NCML

Neutrosophic Computing and Machine Learning, 2025

University of New Mexico

Actor Hypernetworks and Urban Road Hypernetworks with Real-life Applications

Takaaki Fujita[1] * *and Arif Mehmood* [2]

[1] Independent Researcher, Tokyo, Japan. Takaaki.fujita060@gmail.com

[2] Department of Mathematics, Institute of Numerical Sciences, Gomal University, Dera Ismail Khan 29050, KPK, Pakistan (mehdaniyal@gmail.com)

*Correspondence: Takaaki.fujita060@gmail.com

Abstract. Hypergraphs extend classical graphs by allowing hyperedges to connect arbitrary subsets of vertices, while superhypergraphs further enrich this structure through iterated powersets that capture hierarchical and self-referential relationships. An actor network links heterogeneous actants—humans, artifacts, texts, and rules—through directed associations, emphasizing relational materiality and performative agency. An urban road network is a directed, weighted graph of intersections and road segments, modeling connectivity, capacities, and travel dynamics. In this paper, we extend actor networks and urban road networks by employing HyperGraphs and SuperHyperGraphs. These extensions are expected to provide clearer and more expressive representations of hierarchical structures in real-world actor networks and urban road networks.

Keywords: Superhypergraph, Hypergraph, Actor Networks, Actor Hypernetworks, Actor Superhypernetworks, Hypernetworks, Superhypernetworks, Urban Road Networks

1. **Preliminaries**

This section gathers the basic notions and notation used throughout the paper. Unless explicitly stated otherwise, all sets are finite.

1.1. *Power Set and n-th Power Set*

For a set S, the *power set* $\mathcal{P}(S)$ is the family of all subsets of S, including $\varnothing$ and S itself. The *n-fold* (or *n-th*) iterated power set is obtained by re-applying the power-set operator n times, starting from S [1–4].

Definition 1.1 (Base Set)**.** A *base set* is any subset $S \subseteq U$ from which further constructions (e.g., power sets, hyperstructures) are generated.

Definition 1.2 (Power Set)**.** [5,6] The *power set* of S is

$$\mathcal{P}(S) \;=\; \{\, X \mid X \subseteq S \,\}.$$

Definition 1.3 (Nonempty Iterated Power Set)**.** [2,4] Define $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\varnothing\}$ for any set X, and set

$$\mathcal{P}_1^*(S) := \mathcal{P}(S) \setminus \{\varnothing\}, \qquad \mathcal{P}_{k+1}^*(S) := \mathcal{P}^*\big(\mathcal{P}_k^*(S)\big) \quad (k \geq 1).$$

Example 1.4 (Parliamentary coalition building)**.** Let $S = \{P_A, P_B, P_C\}$ denote three political parties in a legislature. The nonempty coalitions are

$$\mathcal{P}_1^*(S) = \big\{\{P_A\}, \{P_B\}, \{P_C\}, \{P_A, P_B\}, \{P_A, P_C\}, \{P_B, P_C\}, \{P_A, P_B, P_C\}\big\}.$$

Elements of $\mathcal{P}_2^*(S)$ are nonempty sets of such coalitions ("coalitions of coalitions"). For instance,

$$\mathcal{C}_{\text{gov}} = \{P_A, P_B\}, \qquad \mathcal{C}_{\text{sup}} = \{P_C\}, \qquad \mathcal{B} = \big\{\mathcal{C}_{\text{gov}},\ \mathcal{C}_{\text{sup}}\big\} \in \mathcal{P}_2^*(S).$$

Here, $\mathcal{B}$ models a governing bloc composed of a governing coalition and a confidence-and-supply partner. Alternative blocs, e.g. $\mathcal{B}' = \big\{\{P_A, P_C\}, \{P_B\}\big\}$, represent competing alliance structures. Thus $\mathcal{P}_1^*(S)$ captures all feasible coalitions, while $\mathcal{P}_2^*(S)$ captures umbrella arrangements of those coalitions, useful for coalition formation and voting analysis in comparative politics.

Example 1.5 (Inter-organizational research consortia)**.** Let $S = \{U, N, G, I\}$ represent a university U, an NGO N, a government agency G, and an industry partner I. Project teams are nonempty subsets of S, i.e. elements of $\mathcal{P}_1^*(S)$. Consider the teams

$$T_1 = \{U, I\}, \qquad T_2 = \{N, G\}, \qquad T_3 = \{U, N\} \;\in\; \mathcal{P}_1^*(S).$$

A multi-project consortium (e.g., for a large societal grant) is naturally modeled as

$$\Omega = \{T_1, T_2\} \;\in\; \mathcal{P}_2^*(S),$$

a nonempty set of nonempty teams. Enlarging the consortium to $\Omega' = \{T_1, T_2, T_3\}$ captures broader stakeholder participation. Thus $\mathcal{P}_1^*(S)$ encodes all possible collaborative teams, while $\mathcal{P}_2^*(S)$ encodes consortia-of-teams, supporting analysis of governance, resource sharing, and coordination across organizations in the social sciences.

1.2. *Hypergraphs and SuperHypergraphs*

Graph theory has been widely applied across diverse scientific domains [7,8]. Hypergraphs extend ordinary graphs by allowing each *hyperedge* to be any nonempty subset of vertices, thereby encoding higher–order relations among entities [9,10]. As extended concepts, Fuzzy HyperGraphs [11,12] and Neutrosophic HyperGraphs [13,14] are well known. A *SuperHyperGraph* goes further by using iterated power sets so that hyperedges can connect objects drawn from higher ranks, enabling hierarchical and self-referential connectivity [15,16]. Recent work underscores the growing importance and breadth of applications of SuperHyperGraphs [17–20].

Definition 1.6 (Hypergraph)**.** [9,21] Given a finite vertex set V, a *hypergraph* is a pair

$$H = (V, E), \qquad E \subseteq \mathcal{P}(V) \setminus \{\varnothing\},$$

whose members $e \in E$ are *hyperedges*. Apart from being nonempty, no constraint is placed on the size of a hyperedge.

Definition 1.7 (n-SuperHyperGraph)**.** (cf. [16]) Let V_0 be a finite, nonempty base set and define

$$\mathcal{P}^0(X) := X, \qquad \mathcal{P}^{k+1}(X) := \mathcal{P}\big(\mathcal{P}^k(X)\big) \quad (k \geq 0).$$

Fix $n \geq 1$. An *n-SuperHyperGraph* is a pair

$$\mathrm{SHG}^{(n)} = (V, E),$$

where the *n-supervertex set* satisfies $\varnothing \neq V \subseteq \mathcal{P}^n(V_0)$, and the *$n$-superedge set* is a nonempty family of nonempty subsets of V:

$$\varnothing \neq E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}.$$

Equivalently, each $e \in E$ is a finite nonempty collection of n-supervertices. When $n = 0$, this reduces to an ordinary hypergraph on V_0.

Example 1.8 (Hypergraph in social science: bill co-sponsorship)**.** Let the vertex set be five legislators

$$V = \{L_1, L_2, L_3, L_4, L_5\}.$$

Create one hyperedge for each bill, containing exactly those legislators who co-sponsor it:

$$e_{\mathrm{B1}} = \{L_1, L_2, L_3\},$$
$$e_{\mathrm{B2}} = \{L_2, L_4\},$$
$$e_{\mathrm{B3}} = \{L_1, L_3, L_5\}.$$

Then

$$H = (V, E), \qquad E = \{e_{\mathrm{B1}}, e_{\mathrm{B2}}, e_{\mathrm{B3}}\} \subseteq \mathcal{P}(V) \setminus \{\varnothing\},$$

is a hypergraph of co-sponsorship. Each hyperedge collects all co-sponsors of a bill, so edges naturally have size > 2 when bills have many supporters. This representation captures higher-order collaboration patterns that ordinary graphs (pairwise ties) cannot.

Example 1.9 (n=2 SuperHyperGraph in social science: multi-level civil-society coalition)**.** Let the base set be individuals

$$V_0 = \{i_1, i_2, i_3, i_4, i_5, i_6\}.$$

First-level groups (organizations) are subsets of V_0:

$$O_A = \{i_1, i_2\}, \quad O_B = \{i_2, i_3, i_4\}, \quad O_C = \{i_5, i_6\} \ \in \mathcal{P}(V_0).$$

Second-level units (consortia) are sets of organizations, i.e. elements of $\mathcal{P}(\mathcal{P}(V_0)) = \mathcal{P}^2(V_0)$:

$$X = \{O_A, O_B\}, \qquad Y = \{O_B, O_C\} \ \in \mathcal{P}^2(V_0).$$

Define the 2-supervertex set and superedges by

$$V = \{X, Y\} \subseteq \mathcal{P}^2(V_0), \qquad E = \big\{ \{X\}, \ \{X, Y\} \big\} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}.$$

The pair $\mathrm{SHG}^{(2)} = (V, E)$ is a 2-SuperHyperGraph: vertices are consortia (sets of organizations), while superedges record joint umbrella campaigns among consortia (e.g. $\{X, Y\}$ denotes a cross-consortium policy alliance). Thus the construction models hierarchical collective action: individuals $\to$ organizations $\to$ consortia, with relations captured at the consortia level.

2. Main Results

This section presents the main results of the paper.

2.1. *Actor network*

An actor network links heterogeneous actants—humans, artifacts, texts, rules—through directed associations, emphasizing relational materiality and performative agency (cf. [22–24]).

Notation 1 (Actants, labels, roles, signs, and weights)**.** *Let $\mathcal{A}$ be a finite nonempty set of* actants *(heterogeneous entities: people, tools, devices, documents, microbes, regulations,* etc.*). Let $\mathcal{L}$ be a finite set of* association labels *(e.g.,* `enrols`*,* `binds`*,* `mediates`*,* `inscribes`*). Let $\mathcal{R}$ be a finite set of* role labels *together with an* orientation partition

$$\pi : \ \mathcal{R} \longrightarrow \{\mathrm{in}, \mathrm{out}, \mathrm{neu}\},$$

which determines, for each role, whether it contributes to an input *side, an* output *side, or is* neutral *within a joint association. We also allow a* sign *map σ with codomain* $\{-1, 0, +1\}$ *(e.g., hindering/neutral/enabling) and a nonnegative* weight *w (e.g., strength, confidence, rate scale). Throughout, all sets are finite and nonempty.*

Definition 2.1 (Actor network)**.** An *actor network* is a tuple

$$\mathcal{N} \ = \ (\mathcal{A}, \, E_{\to}, \, \tau, \, \rho, \, \sigma, \, w),$$

where $\mathcal{A}$ is the set of actants; $E_{\to} \subseteq \mathcal{A} \times \mathcal{A}$ is a set of *directed* associations (arcs); $\tau : \mathcal{A} \to \mathrm{types}$ assigns actant types; $\rho : E_{\to} \to \mathcal{L}$ assigns association labels; $\sigma : E_{\to} \to \{-1, 0, +1\}$ and $w : E_{\to} \to \mathbb{R}_{\geq 0}$ assign sign and weight. Semantically, $(u \to v) \in E_{\to}$ denotes that actant u affects v along relation $\rho(u \to v)$ with intensity $w(u \to v)$ and sign $\sigma(u \to v)$.

Example 2.2 (Actor network in disaster relief coordination)**.** Let the actant set be

$$\mathcal{A} = \{\, G \text{ (gov. agency)},\ N \text{ (NGO)},\ L \text{ (logistics platform)},\ D \text{ (donors)},$$

$$C \text{ (affected community)},\ M \text{ (media outlet)},\ R \text{ (rumor post)},\ F \text{ (fact-checking group)},\ \mathrm{Reg} \text{ (emergency decree)}\,\}.$$

Types are assigned by

$$\tau(G) = \tau(N) = \tau(F) = \text{“organization”}, \quad \tau(L) = \text{“platform”}, \quad \tau(D) = \text{“stakeholder”},$$

$$\tau(C) = \text{“community”}, \quad \tau(M) = \text{“media”}, \quad \tau(R) = \text{“text”}, \quad \tau(\mathrm{Reg}) = \text{“rule”}.$$

Directed associations are

$$E_{\to} = \{G\to N,\ G\to L,\ N\to C,\ D\to N,\ M\to C,\ R\to C,\ F\to R,\ \mathrm{Reg}\to G,\ G\to M\}.$$

For each arc $a \in E_{\to}$, define the label $\rho(a)$, sign $\sigma(a) \in \{-1, 0, +1\}$, and weight $w(a) \in \mathbb{R}_{\geq 0}$ as:

$$\begin{aligned}
\rho(G\to N) &= \text{“grants”}, & \sigma(G\to N) &= +1, & w(G\to N) = 0.8,\\
\rho(G\to L) &= \text{“deploys”}, & \sigma(G\to L) &= +1, & w(G\to L) = 0.9,\\
\rho(N\to C) &= \text{“delivers_aid”}, & \sigma(N\to C) &= +1, & w(N\to C) = 1.0,\\
\rho(D\to N) &= \text{“donates”}, & \sigma(D\to N) &= +1, & w(D\to N) = 0.7,\\
\rho(M\to C) &= \text{“broadcasts_warning”}, & \sigma(M\to C) &= +1, & w(M\to C) = 0.6,\\
\rho(R\to C) &= \text{“misinforms”}, & \sigma(R\to C) &= -1, & w(R\to C) = 0.6,\\
\rho(F\to R) &= \text{“debunks”}, & \sigma(F\to R) &= -1, & w(F\to R) = 0.9,\\
\rho(\mathrm{Reg}\to G) &= \text{“authorizes”}, & \sigma(\mathrm{Reg}\to G) &= +1, & w(\mathrm{Reg}\to G) = 0.8,\\
\rho(G\to M) &= \text{“issues_guideline”}, & \sigma(G\to M) &= +1, & w(G\to M) = 0.7.
\end{aligned}$$

The tuple $\mathcal{N} = (\mathcal{A}, E_{\to}, \tau, \rho, \sigma, w)$ is an actor network capturing how rules, organizations, media, and information (including misinformation) interact during emergency response.

Example 2.3 (Actor network in university technology transfer)**.** Let

$$\mathcal{A} = \{\, U \text{ (university TTO)},\ \mathrm{Lab} \text{ (research lab)},\ P \text{ (patent)},\ S \text{ (startup)},$$

$$\mathrm{VC} \text{ (venture capital)},\ \mathrm{Rg} \text{ (regulator)},\ \mathrm{Std} \text{ (industry standard)},\ \mathrm{Cust} \text{ (customers)}\,\}.$$

Actant types are

$$\tau(U) = \text{“university”}, \quad \tau(\mathrm{Lab}) = \text{“lab”}, \quad \tau(P) = \text{“artifact”},$$

$$\tau(S) = \text{“firm”}, \quad \tau(\mathrm{VC}) = \text{“investor”},$$

$$\tau(\mathrm{Rg}) = \text{“regulator”}, \quad \tau(\mathrm{Std}) = \text{“rule”}, \quad \tau(\mathrm{Cust}) = \text{“market”}.$$

Directed associations:

$$E_{\to} = \{\, \mathrm{Lab}\to U,\ U\to P,\ P\to S,\ U\to S,\ \mathrm{VC}\to S,\ \mathrm{Rg}\to S,\ \mathrm{Std}\to S,\ S\to \mathrm{Cust},\ \mathrm{Cust}\to S\,\}.$$

Labels, signs, and weights:

$$\begin{aligned}
\rho(\text{Lab}\to U) &= \text{“invention_disclosure”}, & \sigma(\text{Lab}\to U) &= +1, & w(\text{Lab}\to U) = 0.8,\\
\rho(U\to P) &= \text{“files_patent”}, & \sigma(U\to P) &= +1, & w(U\to P) = 0.9,\\
\rho(P\to S) &= \text{“exclusivity”}, & \sigma(P\to S) &= +1, & w(P\to S) = 0.7,\\
\rho(U\to S) &= \text{“licenses”}, & \sigma(U\to S) &= +1, & w(U\to S) = 0.8,\\
\rho(\text{VC}\to S) &= \text{“funds”}, & \sigma(\text{VC}\to S) &= +1, & w(\text{VC}\to S) = 0.9,\\
\rho(\text{Rg}\to S) &= \text{“compliance_cost”}, & \sigma(\text{Rg}\to S) &= -1, & w(\text{Rg}\to S) = 0.6,\\
\rho(\text{Std}\to S) &= \text{“reference_standard”}, & \sigma(\text{Std}\to S) &= +1, & w(\text{Std}\to S) = 0.5,\\
\rho(S\to \text{Cust}) &= \text{“sells”}, & \sigma(S\to \text{Cust}) &= +1, & w(S\to \text{Cust}) = 1.0,\\
\rho(\text{Cust}\to S) &= \text{“feedback”}, & \sigma(\text{Cust}\to S) &= +1, & w(\text{Cust}\to S) = 0.4.
\end{aligned}$$

Thus $\mathcal{N} = (\mathcal{A}, E_{\to}, \tau, \rho, \sigma, w)$ encodes the flow from discovery to market, including enabling artifacts (patents), constraints (regulation), and resource exchanges (funding, licensing).

2.2. *Actor hypernetwork*

An actor hypernetwork extends actor networks with hyperedges, representing multi-actor associations where groups of heterogeneous actants jointly perform actions or relations.

Definition 2.4 (Actor hypernetwork)**.** An *actor hypernetwork* is the typed, signed, weighted, role-annotated hypergraph

$$\mathcal{H} = \big(\mathcal{A},\, E,\, \tau,\, \rho,\, \{\lambda_e\}_{e\in E},\, \sigma,\, w\big),$$

where:

- $\mathcal{A}$ is the finite nonempty set of actants (the vertex set);
- $E \subseteq \mathcal{P}(\mathcal{A}) \setminus \{\varnothing\}$ is a nonempty family of *hyperedges* (joint associations);
- $\tau : \mathcal{A} \to$ types assigns a type to each actant;
- $\rho : E \to \mathcal{L}$ assigns an association label to each hyperedge e;
- for each $e \in E$, the *role map* $\lambda_e : e \to \mathcal{R}$ annotates the role played by each incident actant;
- $\sigma : E \to \{-1, 0, +1\}$ and $w : E \to \mathbb{R}_{\geq 0}$ assign sign and weight to hyperedges.

Write the *role-induced partitions* of e as

$$\text{In}(e) := \{u \in e : \pi(\lambda_e(u)) = \text{in}\}, \quad \text{Out}(e) := \{u \in e : \pi(\lambda_e(u)) = \text{out}\}, \quad \text{Neu}(e) := e\setminus(\text{In}(e)\cup\text{Out}(e)).$$

Intuitively, each e captures a single *joint* association involving multiple actants with specified roles.

Example 2.5 (Participatory budgeting as an actor hypernetwork)**.** Let the actant set be

$$\mathcal{A} = \{\,\text{Cit (citizens' assembly)},\ \text{Cou (city council)},\ \text{Plat (online platform)},$$

$$\text{Data (evidence dataset)},\ \text{Ord (budget ordinance)},\ \text{Aud (auditor)}\,\}.$$

We use the orientation partition $\pi : \mathcal{R} \to \{\mathrm{in}, \mathrm{out}, \mathrm{neu}\}$:

$$\pi(\texttt{proposer}) = \mathrm{in}, \quad \pi(\texttt{adopter}) = \mathrm{out}, \quad \pi(\texttt{enactor}) = \mathrm{in},$$

$$\pi(\texttt{artifact}) = \mathrm{out}, \quad \pi(\texttt{validator}) = \mathrm{in}, \quad \pi(\texttt{audited}) = \mathrm{out}, \quad \pi(\texttt{medium}) = \pi(\texttt{evidence}) = \mathrm{neu}.$$

Define three hyperedges $E = \{e_{\mathrm{delib}}, e_{\mathrm{enact}}, e_{\mathrm{audit}}\}$ with labels, signs, and weights:

$$e_{\mathrm{delib}} = \{\mathrm{Cit}, \mathrm{Cou}, \mathrm{Plat}, \mathrm{Data}\}, \quad \rho(e_{\mathrm{delib}}) = \texttt{co\_deliberates}, \quad \sigma(e_{\mathrm{delib}}) = +1, \quad w(e_{\mathrm{delib}}) = 0.8,$$

$$e_{\mathrm{enact}} = \{\mathrm{Cou}, \mathrm{Ord}, \mathrm{Aud}\}, \quad \rho(e_{\mathrm{enact}}) = \texttt{enacts}, \quad \sigma(e_{\mathrm{enact}}) = +1, \quad w(e_{\mathrm{enact}}) = 0.9,$$

$$e_{\mathrm{audit}} = \{\mathrm{Aud}, \mathrm{Cou}, \mathrm{Data}\}, \quad \rho(e_{\mathrm{audit}}) = \texttt{audits}, \quad \sigma(e_{\mathrm{audit}}) = +1, \quad w(e_{\mathrm{audit}}) = 0.6.$$

Role maps $\lambda_e : e \to \mathcal{R}$:

$$\begin{aligned}
&\lambda_{e_{\mathrm{delib}}}(\mathrm{Cit}) = \texttt{proposer}, && \lambda_{e_{\mathrm{delib}}}(\mathrm{Cou}) = \texttt{adopter}, \\
&\lambda_{e_{\mathrm{delib}}}(\mathrm{Plat}) = \texttt{medium}, && \lambda_{e_{\mathrm{delib}}}(\mathrm{Data}) = \texttt{evidence}, \\
&\lambda_{e_{\mathrm{enact}}}(\mathrm{Cou}) = \texttt{enactor}, && \lambda_{e_{\mathrm{enact}}}(\mathrm{Ord}) = \texttt{artifact}, && \lambda_{e_{\mathrm{enact}}}(\mathrm{Aud}) = \texttt{medium}, \\
&\lambda_{e_{\mathrm{audit}}}(\mathrm{Aud}) = \texttt{validator}, && \lambda_{e_{\mathrm{audit}}}(\mathrm{Cou}) = \texttt{audited}, && \lambda_{e_{\mathrm{audit}}}(\mathrm{Data}) = \texttt{evidence}.
\end{aligned}$$

Then

$$\mathcal{H} = (\mathcal{A}, E, \tau, \rho, \{\lambda_e\}, \sigma, w)$$

is an actor hypernetwork in which

$$\mathrm{In}(e_{\mathrm{delib}}) = \{\mathrm{Cit}\}$$

$$\mathrm{Out}(e_{\mathrm{delib}}) = \{\mathrm{Cou}\}$$

,

$$\mathrm{In}(e_{\mathrm{enact}}) = \{\mathrm{Cou}\}$$

$$\mathrm{Out}(e_{\mathrm{enact}}) = \{\mathrm{Ord}\}$$

,

$$\mathrm{In}(e_{\mathrm{audit}}) = \{\mathrm{Aud}\}$$

, and

$$\mathrm{Out}(e_{\mathrm{audit}}) = \{\mathrm{Cou}\}$$

. The role-aware 2-section projects triadic (and higher) joint actions to directed influence among actants.

Example 2.6 (Platform content moderation and appeals as an actor hypernetwork)**.** Let

$$\mathcal{A} = \{\, \mathrm{Plat\ (platform)},\ \mathrm{User},\ \mathrm{Post},$$

$$\mathrm{Mod\ (human\ moderator)},\ \mathrm{AI\ (classifier)},$$

$$\mathrm{Pol\ (policy\ document)},\ \mathrm{Act\ (moderation\ action)},\ \mathrm{Omb\ (ombudsperson)}\,\}.$$

Use π with

$$\pi(\texttt{source}) = \text{in}, \quad \pi(\texttt{target}) = \text{out}, \quad \pi(\texttt{appellant}) = \text{in},$$

$$\pi(\texttt{decision}) = \text{out},$$

$$\pi(\texttt{medium}) = \pi(\texttt{policy}) = \pi(\texttt{tool}) = \text{neu}.$$

Define two hyperedges $E = \{e_{\text{mod}}, e_{\text{app}}\}$:

$$e_{\text{mod}} = \{\text{Post}, \text{Act}, \text{Mod}, \text{AI}, \text{Pol}, \text{Plat}\}, \quad \rho(e_{\text{mod}}) = \texttt{moderates}, \quad \sigma(e_{\text{mod}}) = -1, \quad w(e_{\text{mod}}) = 0.7,$$

$$e_{\text{app}} = \{\text{User}, \text{Act}, \text{Omb}, \text{Plat}, \text{Pol}\}, \quad \rho(e_{\text{app}}) = \texttt{appeals}, \quad \sigma(e_{\text{app}}) = +1, \quad w(e_{\text{app}}) = 0.5.$$

Roles:

$$\begin{array}{llll}
\lambda_{e_{\text{mod}}}(\text{Post}) = \texttt{source}, & \lambda_{e_{\text{mod}}}(\text{Act}) = \texttt{target}, & \lambda_{e_{\text{mod}}}(\text{Mod}) = \texttt{medium}, & \lambda_{e_{\text{mod}}}(\text{AI}) = \texttt{tool}, \\
\lambda_{e_{\text{mod}}}(\text{Pol}) = \texttt{policy}, & \lambda_{e_{\text{mod}}}(\text{Plat}) = \texttt{medium}, & & \\
\lambda_{e_{\text{app}}}(\text{User}) = \texttt{appellant}, & \lambda_{e_{\text{app}}}(\text{Act}) = \texttt{decision}, & \lambda_{e_{\text{app}}}(\text{Omb}) = \texttt{medium}, & \lambda_{e_{\text{app}}}(\text{Plat}) = \texttt{medium}, \\
\lambda_{e_{\text{app}}}(\text{Pol}) = \texttt{policy}. & & &
\end{array}$$

Thus $\text{In}(e_{\text{mod}}) = \{\text{Post}\}$, $\text{Out}(e_{\text{mod}}) = \{\text{Act}\}$ capture the multi-actor decision event, while $\text{In}(e_{\text{app}}) = \{\text{User}\}$, $\text{Out}(e_{\text{app}}) = \{\text{Act}\}$ encode joint appeals with institutional intermediaries.

Theorem 2.7 (Actor hypernetworks are hypergraphs)**.** *Let $\mathcal{H} = (\mathcal{A}, E, \tau, \rho, \{\lambda_e\}, \sigma, w)$ be an actor hypernetwork. Then the underlying pair $(\mathcal{A}, E)$ is a hypergraph.*

Proof. By Definition 2.4, $E \subseteq \mathcal{P}(\mathcal{A}) \setminus \{\varnothing\}$, i.e., each $e \in E$ is a nonempty subset of the vertex set $\mathcal{A}$. Hence $(\mathcal{A}, E)$ satisfies the axioms of a hypergraph. □

Definition 2.8 (Role-aware directed 2-section (projection to an actor network))**.** Given an actor hypernetwork $\mathcal{H}$ and the orientation partition π, define the projected actor network

$$\mathcal{G}_\pi(\mathcal{H}) \;=\; \big(\mathcal{A},\, E_\to,\, \tau,\, \rho_\to,\, \sigma_\to,\, w_\to\big),$$

with vertex set $\mathcal{A}$ and arc set

$$E_\to \;:=\; \big\{(u,v) \in \mathcal{A} \times \mathcal{A} :\; \exists\, e \in E,\; u \in \text{In}(e),\; v \in \text{Out}(e)\big\},$$

and weights aggregated by

$$w_\to(u,v) \;:=\; \sum_{\substack{e \in E:\\ u \in \text{In}(e),\; v \in \text{Out}(e)}} \frac{w(e)}{Z_e}, \qquad Z_e := \max\{1,\; |\text{In}(e)| \cdot |\text{Out}(e)|\}.$$

Labels/signs are inherited from contributing hyperedges (ties resolved by confidence), e.g., $\rho_\to(u,v) := \rho(e)$ and $\sigma_\to(u,v) := \sigma(e)$ for each contributing e.

Example 2.9 (Role-aware directed 2-section (projection to an actor network))**.** Let the actor hypernetwork be

$$\mathcal{H} = (\mathcal{A}, E, \tau, \rho, \{\lambda_e\}_{e\in E}, \sigma, w), \quad \mathcal{A} = \{\text{Scientist (S)}, \text{Lab (L)}, \text{Journal (J)}, \text{Paper (P)}, \text{Funder (F)}\}.$$

Use the orientation partition π with $\pi(\texttt{source}) = \text{in}$, $\pi(\texttt{target}) = \text{out}$, and $\pi(\texttt{mediator}) = \text{neu}$. Consider four hyperedges (joint associations), each with sign $\sigma(e) = +1$ and weight $w(e)$, together with role maps λ_e on their supports:

$$\begin{aligned}
e_{\text{write}} &: \text{supp} = \{S, L, P\},\ \lambda(S) = \texttt{source},\ \lambda(P) = \texttt{target},\ \lambda(L) = \texttt{mediator},\ \rho = \texttt{inscribes},\ w = 1.0;\\
e_{\text{submit}} &: \text{supp} = \{S, J, P\},\ \lambda(S) = \texttt{source},\ \lambda(J) = \texttt{target},\ \lambda(P) = \texttt{mediator},\ \rho = \texttt{submits},\ w = 0.6;\\
e_{\text{grant}} &: \text{supp} = \{F, L, S\},\ \lambda(F) = \texttt{source},\ \lambda(S) = \texttt{target},\ \lambda(L) = \texttt{mediator},\ \rho = \texttt{funds},\ w = 0.8;\\
e_{\text{collab}} &: \text{supp} = \{S, L, P, J\},\ \lambda(S) = \texttt{source},\ \lambda(L) = \texttt{source},\\
&\quad \lambda(P) = \texttt{target},\ \lambda(J) = \texttt{target},\ \rho = \texttt{collaborates},\ w = 1.4.
\end{aligned}$$

By Definition 2.8, the projected actor network

$$\mathcal{G}_\pi(\mathcal{H}) = \big(\mathcal{A}, E_\to, \tau, \rho_\to, \sigma_\to, w_\to\big)$$

has arc set $E_\to = \{(u, v) : \exists e \in E,\ u \in \text{In}(e),\ v \in \text{Out}(e)\}$, and weights

$$w_\to(u, v) = \sum_{\substack{e\in E:\\ u\in\text{In}(e),\ v\in\text{Out}(e)}} \frac{w(e)}{Z_e}, \qquad Z_e = \max\{1,\ |\text{In}(e)| \cdot |\text{Out}(e)|\}.$$

Per-hyperedge contributions.

- e_{write}: In $= \{S\}$, Out $= \{P\}$, so $Z = 1$. $(S \to P) :\ 1.0$.
- e_{submit}: In $= \{S\}$, Out $= \{J\}$, so $Z = 1$. $(S \to J) :\ 0.6$.
- e_{grant}: In $= \{F\}$, Out $= \{S\}$, so $Z = 1$. $(F \to S) :\ 0.8$.
- e_{collab}: In $= \{S, L\}$, Out $= \{P, J\}$, so $Z = 2 \cdot 2 = 4$. $(S \to P) : \frac{1.4}{4} = 0.35$, $(S \to J) : 0.35$, $(L \to P) : 0.35$, $(L \to J) : 0.35$.

Aggregated arcs and weights. Summing identical arcs yields

$$\begin{aligned}
&w_\to(S, P) = 1.0 + 0.35 = 1.35, && w_\to(S, J) = 0.6 + 0.35 = 0.95,\\
&w_\to(L, P) = 0.35, && w_\to(L, J) = 0.35,\\
&w_\to(F, S) = 0.8.
\end{aligned}$$

Assign $\sigma_\to(u, v) = +1$ for all arcs (all contributing hyperedges are positive). For labels, choose the dominant contributor (largest single contribution) per arc:

$$\rho_\to(S, P) = \texttt{inscribes}\ (1.0 > 0.35), \quad \rho_\to(S, J) = \texttt{submits}\ (0.6 > 0.35),$$

$$\rho_\to(L, P) = \rho_\to(L, J) = \texttt{collaborates}, \quad \rho_\to(F, S) = \texttt{funds}.$$

Result. The role-aware directed 2-section produces a typed, signed, weighted actor network on $\mathcal{A}$ with arcs

$$(S\to P),\ (S\to J),\ (L\to P),\ (L\to J),\ (F\to S),$$

and weights $(1.35,\ 0.95,\ 0.35,\ 0.35,\ 0.8)$ respectively, summarizing the multi-actor joint relations into pairwise influences while accounting for role multiplicities via Z_e.

Theorem 2.10 (Actor hypernetworks strictly generalize actor networks)**.** *Let $\mathcal{H}$ be an actor hypernetwork. Then $\mathcal{G}_\pi(\mathcal{H})$ in Definition 2.8 is an actor network. Moreover, if every hyperedge has arity* 2 *and carries exactly one* in *and one* out *role, then $\mathcal{G}_\pi(\mathcal{H})$ reduces to the usual actor network obtained by interpreting each $\{u, v\} \in E$ as a single directed arc $(u\to v)$ with the same label, sign, and weight.*

Proof. (*Projection is a network.*) By construction, $\mathcal{G}_\pi(\mathcal{H})$ has vertex set $\mathcal{A}$ and a set of directed arcs $E_\to \subseteq \mathcal{A} \times \mathcal{A}$, with types τ, labels $\rho_\to$, signs $\sigma_\to$, and nonnegative weights $w_\to$. Hence $\mathcal{G}_\pi(\mathcal{H})$ is a typed, signed, weighted digraph—i.e., an actor network.
(*Reduction under pairwise-only hyperedges.*) Assume every $e \in E$ has the form $e = \{u, v\}$ with $\pi(\lambda_e(u)) = \text{in}$ and $\pi(\lambda_e(v)) = \text{out}$. Then $Z_e = 1$ and $E_\to = \{(u, v) : \{u, v\} \in E\}$ with $w_\to(u, v) = \sum_{e:\, e=\{u,v\}} w(e)$. If the actor hypernetwork is simple (at most one such e per unordered pair), then $w_\to(u, v) = w(\{u, v\})$ and labels/signs match those of e, giving exact equality with the usual actor network constructed from these arcs. Therefore actor networks are a special case of actor hypernetworks, and actor hypernetworks strictly generalize them by allowing hyperedges of arity ≥ 3. □

2.3. *Actor Superhypernetwork*

An actor superhypernetwork generalizes hypernetworks using supervertices and superedges from iterated powersets, capturing hierarchical multi-level associations across nested, heterogeneous actor configurations.

Notation 2 (Base actants, iterated powersets, flattening, and roles)**.** *Let $\mathcal{A}_0$ be a finite nonempty set of* base actants *(people, devices, documents, institutions, biological samples,* etc.*). For a set X, define $\mathcal{P}^0(X) := X$ and $\mathcal{P}^{k+1}(X) := \mathcal{P}\big(\mathcal{P}^k(X)\big)$ $(k \geq 0)$.*
For any $Y \in \mathcal{P}^k(\mathcal{A}_0)$ $(k \geq 1)$, define the flattening *map recursively by*

$$\operatorname{flat}(u) = \{u\}\ (u \in \mathcal{A}_0), \qquad \operatorname{flat}(Y) = \bigcup_{Z\in Y} \operatorname{flat}(Z),$$

and extend to $S \subseteq \mathcal{P}^k(\mathcal{A}_0)$ by $\operatorname{flat}(S) := \bigcup_{Y\in S} \operatorname{flat}(Y)$. Thus $\operatorname{flat}(Y) \subseteq \mathcal{A}_0$ always.

Let $\mathcal{L}$ *be a finite set of* association labels *(e.g.,* ***inscribes****,* ***enrols****,* ***mediates****,* ***authorizes****).* *Let* $\mathcal{R}$ *be a finite set of* role labels *equipped with an* orientation partition

$$\pi : \ \mathcal{R} \longrightarrow \{\text{in}, \text{out}, \text{neu}\}.$$

We also allow a sign set $\{-1, 0, +1\}$ *(hindering/neutral/enabling) and nonnegative weights in* $\mathbb{R}_{\geq 0}$*. A type map* $\tau : \mathcal{A}_0 \to$ *types assigns categories to base actants. All sets are finite throughout.*

Definition 2.11 (*Actor superhypernetwork* at level n)**.** Fix $n \geq 1$. An *actor superhypernetwork* is the tuple

$$\mathcal{SH}^{(n)} \ = \ \big(V, \ E, \ \tau, \ \rho, \ \{\lambda_e\}_{e \in E}, \ \sigma, \ w\big),$$

with the following components.

- V is a nonempty set of n-*supervertices* such that $V \subseteq \mathcal{P}^n(\mathcal{A}_0)$.
- E is a nonempty family of n-*superedges*, with $E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$.
- $\rho : E \to \mathcal{L}$ assigns an association label to each superedge.
- For each $e \in E$, define its *base support* $\text{supp}(e) := \text{flat}(e) \subseteq \mathcal{A}_0$ and a *role map* λ_e : $\text{supp}(e) \to \mathcal{R}$. Write the role-induced partition of base actants as

$$\text{In}(e) := \{u \in \text{supp}(e) : \pi(\lambda_e(u)) = \text{in}\}, \quad \text{Out}(e) := \{u : \pi(\lambda_e(u)) = \text{out}\},$$

$$\text{Neu}(e) := \text{supp}(e) \setminus (\text{In}(e) \cup \text{Out}(e)).$$

- $\sigma : E \to \{-1, 0, +1\}$ assigns a sign; $w : E \to \mathbb{R}_{\geq 0}$ assigns a nonnegative weight.

Semantics: each $e \in E$ denotes a *single joint association* that may tie together collections of collections of actants (through $V \subseteq \mathcal{P}^n(\mathcal{A}_0)$), while roles λ_e specify how base actants participate in that joint event.

Example 2.12 (Public health vaccination rollout as an actor superhypernetwork at level n=2)**. Base actants.** Let the base set be

$$\mathcal{A}_0 = \{\text{MoH}, \ \text{StateH}, \ \text{Reg}, \ \text{Hosp}, \ \text{Clinic}, \ \text{NGO}, \ \text{Logi}, \ \text{Ware}, \ \text{Data}\}.$$

Orientation partition. Use $\pi : \mathcal{R} \to \{\text{in}, \text{out}, \text{neu}\}$ with

$$\pi(\texttt{source}) = \text{in}, \quad \pi(\texttt{target}) = \text{out}, \quad \pi(\texttt{medium}) = \text{neu}.$$

Rank-1 groups (subsets of $\mathcal{A}_0$).

$G_{\text{gov}} = \{\text{MoH}, \text{StateH}, \text{Reg}\}$, $G_{\text{prov}} = \{\text{Hosp}, \text{Clinic}, \text{NGO}\}$, $G_{\text{sup}} = \{\text{Logi}, \text{Ware}\}$, $G_{\text{it}} = \{\text{Data}\}$.

Level-2 supervertices $V \subseteq \mathcal{P}^2(\mathcal{A}_0)$.

$$C_{\text{delivery}} = \{G_{\text{gov}}, G_{\text{prov}}, G_{\text{sup}}\}, \quad C_{\text{report}} = \{G_{\text{gov}}, G_{\text{it}}, G_{\text{prov}}\}, \quad C_{\text{approval}} = \{G_{\text{gov}}, G_{\text{it}}\},$$

and set $V = \{C_{\text{delivery}}, C_{\text{report}}, C_{\text{approval}}\}$.

Level-2 superedges $E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$.

$$e_{\text{rollout}} = \{C_{\text{delivery}}, C_{\text{report}}\}, \qquad e_{\text{regulate}} = \{C_{\text{approval}}, C_{\text{delivery}}\},$$

with labels and signs/weights

$$\rho(e_{\text{rollout}}) = \texttt{rollout\_and\_report}, \quad \sigma(e_{\text{rollout}}) = +1, \quad w(e_{\text{rollout}}) = 0.9,$$

$$\rho(e_{\text{regulate}}) = \texttt{authorize\_and\_constrain}, \quad \sigma(e_{\text{regulate}}) = +1, \quad w(e_{\text{regulate}}) = 0.7.$$

Role maps on base supports. The base support of e_{rollout} is

$$\operatorname{supp}(e_{\text{rollout}}) = \{\text{MoH}, \text{StateH}, \text{Reg}, \text{Hosp}, \text{Clinic}, \text{NGO}, \text{Logi}, \text{Ware}, \text{Data}\}.$$

Define $\lambda_{e_{\text{rollout}}}$ by

$$\lambda(\text{MoH}) = \lambda(\text{StateH}) = \lambda(\text{Logi}) = \texttt{source},$$

$$\lambda(\text{Hosp}) = \lambda(\text{Clinic}) = \lambda(\text{NGO}) = \lambda(\text{Data}) = \texttt{target},$$

$$\lambda(\text{Ware}) = \lambda(\text{Reg}) = \texttt{medium}.$$

Thus $\text{In}(e_{\text{rollout}}) = \{\text{MoH}, \text{StateH}, \text{Logi}\}$, $\text{Out}(e_{\text{rollout}}) = \{\text{Hosp}, \text{Clinic}, \text{NGO}, \text{Data}\}$, and $\text{Neu}(e_{\text{rollout}}) = \{\text{Ware}, \text{Reg}\}$.

For e_{regulate},

$$\operatorname{supp}(e_{\text{regulate}}) = \{\text{MoH}, \text{StateH}, \text{Reg}, \text{Hosp}, \text{Clinic}, \text{NGO}, \text{Logi}, \text{Ware}, \text{Data}\},$$

with

$$\lambda(\text{Reg}) = \lambda(\text{MoH}) = \texttt{source},$$

$$\lambda(\text{StateH}) = \lambda(\text{Hosp}) = \lambda(\text{Clinic}) = \lambda(\text{NGO}) = \texttt{target},$$

$$\lambda(\text{Logi}) = \lambda(\text{Ware}) = \lambda(\text{Data}) = \texttt{medium}.$$

Then $\text{In}(e_{\text{regulate}}) = \{\text{Reg}, \text{MoH}\}$, $\text{Out}(e_{\text{regulate}}) = \{\text{StateH}, \text{Hosp}, \text{Clinic}, \text{NGO}\}$, and the remaining actants are neutral. This instantiates an actor superhypernetwork at level 2.

Example 2.13 (Climate agreement implementation as an actor superhypernetwork at level n=2)**. Base actants.**

$$\mathcal{A}_0 = \{\text{AEnv}, \ \text{BEnv}, \ \text{Secret}, \ \text{NGO}, \ \text{Regis}, \ \text{Sat}, \ \text{Bank}\}.$$

Orientation partition. Use π with

$$\pi(\texttt{submitter}) = \pi(\texttt{funder}) = \text{in},$$

$$\pi(\texttt{recipient}) = \pi(\texttt{report}) = \text{out},$$

$$\pi(\texttt{facilitator}) = \pi(\texttt{monitor}) = \text{neu}.$$

Rank-1 groups.

$$G_{\text{parties}} = \{\text{AEnv}, \text{BEnv}, \text{Secret}\},$$

$$G_{\mathrm{mrv}} = \{\mathrm{Regis}, \mathrm{Sat}\},$$

$$G_{\mathrm{cso}} = \{\mathrm{NGO}\}, \quad G_{\mathrm{fin}} = \{\mathrm{Bank}\}.$$

Level-2 supervertices $V \subseteq \mathcal{P}^2(\mathcal{A}_0)$.

$$C_{\mathrm{NDC}} = \{G_{\mathrm{parties}}, G_{\mathrm{mrv}}\},$$

$$C_{\mathrm{finance}} = \{G_{\mathrm{parties}}, G_{\mathrm{fin}}, G_{\mathrm{cso}}\},$$

and $V = \{C_{\mathrm{NDC}}, C_{\mathrm{finance}}\}$.

Level-2 superedges E.

$$e_{\mathrm{report}} = \{C_{\mathrm{NDC}}, C_{\mathrm{finance}}\},$$

$$\rho(e_{\mathrm{report}}) = \texttt{report\_and\_finance\_flow}, \quad \sigma(e_{\mathrm{report}}) = +1, \quad w(e_{\mathrm{report}}) = 0.8,$$

$$e_{\mathrm{review}} = \{C_{\mathrm{NDC}}\}, \qquad \rho(e_{\mathrm{review}}) = \texttt{technical\_review}, \quad \sigma(e_{\mathrm{review}}) = +1, \quad w(e_{\mathrm{review}}) = 0.6.$$

Role maps on base supports. For e_{report}, the base support is

$$\mathrm{supp}(e_{\mathrm{report}}) = \{\mathrm{AEnv}, \mathrm{BEnv}, \mathrm{Secret}, \mathrm{Regis}, \mathrm{Sat}, \mathrm{Bank}, \mathrm{NGO}\}.$$

Set

$$\lambda(\mathrm{AEnv}) = \lambda(\mathrm{BEnv}) = \texttt{submitter},$$

$$\lambda(\mathrm{Regis}) = \lambda(\mathrm{Sat}) = \texttt{report},$$

$$\lambda(\mathrm{Bank}) = \texttt{recipient},$$

$$\lambda(\mathrm{Secret}) = \lambda(\mathrm{NGO}) = \texttt{facilitator}.$$

Hence $\mathrm{In}(e_{\mathrm{report}}) = \{\mathrm{AEnv}, \mathrm{BEnv}\}$, $\mathrm{Out}(e_{\mathrm{report}}) = \{\mathrm{Regis}, \mathrm{Sat}, \mathrm{Bank}\}$, and $\mathrm{Neu}(e_{\mathrm{report}}) = \{\mathrm{Secret}, \mathrm{NGO}\}$.

For e_{review},

$$\mathrm{supp}(e_{\mathrm{review}}) = \{\mathrm{AEnv}, \mathrm{BEnv}, \mathrm{Secret}, \mathrm{Regis}, \mathrm{Sat}\},$$

with

$$\lambda(\mathrm{Regis}) = \lambda(\mathrm{Sat}) = \texttt{monitor},$$

$$\lambda(\mathrm{Secret}) = \texttt{facilitator},$$

$$\lambda(\mathrm{AEnv}) = \lambda(\mathrm{BEnv}) = \texttt{recipient}.$$

Thus $\mathrm{In}(e_{\mathrm{review}}) = \varnothing$ (pure review), $\mathrm{Out}(e_{\mathrm{review}}) = \{\mathrm{AEnv}, \mathrm{BEnv}\}$, and $\mathrm{Neu}(e_{\mathrm{review}}) = \{\mathrm{Secret}, \mathrm{Regis}, \mathrm{Sat}\}$. These constructions realize two actor superhypernetworks at level 2.

Theorem 2.14 (Actor superhypernetworks have SuperHyperGraph structure)**.** *For any actor superhypernetwork $\mathcal{SH}^{(n)}$ in Definition 2.11, the underlying pair (V, E) is an n-SuperHyperGraph.*

Proof. By Definition 2.11, $V \subseteq \mathcal{P}^n(\mathcal{A}_0)$ and $E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$ with both V, E nonempty. This is exactly the axiom of an n-SuperHyperGraph. □

Definition 2.15 (Role-aware directed 2-section on the *base* layer)**.** Given $\mathcal{SH}^{(n)}$, define its base-layer actor network

$$\mathcal{G}_\pi\big(\mathcal{SH}^{(n)}\big) = \big(\mathcal{A}_0,\ E_\to,\ \tau,\ \rho_\to,\ \sigma_\to,\ w_\to\big),$$

with arcs and weights

$$E_\to = \Big\{(u,v) \in \mathcal{A}_0 \times \mathcal{A}_0 :\ \exists e \in E,\ u \in \mathrm{In}(e),\ v \in \mathrm{Out}(e)\Big\},$$

$$w_\to(u,v) = \sum_{\substack{e \in E: \\ u \in \mathrm{In}(e),\ v \in \mathrm{Out}(e)}} \frac{w(e)}{Z_e},$$

$$Z_e := \max\{1,\ |\mathrm{In}(e)| \cdot |\mathrm{Out}(e)|\},$$

and labels/signs inherited from contributing superedges (ties resolved arbitrarily or by confidence). This yields a typed, signed, weighted digraph on base actants.

Example 2.16 (Role-aware directed 2-section on the *base* layer)**.** Consider an actor superhypernetwork $\mathcal{SH}^{(n)}$ (any $n \geq 1$) with base actants

$$\mathcal{A}_0 = \{\mathrm{Lab},\ \mathrm{Company},\ \mathrm{Grant},\ \mathrm{Repo},\ \mathrm{Ethics}\}.$$

Use an orientation partition π with

$$\pi(\texttt{source}) = \mathrm{in}, \qquad \pi(\texttt{target}) = \mathrm{out}, \qquad \pi(\texttt{facilitator}) = \mathrm{neu}.$$

Suppose $\mathcal{SH}^{(n)}$ has four superedges $E = \{e_{\mathrm{fund}}, e_{\mathrm{proj}}, e_{\mathrm{transfer}}, e_{\mathrm{report}}\}$, each equipped with a weight $w(e)$, a sign $\sigma(e) = +1$, and a role map $\lambda_e : \mathrm{supp}(e) \to \{\texttt{source}, \texttt{target}, \texttt{facilitator}\}$ on its base support $\mathrm{supp}(e) \subseteq \mathcal{A}_0$:

$$\begin{aligned}
e_{\mathrm{fund}} &: \mathrm{supp} = \{\mathrm{Grant}, \mathrm{Lab}, \mathrm{Company}\}, \\
&\quad \lambda(\mathrm{Grant}) = \texttt{source}, \quad \lambda(\mathrm{Lab}) = \lambda(\mathrm{Company}) = \texttt{target}, \quad w = 0.9; \\
e_{\mathrm{proj}} &: \mathrm{supp} = \{\mathrm{Lab}, \mathrm{Company}, \mathrm{Repo}, \mathrm{Ethics}\}, \\
&\quad \lambda(\mathrm{Lab}) = \lambda(\mathrm{Company}) = \texttt{source}, \quad \lambda(\mathrm{Repo}) = \texttt{target}, \quad \lambda(\mathrm{Ethics}) = \texttt{facilitator}, \quad w = 0.8; \\
e_{\mathrm{transfer}} &: \mathrm{supp} = \{\mathrm{Company}, \mathrm{Repo}\}, \\
&\quad \lambda(\mathrm{Company}) = \texttt{source}, \quad \lambda(\mathrm{Repo}) = \texttt{target}, \quad w = 0.5; \\
e_{\mathrm{report}} &: \mathrm{supp} = \{\mathrm{Lab}, \mathrm{Grant}, \mathrm{Repo}\}, \\
&\quad \lambda(\mathrm{Lab}) = \texttt{source}, \quad \lambda(\mathrm{Repo}) = \texttt{target}, \quad \lambda(\mathrm{Grant}) = \texttt{facilitator}, \quad w = 0.6.
\end{aligned}$$

Applying the base-layer role-aware directed 2-section (Definition) yields the actor network

$$\mathcal{G}_\pi\big(\mathcal{SH}^{(n)}\big) = \big(\mathcal{A}_0,\ E_\to,\ \tau,\ \rho_\to,\ \sigma_\to,\ w_\to\big),$$

with arc set

$$E_\to = \{(u,v) \in \mathcal{A}_0^2 :\ \exists e \in E \text{ with } u \in \mathrm{In}(e),\ v \in \mathrm{Out}(e)\},$$

and weights

$$w_\to(u,v) = \sum_{\substack{e\in E:\\ u\in\mathrm{In}(e),\, v\in\mathrm{Out}(e)}} \frac{w(e)}{Z_e}, \qquad Z_e = \max\{1,\ |\mathrm{In}(e)|\cdot|\mathrm{Out}(e)|\}.$$

Arc computation.

- e_{fund}: $\mathrm{In} = \{\mathrm{Grant}\}$, $\mathrm{Out} = \{\mathrm{Lab}, \mathrm{Company}\}$, so $Z = 1 \cdot 2 = 2$. Contributions: $(\mathrm{Grant} \to \mathrm{Lab}) = \frac{0.9}{2} = 0.45$, $(\mathrm{Grant} \to \mathrm{Company}) = \frac{0.9}{2} = 0.45$.
- e_{proj}: $\mathrm{In} = \{\mathrm{Lab}, \mathrm{Company}\}$, $\mathrm{Out} = \{\mathrm{Repo}\}$, so $Z = 2 \cdot 1 = 2$. Contributions: $(\mathrm{Lab} \to \mathrm{Repo}) = \frac{0.8}{2} = 0.4$, $(\mathrm{Company} \to \mathrm{Repo}) = \frac{0.8}{2} = 0.4$.
- e_{transfer}: $\mathrm{In} = \{\mathrm{Company}\}$, $\mathrm{Out} = \{\mathrm{Repo}\}$, so $Z = 1$. Contribution: $(\mathrm{Company} \to \mathrm{Repo}) = 0.5$.
- e_{report}: $\mathrm{In} = \{\mathrm{Lab}\}$, $\mathrm{Out} = \{\mathrm{Repo}\}$, so $Z = 1$. Contribution: $(\mathrm{Lab} \to \mathrm{Repo}) = 0.6$.

Aggregated base-layer digraph. Summing contributions for identical arcs gives

$$\begin{aligned} w_\to(\mathrm{Grant}, \mathrm{Lab}) &= 0.45, \\ w_\to(\mathrm{Grant}, \mathrm{Company}) &= 0.45, \\ w_\to(\mathrm{Lab}, \mathrm{Repo}) &= 0.4 + 0.6 = 1.0, \\ w_\to(\mathrm{Company}, \mathrm{Repo}) &= 0.4 + 0.5 = 0.9. \end{aligned}$$

Thus the role-aware directed 2-section on the *base* layer is a typed, signed (+1 here), weighted digraph capturing how sources (e.g., Grant, Lab, Company) direct influence toward targets (e.g., Lab, Company, Repo) after accounting for multi-actor joint associations and role multiplicities.

Theorem 2.17 (Reduction to actor hypernetworks ($n = 1$))**.** *Let $n = 1$ and choose $V = \big\{\{a\} : a \in \mathcal{A}_0\big\} \subseteq \mathcal{P}(\mathcal{A}_0)$. For any family of nonempty subsets $\widehat{E} \subseteq \mathcal{P}(\mathcal{A}_0) \setminus \{\varnothing\}$, set*

$$E\ :=\ \big\{\ e \subseteq V\ \big|\ \mathrm{flat}(e) \in \widehat{E}\ \big\}.$$

Associate to each $e \in E$ the same label/sign/weight as its support $\mathrm{flat}(e)$*, and define λ_e on* $\mathrm{supp}(e) = \mathrm{flat}(e)$*. Then $\mathcal{SH}^{(1)}$ is an* actor hypernetwork *whose vertices are base actants and whose hyperedges are exactly $\widehat{E}$ with the given annotations.*

Proof. For $n = 1$, every supervertex has the form $\{a\}$ with $\mathrm{flat}(\{a\}) = \{a\}$. If $e \subseteq V$ is nonempty, $\mathrm{flat}(e) = \{a : \{a\} \in e\}$ is the base set incident to e. By construction, the family $\{\,\mathrm{flat}(e) : e \in E\,\}$ equals $\widehat{E}$, so $(\mathcal{A}_0, \widehat{E})$ is the underlying hypergraph. The transferred maps

$(\rho, \lambda_e, \sigma, w)$ are defined on flat(e), hence we obtain an actor hypernetwork in the sense of role-annotated hyperedges on $\mathcal{A}_0$. □

Theorem 2.18 (Reduction to actor networks (pairwise case))**.** *Under the setting of Theorem 2.17, further restrict to supports* $|\mathrm{supp}(e)| = 2$ *with exactly one* in *and one* out *role. Then the base-layer* 2*-section* $\mathcal{G}_\pi(\mathcal{SH}^{(1)})$ *is precisely the typed, signed, weighted actor network whose arc set contains* $(u \to v)$ *whenever* $\{u, v\} = \mathrm{supp}(e)$ *with* $u \in \mathrm{In}(e)$ *and* $v \in \mathrm{Out}(e)$*, and whose arc weight equals* $w(e)$*.*

Proof. For any such e, we have $|\mathrm{In}(e)| = |\mathrm{Out}(e)| = 1$, hence $Z_e = 1$. The 2-section contributes the unique arc (u, v) with weight $w(e)$, label $\rho(e)$, and sign $\sigma(e)$. Summing over hyperedges yields the stated actor network. □

2.4. *Urban Road Network*

We formalize an urban road network as a finite, directed, weighted multigraph equipped with standard traffic–engineering attributes (cf. [25, 26]). An urban road network is a directed, weighted graph of intersections and road segments, modeling connectivity, capacities, and travel dynamics.

Definition 2.19 (Urban road network (primal, directed, lane/segment level))**.** (cf. [27–29]) Let I be a finite nonempty set of *intersections* (junctions) and let A be a finite set of *directed road segments* (links). Each $a \in A$ is an oriented piece of roadway that carries traffic in one direction between two intersections; two-way streets are represented by a pair of antiparallel segments. The incidence maps are

$$\mathrm{tail}, \mathrm{head} : A \to I, \qquad a : u \to v \iff \mathrm{tail}(a) = u,\ \mathrm{head}(a) = v.$$

Each segment a is endowed with attributes:

$$\ell_a > 0 \text{ (length)}, \qquad C_a > 0 \text{ (capacity)}, \qquad v_a^{\mathrm{ff}} > 0 \text{ (free–flow speed)}.$$

Define the free–flow travel time $t_a^{\mathrm{ff}} := \ell_a / v_a^{\mathrm{ff}}$. The tuple

$$\mathcal{R} := (I, A, \ell, C, v^{\mathrm{ff}})$$

is an *urban road network (URN)*.

Example 2.20 (Toy URN with three intersections and five directed segments (for Def. 2.19))**.** Let the intersection set be $I = \{J_1, J_2, J_3\}$ and the directed segment set

$$A = \{a_1{:}J_1 \to J_2,\ a_2{:}J_2 \to J_1,\ a_3{:}J_2 \to J_3,\ a_4{:}J_3 \to J_2,\ a_5{:}J_1 \to J_3\}.$$

Here (a_1, a_2) model a two-way street via antiparallel links. Assign attributes (length ℓ_a in km, capacity C_a in veh/h, free-flow speed v_a^{ff} in km/h):

link a	a_1	a_2	a_3	a_4	a_5
$(\mathrm{tail}(a), \mathrm{head}(a))$	(J_1, J_2)	(J_2, J_1)	(J_2, J_3)	(J_3, J_2)	(J_1, J_3)
ℓ_a [km]	1.2	1.2	0.8	0.8	2.0
C_a [veh/h]	1800	1800	1500	1500	2000
v_a^{ff} [km/h]	60	60	50	50	70

Then the free-flow times $t_a^{\mathrm{ff}} := \ell_a / v_a^{\mathrm{ff}}$ (hours) are

$$\begin{aligned} t_{a_1}^{\mathrm{ff}} &= 1.2/60 = 0.020 \text{ h} = 1.200 \text{ min}, \quad t_{a_2}^{\mathrm{ff}} = 0.020 \text{ h} = 1.200 \text{ min}, \\ t_{a_3}^{\mathrm{ff}} &= 0.8/50 = 0.016 \text{ h} = 0.960 \text{ min}, \quad t_{a_4}^{\mathrm{ff}} = 0.016 \text{ h} = 0.960 \text{ min}, \\ t_{a_5}^{\mathrm{ff}} &= 2.0/70 = \tfrac{1}{35} \approx 0.028571 \text{ h} \approx 1.714286 \text{ min}. \end{aligned}$$

Thus $\mathcal{R} = (I, A, \ell, C, v^{\mathrm{ff}})$ is a concrete URN with explicit incidence, geometry, and capacities.

Definition 2.21 (Flows, VOC, and link performance)**.** Let $x_a \geq 0$ denote the (hourly) flow on link a. Its *volume–over–capacity* is $\mathrm{VOC}_a := x_a / C_a$. A standard link travel–time function is the BPR form

$$t_a(x_a) \;=\; t_a^{\mathrm{ff}} \Big(1 + \alpha\, \mathrm{VOC}_a^{\beta}\Big) \quad (\alpha > 0,\ \beta > 1).$$

Example 2.22 (Computing VOC and BPR travel time (for Def. 2.21))**.** On the URN above, suppose hourly link flows (veh/h)

$$x_{a_1} = 1200, \quad x_{a_2} = 600, \quad x_{a_3} = 1400, \quad x_{a_4} = 800, \quad x_{a_5} = 900,$$

and use the BPR form $t_a(x_a) = t_a^{\mathrm{ff}}\big(1 + \alpha\, \mathrm{VOC}_a^{\beta}\big)$ with $(\alpha, \beta) = (0.15, 4)$ and $\mathrm{VOC}_a := x_a / C_a$.

Step-by-step derivations (two links).

$$\begin{aligned} \textbf{Link } a_1: \quad & \mathrm{VOC}_{a_1} = \frac{1200}{1800} = \frac{2}{3}, \quad \mathrm{VOC}_{a_1}^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{81} \approx 0.197530864, \\ & 1 + \alpha\, \mathrm{VOC}_{a_1}^4 = 1 + 0.15 \cdot \frac{16}{81} = 1 + \frac{2.4}{81} = 1.0296296296, \\ & t_{a_1}(x_{a_1}) = t_{a_1}^{\mathrm{ff}} \cdot 1.0296296296 = 0.020 \times 1.0296296296 = 0.0205925926 \text{ h} = 1.235555556 \text{ min}. \end{aligned}$$

$$\begin{aligned} \textbf{Link } a_3: \quad & \mathrm{VOC}_{a_3} = \frac{1400}{1500} = \frac{14}{15}, \quad \mathrm{VOC}_{a_3}^4 = \left(\frac{14}{15}\right)^4 = \frac{38416}{50625} \approx 0.758834568, \\ & 1 + \alpha\, \mathrm{VOC}_{a_3}^4 = 1 + 0.15 \cdot \frac{38416}{50625} = 1 + \frac{5762.4}{50625} = 1.113825185, \\ & t_{a_3}(x_{a_3}) = t_{a_3}^{\mathrm{ff}} \cdot 1.113825185 = 0.016 \times 1.113825185 = 0.017821203 \text{ h} = 1.069272178 \text{ min}. \end{aligned}$$

All links (summary). With the same procedure,

link a	a_1	a_2	a_3	a_4	a_5
$\mathrm{VOC}_a = x_a/C_a$	0.666667	0.333333	0.933333	0.533333	0.450000
$1 + 0.15\,\mathrm{VOC}_a^4$	1.029630	1.001852	1.113825	1.012136	1.006151
$t_a(x_a)$ [min]	1.235556	1.202222	1.069272	0.971651	1.724830

Each $t_a(x_a)$ equals t_a^{ff} multiplied by $1 + 0.15\,\mathrm{VOC}_a^4$ as required by the BPR performance function.

Remark 2.23 (Monotonicity, convexity, and concrete values)**.** For BPR, $t_a'(x_a) = t_a^{\mathrm{ff}}\,\alpha\,\beta\,C_a^{-1}\big(x_a/C_a\big)^{\beta-1} > 0$ and $t_a''(x_a) = t_a^{\mathrm{ff}}\,\alpha\,\beta(\beta-1)\,C_a^{-2}\big(x_a/C_a\big)^{\beta-2} > 0$, hence t_a is strictly increasing and convex. With the commonly used $(\alpha,\beta) = (0.15, 4)$,

$$\mathrm{VOC}_a = 0.8 \;\Rightarrow\; t_a = t_a^{\mathrm{ff}}\big(1 + 0.15 \cdot 0.8^4\big) = t_a^{\mathrm{ff}}(1 + 0.06144) = 1.06144\,t_a^{\mathrm{ff}},$$

$$\mathrm{VOC}_a = 1.2 \;\Rightarrow\; t_a = t_a^{\mathrm{ff}}\big(1 + 0.15 \cdot 1.2^4\big) = t_a^{\mathrm{ff}}(1 + 0.31104) = 1.31104\,t_a^{\mathrm{ff}}.$$

Definition 2.24 (OD pairs, paths, feasible flows)**.** Let $W \subseteq I \times I$ be the set of origin–destination (OD) pairs with demands $q_w \geq 0$ $(w \in W)$. A (simple) *path* p is a directed sequence of links from o to d. Let $\mathcal{P}_w$ be the set of $o \to d$ paths for $w = (o, d)$ and let $\mathcal{P} := \bigcup_w \mathcal{P}_w$. Path flows $f_p \geq 0$ satisfy, for each w,

$$\sum_{p \in \mathcal{P}_w} f_p \;=\; q_w, \qquad x_a \;=\; \sum_{p \in \mathcal{P}} \delta_{ap} f_p \quad (a \in A),$$

where $\delta_{ap} = 1$ if link a lies on path p and 0 otherwise. The vector $x = (x_a)_{a \in A}$ is a *feasible link–flow.*

Example 2.25 (Enumerating OD pairs, paths, and a feasible flow (for Def. 2.24))**.** Consider the toy URN with intersections $I = \{J_1, J_2, J_3\}$ and directed links

$$a_1 : J_1 \to J_2, \qquad a_3 : J_2 \to J_3, \qquad a_5 : J_1 \to J_3.$$

Let the OD set be $W = \{w\}$ with $w = (J_1, J_3)$ and demand $q_w = 1000$ veh/h. There are two simple $J_1 \to J_3$ paths:

$$P_1 = (a_5), \qquad P_2 = (a_1, a_3).$$

Choose nonnegative path flows

$$f_{P_1} = 700, \qquad f_{P_2} = 300,$$

which satisfy the OD constraint $\sum_{p \in \{P_1, P_2\}} f_p = 700 + 300 = 1000 = q_w$. The implied link flows, using $x_a = \sum_p \delta_{ap} f_p$, are

$$x_{a_5} = \delta_{a_5, P_1} f_{P_1} + \delta_{a_5, P_2} f_{P_2} = 1 \cdot 700 + 0 \cdot 300 = 700,$$

$$x_{a_1} = 0 \cdot 700 + 1 \cdot 300 = 300, \qquad x_{a_3} = 0 \cdot 700 + 1 \cdot 300 = 300.$$

Thus $x = (x_{a_1}, x_{a_3}, x_{a_5}) = (300, 300, 700)$ is a *feasible* link–flow vector induced by $(f_{P_1}, f_{P_2}) = (700, 300)$.

Definition 2.26 (Path costs and Wardrop user equilibrium)**.** Define the link costs $t_a(x_a)$ and the cost of path p as $c_p(x) := \sum_a \delta_{ap}\, t_a(x_a)$. A feasible flow (f, x) is a *Wardrop user equilibrium (UE)* if for each w and all $p \in \mathcal{P}_w$,

$$f_p > 0 \;\Rightarrow\; c_p(x) = \min_{p' \in \mathcal{P}_w} c_{p'}(x), \qquad f_p = 0 \;\Rightarrow\; c_p(x) \geq \min_{p' \in \mathcal{P}_w} c_{p'}(x).$$

If each $t_a(\cdot)$ is continuous, strictly increasing, the UE exists and (with BPR) is characterized by the Beckmann program

$$\min_{x,f} \; \Phi(x) := \sum_{a \in A} \int_0^{x_a} t_a(s)\, ds \quad \text{s.t. the feasibility constraints in Def. 2.24.}$$

For BPR,

$$\int_0^{x_a} t_a(s)\, ds = t_a^{\mathrm{ff}} \left[x_a + \frac{\alpha\, C_a}{\beta + 1} \left(\frac{x_a}{C_a} \right)^{\beta+1} \right].$$

Example 2.27 (Checking a Wardrop user equilibrium with BPR costs (for Def. 2.26))**.** On the same network, let $q_{(J_1,J_3)} = 1000$ veh/h and adopt BPR link costs

$$t_a(x_a) = t_a^{\mathrm{ff}} \Big(1 + \alpha\, (x_a/C_a)^{\beta} \Big), \qquad (\alpha, \beta) = (0.15, 4),$$

with free–flow times (minutes) and capacities (veh/h)

$$t_{a_1}^{\mathrm{ff}} = 1.20,\; C_{a_1} = 1800; \quad t_{a_3}^{\mathrm{ff}} = 0.96,\; C_{a_3} = 1500; \quad t_{a_5}^{\mathrm{ff}} = 1.7142857,\; C_{a_5} = 2000.$$

Consider the candidate flow where all users take the direct path $P_1 = (a_5)$:

$$f_{P_1} = 1000, \qquad f_{P_2} = 0 \quad \Rightarrow \quad x_{a_5} = 1000,\; x_{a_1} = x_{a_3} = 0.$$

Path costs. Compute $\mathrm{VOC}_{a_5} = x_{a_5}/C_{a_5} = 1000/2000 = 0.5$ and

$$1 + \alpha\, \mathrm{VOC}_{a_5}^{\beta} = 1 + 0.15 \cdot (0.5)^4 = 1 + 0.15 \cdot 0.0625 = 1 + 0.009375 = 1.009375.$$

Hence

$$c_{P_1} = t_{a_5}(1000) = t_{a_5}^{\mathrm{ff}} \cdot 1.009375 = 1.7142857 \times 1.009375 = 1.7303571 \text{ min}.$$

For the unused path $P_2 = (a_1, a_3)$ we have $x_{a_1} = x_{a_3} = 0$, so

$$c_{P_2} = t_{a_1}(0) + t_{a_3}(0) = t_{a_1}^{\mathrm{ff}} + t_{a_3}^{\mathrm{ff}} = 1.20 + 0.96 = 2.16 \text{ min}.$$

Wardrop conditions. The used path P_1 has cost $c_{P_1} = 1.7303571$ which equals the minimum path cost, and the unused path satisfies $c_{P_2} = 2.16 \geq c_{P_1}$. Therefore

$$f_{P_1} > 0 \;\Rightarrow\; c_{P_1} = \min\{c_{P_1}, c_{P_2}\}, \qquad f_{P_2} = 0 \;\Rightarrow\; c_{P_2} \geq \min\{c_{P_1}, c_{P_2}\},$$

so (f, x) is a Wardrop user equilibrium for this OD and cost specification.

2.5. *Urban Road HyperNetwork*

Signal phases, merging/diverging maneuvers, and coordinated movements naturally involve *sets of incoming links jointly producing sets of outgoing links.* This is captured by a role–annotated hypergraph on the same base A (links) or on the set of *movements* $\mathcal{M} \subseteq I_{\text{in}} \times I_{\text{out}}$.

Definition 2.28 (Urban road hypernetwork (movement hyperedges))**.** Fix the URN $\mathcal{R} = (I, A, \ldots)$. An *urban road hypernetwork* is

$$\mathcal{H}_{\text{road}} = \big(V,\ E,\ \{\lambda_e\}_{e\in E},\ \rho,\ \sigma,\ w\big),$$

where V is either A (links) or a set $\mathcal{M}$ of atomic movements; $E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$ is a nonempty family of *hyperedges* (joint operations, e.g. a signal phase); each $e \in E$ carries a role map $\lambda_e : e \to \{\text{in}, \text{out}, \text{neu}\}$ that partitions

$$\text{In}(e) := \lambda_e^{-1}(\text{in}), \quad \text{Out}(e) := \lambda_e^{-1}(\text{out}), \quad \text{Neu}(e) := \lambda_e^{-1}(\text{neu}),$$

and annotations: label $\rho(e)$ (e.g. `phase_1`, `merge`, `diverge`), sign $\sigma(e) \in \{-1, 0, +1\}$ (hindering/neutral/enabling), weight $w(e) \geq 0$ (e.g. green–time share or saturation).

Definition 2.29 (Role–aware directed 2–section (movement projection))**.** Given $\mathcal{H}_{\text{road}}$, define a directed, weighted graph on V with arc set

$$E_{\to} \ = \ \big\{(u, v) \in V^2 :\ \exists e \in E,\ u \in \text{In}(e),\ v \in \text{Out}(e)\big\},$$

and weights

$$w_{\to}(u, v) \ = \ \sum_{\substack{e\in E:\\ u\in \text{In}(e),\, v\in\text{Out}(e)}} \frac{w(e)}{\max\{1,\ |\text{In}(e)|\,|\text{Out}(e)|\}}.$$

This recovers a digraph of *effective pairwise influences* induced by joint operations.

Example 2.30 (Emergency vehicle preemption at a 4-leg signal (urban road hypernetwork))**.** Let $V = A$ (links) with approaches $\{a_{Nin}, a_{Ein}, a_{Sin}, a_{Win}\}$ and departures $\{a_{Nout}, a_{Eout}, a_{Sout}, a_{Wout}\}$. When a northbound ambulance triggers preemption, the controller opens straight and right from the north approach while holding conflicts. Model this as the single hyperedge e_{pre} with

$$\text{In}(e_{\text{pre}}) = \{a_{Nin}\}, \qquad \text{Out}(e_{\text{pre}}) = \{a_{Sout}, a_{Eout}\}, \qquad \rho(e_{\text{pre}}) = \texttt{preemption},\ \ \sigma(e_{\text{pre}}) = +1,\ \ w(e_{\text{pre}}) = 0.90.$$

Role–aware 2–section weight per ordered pair uses

$$Z_{e_{\text{pre}}} = \max\{1, |\text{In}| \cdot |\text{Out}|\} = \max\{1, 1 \cdot 2\} = 2, \qquad w_{\to}(u, v) = \frac{w(e_{\text{pre}})}{Z_{e_{\text{pre}}}}.$$

Hence each permitted movement gets

$$w_{\to}(a_{Nin}, a_{Sout}) = w_{\to}(a_{Nin}, a_{Eout}) = \frac{0.90}{2} = 0.45.$$

All other pairs receive no contribution from e_{pre}.

Example 2.31 (Two on-ramps metered into a freeway mainline (urban road hypernetwork))**.** Let $V = A$ with links $\{a_{\text{Mup}}, a_{\text{R1}}, a_{\text{R2}}, a_{\text{Mdown}}\}$ denoting upstream mainline, two on-ramps, and the downstream mainline. Coordinated ramp metering allocates a fixed share of discharge toward the downstream bottleneck. Represent this with one hyperedge e_{meter}:

$$\mathrm{In}(e_{\text{meter}}) = \{a_{\text{Mup}}, a_{\text{R1}}, a_{\text{R2}}\},$$

$$\mathrm{Out}(e_{\text{meter}}) = \{a_{\text{Mdown}}\},$$

$$\rho(e_{\text{meter}}) = \texttt{coordinated\_metering}, \quad \sigma(e_{\text{meter}}) = +1, \quad w(e_{\text{meter}}) = 0.75.$$

Then

$$Z_{e_{\text{meter}}} = \max\{1, 3 \cdot 1\} = 3,$$

$$w_{\to}(u, v) = \frac{0.75}{3} = 0.25 \quad \text{for } (u, v) \in \{(a_{\text{Mup}}, a_{\text{Mdown}}), (a_{\text{R1}}, a_{\text{Mdown}}), (a_{\text{R2}}, a_{\text{Mdown}})\}.$$

Thus each contributing input (mainline and both ramps) receives a projected influence weight 0.25 toward the downstream mainline under this joint operation.

Example 2.32 (Single-lane 4-leg roundabout (urban road hypernetwork))**.** Let $V = A$ with entries

$$\{a_{Nin}, a_{Ein}, a_{Sin}, a_{Win}\}$$

and exits

$$\{a_{Nout}, a_{Eout}, a_{Sout}, a_{Wout}\}$$

. Gap-acceptance and circulating priority can be summarized as a joint operation that *enables* entries to feed exits over a cycle. Model one cycle by the hyperedge e_{rb}:

$$\mathrm{In}(e_{\text{rb}}) = \{a_{Nin}, a_{Ein}, a_{Sin}, a_{Win}\},$$

$$\mathrm{Out}(e_{\text{rb}}) = \{a_{Nout}, a_{Eout}, a_{Sout}, a_{Wout}\},$$

$$\rho(e_{\text{rb}}) = \texttt{circulating\_priority},$$

$$\sigma(e_{\text{rb}}) = +1, \qquad w(e_{\text{rb}}) = 0.90.$$

The role–aware projection gives

$$Z_{e_{\text{rb}}} = \max\{1, 4 \cdot 4\} = 16,$$

$$w_{\to}(u, v) = \frac{0.90}{16} = 0.05625 \quad \text{for each } u \in \mathrm{In}(e_{\text{rb}}),\ v \in \mathrm{Out}(e_{\text{rb}}).$$

For instance,

$$w_{\to}(a_{Nin}, a_{Eout}) = w_{\to}(a_{Nin}, a_{Sout}) = w_{\to}(a_{Nin}, a_{Wout}) = 0.05625,$$

and analogously for all other entry–exit pairs in the cycle (pairs not permitted in a particular jurisdiction can be removed by excluding those v from $\mathrm{Out}(e_{\text{rb}})$).

Theorem 2.33 (Hypernetwork generalizes pairwise control)**.** *If every $e \in E$ has $|\mathrm{In}(e)| = |\mathrm{Out}(e)| = 1$, then the 2–section recovers an ordinary directed graph on V; thus hypernetworks strictly generalize pairwise linking of movements.*

Proof. Immediate from the definition of $E_{\to}$ and the denominator $\max\{1, |\mathrm{In}(e)|\,|\mathrm{Out}(e)|\}$. □

2.6. *Urban Road SuperHyperNetwork*

Corridors, subnetworks, and control groups naturally form *collections of links/movements*. Superhypernetworks model joint operations across such collections using iterated powersets.

Definition 2.34 (Urban road superhypernetwork at level n)**.** Let the base set be $V_0 := A$ (links) or $V_0 := \mathcal{M}$ (movements). For $n \geq 1$, an *n–level urban road superhypernetwork* is

$$\mathcal{SH}^{(n)}_{\text{road}} = \big(V,\ E,\ \{\lambda_e\}_{e\in E},\ \rho,\ \sigma,\ w\big),$$

with $V \subseteq \mathcal{P}^n(V_0)$ nonempty (supervertices are groups, corridors, or clusters at rank n) and $E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$ nonempty (superedges are joint corridor–level operations). For each $e \in E$, let $\mathrm{supp}(e) \subseteq V_0$ be the union of base elements inside all supervertices of e; the role map $\lambda_e : \mathrm{supp}(e) \to \{\mathrm{in}, \mathrm{out}, \mathrm{neu}\}$ partitions base links into sources, sinks, and neutral participants. Signs and weights annotate control/enabling strength at the group level.

Definition 2.35 (Base–layer 2–section)**.** Project $\mathcal{SH}^{(n)}_{\text{road}}$ to the base layer V_0 by adding an arc (u, v) for every $e \in E$ with $u \in \mathrm{In}(e)$, $v \in \mathrm{Out}(e)$, and aggregating weights as in the hypernetwork projection.

Example 2.36 (Special-event detours (level n=2 urban road superhypernetwork))**.** Let the base set be links $V_0 = A = \{a_1, \ldots, a_8\}$ in a CBD. Form rank-1 groups (corridor/link clusters):

$$G_{\text{core}} = \{a_1, a_2, a_3\}, \quad G_{\text{detour}} = \{a_4, a_5, a_6\}, \quad G_{\text{transit}} = \{a_7, a_8\}.$$

Form rank-2 supervertices

$$C_{\text{event}} = \{G_{\text{core}}, G_{\text{transit}}\}, \qquad C_{\text{detour}} = \{G_{\text{detour}}\},$$

and set $V = \{C_{\text{event}}, C_{\text{detour}}\} \subseteq \mathcal{P}^2(V_0)$. Define a superedge $e_{\text{event}} = \{C_{\text{event}}, C_{\text{detour}}\} \in E$ with

$$\rho(e_{\text{event}}) = \texttt{marathon\_control}, \quad \sigma(e_{\text{event}}) = +1, \quad w(e_{\text{event}}) = 0.75.$$

Role map $\lambda_{e_{\text{event}}}$ on the base support $\mathrm{supp}(e_{\text{event}}) = G_{\text{core}} \cup G_{\text{detour}} \cup G_{\text{transit}}$ is

$$\mathrm{In}(e_{\text{event}}) = G_{\text{core}} = \{a_1, a_2, a_3\},$$

$$\mathrm{Out}(e_{\text{event}}) = G_{\text{detour}} = \{a_4, a_5, a_6\},$$

$$\mathrm{Neu}(e_{\text{event}}) = G_{\text{transit}} = \{a_7, a_8\}.$$

Base-layer 2-section (pairwise projection). Using

$$Z_e := \max\{1, |\mathrm{In}(e)|\,|\mathrm{Out}(e)|\}, \qquad w_{\to}(u,v) = \sum_e \frac{w(e)\,\mathbf{1}_{\{u\in\mathrm{In}(e),\, v\in\mathrm{Out}(e)\}}}{Z_e},$$

we have $Z_{e_{\mathrm{event}}} = 3\cdot 3 = 9$ and each in–out pair receives

$$\frac{w(e_{\mathrm{event}})}{Z_{e_{\mathrm{event}}}} = \frac{0.75}{9} = \frac{1}{12} = 0.083333\ldots.$$

Thus the 9 arcs $a_i \to a_j$ ($a_i \in \{a_1,a_2,a_3\}$, $a_j \in \{a_4,a_5,a_6\}$) each get weight 1/12; e.g.,

$$w_{\to}(a_1,a_4) = w_{\to}(a_2,a_5) = w_{\to}(a_3,a_6) = \frac{1}{12}.$$

Operationally, the superedge encodes a special-event plan that throttles *core* links (in) and diverts flow to designated *detour* links (out), while *transit* links remain neutral.

Example 2.37 (District-scale green waves with ramp metering (level $n{=}3$ urban road super-hypernetwork))**.** Let the base set be movements $V_0 = \mathcal{M} = \{b_1,\ldots,b_{12}\}$ along two opposed arterials with ramps and cross-streets. Rank-1 groups (segment-level movement sets):

$$S_\uparrow = \{b_1,b_2,b_3,b_4\}, \quad S_\downarrow = \{b_5,b_6,b_7\},$$

$$S_\times = \{b_8,b_9\}, \quad S_{\mathrm{ramp}} = \{b_{10},b_{11},b_{12}\}.$$

Rank-2 supervertices (corridor level):

$$\mathrm{Cor}_N = \{S_\uparrow, S_\times\}, \qquad \mathrm{Cor}_S = \{S_\downarrow, S_{\mathrm{ramp}}\}.$$

Rank-3 supervertices (district level):

$$D_A = \{\mathrm{Cor}_N\}, \qquad D_B = \{\mathrm{Cor}_S\},$$

and $V = \{D_A, D_B\} \subseteq \mathcal{P}^3(V_0)$.

Two superedges model coordinated operations:

$$\begin{aligned} e_{\mathrm{wave}} &= \{D_A, D_B\}, & \rho(e_{\mathrm{wave}}) &= \texttt{green\_wave}, & \sigma(e_{\mathrm{wave}}) &= +1, & w(e_{\mathrm{wave}}) &= 0.60,\\ e_{\mathrm{meter}} &= \{D_B\}, & \rho(e_{\mathrm{meter}}) &= \texttt{ramp\_metering}, & \sigma(e_{\mathrm{meter}}) &= +1, & w(e_{\mathrm{meter}}) &= 0.40. \end{aligned}$$

Role maps on base supports:

$$\mathrm{In}(e_{\mathrm{wave}}) = S_\uparrow = \{b_1,b_2,b_3,b_4\}, \quad \mathrm{Out}(e_{\mathrm{wave}}) = S_\downarrow = \{b_5,b_6,b_7\},$$

$$\mathrm{Neu}(e_{\mathrm{wave}}) = S_\times \cup S_{\mathrm{ramp}} = \{b_8,b_9,b_{10},b_{11},b_{12}\}.$$

$$\mathrm{In}(e_{\mathrm{meter}}) = S_{\mathrm{ramp}} = \{b_{10},b_{11},b_{12}\}, \quad \mathrm{Out}(e_{\mathrm{meter}}) = S_\downarrow = \{b_5,b_6,b_7\},$$

$$\mathrm{Neu}(e_{\mathrm{meter}}) = S_\uparrow \cup S_\times.$$

Base-layer 2-section weights (same aggregation as in Example 1). For e_{wave}: $|\mathrm{In}| = 4$, $|\mathrm{Out}| = 3$, hence $Z_{e_{\mathrm{wave}}} = 12$ and each upstream–downstream pair gets

$$\frac{w(e_{\mathrm{wave}})}{Z_{e_{\mathrm{wave}}}} = \frac{0.60}{12} = \frac{1}{20} = 0.05.$$

There are $4 \times 3 = 12$ such arcs; e.g.,

$$w_{\to}(b_1, b_5) = w_{\to}(b_2, b_6) = w_{\to}(b_4, b_7) = 0.05.$$

For e_{meter}: $|\text{In}| = 3$, $|\text{Out}| = 3$, so $Z_{e_{\text{meter}}} = 9$ and each ramp–to–downstream pair gets

$$\frac{w(e_{\text{meter}})}{Z_{e_{\text{meter}}}} = \frac{0.40}{9} = \frac{2}{45} \approx 0.044444\ldots,$$

across $3 \times 3 = 9$ arcs; e.g.,

$$w_{\to}(b_{10}, b_5) = w_{\to}(b_{11}, b_6) = w_{\to}(b_{12}, b_7) = \frac{2}{45}.$$

Interpretation. The rank-3 *green-wave* superedge coordinates offsets so upstream platoons (in) feed downstream segments (out). The *ramp-metering* superedge throttles ramps (in) toward the same downstream set (out). Their induced pairwise weights sum on common ordered pairs—e.g., if a ramp movement also contributes via the platoon, the final arc weight is $0.05 + \frac{2}{45}$.

Theorem 2.38 (Reductions)**.** *For $n = 1$ with $V = \{\{a\} : a \in V_0\}$, $\mathcal{SH}^{(1)}_{\text{road}}$ reduces to an urban road hypernetwork on V_0. If, further, every superedge has exactly one* in *and one* out *base link, the base–layer 2–section is precisely a directed graph on V_0.*

Proof. Identical to the reductions proved earlier for generic (super)hypernetworks, applied with $V_0 = A$ or $V_0 = \mathcal{M}$. □

3. **Conclusion**

In this paper, we extended actor networks by employing HyperGraphs and SuperHyperGraphs. Looking ahead, we hope that future research will further develop extended frameworks based on Fuzzy Graphs [30], Intuitionistic Fuzzy Graphs [31,32], Hesitant Fuzzy Graphs [33], Picture Fuzzy Graphs [34], Neutrosophic Graphs [35, 36], and Plithogenic Graphs [37]. In addition, for the concepts defined in this paper, we expect future progress in developing programming tools, conducting quantitative computational analyses, performing experimental validations, and investigating their mathematical properties.

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Ajoy Kanti Das, Rajat Das, Suman Das, Bijoy Krishna Debnath, Carlos Granados, Bimal Shil, and Rakhal Das. A comprehensive study of neutrosophic superhyper bci-semigroups and their algebraic significance. *Transactions on Fuzzy Sets and Systems*, 8(2):80, 2025.

[2] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[3] Florentin Smarandache. Foundation of revolutionary topologies: An overview, examples, trend analysis, research issues, challenges, and future directions. 2024.

[4] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[5] Melody Mae Cabigting Lunar and Renson Aguilar Robles. Characterization and structure of a power set graph. *International Journal of Advanced Research and Publications*, 3(6):1–4, 2019.

[6] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[7] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[8] Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications*. Chapman and Hall/CRC, 2018.

[9] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[10] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[11] KK Myithili and C Nandhini. Analysis of hub parameters in fuzzy hypergraphs extending to intuitionistic fuzzy threshold hypergraphs: Applications in designing transport networks in amusement parks using hub hyperpaths. 2024.

[12] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[13] Muhammad Aslam Malik, Ali Hassan, Said Broumi, Assia Bakali, Mohamed Talea, and Florentin Smarandache. Isomorphism of bipolar single valued neutrosophic hypergraphs. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 72, 2022.

[14] Muhammad Akram and Anam Luqman. Intuitionistic single-valued neutrosophic hypergraphs. *Opsearch*, 54:799–815, 2017.

[15] Takaaki Fujita and Florentin Smarandache. Random n-superhypergraphs: A probabilistic model and generation algorithm for hierarchical networks. *European Journal of Pure and Applied Mathematics*, 18(4):6835–6835, 2025.

[16] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[17] Eduardo Martín Campoverde Valencia, Jessica Paola Chuisaca Vásquez, and Francisco Ángel Becerra Lois. Multineutrosophic analysis of the relationship between survival and business growth in the manufacturing sector of azuay province, 2020–2023, using plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84(1):28, 2025.

[18] Takaaki Fujita and Florentin Smarandache. Property hypergraphs and property superhypergraphs for data analysis. *European Journal of Pure and Applied Mathematics*, 18(4):6729–6729, 2025.

[19] Nelly Hodelín Amable, Elizabeth Esther Vergel De Salazar, Martha Gloria Martínez Isaac, Olivia Catalina Olavarría Sánchez, and Johanna Mariuxi Solís Palma. Representation of motivational dynamics in school environments through plithogenic n-superhypergraphs with family participation. *Neutrosophic Sets and Systems*, 92:570–583, 2025.

[20] Eduardo Luciano Hernandez Ramos, Luis Ramiro Ayala Ayala, and Kevin Alexander Samaniego Macas. Study of factors that influence a victim's refusal to testify for sexual reasons due to external influence using plithogenic n-superhypergraphs. *Operational Research Journal*, 46(2):328–337, 2025.

[21] Muhammad Akram and Gulfam Shahzadi. Hypergraphs in m-polar fuzzy environment. *Mathematics*, 6(2):28, 2018.

[22] Michel Callon. The sociology of an actor-network: The case of the electric vehicle. In *Mapping the dynamics of science and technology: Sociology of science in the real world*, pages 19–34. Springer, 1986.

[23] Sophie Mützel. Networks as culturally constituted processes: a comparison of relational sociology and actor-network theory. *Current sociology*, 57(6):871–887, 2009.

[24] Michel Callon. Actor-network theory-the market test. *The sociological review*, 47(1_suppl):181–195, 1999.

[25] Ge Shi, Jie Shan, Liang Ding, Peng Ye, Yang Li, and Nan Jiang. Urban road network expansion and its driving variables: a case study of nanjing city. *International journal of environmental research and public health*, 16(13):2318, 2019.

[26] Farhad Ahmadzai, KM Lakshmana Rao, and Shahzada Ulfat. Assessment and modelling of urban road networks using integrated graph of natural road network (a gis-based approach). *Journal of Urban Management*, 8(1):109–125, 2019.

[27] Kezhi Liu, Changhai Zhai, You Dong, and Xianghai Meng. Post-earthquake functionality assessment of urban road network considering emergency response. *Journal of Earthquake Engineering*, 27(9):2406–2431, 2023.

[28] BingFeng Si, JianCeng Long, and ZiYou Gao. Optimization model and algorithm for mixed traffic of urban road network with flow interference. *Science in China Series E: Technological Sciences*, 51(12):2223–2232, 2008.

[29] Jianbin Chen, Demin Li, Guanglin Zhang, and Xiaolu Zhang. Localized space-time autoregressive parameters estimation for traffic flow prediction in urban road networks. *Applied Sciences*, 8(2):277, 2018.

[30] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[31] Sankar Sahoo and Madhumangal Pal. Product of intuitionistic fuzzy graphs and degree. *Journal of Intelligent & Fuzzy Systems*, 32(1):1059–1067, 2017.

[32] Muhammad Akram, Bijan Davvaz, and Feng Feng. Intuitionistic fuzzy soft k-algebras. *Mathematics in Computer Science*, 7:353–365, 2013.

[33] Eman A AbuHijleh. Complex hesitant fuzzy graph. *Fuzzy Information and Engineering*, 15(2):149–161, 2023.

[34] Waheed Ahmad Khan, Babir Ali, and Abdelghani Taouti. Bipolar picture fuzzy graphs with application. *Symmetry*, 13:1427, 2021.

[35] Said Broumi, Shio Gai Quek, Ganeshsree Selvachandran, Florentin Smarandache, Assia Bakali, and Mohamed Talea. Finding the shortest path with neutrosophic theory: A tool for network optimization. In *Neutrosophic Graph Theory and Algorithms*, pages 1–32. IGI Global, 2020.

[36] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.

[37] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Plithogenic Graphs*. Infinite Study, 2020.