mathcal{P}(V)$ is a finite family of *n-superedges* (incidence constraint: every edge $R \in E$ satisfies $R \subseteq V$),
- $\ell_V : V \to L_V$ assigns a label to each supervertex,
- $\ell_E : E \to L_E$ assigns a label to each superedge.

Equivalently, edges are subsets of the vertex set at the same level: $E \subseteq \mathcal{P}(V)$.

Example 3.2 (Smart City Cognitive 2-SuperHyperGraph)**.** Consider a small smart city divided into four *blocks*:

$$S = \{\text{Block A}, \text{Block B}, \text{Block C}, \text{Block D}\}.$$

First level $P^1(S) = \mathcal{P}(S)$: neighborhoods

$$N_1 = \{\text{Block A}, \text{Block B}\}, \quad N_2 = \{\text{Block C}, \text{Block D}\}, \quad N_3 = \{\text{Block B}, \text{Block C}\}.$$

Second level $P^2(S) = \mathcal{P}\big(P^1(S)\big)$: districts

$$D_1 = \{N_1, N_2\}, \qquad D_2 = \{N_2, N_3\}.$$

Define the 2-supervertices and 2-superedges by

$$V = \{D_1, D_2\} \subseteq P^2(S), \qquad R_1 = \{D_1\},\ R_2 = \{D_1, D_2\}, \qquad E = \{R_1, R_2\} \subseteq \mathcal{P}(V).$$

Labels:

$$\ell_V(D_1) = \text{“Commercial District”}, \quad \ell_V(D_2) = \text{“Residential District”},$$

$$\ell_E(R_1) = \text{“Downtown Zone”}, \quad \ell_E(R_2) = \text{“Greater Metro Area”}.$$

Thus $\mathcal{C}\mathrm{SH}^{(2)} = (V, E, \ell_V, \ell_E)$ respects $E \subseteq \mathcal{P}(V)$.

Example 3.3 (University Course Planning as a Cognitive 2-SuperHyperGraph)**.** Let the base set of modules be

$$S = \{\text{IntroCS}, \text{DataStr}, \text{CalcI}, \text{CalcII}, \text{PhysI}, \text{PhysII}\}.$$

Level 1 (paths):

$$P_{\text{Core}} = \{\text{IntroCS}, \text{DataStr}, \text{CalcI}\}, \quad P_{\text{Sci}} = \{\text{CalcII}, \text{PhysI}, \text{PhysII}\}.$$

Level 2 (tracks):

$$\text{Track}_{\text{Eng}} = \{P_{\text{Core}}, P_{\text{Sci}}\}, \qquad \text{Track}_{\text{Sci}} = \{P_{\text{Sci}}\}.$$

Take

$$V = \{\text{Track}_{\text{Eng}},\ \text{Track}_{\text{Sci}}\} \subseteq P^2(S),$$

and define degree programs as edges on V:

$$\text{Degree}_{\text{BScCS}} = \{\text{Track}_{\text{Eng}}\}, \quad \text{Degree}_{\text{BScPhys}} = \{\text{Track}_{\text{Sci}}\},$$

$$E = \{\text{Degree}_{\text{BScCS}},\ \text{Degree}_{\text{BScPhys}}\} \subseteq \mathcal{P}(V).$$

Labels:

$$\ell_V(\text{Track}_{\text{Eng}}) = \text{“CS + Science Track”}, \quad \ell_V(\text{Track}_{\text{Sci}}) = \text{“Science Track”},$$

$$\ell_E(\text{Degree}_{\text{BScCS}}) = \text{“Bachelor of Science in CS”},$$

$$\ell_E(\text{Degree}_{\text{BScPhys}}) = \text{“Bachelor of Science in Physics”}.$$

Example 3.4 (Global Spatial Cognitive 3-SuperHyperGraph)**.** Let

$$S = \{\text{Tokyo}, \text{Osaka}, \text{Kyoto}, \text{New York}, \text{Los Angeles}, \text{London}, \text{Paris}\}.$$

Build

$P^1(S):\ C_1 = \{\text{Tokyo}, \text{Osaka}, \text{Kyoto}\},\ C_2 = \{\text{New York}, \text{Los Angeles}\},\ C_3 = \{\text{London}, \text{Paris}\};$

$P^2(S):\ K_1 = \{C_1\},\ K_2 = \{C_2\},\ K_3 = \{C_3\};$

$P^3(S):\ D_1 = \{K_1, K_2\},\ D_2 = \{K_3\}.$

Set

$$V = \{D_1, D_2\} \subseteq P^3(S), \qquad R = \{D_1, D_2\}, \quad E = \{R\} \subseteq \mathcal{P}(V).$$

Labels:

$\ell_V(D_1) =$ "Asia–Americas Cluster", $\ell_V(D_2) =$ "Europe Cluster", $\ell_E(R) =$ "Global Network".

Example 3.5 (Corporate Organizational Structure as a 3-SuperHyperGraph)**.** Let

$$S = \{\text{Ayano}, \text{Taro}, \text{Hiroko}, \text{Dave}, \text{Eve}, \text{Frank}\}.$$

Teams ($P^1(S)$):

$$T_1 = \{\text{Ayano}, \text{Taro}, \text{Hiroko}\}, \quad T_2 = \{\text{Dave}, \text{Eve}, \text{Frank}\}.$$

Departments ($P^2(S)$):

$$D_{\text{Tech}} = \{T_1\}, \qquad D_{\text{Ops}} = \{T_2\}.$$

Divisions ($P^3(S)$):

$$\text{Div}_{\text{Tech}} = \{D_{\text{Tech}}\},$$

$$\text{Div}_{\text{Ops}} = \{D_{\text{Ops}}\}.$$

Then

$$V = \{\text{Div}_{\text{Tech}},\ \text{Div}_{\text{Ops}}\} \subseteq P^3(S),$$

$$C = \{\text{Div}_{\text{Tech}}, \text{Div}_{\text{Ops}}\},$$

$$E = \{C\} \subseteq \mathcal{P}(V).$$

Labels:

$$\ell_V(\text{Div}_{\text{Tech}}) = \text{"Technology Division"},$$

$$\ell_V(\text{Div}_{\text{Ops}}) = \text{"Operations Division"},$$

$$\ell_E(C) = \text{"Corporate Structure"}.$$

Theorem 3.6 (Reduction to Cognitive HyperGraph and Cognitive Graph)**.** *Let* $\mathcal{C}\mathrm{SH}^{(n)} = (V, E, \ell_V, \ell_E)$ *be a Cognitive* n*-SuperHyperGraph on* S*.*

(i) If $n = 1$*, then* $P^1(S) = \mathcal{P}(S)$ *and* $\mathcal{C}\mathrm{SH}^{(1)}$ *is a Cognitive HyperGraph (with* $E \subseteq \mathcal{P}(V)$*).*

(ii) If $n = 0$ and, in addition, $E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$, then $\mathcal{C}\mathrm{SH}^{(0)}$ is a Cognitive Graph.

Proof. (i) When $n = 1$, vertices lie in $\mathcal{P}(S)$ and edges are subsets of V, which is the hypergraph case with labels. (ii) When $n = 0$, $V \subseteq S$ and edges are unordered pairs of distinct vertices, yielding a (labeled) graph. □

Theorem 3.7 (Underlying n-SuperHyperGraph)**.** *If $\mathcal{C}\mathrm{SH}^{(n)} = (V, E, \ell_V, \ell_E)$ is a Cognitive n-SuperHyperGraph on S, then (V, E) is an n-SuperHyperGraph on S (with $E \subseteq \mathcal{P}(V)$).*

Proof. By definition $V \subseteq P^n(S)$ and $E \subseteq \mathcal{P}(V)$; hence (V, E) satisfies the n-SuperHyperGraph axioms. □

Theorem 3.8 (Intersection of Cognitive n-SuperHyperGraphs)**.** *Let $\{\mathcal{C}\mathrm{SH}^{(n)}_\alpha = (V_\alpha, E_\alpha, \ell_{V,\alpha}, \ell_{E,\alpha})\}_{\alpha \in A}$ be Cognitive n-SuperHyperGraphs on the same base set S. Define*

$$V := \bigcap_{\alpha \in A} V_\alpha, \qquad E := \bigcap_{\alpha \in A} E_\alpha,$$

and let ℓ_V, ℓ_E be the restrictions of the labelings to V, E, respectively, assuming label agreement on overlaps. Then $\mathcal{C}\mathrm{SH}^{(n)} = (V, E, \ell_V, \ell_E)$ is a Cognitive n-SuperHyperGraph.

Proof. For any $R \in E$, we have $R \in E_\alpha$ for all α, hence $R \subseteq V_\alpha$ for all α. Therefore $R \subseteq \bigcap_\alpha V_\alpha = V$, so $E \subseteq \mathcal{P}(V)$. Since $V \subseteq P^n(S)$, the claim follows. □

Theorem 3.9 (Homomorphic Image)**.** *Let $\mathcal{C}\mathrm{SH}^{(n)}_S = (V_S, E_S, \ell_{V,S}, \ell_{E,S})$ on S and $\mathcal{C}\mathrm{SH}^{(n)}_T = (V_T, E_T, \ell_{V,T}, \ell_{E,T})$ on T. A* homomorphism *is a map $f\colon V_S \to V_T$ such that the induced map on edges*

$$f_*\colon \mathcal{P}(V_S) \to \mathcal{P}(V_T), \qquad f_*(R) := \{\, f(v) \mid v \in R \,\},$$

satisfies $f_*(E_S) \subseteq E_T$, and labels are preserved: $\ell_{V,T} \circ f = \ell_{V,S}$ on V_S, and $\ell_{E,T} \circ f_* = \ell_{E,S}$ on E_S. Then $\big(f(V_S),\, f_*(E_S)\big)$ with the inherited labels is a Cognitive n-SuperHyperGraph on T.

Proof. Because $f(V_S) \subseteq V_T$ and $f_*(E_S) \subseteq \mathcal{P}(f(V_S)) \subseteq \mathcal{P}(V_T)$, the image respects the incidence constraint. Label compatibility gives well-defined labels on the image. □

Theorem 3.10 (Hierarchy of Levels)**.** *Let $\mathcal{C}\mathrm{SH}^{(n)} = (V, E, \ell_V, \ell_E)$ be a Cognitive n-SuperHyperGraph on S. For each k with $0 \leq k \leq n$, set*

$$V_k := V \cap P^k(S), \qquad E_k := \{\, R \in E \mid R \subseteq V_k \,\},$$

and define $\ell_{V,k} := \ell_V|_{V_k}$, $\ell_{E,k} := \ell_E|_{E_k}$. Then $\mathcal{C}\mathrm{SH}^{(k)} = (V_k, E_k, \ell_{V,k}, \ell_{E,k})$ is a Cognitive k-SuperHyperGraph on S.

Proof. Clearly $V_k \subseteq P^k(S)$. By definition of E_k, every edge in E_k is a subset of V_k, hence $E_k \subseteq \mathcal{P}(V_k)$. Label restrictions are well-defined, so the claim follows. □

4. Conclusion

In this paper, we introduced two extended models: the *Cognitive HyperGraph* and the *Cognitive SuperHyperGraph*, which generalize the classical cognitive graph framework by employing hypergraph and superhypergraph theory (cf. [7]).

For future research, we aim to explore the design of graph algorithms tailored to these new structures, as well as potential extensions by incorporating other advanced mathematical frameworks such as MetaGraphs [29, 30], Fuzzy Sets [31–33], Bidirected Graphs [34–36], Neutrosophic Sets [37–39], Hyperfuzzy Sets [40–43], Soft Sets [44,45], Near Sets [46–48], Plithogenic Sets [49–51], and Intuitionistic Fuzzy Sets [52, 53].

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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