NCML

Neutrosophic Computing and Machine Learning, Vol. **44**, 2026

University of New Mexico

Basic Theory of Hierarchical SuperHyperGraphs: Cross-Level Incidence, Direction, and Bidirection

Takaaki Fujita[1,*], Ajoy Kanti Das[2], Suman Das[3], Sankar Prasad Mondal[4]

[1*] Independent Researcher, Tokyo, Japan. (Takaaki.fujita060@gmail.com).
https://orcid.org/0000-0002-9380-386X.

[2] Associate Professor, Department of Mathematics, Tripura University, Agartala-799022, Tripura, India.
(ajoykantidas@gmail.com). https://orcid.org/0000-0002-9326-1677.

[3] Assistant Professor Grade II (Mathematics), Department of Education, National Institute of Technology Calicut, Kozhikode-673601, Kerala, India. (dr.sumandas1995@gmail.com).
https://orcid.org/0000-0001-5682-9334.

[4] Department of Applied Mathematics, Maulana Abul Kalam Azad University of Technology, West Bengal, Haringhata-741249, West Bengal, India. (sankar.mondal02@gmail.com) .
https://orcid.org/0000-0003-4690-2598.

*Correspondence: Takaaki.fujita060@gmail.com

Abstract. Hypergraphs extend ordinary graphs by permitting a single hyperedge to connect any nonempty set of vertices, thereby representing intrinsically multiway interactions. To capture hierarchical or multi-layer organization, SuperHyperGraphs further iterate the powerset construction, so that aggregates formed at one level can themselves act as vertices or edge endpoints at higher levels. While this modeling perspective is increasingly relevant, a systematic treatment of *hierarchical* SuperHyperGraphs remains at an early stage and many foundational notions are not yet widely standardized. Addressing this gap, we introduce and investigate three hierarchical extensions: *Hierarchical SuperHyperGraphs*, *Hierarchical Directed SuperHyperGraphs*, and *Hierarchical Bidirected SuperHyperGraphs*.

Keywords: SuperHyperGraph, Hierarchical Superhypergraphs, Hierarchical Directed SuperHyperGraphs, Hierarchical Bidirected SuperHyperGraphs

---

1. Introduction

1.1. *SuperHyperGraph theory*

Graphs are a classical and widely used abstraction for relational data: vertices represent entities, and edges encode *binary* relations between pairs of entities [1]. Various related extensions, including *Fuzzy Graphs* [2, 3] and *Neutrosophic Graphs* [4–6], are also known. In many contemporary applications, however, interactions are not merely pairwise. Collaborative teams, multi-agent protocols, and shared-resource systems routinely exhibit *higher-order* dependence, where a single interaction simultaneously involves three or more participants. In such settings, ordinary graphs may be structurally inadequate. A finite *hypergraph* addresses this limitation by allowing a *hyperedge* to join an arbitrary nonempty subset of vertices, thereby representing genuinely multiway relations within a single combinatorial object [7–9]. A wide variety of extensions and related hypergraph structures have also been studied, including *Fuzzy Hypergraphs* [10, 11], *Intuitionistic Fuzzy Hypergraphs* [12–14], *Hypertrees* [15–17], *Soft Hypergraphs* [18, 19], *Meta-HyperGraphs* [20], *Neutrosophic Hypergraphs* [21, 22], and *Directed Hypergraphs* [23, 24]. Hypergraphs are also widely used in applications such as *Hypergraph Partitioning* [25–27] and *Hypergraph Neural Networks* [8, 28, 29], further highlighting the importance of research on hypergraph structures and their theoretical and computational properties.

Higher arity alone does not capture another ubiquitous feature of modern data: *hierarchical* or *multi-scale* organization. Typical examples include modules composed of submodules, communities nested within larger communities, and layered service architectures. A natural set-theoretic mechanism for modeling such nesting is to iterate the powerset operator. This idea motivates finite *SuperHyperGraphs*, in which set-valued aggregates produced at one level may serve as vertices (and participate in edges) at higher levels. Consequently, one obtains supervertices and superhyperedges that encode both multiway interaction (as in hypergraphs) and aggregation across scales (groups, groups of groups, and so forth) within a unified framework [30–32]. Related hierarchical encodings also appear in complex-network modeling and applied sciences [33, 34]. Unless stated otherwise, the index n in $\mathcal{P}^n(\cdot)$ and the level of an n-SuperHyperGraph are taken to be nonnegative integers. A comparison of Graphs, HyperGraphs, and SuperHyperGraphs is presented in Table 1.

A further direction studied in the literature is that of *hierarchical superhypergraphs* [32, 35–37]. Informally, these are superhypergraphs in which vertices may inhabit multiple powerset levels and superhyperedges are permitted to connect *mixed-level* supervertices. To maintain coherence across levels, one typically imposes a downward-closure condition: whenever a higher-level aggregate is included as a vertex, its lower-level constituents are included as well. This requirement prevents multilevel objects from being introduced in isolation and ensures consistency

Table 1. Comparison of Graphs, HyperGraphs, and SuperHyperGraphs.

Aspect	**Graph**	**HyperGraph**	**SuperHyperGraph**
Vertices	Atomic vertices.	Atomic vertices.	Set-valued or higher-level supervertices.
Edges	Connect pairs of vertices.	Connect arbitrary nonempty sets of vertices.	Connect supervertices, including higher-level aggregates.
Main structure	Pairwise relations.	Higher-order relations.	Higher-order and hierarchical relations.
Typical representation	$G = (V, E)$.	$H = (V, E)$, $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$.	$\mathrm{SHG}^{(n)} = (V, E)$, $V \subseteq \mathcal{P}^n(V_0)$.

with their internal decomposition. For reference, the comparison between (finite) superhypergraphs and hierarchical superhypergraphs is summarized in Table 2.

1.2. *Directed Graphs and HyperGraphs*

Directed Graphs, Directed HyperGraphs, and Directed SuperHyperGraphs are also well-known graph-theoretic structures. These models are frequently used when directional information needs to be explicitly incorporated into a graph-based representation.

A directed graph uses oriented edges between vertex pairs, representing asymmetric relationships, flows, dependencies, or transitions from sources to targets [38,39]. A directed hypergraph uses directed hyperedges connecting multiple vertices, representing higher-order interactions with specified source sets and target sets simultaneously [40, 41]. A Directed SuperHyperGraph extends directed hypergraphs by using set-valued supervertices and directed superedges, thereby representing hierarchical, higher-order, and directional interactions across nested levels [32]. As in the undirected setting, these structures have also been extended and studied under various uncertainty-aware frameworks, including *Fuzzy Directed Graphs* [42, 43] and *Neutrosophic Directed Graphs* [44, 45]. Related graph concepts that extend directed graphs, such as *Bidirected Graphs* [46,47], are also known. A comparison of Directed Graphs, Directed HyperGraphs, and Directed SuperHyperGraphs is presented in Table 3.

1.3. *Our contributions*

The considerations above highlight the importance of SuperHyperGraph-based modeling. However, systematic work on *hierarchical* SuperHyperGraphs is still in its early stages, and many basic notions remain relatively unfamiliar to a broader audience. Motivated by this gap, we introduce and study three hierarchical extensions: *Hierarchical SuperHyperGraphs*,

TABLE 2. Concise comparison between (finite) superhypergraphs and hierarchical superhypergraphs.

Aspect	**Superhypergraph (e.g., n-SHG)**	**Hierarchical superhypergraph (height r)**
Vertex domain	$V \subseteq \mathcal{P}^n(V_0)$ (single fixed powerset level)	$V \subseteq \bigcup_{k=0}^{r} \mathcal{P}^{\langle k\rangle}(V_0)$ (multiple levels allowed)
Vertex levels	Uniform: all vertices have the same level n	Non-uniform: vertices may have mixed levels $\ell(x) \in \{0, \dots, r\}$
Edges	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Same form, but edges may be *cross-level* (contain vertices of different levels)
Hierarchy/coherence constraint	Not required by definition	Typically required: downward closure (if $X \in V$ with $\ell(X) \geq 1$, then its constituents are in V)
Interpretation	Multiway relations among same-level aggregates	Multiway relations plus explicit *multi-scale* aggregation (objects, groups, groups of groups, ...)
Typical use	Higher-order interaction at a fixed abstraction level	Joint modeling of interaction *and* hierarchical organization across scales

Hierarchical Directed SuperHyperGraphs, and *Hierarchical Bidirected SuperHyperGraphs*. For each structure, we provide precise definitions and investigate foundational properties, with the aim of clarifying their set-theoretic organization and their ability to represent cross-level interactions in a coherent way. A comparison of Directed SuperHyperGraphs and Hierarchical Directed SuperHyperGraphs is presented in Table 4.

2. **Preliminaries**

This section establishes notation and summarizes the set-theoretic and combinatorial notions needed in the sequel. Throughout, all sets that serve as vertex domains are assumed to be finite, and we write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

TABLE 3. Comparison of Directed Graphs, Directed HyperGraphs, and Directed SuperHyperGraphs.

Aspect	**Directed Graph**	**Directed HyperGraph**	**Directed SuperHyperGraph**
Vertices	Atomic vertices.	Atomic vertices.	Set-valued supervertices.
Directed relations	Connect ordered pairs of vertices.	Connect source and target sets of vertices.	Connect source and target sets of supervertices.
Interaction type	Pairwise and directional.	Higher-order and directional.	Hierarchical, higher-order, and directional.
Main feature	Directed binary relations.	Directed multiway relations.	Directed multiway relations among higher-level aggregates.

TABLE 4. Comparison of Directed SuperHyperGraphs and Hierarchical Directed SuperHyperGraphs.

Aspect	**Directed SuperHyperGraph**	**Hierarchical Directed SuperHyperGraph**
Vertex structure	Supervertices belong to a fixed powerset level.	Supervertices may belong to multiple hierarchical levels.
Directed edges	Directed superedges connect same-level supervertices.	Directed superedges may connect different levels.
Hierarchy	No explicit hierarchy condition is required.	Downward closure preserves hierarchical coherence.
Main feature	Directed higher-order interactions.	Directed higher-order and cross-level interactions.

2.1. *SuperHyperGraphs*

We begin with the set-theoretic constructions underlying SuperHyperGraphs and then recall the standard hypergraph terminology.

Definition 2.1 (Base set)**.** A *base set* is a fixed nonempty set S whose elements represent ground-level objects. Higher-level objects are obtained from S by applying the powerset operator, possibly iterated.

Definition 2.2 (Powerset)**.** (see, e.g., [48]) For a set S, the *powerset* of S is

$$\mathcal{P}(S) := \{ A \mid A \subseteq S \}.$$

In particular, $\emptyset \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$.

Definition 2.3 (n-fold iterated powerset)**.** [49] Let X be a set. Define $\mathcal{P}^0(X) := X$, and for $n \in \mathbb{N}_0$ set

$$\mathcal{P}^{n+1}(X) \;:=\; \mathcal{P}\big(\mathcal{P}^n(X)\big).$$

When we exclude the empty set at the outermost stage, we write

$$\mathcal{P}^n_*(X) \;:=\; \mathcal{P}^n(X) \setminus \{\emptyset\}.$$

Definition 2.4 (Hypergraph)**.** [50,51] A *hypergraph* is an ordered pair $H = (V, E)$ such that

- V is a finite set whose elements are called *vertices*, and
- E is a finite set of nonempty subsets of V, i.e.,

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

  Elements of E are called *hyperedges*.

Hence a hyperedge may contain more than two vertices, allowing the representation of multiway relations.

Example 2.5 (A concrete hypergraph)**.** Consider the hypergraph

$$H = (V, E),$$

where

$$V = \{v_1, v_2, v_3, v_4\}$$

and

$$E = \{e_1, e_2, e_3\},$$

with

$$e_1 = \{v_1, v_2, v_3\}, \qquad e_2 = \{v_2, v_4\}, \qquad e_3 = \{v_1, v_3, v_4\}.$$

Then each e_i is a nonempty subset of V, so indeed

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Hence $H = (V, E)$ is a hypergraph.

This example illustrates the difference between ordinary graphs and hypergraphs. In particular, the hyperedges e_1 and e_3 each join three vertices at once, so they represent genuinely multiway relations rather than merely pairwise ones.

Figure 1 shows an incidence representation of this hypergraph.

For example, since

$$v_1, v_2, v_3 \in e_1,$$

the hyperedge e_1 connects the three vertices v_1, v_2, v_3 simultaneously. Likewise,

$$v_2, v_4 \in e_2 \qquad \text{and} \qquad v_1, v_3, v_4 \in e_3.$$

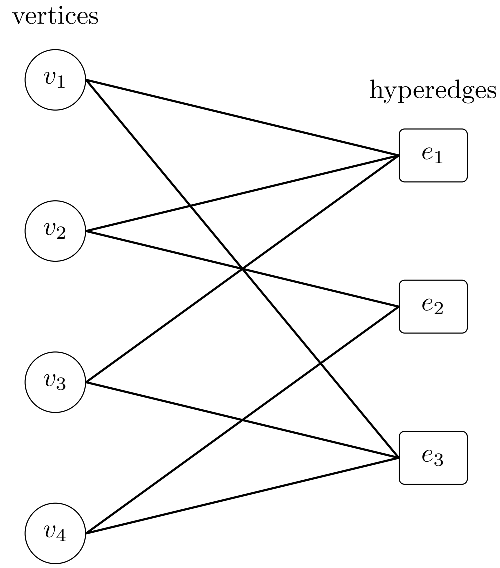

FIGURE 1. An incidence representation of the hypergraph $H = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$, $e_1 = \{v_1, v_2, v_3\}$, $e_2 = \{v_2, v_4\}$, and $e_3 = \{v_1, v_3, v_4\}$.

Therefore, this hypergraph contains both a two-vertex hyperedge and three-vertex hyperedges, illustrating the flexibility of the hypergraph model.

Definition 2.6 (n-SuperHyperGraph)**.** (see [30]) Let V_0 be a finite nonempty base set and let $n \in \mathbb{N}_0$. An *n-SuperHyperGraph on* V_0 is a pair

$$\mathrm{SHG}^{(n)} = (V, E)$$

satisfying

$$V \subseteq \mathcal{P}^n(V_0) \qquad \text{and} \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Elements of V are called *n-supervertices*, and elements of E are called *superhyperedges* (equivalently, each superhyperedge is a nonempty subset of the supervertex set V).

Example 2.7 (A concrete 2-SuperHyperGraph)**.** Let $V_0 = \{1, 2, 3\}$ and take $n = 2$. Then $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$ and $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$.
Define five 2-supervertices (each is a set of subsets of V_0):

$$v_1 = \{\{1\}\}, \quad v_2 = \{\{2\}\}, \quad v_3 = \{\{1\}, \{2\}\}, \quad v_4 = \{\{2, 3\}\}, \quad v_5 = \{\{1, 2, 3\}\}.$$

Set

$$V = \{v_1, v_2, v_3, v_4, v_5\} \subseteq \mathcal{P}^2(V_0).$$

Define two superhyperedges

$$e_1 = \{v_1, v_2, v_3\}, \qquad e_2 = \{v_3, v_4, v_5\},$$

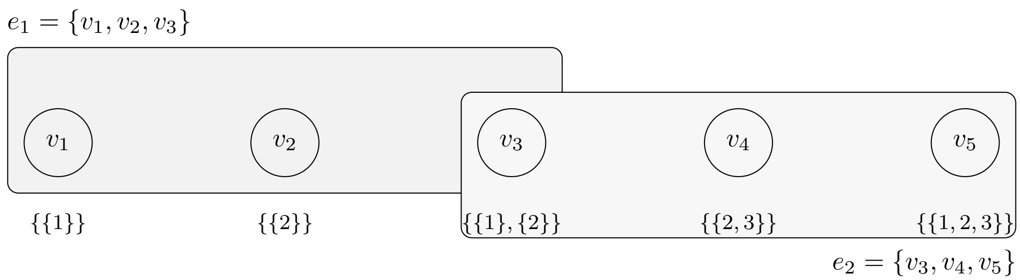

FIGURE 2. A concrete 2-SuperHyperGraph $\mathrm{SHG}^{(2)} = (V, E)$ on $V_0 = \{1, 2, 3\}$. Vertices are 2-supervertices in $\mathcal{P}^2(V_0)$, and each superhyperedge is a nonempty subset of V.

and set $E = \{e_1, e_2\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Then $\mathrm{SHG}^{(2)} = (V, E)$ is a 2-SuperHyperGraph on V_0. A schematic diagram is given in Figure 2.

2.2. *Directed SuperHyperGraph and Bidirected SuperHyperGraph*

A directed superhypergraph uses ordered superedges (Tail, Head) to model directional higher-order interactions from a tail set of supervertices to a head set, whereas a bidirected superhypergraph assigns each supervertex–superedge incidence a sign in $\{-1, 0, 1\}$ and requires every superedge to satisfy a balancing condition (signed sum = 0).

Definition 2.8 (Directed n-SuperHyperGraph)**.** (cf. [30]) Let S be a nonempty *base set* and let $n \geq 0$ be an integer. Define the iterated powersets by

$$\mathcal{P}^0(S) \coloneqq S, \qquad \mathcal{P}^{k+1}(S) \coloneqq \mathcal{P}\big(\mathcal{P}^k(S)\big) \quad (k \geq 0).$$

A *directed n-SuperHyperGraph* is a pair

$$\mathrm{DSHG}^{(n)} = (V, E),$$

where $V \subseteq \mathcal{P}^n(S)$ is the set of *n-supervertices*, and E is a set of *directed n-superedges*. Each directed n-superedge $e \in E$ is an ordered pair

$$e = (T, H),$$

where $T, H \subseteq V$. We write $\mathrm{Tail}(e) \coloneqq T$ and $\mathrm{Head}(e) \coloneqq H$, and interpret e as carrying a directed interaction (or "flow") from the entire tail set $\mathrm{Tail}(e)$ into the head set $\mathrm{Head}(e)$. In many applications one additionally assumes $\mathrm{Tail}(e) \neq \emptyset$ and $\mathrm{Head}(e) \neq \emptyset$.

Example 2.9 (A concrete directed 1-SuperHyperGraph)**.** Let $S = \{1, 2, 3, 4\}$ and take $n = 1$. Then $\mathcal{P}^1(S) = \mathcal{P}(S)$. Define the vertex set of 1-supervertices by

$$V = \big\{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}\big\} \subseteq \mathcal{P}(S).$$

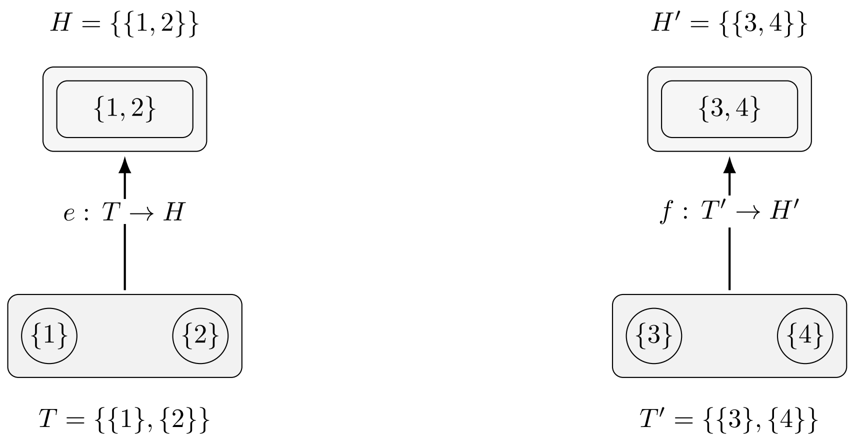

FIGURE 3. A directed 1-SuperHyperGraph $\mathrm{DSHG}^{(1)} = (V, E)$ with two directed superedges $e = (T, H)$ and $f = (T', H')$, where arrows indicate directed aggregation from tail sets to head sets.

Consider the directed 1-superedge

$$e = (T, H), \qquad T = \big\{\{1\}, \{2\}\big\} \subseteq V, \quad H = \big\{\{1, 2\}\big\} \subseteq V,$$

which represents a directed aggregation from the tail set $\{\,\{1\}, \{2\}\,\}$ into the head set $\{\,\{1, 2\}\,\}$. Add another directed 1-superedge

$$f = (T', H'), \qquad T' = \big\{\{3\}, \{4\}\big\} \subseteq V, \quad H' = \big\{\{3, 4\}\big\} \subseteq V.$$

Then $\mathrm{DSHG}^{(1)} = (V, E)$ with $E = \{e, f\}$ is a directed 1-SuperHyperGraph. The concrete example is illustrated in Figure 3.

Definition 2.10 (Bidirected n-SuperHyperGraph)**.** Let S be a nonempty *base set* and let $n \geq 0$ be an integer. Define the iterated powersets by

$$\mathcal{P}^0(S) := S, \qquad \mathcal{P}^{k+1}(S) := \mathcal{P}\big(\mathcal{P}^k(S)\big) \quad (k \geq 0).$$

Let $V \subseteq \mathcal{P}^n(S)$ be a nonempty set of *n-supervertices*, and let

$$\mathcal{P}^*(V) := \mathcal{P}(V) \setminus \{\emptyset\}$$

denote the nonempty powerset of V. A *bidirected n-SuperHyperGraph* is a triple with incidence signs

$$\mathcal{H}^{(n)} = (V, E, \tau),$$

where $E \subseteq \mathcal{P}^*(V)$ is a family of *n-superedges* and $\tau : V \times E \to \{-1, 0, 1\}$ is a *bidirection function* such that

$$\tau(v, e) = 0 \iff v \notin e,$$

and for every superedge $e \in E$ the *balancing condition*

$$\sum_{v \in e} \tau(v, e) = 0$$

holds. (Here $\tau(v, e) = +1$ and $\tau(v, e) = -1$ may be interpreted as opposite local incidence orientations of v at e.)

Example 2.11 (A concrete bidirected 2-SuperHyperGraph)**.** Let $S = \{1, 2\}$ and $n = 2$. Then $\mathcal{P}^1(S) = \mathcal{P}(S)$ and $\mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S))$. Choose the 2-supervertex set

$$V = \{v_1, v_2, v_3, v_4\} \subseteq \mathcal{P}^2(S),$$

where

$$v_1 = \{\{1\}\}, \quad v_2 = \{\{2\}\}, \quad v_3 = \{\{1\}, \{2\}\}, \quad v_4 = \{\{1, 2\}\}.$$

Define two 2-superedges

$$e = \{v_1, v_2, v_3, v_4\}, \qquad f = \{v_1, v_3\},$$

so $E = \{e, f\} \subseteq \mathcal{P}^*(V)$. Define $\tau : V \times E \to \{-1, 0, 1\}$ by

$$\tau(v_1, e) = +1,\ \tau(v_2, e) = +1,\ \tau(v_3, e) = -1,\ \tau(v_4, e) = -1,$$

$$\tau(v_1, f) = +1,\ \tau(v_3, f) = -1,$$

and $\tau(v, g) = 0$ for all remaining pairs (v, g) (equivalently, whenever $v \notin g$). Then the balancing condition holds:

$$\sum_{v \in e} \tau(v, e) = 1 + 1 - 1 - 1 = 0, \qquad \sum_{v \in f} \tau(v, f) = 1 - 1 = 0.$$

Hence $\mathcal{H}^{(2)} = (V, E, \tau)$ is a bidirected 2-SuperHyperGraph.

3. **Review: Hierarchical Undirected SuperHyperGraphs**

A hierarchical SuperHyperGraph permits supervertices from several levels of an iterated powerset hierarchy and allows a superhyperedge to contain supervertices from different levels, while imposing a downward-closure condition that preserves the internal hierarchical structure.

Definition 3.1 (Hierarchical SuperHyperGraph of height r)**.** Let S be a finite nonempty base set and let

$$r \in \mathbb{N}_0.$$

Define the *nonempty iterated powerset tower*

$$\left(\mathcal{P}^{\langle k \rangle}(S)\right)_{k=0}^{r}$$

recursively by

$$\mathcal{P}^{\langle 0 \rangle}(S) := S,$$

and

$$\mathcal{P}^{\langle k+1\rangle}(S) := \mathcal{P}\Big(\mathcal{P}^{\langle k\rangle}(S)\Big) \setminus \{\emptyset\} \qquad (0 \le k < r).$$

Thus every element of $\mathcal{P}^{\langle k+1\rangle}(S)$ is a nonempty set of elements of $\mathcal{P}^{\langle k\rangle}(S)$.

Define the *hierarchical universe of height* r by

$$\mathcal{U}_r(S) := \bigcup_{k=0}^{r} \mathcal{P}^{\langle k\rangle}(S).$$

For each

$$x \in \mathcal{U}_r(S),$$

define its *level* by

$$\ell(x) := \min\Big\{k \in \{0, 1, \ldots, r\} : x \in \mathcal{P}^{\langle k\rangle}(S)\Big\}.$$

A *hierarchical SuperHyperGraph of height* r *on* S is a pair

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

satisfying the following conditions.

(H1) (Hierarchical vertex set) The set V is a finite nonempty subset of the hierarchical universe:

$$\emptyset \neq V \subseteq \mathcal{U}_r(S).$$

Elements of V are called *hierarchical supervertices*.

(H2) (Cross-level superhyperedges) The superhyperedge family E satisfies

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Thus every

$$e \in E$$

is a nonempty subset of V. Elements of E are called *hierarchical superhyperedges*.

In particular, a superhyperedge may contain supervertices belonging to different hierarchical levels.

(H3) (Downward-closure coherence) If

$$X \in V \qquad \text{and} \qquad \ell(X) \ge 1,$$

then every immediate constituent of X is also a hierarchical supervertex. Equivalently,

$$X \subseteq V.$$

For every

$$k \in \{0, 1, \ldots, r\},$$

define the k-th vertex layer by

$$V_k := \{x \in V : \ell(x) = k\}.$$

Then

$$V = \dot{\bigcup}_{k=0}^{r} V_k,$$

where the dot denotes disjoint union.

Remark 3.2 (Possible level ambiguity)**.** Because the objects under consideration are sets, it is possible in principle that the same set-theoretic object belongs to more than one member of the powerset tower. The level function

$$\ell(x) = \min\left\{k : x \in \mathcal{P}^{\langle k\rangle}(S)\right\}$$

resolves such an ambiguity by assigning x to its smallest admissible level.
If complete separation of the hierarchical levels is required, one may instead replace the levels by tagged copies

$$\{k\} \times \mathcal{P}^{\langle k\rangle}(S),$$

which makes the levels pairwise disjoint by construction.

Example 3.3 (Grocery kits and weekly plans as a hierarchical SuperHyperGraph)**.** Consider a grocery-delivery platform that sells individual items, curated kits, and weekly meal plans. Let the base set be

$$S = \{m, c, p, s\},$$

where

$$m = \text{“milk”}, \qquad c = \text{“cereal”}, \qquad p = \text{“pasta”}, \qquad s = \text{“sauce”}.$$

Take

$$r = 2.$$

Define two level-1 supervertices

$$b := \{m, c\} \in \mathcal{P}^{\langle 1\rangle}(S), \qquad d := \{p, s\} \in \mathcal{P}^{\langle 1\rangle}(S),$$

representing two grocery kits.
Define the level-2 supervertex

$$w := \{b, d\} \in \mathcal{P}^{\langle 2\rangle}(S),$$

representing a weekly plan consisting of the two kits.
Set

$$V = \{m, c, p, s, b, d, w\} \subseteq \mathcal{U}_2(S).$$

The vertex layers are

$$V_0 = \{m, c, p, s\}, \qquad V_1 = \{b, d\}, \qquad V_2 = \{w\}.$$

Define

$$e_{\text{bk}} := \{m, c, b\}, \qquad e_{\text{pa}} := \{p, s, d\},$$

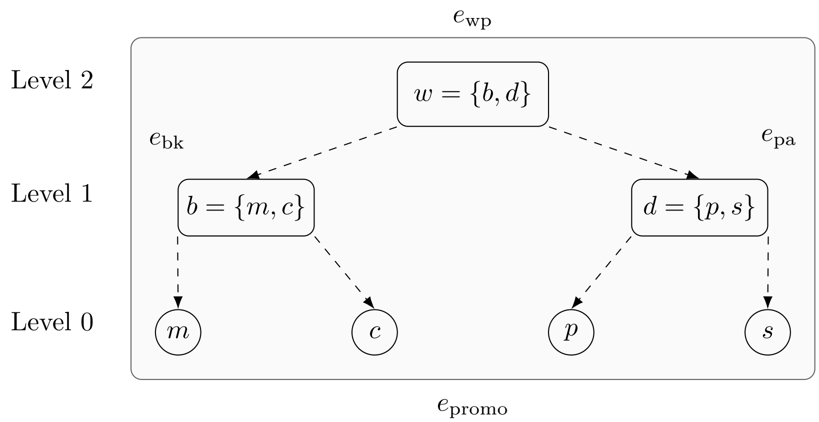

FIGURE 4. A hierarchical SuperHyperGraph of height 2 representing grocery items at level 0, grocery kits at level 1, and a weekly plan at level 2. Dashed arrows indicate immediate-constituent relations, while rounded boxes represent cross-level superhyperedges.

$$e_{\mathrm{wp}} := \{b, d, w\}, \qquad e_{\mathrm{promo}} := \{m, c, p, s, b, d, w\},$$

and let

$$E = \{e_{\mathrm{bk}}, e_{\mathrm{pa}}, e_{\mathrm{wp}}, e_{\mathrm{promo}}\}.$$

Clearly,

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

The downward-closure condition is satisfied because

$$b = \{m, c\} \subseteq V, \qquad d = \{p, s\} \subseteq V,$$

and

$$w = \{b, d\} \subseteq V.$$

Therefore

$$\mathbb{H}^{\langle 2 \rangle} = (V, E)$$

is a hierarchical SuperHyperGraph of height 2.

The superhyperedges are genuinely cross-level. For example,

$$e_{\mathrm{bk}} = \{m, c, b\}$$

contains vertices from levels 0 and 1, while

$$e_{\mathrm{promo}} = \{m, c, p, s, b, d, w\}$$

contains vertices from all three levels.

A schematic representation is shown in Figure 4.

4. **Review: Hierarchical Directed SuperHyperGraphs**

A hierarchical directed SuperHyperGraph extends the preceding hierarchical framework by assigning an ordered tail set and head set to each superhyperedge. Both the tail and the head may contain supervertices from different hierarchical levels, thereby allowing directed interactions to propagate across the hierarchy.

Definition 4.1 (Hierarchical directed SuperHyperGraph of height r)**.** Let S be a finite nonempty base set and let

$$r \in \mathbb{N}_0.$$

Use the same nonempty iterated powerset tower as in Definition 3.1:

$$\mathcal{P}^{\langle 0\rangle}(S) := S,$$

and

$$\mathcal{P}^{\langle k+1\rangle}(S) := \mathcal{P}\Big(\mathcal{P}^{\langle k\rangle}(S)\Big) \setminus \{\emptyset\} \qquad (0 \le k < r).$$

Define

$$\mathcal{U}_r(S) := \bigcup_{k=0}^{r} \mathcal{P}^{\langle k\rangle}(S),$$

and, for

$$x \in \mathcal{U}_r(S),$$

define

$$\ell(x) := \min\Big\{k \in \{0,1,\ldots,r\} : x \in \mathcal{P}^{\langle k\rangle}(S)\Big\}.$$

A *hierarchical directed SuperHyperGraph of height r on S* is a quadruple

$$\mathbb{H}^{\langle r\rangle}_{\rightarrow} = (V, E, \partial^-, \partial^+)$$

satisfying the following conditions.

(HD1) (Hierarchical vertex set) The set V is a finite nonempty subset of the hierarchical universe:

$$\emptyset \neq V \subseteq \mathcal{U}_r(S).$$

(HD2) (Directed cross-level superedges) The set E is a finite set of directed-superedge identifiers, and

$$\partial^-, \partial^+ : E \longrightarrow \mathcal{P}(V) \setminus \{\emptyset\}$$

are respectively the *tail-incidence map* and *head-incidence map*.

For every

$$e \in E,$$

the sets

$$\partial^-(e) \subseteq V \qquad \text{and} \qquad \partial^+(e) \subseteq V$$

are nonempty and are called respectively the *tail* and *head* of e.

No requirement is imposed that all members of either $\partial^-(e)$ or $\partial^+(e)$ have the same level. Consequently, a directed superedge may represent a genuinely cross-level directed interaction.

(HD3) (Downward-closure coherence) If

$$X \in V \qquad \text{and} \qquad \ell(X) \geq 1,$$

then every immediate constituent of X also belongs to V:

$$X \subseteq V.$$

Remark 4.2 (Tail–head overlap)**.** Definition 4.1 permits

$$\partial^-(e) \cap \partial^+(e) \neq \emptyset.$$

Thus it adopts a general incidence-map convention for directed hypergraphs.
If a particular application requires disjoint tails and heads, one may add the condition

$$\partial^-(e) \cap \partial^+(e) = \emptyset \qquad (\forall\, e \in E).$$

Remark 4.3 (Underlying directed hypergraph)**.** If the hierarchical level information is forgotten, then

$$\mathbb{H}^{\langle r \rangle}_{\rightarrow} = (V, E, \partial^-, \partial^+)$$

is a directed hypergraph in the incidence-map sense adopted above: every $e \in E$ has a nonempty tail $\partial^-(e) \subseteq V$ and a nonempty head $\partial^+(e) \subseteq V$. Thus the additional structure is precisely the placement of the vertices in the hierarchical universe together with the downward-closure condition.

Example 4.4 (Multi-level incident response with directed escalation)**.** Consider a cybersecurity incident-response workflow in which low-level alerts are aggregated into cases and cases are subsequently escalated to higher-level response queues.

Base set and height. Let

$$S = \{a, p, e, c\},$$

where

$$a = \text{"unusual login"}, \qquad p = \text{"port scan"},$$

$$e = \text{"malware email"}, \qquad c = \text{"critical host touched"}.$$

Take

$$r = 2.$$

Then

$$\mathcal{U}_2(S) = \mathcal{P}^{\langle 0 \rangle}(S) \cup \mathcal{P}^{\langle 1 \rangle}(S) \cup \mathcal{P}^{\langle 2 \rangle}(S)$$

contains atomic alerts, nonempty sets of alerts, and nonempty sets of such sets.

Hierarchical vertex set. Define the level-1 incident cases

$$C_1 := \{a, p\} \in \mathcal{P}^{\langle 1\rangle}(S),$$

and

$$C_2 := \{e, c\} \in \mathcal{P}^{\langle 1\rangle}(S).$$

Define the level-2 response queue

$$Q := \{C_1, C_2\} \in \mathcal{P}^{\langle 2\rangle}(S).$$

Set

$$V = \{a, p, e, c, C_1, C_2, Q\} \subseteq \mathcal{U}_2(S).$$

Its layers are

$$V_0 = \{a, p, e, c\}, \qquad V_1 = \{C_1, C_2\}, \qquad V_2 = \{Q\}.$$

The downward-closure condition holds because

$$C_1 = \{a, p\} \subseteq V, \qquad C_2 = \{e, c\} \subseteq V,$$

and

$$Q = \{C_1, C_2\} \subseteq V.$$

Directed cross-level superedges. Let

$$E = \{e_1, e_2, e_3\}$$

be a set of directed-superedge identifiers and define

$$\partial^-, \partial^+ : E \longrightarrow \mathcal{P}(V) \setminus \{\emptyset\}$$

by

$$\partial^-(e_1) = \{a, p\}, \qquad \partial^+(e_1) = \{C_1\},$$
$$\partial^-(e_2) = \{e, c\}, \qquad \partial^+(e_2) = \{C_2\},$$

and

$$\partial^-(e_3) = \{C_1, C_2\}, \qquad \partial^+(e_3) = \{Q\}.$$

Thus e_1 and e_2 represent aggregation from level 0 to level 1:

$$\{a, p\} \longrightarrow \{C_1\}, \qquad \{e, c\} \longrightarrow \{C_2\},$$

while e_3 represents escalation from level 1 to level 2:

$$\{C_1, C_2\} \longrightarrow \{Q\}.$$

Consequently,

$$\mathbb{H}^{\langle 2\rangle}_{\rightarrow} = (V, E, \partial^-, \partial^+)$$

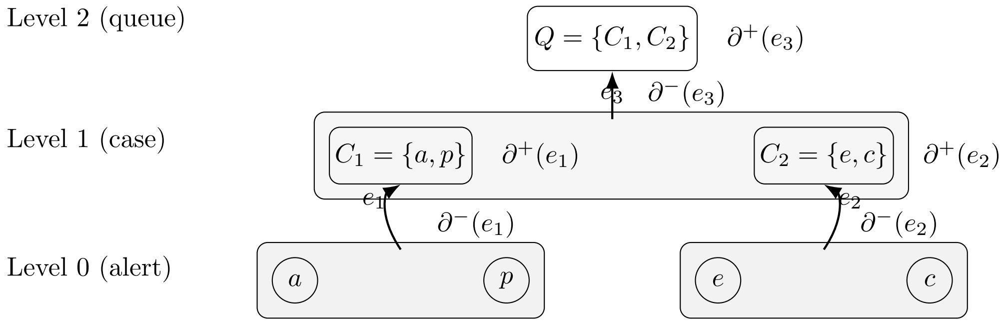

FIGURE 5. A hierarchical directed SuperHyperGraph of height 2 modeling incident-response aggregation and escalation. The directed superedges e_1, e_2, e_3 are represented by the tail and head incidence maps ∂^- and ∂^+.

is a hierarchical directed SuperHyperGraph of height 2 on S.
A schematic representation is shown in Figure 5.

5. Results: Hierarchical Bidirected SuperHyperGraphs

In this section, we introduce a hierarchical bidirected n-SuperHyperGraph as a family of bidirected n-SuperHyperGraphs organized by refinement and coarsening maps. Unlike the powerset-level hierarchy considered in the preceding sections, the hierarchical index used here does not represent the depth of the iterated powerset construction. Rather, all supervertices belong to the fixed ambient space $\mathcal{P}^n(S)$, while the hierarchy is induced by parent maps between successive refinement layers.

Definition 5.1 (Hierarchical bidirected n-SuperHyperGraph)**.** Let S be a finite nonempty base set and let $n \geq 1$ be an integer. Define the iterated powersets recursively by

$$\mathcal{P}^0(S) := S, \qquad \mathcal{P}^{k+1}(S) := \mathcal{P}\big(\mathcal{P}^k(S)\big) \qquad (k \geq 0),$$

and write

$$\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$$

for the family of all nonempty subsets of X.
A *hierarchical bidirected n-SuperHyperGraph of height* $L \in \mathbb{N}_0$ is a tuple

$$\mathcal{H}^{(n)}_{\text{hier}} = \Big((V_\ell, E_\ell, \tau_\ell)_{\ell=0}^{L}, (\pi_\ell)_{\ell=1}^{L}\Big)$$

satisfying the following axioms.

(H1) (Bidirected structure at each hierarchical layer) For every

$$\ell \in \{0, 1, \dots, L\},$$

the set

$$V_\ell \subseteq \mathcal{P}^n(S)$$

is nonempty,

$$E_\ell \subseteq \mathcal{P}^*(V_\ell),$$

and

$$\tau_\ell : V_\ell \times E_\ell \longrightarrow \{-1, 0, 1\}$$

satisfies

$$\tau_\ell(v, e) = 0 \iff v \notin e$$

for all $v \in V_\ell$ and $e \in E_\ell$, together with the balancing condition

$$\sum_{v \in e} \tau_\ell(v, e) = 0 \qquad (\forall\, e \in E_\ell).$$

(H2) (Vertex refinement and parent maps) For every

$$\ell \in \{1, \ldots, L\},$$

there exists a surjective map

$$\pi_\ell : V_\ell \twoheadrightarrow V_{\ell-1},$$

called the *parent map*, such that

$$v \subseteq \pi_\ell(v) \qquad (\forall\, v \in V_\ell),$$

and

$$u = \bigcup_{v \in \pi_\ell^{-1}(u)} v \qquad (\forall\, u \in V_{\ell-1}).$$

Thus every supervertex in the coarser layer $V_{\ell-1}$ is exactly the union of its children in the finer layer V_ℓ.

(H3) (Edge projection and incidence-sign compatibility) For every

$$\ell \in \{1, \ldots, L\}$$

and every $e \in E_\ell$, define

$$\pi_\ell(e) := \{\pi_\ell(v) \mid v \in e\} \subseteq V_{\ell-1}.$$

Then

$$\pi_\ell(e) \in E_{\ell-1}.$$

Moreover, for every incident pair $v \in e$, the incidence sign is preserved under the parent map:

$$\tau_\ell(v, e) = \tau_{\ell-1}\big(\pi_\ell(v), \pi_\ell(e)\big) \qquad (\forall\, v \in e).$$

Remark 5.2 (Nature of the hierarchical index)**.** The index ℓ in Definition 5.1 is a *refinement-layer index*, not an iterated-powerset level. Indeed,

$$V_\ell \subseteq \mathcal{P}^n(S) \qquad (\ell = 0, \dots, L),$$

so all layers consist of n-supervertices. The hierarchy is generated by the maps

$$V_L \xrightarrow{\pi_L} V_{L-1} \xrightarrow{\pi_{L-1}} \cdots \xrightarrow{\pi_2} V_1 \xrightarrow{\pi_1} V_0.$$

Thus increasing ℓ corresponds to moving toward finer representations, whereas decreasing ℓ corresponds to coarsening.

Remark 5.3 (Compatibility of the balancing conditions)**.** Axiom (H1) requires balancing independently at every hierarchical layer, while (H3) requires incidence signs to be inherited from the projected edge.

For $e \in E_\ell$ and $u \in \pi_\ell(e)$, define

$$m_e(u) := \big|\{v \in e : \pi_\ell(v) = u\}\big|.$$

Then (H3) implies

$$\sum_{v \in e} \tau_\ell(v, e) = \sum_{u \in \pi_\ell(e)} m_e(u)\, \tau_{\ell-1}\big(u, \pi_\ell(e)\big).$$

Consequently, the fine-layer balancing condition requires

$$\sum_{u \in \pi_\ell(e)} m_e(u)\, \tau_{\ell-1}\big(u, \pi_\ell(e)\big) = 0,$$

whereas the coarse-layer balancing condition gives

$$\sum_{u \in \pi_\ell(e)} \tau_{\ell-1}\big(u, \pi_\ell(e)\big) = 0.$$

Thus the two balancing conditions are compatible constraints rather than redundant requirements.

Example 5.4 (A two-level hierarchical bidirected 1-SuperHyperGraph)**.** Let

$$S = \{1, 2, 3, 4\}$$

and let $n = 1$, so that

$$\mathcal{P}^1(S) = \mathcal{P}(S).$$

Level 1 **(fine layer).** Let

$$V_1 = \big\{\{1\}, \{2\}, \{3\}, \{4\}\big\} \subseteq \mathcal{P}(S),$$

and define

$$e_1 = \big\{\{1\}, \{2\}, \{3\}, \{4\}\big\} \in \mathcal{P}^*(V_1), \qquad E_1 = \{e_1\}.$$

Define

$$\tau_1 : V_1 \times E_1 \longrightarrow \{-1, 0, 1\}$$

by

$$\tau_1(\{1\}, e_1) = +1, \qquad \tau_1(\{2\}, e_1) = +1,$$
$$\tau_1(\{3\}, e_1) = -1, \qquad \tau_1(\{4\}, e_1) = -1.$$

Thus

$$\sum_{v \in e_1} \tau_1(v, e_1) = 1 + 1 - 1 - 1 = 0.$$

Hence

$$(V_1, E_1, \tau_1)$$

is a bidirected 1-SuperHyperGraph.

Level 0 (coarse layer). Let

$$V_0 = \big\{\{1, 2\}, \{3, 4\}\big\} \subseteq \mathcal{P}(S),$$

and define

$$e_0 = \big\{\{1, 2\}, \{3, 4\}\big\} \in \mathcal{P}^*(V_0), \qquad E_0 = \{e_0\}.$$

Define

$$\tau_0 : V_0 \times E_0 \longrightarrow \{-1, 0, 1\}$$

by

$$\tau_0(\{1, 2\}, e_0) = +1, \qquad \tau_0(\{3, 4\}, e_0) = -1.$$

Then

$$\sum_{u \in e_0} \tau_0(u, e_0) = 1 - 1 = 0,$$

so

$$(V_0, E_0, \tau_0)$$

is also a bidirected 1-SuperHyperGraph.

Hierarchy and compatibility. Define the parent map

$$\pi_1 : V_1 \twoheadrightarrow V_0$$

by

$$\pi_1(\{1\}) = \pi_1(\{2\}) = \{1, 2\},$$

and

$$\pi_1(\{3\}) = \pi_1(\{4\}) = \{3, 4\}.$$

The map π_1 is surjective, and every fine supervertex is contained in its parent:

$$\{1\}, \{2\} \subseteq \{1, 2\}, \qquad \{3\}, \{4\} \subseteq \{3, 4\}.$$

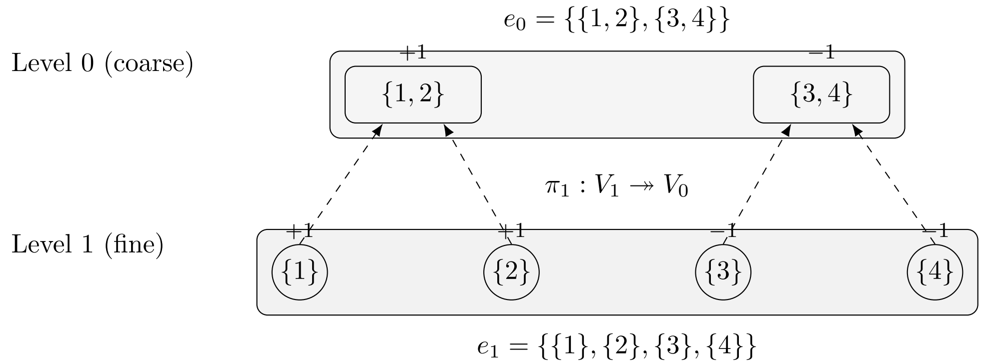

FIGURE 6. A two-level hierarchical bidirected 1-SuperHyperGraph. Rounded boxes depict the superedges e_0 and e_1. The labels $+1$ and -1 indicate the values of the bidirection functions τ_0 and τ_1, respectively. Dashed arrows represent the parent map π_1.

Furthermore,

$$\bigcup_{v\in\pi_1^{-1}(\{1,2\})} v = \{1\}\cup\{2\} = \{1,2\},$$

and

$$\bigcup_{v\in\pi_1^{-1}(\{3,4\})} v = \{3\}\cup\{4\} = \{3,4\}.$$

Hence Axiom (H2) is satisfied.

The projected fine edge is

$$\begin{aligned}\pi_1(e_1) &= \{\pi_1(v) \mid v \in e_1\}\\ &= \big\{\{1,2\},\{3,4\}\big\}\\ &= e_0 \in E_0.\end{aligned}$$

Moreover,

$$\tau_1(\{1\},e_1) = \tau_0(\{1,2\},e_0) = +1,$$

$$\tau_1(\{2\},e_1) = \tau_0(\{1,2\},e_0) = +1,$$

and

$$\tau_1(\{3\},e_1) = \tau_0(\{3,4\},e_0) = -1,$$

$$\tau_1(\{4\},e_1) = \tau_0(\{3,4\},e_0) = -1.$$

Thus Axiom (H3) also holds.

Consequently,

$$\mathcal{H}^{(1)}_{\mathrm{hier}} = \Big((V_0,E_0,\tau_0),(V_1,E_1,\tau_1),\pi_1\Big)$$

is a hierarchical bidirected 1-SuperHyperGraph of height 1.

A schematic representation is shown in Figure 6.

Theorem 5.5 (Bidirected n-SuperHyperGraphs as the height-zero case)**.** *Let S be a finite nonempty base set and let $n \geq 1$. Every bidirected n-SuperHyperGraph*

$$\mathcal{H}^{(n)} = (V, E, \tau)$$

has a canonical realization as a hierarchical bidirected n-SuperHyperGraph of height $L = 0$. Conversely, every hierarchical bidirected n-SuperHyperGraph of height $L = 0$ is naturally identified with a bidirected n-SuperHyperGraph.

Proof. Let

$$\mathcal{H}^{(n)} = (V, E, \tau)$$

be a bidirected n-SuperHyperGraph. Thus

$$\emptyset \neq V \subseteq \mathcal{P}^n(S), \qquad E \subseteq \mathcal{P}^*(V),$$

and

$$\tau : V \times E \longrightarrow \{-1, 0, 1\}$$

satisfies

$$\tau(v, e) = 0 \iff v \notin e$$

for all $v \in V$ and $e \in E$, together with

$$\sum_{v \in e} \tau(v, e) = 0 \qquad (\forall\, e \in E).$$

Define a hierarchical object of height $L = 0$ by setting

$$(V_0, E_0, \tau_0) := (V, E, \tau).$$

Since $L = 0$, there are no parent maps

$$\pi_\ell, \qquad \ell \in \{1, \ldots, L\},$$

to specify. Axiom (H1) holds exactly because (V, E, τ) is a bidirected n-SuperHyperGraph, whereas Axioms (H2) and (H3) are vacuously satisfied. Hence

$$\mathcal{H}^{(n)}_{\text{hier}} = \big((V_0, E_0, \tau_0)\big)$$

is a hierarchical bidirected n-SuperHyperGraph of height 0.

Conversely, suppose that

$$\mathcal{H}^{(n)}_{\text{hier}}$$

has height $L = 0$. Then the hierarchical structure consists only of the triple

$$(V_0, E_0, \tau_0).$$

By Axiom (H1),

$$\emptyset \neq V_0 \subseteq \mathcal{P}^n(S), \qquad E_0 \subseteq \mathcal{P}^*(V_0),$$

and

$$\tau_0 : V_0 \times E_0 \longrightarrow \{-1, 0, 1\}$$

satisfies

$$\tau_0(v, e) = 0 \iff v \notin e$$

and

$$\sum_{v \in e} \tau_0(v, e) = 0 \qquad (\forall\, e \in E_0).$$

These are precisely the defining conditions of a bidirected n-SuperHyperGraph. Therefore

$$(V_0, E_0, \tau_0)$$

is a bidirected n-SuperHyperGraph.

Hence the class of bidirected n-SuperHyperGraphs is exactly the height-zero subclass of hierarchical bidirected n-SuperHyperGraphs. ∎

6. **Conclusion**

In this paper, we introduced and studied three hierarchical extensions: *Hierarchical SuperHyperGraphs*, *Hierarchical Directed SuperHyperGraphs*, and *Hierarchical Bidirected SuperHyperGraphs*. These frameworks extend the conventional SuperHyperGraph setting by incorporating explicit hierarchical relationships across multiple levels. In particular, Hierarchical SuperHyperGraphs allow cross-level superhyperedges while maintaining coherence through downward closure, whereas Hierarchical Directed SuperHyperGraphs incorporate directional cross-level interactions through tail and head incidence maps. For Hierarchical Bidirected SuperHyperGraphs, we formulated level-wise bidirected structures together with parent maps, edge compatibility, and consistency of incidence signs across adjacent levels. We also showed that the ordinary bidirected n-SuperHyperGraph is recovered as the height-0 special case of the hierarchical bidirected framework.

These constructions provide a basic mathematical foundation for representing systems that simultaneously exhibit higher-order interactions, hierarchical aggregation, and directional or bidirectional relationships. Future work may investigate further structural properties, including connectivity, paths, cycles, matrix representations, and related invariants. We also expect that uncertainty-aware extensions based on fuzzy graphs [3], neutrosophic graphs [6, 52], and plithogenic graphs [53, 54] will provide useful directions for further generalization. Algorithmic aspects, including the design and analysis of practical algorithms, as well as experimental and empirical investigations of the proposed models in real-world hierarchical systems, also remain important topics for future research.

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Use of Generative AI and AI-Assisted Tools

We used generative AI and AI-assisted tools solely for English grammar checking and stylistic refinement. We did not use these tools to generate scientific content, results, proofs, or citations, and we complied with applicable ethical standards.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications*. Chapman and Hall/CRC, 2018.

[2] Sunil MP and J Suresh Kumar. On beta product of hesitancy fuzzy graphs and intuitionistic hesitancy fuzzy graphs. *Korean Journal of Mathematics*, 31(4):485–494, 2023.

[3] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[4] Mohsin Khan, Sadaf Umar, and Said Broumi. Laplacian energy of a complex neutrosophic graph. *Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets*, 2018.

[5] Juanjuan Ding, Wenhui Bai, and Chao Zhang. A new multi-attribute decision making method with single-valued neutrosophic graphs. *International Journal of Neutrosophic Science*, 11(2):76–86, 2020.

[6] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.

[7] Yue Gao, Zizhao Zhang, Haojie Lin, Xibin Zhao, Shaoyi Du, and Changqing Zou. Hypergraph learning: Methods and practices. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 44(5):2548–2566, 2020.

[8] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.

[9] George Karypis. Multilevel hypergraph partitioning. In *Multilevel Optimization in VLSICAD*, pages 125–154. Springer, 2003.

[10] Tamunokuro Opubo William-West and Dasharath Singh. Information granulation for rough fuzzy hypergraphs. *Granular Computing*, 3(1):75–92, 2018.

[11] KK Myithili and C Nandhini. Analysis of hub parameters in fuzzy hypergraphs extending to intuitionistic fuzzy threshold hypergraphs: Applications in designing transport networks in amusement parks using hub hyperpaths. *Notes on Intuitionistic Fuzzy Sets*, 2024.

[12] R Parvathi, S Thilagavathi, and MG Karunambigai. Intuitionistic fuzzy hypergraphs. *Cybernetics and Information Technologies*, 9(2):46–53, 2009.

[13] P Aruna Sri, N Thamaraikannan, Karuppusamy Loganathan, and Dinesh Kumar Chaudhary. Double domination and regular domination in intuitionistic fuzzy hypergraph. *Journal of Mathematics*, 2022(1):1436194, 2022.

[14] Rangasamy Parvathi, S. Thilagavathi, and M. G. Karunambigai. Operations on intuitionistic fuzzy hypergraphs. *International Journal of Computer Applications*, 51:46–54, 2012.

[15] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions and tractable queries. In *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 21–32, 1999.

[16] Isolde Adler, Tomáš Gavenčiak, and Tereza Klimošová. Hypertree-depth and minors in hypergraphs. *Theoretical Computer Science*, 463:84–95, 2012.

[17] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions: A survey. In *Mathematical Foundations of Computer Science 2001: 26th International Symposium, MFCS 2001 Mariánské Lázne, Czech Republic, August 27–31, 2001 Proceedings 26*, pages 37–57. Springer, 2001.

[18] Muhammad Akram and Hafiza Saba Nawaz. Implementation of single-valued neutrosophic soft hypergraphs on human nervous system. *Artificial Intelligence Review*, 56(2):1387–1425, 2023.

[19] R Keerthana and S Venkatesh. A systematic study of intercropping agricultural system using fuzzy and inverse fuzzy hypersoft hypergraph. *Journal of Applied Mathematics and Computing*, 71(3):4067–4094, 2025.

[20] Takaaki Fujita. Metahypergraphs, metasuperhypergraphs, and iterated metagraphs: Modeling graphs of graphs, hypergraphs of hypergraphs, superhypergraphs of superhypergraphs, and beyond. *Current Research in Interdisciplinary Studies*, 4(5):1–23, 2025.

[21] T Fujita. On spherical neutrosophic superhypergraphs and complex spherical neutrosophic superhypergraphs. *Int. J. Open Problems Complex Analysis*, 18(1), 2026.

[22] Mehbooba P Shareef, Babita Roslind Jose, Jimson Mathew, and Ramkumar PB. Indeterminacy-driven trade-off in reinforcement learning on neutrosophic fuzzy hypergraphs for explainable item recommendation with path-compliant rewards. *IEEE Transactions on Emerging Topics in Computational Intelligence*, 2025.

[23] Jinshan Xie and Liqun Qi. Spectral directed hypergraph theory via tensors. *Linear and Multilinear Algebra*, 64(4):780–794, 2016.

[24] Spencer Krieger and John Kececioglu. Shortest hyperpaths in directed hypergraphs for reaction pathway inference. *Journal of Computational Biology*, 30(11):1198–1225, 2023.

[25] George Karypis and Vipin Kumar. Multilevel k-way hypergraph partitioning. In *Proceedings of the 36th annual ACM/IEEE design automation conference*, pages 343–348, 1999.

[26] George Karypis, Rajat Aggarwal, Vipin Kumar, and Shashi Shekhar. Multilevel hypergraph partitioning: Application in vlsi domain. In *Proceedings of the 34th annual Design Automation Conference*, pages 526–529, 1997.

[27] Sebastian Schlag, Tobias Heuer, Lars Gottesbüren, Yaroslav Akhremtsev, Christian Schulz, and Peter Sanders. High-quality hypergraph partitioning. *ACM Journal of Experimental Algorithmics*, 27:1–39, 2023.

[28] Yuxin Wang, Quan Gan, Xipeng Qiu, Xuanjing Huang, and David Wipf. From hypergraph energy functions to hypergraph neural networks. In *International Conference on Machine Learning*, pages 35605–35623. PMLR, 2023.

[29] Tao Yin, Weiping Ding, Hengrong Ju, Jiashuang Huang, and Yuepeng Chen. The fuzzy hypergraph neural network model based on sparse k-nearest neighborhood granule. *Applied Soft Computing*, 170:112721, 2025.

[30] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[31] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications (II): Graph Property and Parameter*, volume II of *HyperGraph and SuperHyperGraph Theory with Applications*. Neutrosophic Science International Association (NSIA) Publishing House, 1.0 edition, 2026.

[32] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[33] Berrocal Villegas Salomón Marcos, Montalvo Fritas Willner, Berrocal Villegas Carmen Rosa, Flores Fuentes Rivera María Yissel, Espejo Rivera Roberto, Laura Daysi Bautista Puma, and Dante Manuel Macazana Fernández. Using plithogenic n-superhypergraphs to assess the degree of relationship between information skills and digital competencies. *Neutrosophic Sets and Systems*, 84:513–524, 2025.

[34] Nelly Hodelín Amable, Elizabeth Esther Vergel De Salazar, Martha Gloria Martínez Isaac, Olivia Catalina Olavarría Sánchez, and Johanna Mariuxi Solís Palma. Representation of motivational dynamics in school environments through plithogenic n-superhypergraphs with family participation. *Neutrosophic Sets and Systems*, 92:570–583, 2025.

[35] Takaaki Fujita and Florentin Smarandache. *Representing Higher-Order Networks: A Survey of Graph-Based Frameworks*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[36] T.Fujita. Short introduction to superhypergraph theory with some applications. *International Journal of Open Problems in Computer Science and Mathematics*, 19(2):74–86, 2026.

[37] Takaaki Fujita and Florentin Smarandache. *Hyperstructures and Superhyperstructures: Theory and Selected Applications*. Infinite Study, 2026.

[38] Yixuan He, Gesine Reinert, David Wipf, and Mihai Cucuringu. Robust angular synchronization via directed graph neural networks. *arXiv preprint arXiv:2310.05842*, 2023.

[39] Yixuan He, Xitong Zhang, Junjie Huang, Benedek Rozemberczki, Mihai Cucuringu, and Gesine Reinert. Pytorch geometric signed directed: a software package on graph neural networks for signed and directed graphs. In *Learning on Graphs Conference*, pages 12–1. PMLR, 2024.

[40] Daniele Pretolani. Finding hypernetworks in directed hypergraphs. *European Journal of Operational Research*, 230(2):226–230, 2013.

[41] Daniele Pretolani. A directed hypergraph model for random time dependent shortest paths. *European Journal of Operational Research*, 123(2):315–324, 2000.

[42] Enrico L. Enriquez, Grace M. Estrada, Carmelita Loquias, Reuella J. Bacalso, and Lanndon A. Ocampo. Domination in fuzzy directed graphs. *Mathematics*, 2021.

[43] Tahira Batool and Uzma Ahmad. A new approximation technique based on central bodies of rough fuzzy directed graphs for agricultural development. *Journal of Applied Mathematics and Computing*, 70(2):1673–1705, 2024.

[44] T. Fujita and Florentin Smarandache. *Fuzzy, Neutrosophic, and Uncertain Graph Theory: Properties and Applications*. Neutrosophic Science International Association (NSIA) Publishing House, 1.0 edition, 2026.

[45] Takaaki Fujita. Review of rough turiyam neutrosophic directed graphs and rough pentapartitioned neutrosophic directed graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:48–79, 2025.

[46] Laura Gellert and Raman Sanyal. On degree sequences of undirected, directed, and bidirected graphs. *European Journal of Combinatorics*, 64:113–124, 2017.

[47] Rui Xu and Cun-Quan Zhang. On flows in bidirected graphs. *Discrete mathematics*, 299(1-3):335–343, 2005.

[48] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[49] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[50] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[51] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[52] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, 10:86–101, 2016.

[53] Prem Kumar Singh et al. Dark data analysis using intuitionistic plithogenic graphs. *International Journal of Neutrosophic Sciences*, 16(2):80–100, 2021.

[54] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.