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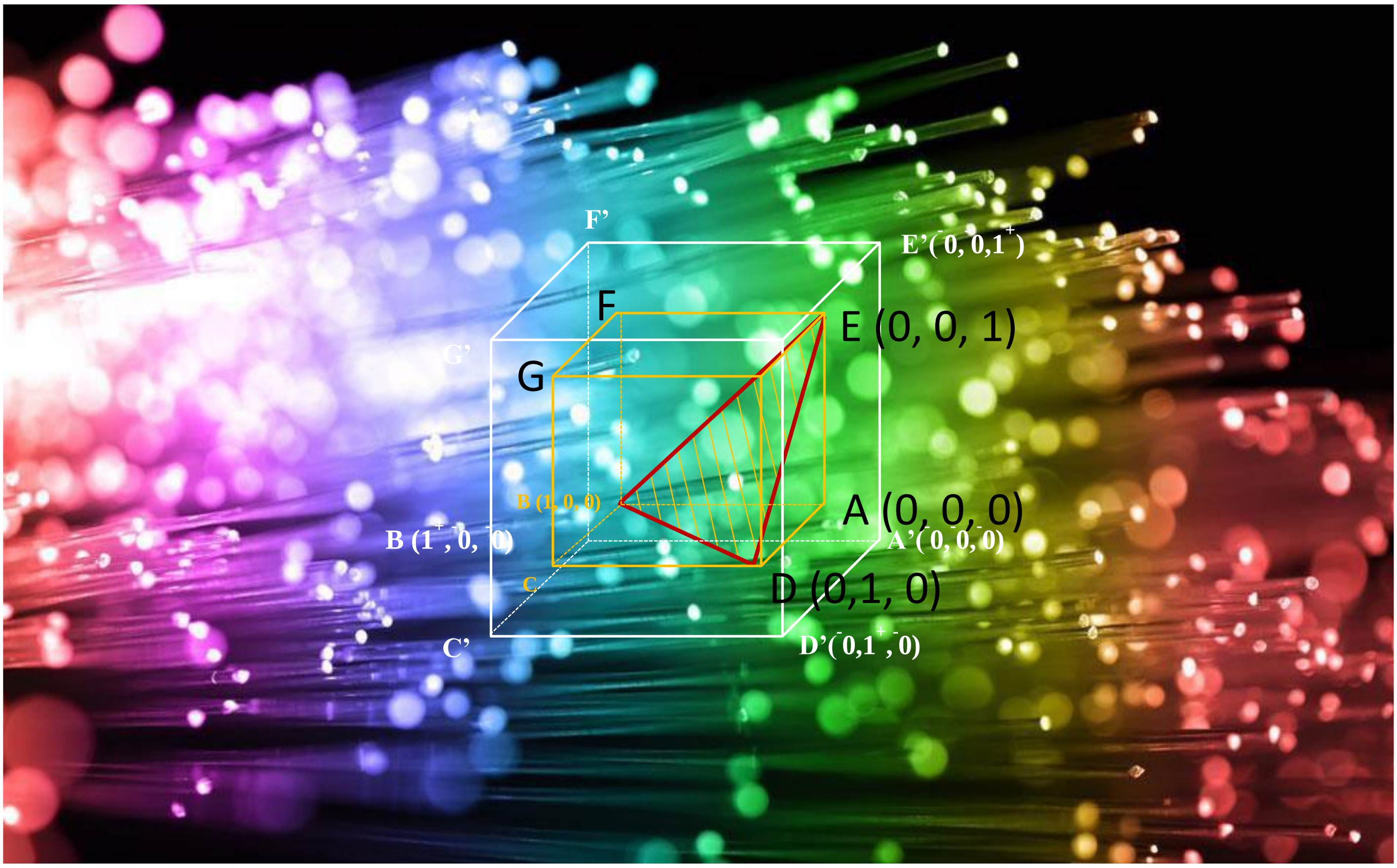

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Salah Bouzina, Florentin Smarandache, Ahmed Hatip

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Neutrosophic Knowledge

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'Neutrosophics Knowledge" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good Arabic, Turkish, English and French containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

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This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Content

Article

Indeterminacy Fuzzy Set coincides with the Picture Fuzzy Set, and both are particular cases of the Refined Neutrosophic Set

Florentin Smarandache[1,*]

[1] University on New Mexico, Gallup Campus, United States
* Correspondence: smarand@unm.edu

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Abstract: In this short note, we analyze the relationship between C. Ardil's Indeterminacy Fuzzy Set (IDFS) and Bui Cong Cuong's Picture Fuzzy Set (PFS). We show that Ardil's "epistemic freedom residual" mathematically coincides with Cuong's "refusal degree," and that the two formalisms satisfy the same admissible simplex constraint. Consequently, IDFS and PFS are mathematically equivalent representations under different notations and interpretations. Furthermore, we show that both structures can be regarded as particular cases of the n-refined neutrosophic set and its generalized extensions.

Keywords: Indeterminacy Fuzzy Set, Picture Fuzzy Set, n-Refined Neutrosophic Set, Nonstandard Refined Neutrosophic Set, Refined Neutrosophic Over/Under/Off-Set, Nonstandard Refined Neutrosophic Over/Under/Off-Set.

1. Introduction

We present, for comparison, the definitions of Indeterminacy Fuzzy Set [1] and Picture Fuzzy Set [2] below, and it is clearly seeing that they are the same. Since the Picture Fuzzy Set was introduced in 2014 by Cuong [2], it may represent an earlier formulation of the same mathematical structure.

The other definitions below (n-Refined Neutrosophic Set, Nonstandard Refined Neutrosophic Set, Refined Neutrosophic Over/Under/Off-Set, Nonstandard Refined Neutrosophic Over/Under/Off-Set) are displayed because they are generalizations of the Indeterminacy Fuzzy Set (Picture Fuzzy Set).

2. Definition 1.

Indeterminacy Fuzzy Set. Let X be a nonempty universe. An IDFS A on X is

$$A = \{\langle x, TA(x), IA(x), FA(x)\rangle \mid x \in X\},$$

where $TA, IA, FA : X \to [0, 1]$ satisfy the simplex constraint $0 \le TA(x) + IA(x) + FA(x) \le 1$ ($\forall x \in X$).

(1) The quantity $\varrho A(x) = 1 - TA(x) - IA(x) - FA(x) \ge 0$ is the <u>epistemic freedom residual</u>.

The admissible state space is the closed simplex

$$\Delta* = \{(T, I, F) \in [0, 1]^3 \mid T + I + F \le 1\}.$$

(2) The boundary $T + I + F = 1$ (i.e., $\varrho = 0$) contains all IFS states embedded by

$$\Phi(\mu, \nu) = (\mu, 1 - \mu - \nu, \nu).$$

Interior points ($\varrho > 0$) are exclusive to IDFS.

The maximum entropy IDFS state (1/4, 1/4, 1/4) with ϱ = 1/4 achieves entropy ln4, strictly exceeding the IFS maximum ln3. [1]

3. Definition 2.

Picture Fuzzy Set A on a universe X is an object in the form of

$$A = \{(x, \mu A(x), \eta A(x), \nu A(x)) \mid x \in X\}$$

where μA (x) ∈ [0, 1] is called the degree of positive membership of x in A, ηA(x) ∈ [0, 1] is called the degree of neutral membership of x in A and νA(x) ∈ [0, 1] is called the degree of negative membership of x in A, and where μA, ηA and νA satisfy the following condition:

$$(\forall x \in X)\ (\mu A(x) + \eta A(x) + \nu A(x) \leq 1).$$

Now (1 − (μA(x) + ηA(x) + νA(x)) could be called the degree of refusal membership of x in A.

Let PFS(X) denote the set of all the picture fuzzy sets on a universe X. Basically, picture fuzzy sets based models may be adequate in situations when we face human opinions involving more answers of types: yes, abstain, no, refusal. [2]

4. Theorem 1 (Equivalence of IDFS and PFS)

Let A be an Indeterminacy Fuzzy Set (IDFS) on universe X:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}$$

such that

$$T_A(x) + I_A(x) + F_A(x) \leq 1.$$

Let B be a Picture Fuzzy Set (PFS):

$$B = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$$

with

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1.$$

Then IDFS and PFS are mathematically equivalent under the bijective mapping:

$$T_A(x) \leftrightarrow \mu_A(x),$$
$$I_A(x) \leftrightarrow \eta_A(x),$$
$$F_A(x) \leftrightarrow \nu_A(x).$$

Moreover,

$$\rho_A(x) = 1 - T_A(x) - I_A(x) - F_A(x)$$

coincides with the Picture Fuzzy Set refusal degree:

$$\pi_A(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x).$$

Therefore,

$$\rho_A(x) = \pi_A(x),$$

which proves the equivalence of the two formalisms.

Proof.

Under the above correspondence between membership components, the admissible state spaces are identical since both satisfy the same simplex constraint. The residual quantity in IDFS equals the refusal degree in PFS through direct substitution. Hence every IDFS induces a unique PFS and vice versa, establishing a one-to-one correspondence. ■

5. Definition 3

Single-Valued n-Refined Neutrosophic Set/Logic/Probability [3], where there are many refinements $T_1, T_2, \ldots T_p$; $I_1, I_2, \ldots I_r$; $F_1, F_2, \ldots, F_s$, with integers p, r, s ≥ 1, and p + r + s = n, and all numerical $T_1, T_2, \ldots T_p$, $I_1, I_2, \ldots I_r$, $F_1, F_2, \ldots, F_s \in [0, 1]$.

And Set-Valued n-Refined Neutrosophic Set/Logic/Probability, where similarly there are many refinements $T_1, T_2, \ldots T_p$; $I_1, I_2, \ldots I_r$; $F_1, F_2, \ldots, F_s$, with integers p, r, s ≥ 1 and p + r + s = n, and all subsets $T_1, T_2, \ldots T_p$, $I_1, I_2, \ldots I_r$, $F_1, F_2, \ldots, F_s \subseteq [0, 1]$.

6. Definition 4

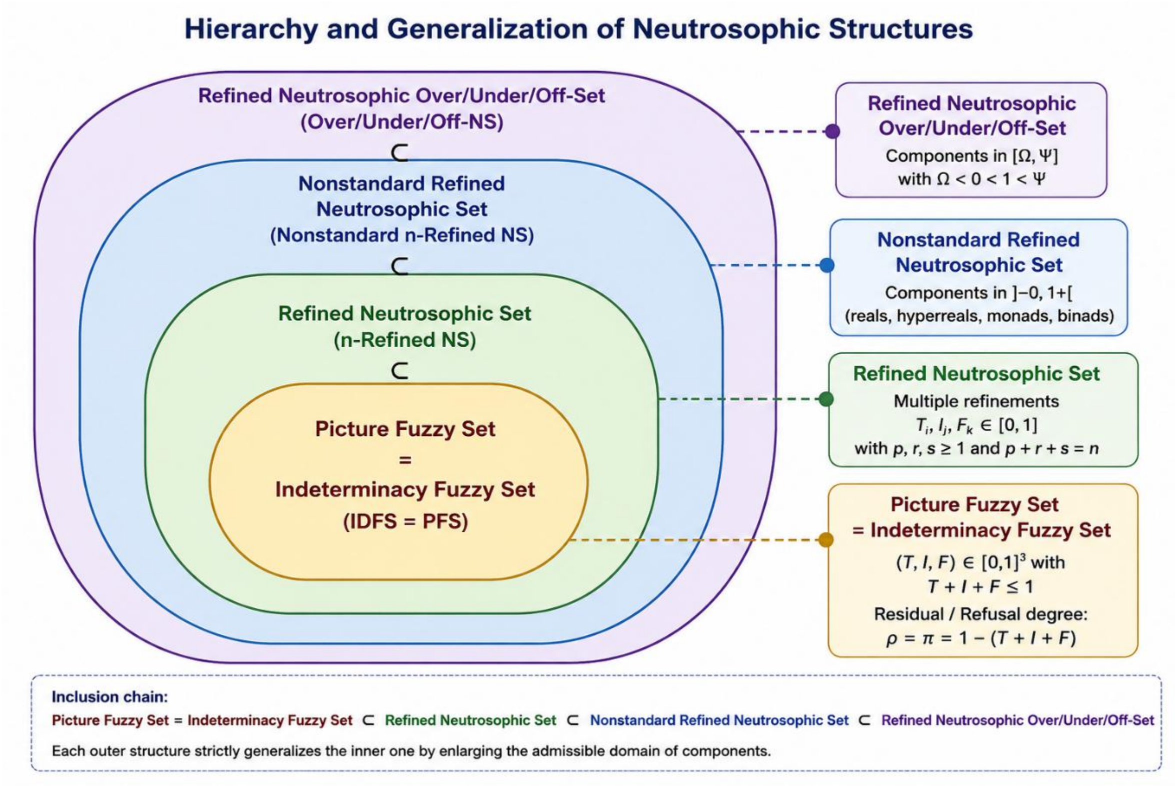
More generalization there is the Nonstandard Refined Neutrosophic Set [4] (within the Extended Nonstandard Analysis), where $T_1, T_2, \ldots T_p,\ \ I_1, I_2, \ldots I_r, F_1, F_2, \ldots, F_s \in\ ]^{-}0, 1^{+}[$ and they may be reals, hyperreals, monads, or binads.

7. Definition 5

Further on, one has the **Refined Neutrosophic Over/Under/Off-Set/Logic/Probability** [3, 4], where all subcomponents $T_1, T_2, \ldots T_p,\ \ I_1, I_2, \ldots I_r, F_1, F_2, \ldots, F_s \in [\Omega, \Psi]$, with $\Omega < 0 < 1 < \Psi$.

8. Definition 6

Also, **Nonstandard Refined Neutrosophic Over/Under/Off-Set/Logic/Probability**, where all subcomponents $T_1, T_2, \ldots T_p,\ \ I_1, I_2, \ldots I_r, F_1, F_2, \ldots, F_s \in [^{-}\Omega, \Psi^{+}]$, with $\Omega < 0 < 1 < \Psi$, and they may be reals, hyperreals, monads, or binads (within Extended NonStandard Analysis). [4, 5]



9. Conclusion

Definitions 1 and 2 coincide, but using different notations. While the mathematical structure of the Indeterminacy Fuzzy Set coincides with that of the Picture Fuzzy Set, one may still distinguish the semantic interpretation of the residual component. In the IDFS framework, the quantity ρ is interpreted as an *epistemic freedom residual*, whereas in the PFS framework it is interpreted as a *refusal degree*. Therefore, the present note does not deny possible semantic reinterpretations, but rather establishes that the underlying mathematical formalism is equivalent.

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Article

Refined Uncertain Set, Refined Nonstandard Uncertain Set, Refined Uncertain Over/Under/Off-Set, Refined Nonstandard Uncertain Over/Under/Off-Set

Florentin Smarandache[1,*]

[1] Math and Science Department, University of New Mexico, Gallup Campus, USA
* Correspondence: smarand@unm.edu

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Abstract: This short note introduces four generalized frameworks for modeling refined uncertainty: Refined Uncertain Set, Refined Nonstandard Uncertain Set, Refined Uncertain Over/Under/Off-Set, and Refined Nonstandard Uncertain Over/Under/Off-Set. These frameworks extend refined neutrosophic structures to a broader class of uncertainty representations, including fuzzy, intuitionistic fuzzy, picture fuzzy, q-rung orthopair fuzzy, spherical fuzzy, plithogenic, and related models. The proposed approach enables the decomposition of uncertainty components into refined subcomponents and further extends them into nonstandard and over/under/off domains. Analogous extensions may also be developed for logic, measure, probability, and statistics.

Keywords: Uncertain set theory; refined uncertainty; nonstandard analysis; neutrosophic set; refined neutrosophic logic; over-/under-/off-set; uncertainty formalism.

MSC 2020: 03E72 (Fuzzy set theory), 03B52 (Fuzzy logic), 28E10 (Fuzzy measure theory), 68T37 (Reasoning under uncertainty), 03H05 (Nonstandard models in mathematics).

1. Introduction and Motivation

Many uncertainty theories have been proposed to represent imprecise, incomplete, inconsistent, or indeterminate information, including fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, picture fuzzy sets, plithogenic sets, spherical fuzzy sets, and their various extensions. However, these theories often remain isolated and use different mathematical structures.

The notion of refinement, introduced in neutrosophic theory, allows uncertainty components to be decomposed into semantically meaningful subcomponents. Such decomposition permits a more nuanced representation of truth, indeterminacy, falsity, or other uncertainty factors.

The purpose of this note is to generalize the refined neutrosophic framework to a broader category called uncertain systems, thereby providing a common conceptual structure applicable to many existing uncertainty theories.

By uncertain, we refer to all fuzzy-based or uncertainty-based representations, including but not limited to: Fuzzy Sets, Intuitionistic Fuzzy Sets, Picture Fuzzy Sets (Ternary Fuzzy Sets), Neutrosophic Sets, Pythagorean Fuzzy Sets, Fermatean Fuzzy Sets, q-Rung Orthopair Fuzzy Sets, Spherical and n-Hyperspherical Fuzzy Sets, Refined Neutrosophic Sets, Plithogenic Sets, and related uncertainty extensions.

2. Refined Uncertain Structures

Definition 1. Refined Uncertain Set / Logic / Measure / Probability

A **Refined Uncertain Set** is an uncertain set whose uncertainty components are decomposed into multiple refined subcomponents.

Suppose an uncertainty representation consists of n subcomponents partitioned into:

$$T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s$$

where

$$p, r, s \geq 1, p + r + s = n, \text{ and } T_i, I_j, F_k \in [0,1].$$

The refinement process decomposes uncertainty information into semantically meaningful categories. For example, truth may be refined into several kinds of evidence, indeterminacy into different sources of ambiguity, and falsity into different categories of contradiction or opposition.

This definition generalizes the concept of the **Refined Neutrosophic Set** [1] to broader uncertainty systems.

Definition 2. Refined Nonstandard Uncertain Set / Logic / Measure / Probability

A further generalization is the **Refined Nonstandard Uncertain Set**, defined within the framework of **Extended Nonstandard Analysis** [2].

In this case,

$$T_i, I_j, F_k \in]^-0, 1^+[$$

where the subcomponents may be: real numbers, hyperreal numbers, monads, or binads.

This extension enables modeling uncertainty beyond standard real-valued boundaries.

Definition 3. Refined Uncertain Over/Under/Off-Set / Logic / Measure / Probability

A **Refined Uncertain Over/Under/Off-Set** extends refined uncertainty beyond the classical interval $[0, 1]$.

The refined subcomponents satisfy:

$$T_i, I_j, F_k \in [\Omega, \Psi]$$

where

$$\Omega < 0 < 1 < \Psi.$$

Such generalization permits:

- **overset values** (> 1), representing over-membership or super-evidence,
- **underset values** (< 0), representing negative evidence or anti-membership,
- **offset structures**, where both under- and over-values coexist.

This extends the notion of neutrosophic over-/under-/off-set [3] to a general uncertainty framework.

Definition 4. Refined Nonstandard Uncertain Over/Under/Off-Set / Logic / Measure / Probability

Finally, the **Refined Nonstandard Uncertain Over/Under/Off-Set** combines refinement, nonstandard analysis, and over/under/off extensions.

The refined subcomponents satisfy:

$$T_i, I_j, F_k \in]\Omega^-, \Psi^+[$$

where

$$\Omega < 0 < 1 < \Psi,$$

and the components may be: real numbers, hyperreal numbers, monads, or binads.

This structure represents the most general refined uncertain framework introduced in this note.

Figure 1: Definitions.

3. Illustrative Example

Consider an environmental assessment problem in which ecological sustainability is evaluated through refined criteria.

Truth-membership may be refined into:

$$T = (T_1, T_2) = (0.8, 0.7)$$

where:

- T_1 represents ecosystem integrity,
- T_2 represents biodiversity resilience.

Indeterminacy may be represented as:

$$I = (0.2)$$

due to incomplete field measurements.

Falsity may be refined into:

$$F = (F_1, F_2) = (0.1, 0.3)$$

where:

- F_1 measures pollution stress,
- F_2 measures habitat fragmentation.

In over/under/off extensions, some values may exceed the classical interval [0, 1], allowing the representation of exceptional support or negative evidence.

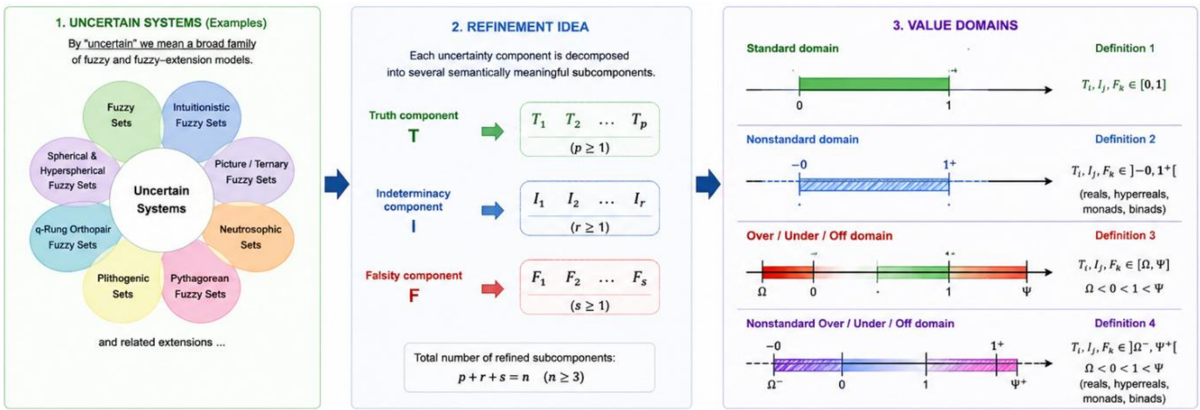

Figure 2: Refined Uncertain Framework.

Figure 3: Applications.

4. Conclusion

This short note introduced four generalized frameworks for representing refined uncertainty: the **Refined Uncertain Set**, **Refined Nonstandard Uncertain Set**, **Refined Uncertain Over/Under/Off-Set**, and **Refined Nonstandard Uncertain Over/Under/Off-Set**. These structures extend the refined neutrosophic paradigm toward a broader class of uncertainty theories, including fuzzy, intuitionistic fuzzy, spherical fuzzy, plithogenic, q-rung orthopair fuzzy, and related systems.

The proposed framework enables the decomposition of uncertainty components into refined subcomponents, thereby allowing a more detailed and semantically meaningful representation of uncertain information. Furthermore, the incorporation of nonstandard analysis and over/under/off domains extends the modeling capability beyond classical intervals, permitting the representation of exceptional, incomplete, contradictory, or nonconventional uncertainty values.

By establishing a unified and extensible conceptual structure, the present work opens the possibility for developing corresponding generalized forms of uncertain logic, measure, probability, statistics, and decision-making models. Future research may focus on formal axiomatization, algebraic properties, aggregation operators, similarity and distance measures, and applications in artificial intelligence, environmental systems, engineering, social sciences, and complex decision environments.

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University of New Mexico

Cartesian Product of Complex Neutrosophic Soft Graphs

Agilarasan K[1*], *Suriyakumar G*[1], *V. J. Sudhakar*[2], *R. Dhivya*[3], *and Takaaki Fujita*[4] *

[1] Research Scholar, Department of Mathematics, Islamiah College (Autonomous), Vaniyambadi, Affiliated to Thiruvalluvar University, Vellore, Tamil Nadu, India. agilking@gmail.com, gansuriyakumar@gmail.com

[2] Department of Mathematics, Islamiah College (Autonomous), Vaniyambadi, Affiliated to Thiruvalluvar University, Vellore, Tamil Nadu, India. vjsvec1@gmail.com

[3] Adhiparasakthi College of Engineering, Kalavai, Tamil Nadu, India dhivyarohi@gmail.com

[4] Independent Researcher, Tokyo, Japan. Takaaki.fujita060@gmail.com

*Correspondence: agilking@gmail.com

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Abstract. The Cartesian product of complex neutrosophic soft graphs is defined and its basic structural aspects are examined in this work. Under the product operation, the membership functions of vertices and edges are examined. The work shows how complex neutrosophic soft graphs can be methodically built from simpler components and provides preservation findings.

Keywords: Soft Graph; Neutrosophic Soft Graph; Complex Neutrosophic Soft Graph.

1. Introduction

Zadeh [1] developed fuzzy set theory to deal with uncertainty. Since then, many researchers have studied fuzzy logic and fuzzy set theory in an attempt to solve a range of real-world problems involving ambiguous circumstances. According to Smarandache [2], neutrosophic sets (NSs) are a useful mathematical tool for dealing with confusing, inconsistent, and incomplete real-world data. Neutrosophic sets are distinguished by a truth-membership function (T), an indeterminacy-membership function (I), and a falsity-membership function (F) that can all have values in the standard or non-standard interval of 0 to 1. Because the concept of a Neutrosophic Set is important, several extended notions of Neutrosophic Sets have also been introduced [20–22].

Akram [3–6] distinguishes between two soft computing methods for conveying ambiguity and uncertainty: fuzzy sets and soft sets. They used soft computing models to examine graph ambiguity and uncertainty. Several regular fuzzy soft graph properties were investigated. He then discussed intuitionistic fuzzy soft graphs, neutrosophic soft graphs, and their uses. In a soft graph, each vertex and edge has a set of parameters. According to Satham Hussain [7], fuzzy sets and neutrosophic soft sets are effective ways to describe data with information. This method was used to create neutrosophic soft graphs. Decision-makers can then customize their findings to fit their particular fields of expertise.

Satham Hussain [8] introduced the ideas of *neutrosophic graphs* and *strong neutrosophic graphs*, which served as a basis for more in-depth research. Neutrosophic graphs and graph components are combined in this new theoretical framework. The total membership values for truth, indeterminacy, and falsity in this case fall between 0 and 2 as truth and falsity are considered dependent variables. Applications in operations research and issues in social networks are the focus of this theory's development. In the latter industry, the rise of phony or false profiles is an especially serious issue. A new solution to these problems is provided by the neutrosophic graph model. Examples were provided to demonstrate the suggested concepts, and strong, complete, and self-complementary neutrosophic graphs were examined.

The complex neutrosophic soft set model, which blends neutrosophic and soft sets, was proposed by Said Broumi [9, 10]. It presents and illustrates the fundamental operations of set theory along with other concepts related to this model structure. The use of this paradigm is demonstrated using a decision-making challenge including subjective and ambiguous information. This illustration shows how well the approach manages ambiguity and encourages well-informed decision-making. It also introduced fermatean neutrosophic graphs and provides opportunities for further research to enhance the methodology and take into account its ramifications in other domains, such as artificial intelligence and decision science.

The concept of complex neutrosophic soft graphs (CNSGs), which provide a sophisticated framework for simulating indeterminacy, ambiguity, and uncertainty in complex systems, is introduced and developed by Suriya [11–14]. High degrees of neutrosophic uncertainty and complex-valued memberships for both edges and vertices under soft set parameters define a strong complex neutrosophic soft graph (SCNSG). We present a novel measure that considers the cumulative strength and uncertainty of connections within graph characteristics, as well as the complements of a complex neutrosophic soft graph and a strong complex neutrosophic soft graph.

The Cartesian product of complex neutrosophic soft graphs is defined and its basic structural characteristics are examined in this work. The Cartesian product procedure is used to analyze vertex and edge membership functions. A number of preservation outcomes concerning graph structure and connectivity are established. The suggested structure shows how smaller graphs can be methodically transformed into complex neutrosophic soft graphs. Additionally, possible uses of complex neutrosophic soft graphs for complex network systems and uncertainty modeling are examined.

2. **Preliminaries**

This section provides a basic description and example to support the key findings.

Definition 2.1. *[10] Assume $U \neq \varphi$, a complex fuzzy set A is an entity with the following structure:*

$$A = \{(x, \rho_A(x)) : x \in U\} = \{x, \mathfrak{X}_A(x)e^{iw_A(x)}\}$$

where $i = \sqrt{-1}, \mathfrak{X}_A(x) \in [0, 1]$ and $0 \leq w_A(x) \leq 2\pi$.

Definition 2.2. *[1] Assume $U \neq \varphi$. A complex intuitionistic fuzzy set A is defined as follows:*

$$A = \{(x, \mathfrak{X}_A(x), \mathfrak{Z}_A(x)) : x \in U\}$$

$$A = \{(x, \mathfrak{X}_A(x)e^{i\alpha_A(x)}, \mathfrak{Z}_A(x)e^{i\gamma_A(x)}) : x \in U\}$$

where $i = \sqrt{-1}, \mathfrak{X}_A(x), \mathfrak{Z}_A(x) \in [0, 1], \alpha_A(x), \gamma_X(x) \in [0, 2\pi]$ and $0^- \leq \mathfrak{X}_A(x) + \mathfrak{Z}_A(x) \leq 1^+$.

Definition 2.3. *[1] Assume $U \neq \varphi$. An object with the shape of a complex neutrosophic set A is*

$$A = \{(x, \mathfrak{X}_A(x), \mathfrak{Y}_A(x), \mathfrak{Z}_F(x)) : x \in U\}$$

$$A = \{(x, \mathfrak{X}_A(x)e^{i\alpha_A(x)}, \mathfrak{Y}_A(x)e^{i\beta_A(x)}, \mathfrak{Z}_A(x)e^{I\gamma_A(x)}) : x \in U\}$$

where $i = \sqrt{-1}, \mathfrak{X}_A(x), \mathfrak{Y}_A(x), \mathfrak{Z}_A(x) \in [0, 1], \alpha_A(x), \beta_A(x), \gamma_X(x) \in [0, 2\pi]$ and $0^- \leq \mathfrak{X}_A(x) + \mathfrak{Y}_A(x) + \mathfrak{Z}_A(x) \leq 3^+$.

Definition 2.4. *([4]) A fuzzy soft graph $G = (G^*, M, N, R)$ is an ordered quadruple that meets the following conditions:*

- *$G^* = (V, E)$ is a simple graph,*
- *R is a non-empty set of parameters,*
- *(M, R) is a fuzzy set defined over the vertex set V,*

- *(N, R) is a fuzzy set defined over the edge set E,*
- *For each $a \in R$, the pair $(M(a), N(a))$ forms a fuzzy graph associated with G^*, such that:*

$$N(a)(xy) \leq \min\{M(a)(x), M(a)(y)\}$$

holds for all $x, y \in V$.

$\mathfrak{H}(a)$ represents the fuzzy graph $(M(a), N(a))$. A fuzzy soft graph can be thought of as a family of fuzzy graphs parameterized by R. $M(G^*)$ denotes the class of all fuzzy soft graphs over G^*.

Definition 2.5. *[3] An intuitionistic fuzzy soft graph $G = (G^*, F, K, Q)$ is an ordered four tuple if it satisfies the following conditions:*

- *$G^* = (V, E)$ is a simple graph,*
- *Q is a non-empty set of parameters,*
- *(F, Q) is an intuitionistic fuzzy set over V,*
- *(K, Q) is an intuitionistic fuzzy set over E,*
- *$(F(a), K(a))$ is an intuitionistic fuzzy graph of G^* for all $a \in Q$. That is*

$$\mathfrak{X}_{K(a)}(ab) \leq \min\{\mathfrak{X}_{F(a)}(a), \mathfrak{X}_{F(a)}(b)\}$$

$$\mathfrak{Z}_{K(a)}(ab) \geq \max\{\mathfrak{Z}_{F(a)}(a), \mathfrak{Z}_{F(a)}(b)\}$$

for all $a \in Q$ and $x, y \in V$. The intuitionistic fuzzy graph $(F(a), K(a))$ is denoted by $H(a)$ for convenience. An intuitionistic fuzzy soft graph is a parameterized family of intuitionistic fuzzy graphs.

Definition 2.6. *[7] A neutrosophic soft graph $G = (G^*, M, N, R)$ is defined as a quadruple satisfying the following conditions:*

- *$G^* = (V, E)$ represents a simple (i.e., undirected and without loops or multiple edges) graph,*
- *R is a non-empty collection of parameters,*
- *(M, R) denotes a neutrosophic set defined over the vertex set V,*
- *(N, R) denotes a neutrosophic set defined over the edge set E,*
- *For every $a \in R$, the pair $(M(a), N(a))$ forms a neutrosophic subgraph of G^*. Specifically, the following conditions hold:*

$$\mathfrak{X}_{N(a)}(xy) \leq \min\{\mathfrak{X}_{M(a)}(x), \mathfrak{X}_{M(a)}(y)\},$$

$$\mathfrak{Y}_{N(a)}(xy) \leq \min\{\mathfrak{Y}_{M(a)}(x), \mathfrak{Y}_{M(a)}(y)\},$$

$$\mathfrak{Z}_{N(a)}(xy) \geq \max\{\mathfrak{Z}_{M(a)}(x), \mathfrak{Z}_{M(a)}(y)\},$$

For ease of use, $\mathfrak{H}(a)$ is used to represent the neutrosophic fuzzy graph $(M(a), N(a))$. A neutrosophic fuzzy soft graph may therefore be thought of as a family of neutrosophic fuzzy *graphs that are parameterized by R. $M(G^*)$ represents the class of all neutrosophic fuzzy soft graphs over G^*, for all $a \in R$ and $x, y \in V$.*

3. **Cartesian Product of Complex Neutrosophic Soft Graphs.**

We explain the Cartesian product of complex neutrosophic soft graphs.

Definition 3.1. *A **Complex Neutrosophic Soft Graph (CNSG)** is defined as an ordered quadruple $G = (G^*, \mathfrak{S}, \mathfrak{K}, L)$ that satisfies the following conditions:*

- *$G^* = (V, E)$ is a simple graph.*
- *L is a non-empty set of parameters.*
- *$(\mathfrak{S}, L)$ constitutes a complex neutrosophic soft (CNS) set over the vertex set V.*
- *$(\mathfrak{K}, L)$ represents a CNS set over the edge set E.*
- *For each parameter $l \in L$, the pair $(\mathfrak{S}(l), \mathfrak{K}(l))$ forms a complex neutrosophic soft graph (CNSG) on G^*.*

1. Let $V = \{l_1, l_2, \cdots, l_n\}$, and consider the functions:

$$\mathfrak{X}_{\mathfrak{S}(l)} : V \to [0,1], \quad \mathfrak{Y}_{\mathfrak{S}(l)} : V \to [0,1], \quad \mathfrak{Z}_{\mathfrak{S}(l)} : V \to [0,1]$$

The degrees of truth-membership, indeterminacy-membership and falsity-membership for each vertex $p_i \in V$ under the parameter $l \in L$ are represented by these functions, respectively. Furthermore, in $[0, 2\pi]$, let $A_{\mathfrak{S}(l)}, B_{\mathfrak{S}(l)}$, and $C_{\mathfrak{S}(l)}$ represent the corresponding argument functions for every membership function. Afterward, for every vertex p_i:

$$0 \leq \mathfrak{X}_{\mathfrak{S}(l)}(p_i)e^{iA_{\mathfrak{S}(l)}(p_i)} + \mathfrak{Y}_{\mathfrak{S}(l)}(p_i)e^{iB_{\mathfrak{S}(l)}(p_i)} + \mathfrak{Z}_{\mathfrak{S}(l)}(p_i)e^{iC_{\mathfrak{S}(l)}(p_i)} \leq 2$$

2. For the edge set $E \subseteq V \bigotimes V$, the corresponding functions are defined as:

$$\mathfrak{X}_{\mathfrak{K}(l)} : E \to [0,1], \quad \mathfrak{Y}_{\mathfrak{K}(l)} : E \to [0,1], \quad \mathfrak{Z}_{\mathfrak{K}(l)} : E \to [0,1]$$

With regard to parameter l, these functions indicate the degrees of truth-membership, indeterminacy-membership and falsity-membership for every edge $p_ip_j \in E$. The functions $A_{\mathfrak{K}(l)}, B_{\mathfrak{K}(l)}$, and $C_{\mathfrak{K}(l)} \in [0, 2\pi]$ are argument functions. These also correspond to the associated membership functions.

For each edge $p_ip_j \in E$, the following conditions must be satisfied:

$$\mathfrak{X}_{\mathfrak{K}(l)}(p_ip_j)e^{iA_{\mathfrak{K}(l)}(p_ip_j)} \leq [\mathfrak{X}_{\mathfrak{S}(l)}(p_i) \wedge \mathfrak{X}_{\mathfrak{S}(l)}(p_j)]e^{i\{A_{\mathfrak{S}(l)}(p_i) \wedge A_{\mathfrak{S}(l)}(p_j)\}}$$

$$\mathfrak{Y}_{\mathfrak{K}(l)}(p_ip_j)e^{iB_{\mathfrak{K}(l)}(p_ip_j)} \leq [\mathfrak{Y}_{\mathfrak{S}(l)}(p_i) \wedge \mathfrak{Y}_{\mathfrak{S}(l)}(p_j)]e^{i\{B_{\mathfrak{S}(l)}(p_i) \wedge B_{\mathfrak{S}(l)}(p_j)\}}$$

$$\mathfrak{Z}_{\mathfrak{K}(l)}(p_ip_j)e^{iC_{\mathfrak{K}(l)}(p_ip_j)} \geq [\mathfrak{Z}_{\mathfrak{S}(l)}(p_i) \vee \mathfrak{Z}_{\mathfrak{S}(l)}(p_j)]e^{i\{C_{\mathfrak{S}(l)}(p_i) \vee C_{\mathfrak{S}(l)}(p_j)\}}$$

Furthermore, each edge satisfies the bounded magnitude condition:

$$0 \leq \mathfrak{X}_{\mathfrak{K}(l)}(p_ip_j)e^{iA_{\mathfrak{K}(l)}(p_ip_j)} + \mathfrak{Y}_{\mathfrak{K}(l)}(p_ip_j)e^{iB_{\mathfrak{K}(l)}(p_ip_j)} + \mathfrak{Z}_{\mathfrak{K}(l)}(p_ip_j)e^{iC_{\mathfrak{K}(l)}(p_ip_j)} \leq 2$$

For simplicity, the pair $(\mathfrak{S}(l), \mathfrak{K}(l))$ *is known as the complex neutrosophic graph* $\mathbb{G}$*. Thus, a CNSG is a parameterized structure that expands on the idea of a complex neutrosophic graph.*

Definition 3.2 (Cartesian product of complex neutrosophic soft graphs)**.** *Let*

$$G_1 = (G_1^*, \mathfrak{S}_1, \mathfrak{K}_1, L_1) \qquad \text{and} \qquad G_2 = (G_2^*, \mathfrak{S}_2, \mathfrak{K}_2, L_2)$$

be two complex neutrosophic soft graphs (CNSGs), where

$$G_1^* = (V_1, E_1), \qquad G_2^* = (V_2, E_2).$$

The Cartesian product *of* G_1 *and* G_2*, denoted by*

$$G_1 \bigotimes G_2,$$

is the CNSG

$$G_1 \bigotimes G_2 = \big(G^*, \mathfrak{S}, \mathfrak{K}, L_1 \times L_2\big),$$

where the underlying crisp graph is

$$G^* = (V_1 \times V_2, E),$$

with

$$E = \big\{(p, q_1)(p, q_2) \mid p \in V_1,\ q_1q_2 \in E_2\big\} \cup \big\{(p_1, q)(p_2, q) \mid p_1p_2 \in E_1,\ q \in V_2\big\}.$$

For each $(l, k) \in L_1 \times L_2$*, the vertex-membership values are defined by*

$$\mathfrak{X}_{\mathfrak{S}_1(l)\otimes\mathfrak{S}_2(k)}(p,q)\, e^{iA_{\mathfrak{S}_1(l)\otimes\mathfrak{S}_2(k)}(p,q)} = \big[\mathfrak{X}_{\mathfrak{S}_1(l)}(p) \wedge \mathfrak{X}_{\mathfrak{S}_2(k)}(q)\big] e^{i\left(A_{\mathfrak{S}_1(l)}(p) \wedge A_{\mathfrak{S}_2(k)}(q)\right)},$$

$$\mathfrak{Y}_{\mathfrak{S}_1(l)\otimes\mathfrak{S}_2(k)}(p,q)\, e^{iB_{\mathfrak{S}_1(l)\otimes\mathfrak{S}_2(k)}(p,q)} = \big[\mathfrak{Y}_{\mathfrak{S}_1(l)}(p) \wedge \mathfrak{Y}_{\mathfrak{S}_2(k)}(q)\big] e^{i\left(B_{\mathfrak{S}_1(l)}(p) \wedge B_{\mathfrak{S}_2(k)}(q)\right)},$$

$$\mathfrak{Z}_{\mathfrak{S}_1(l)\otimes\mathfrak{S}_2(k)}(p,q)\, e^{iC_{\mathfrak{S}_1(l)\otimes\mathfrak{S}_2(k)}(p,q)} = \big[\mathfrak{Z}_{\mathfrak{S}_1(l)}(p) \vee \mathfrak{Z}_{\mathfrak{S}_2(k)}(q)\big] e^{i\left(C_{\mathfrak{S}_1(l)}(p) \vee C_{\mathfrak{S}_2(k)}(q)\right)},$$

for all $(p, q) \in V_1 \times V_2$*.*
The edge-membership values are defined as follows.

(1) *If* $(p, q_1)(p, q_2) \in E$*, where* $p \in V_1$ *and* $q_1q_2 \in E_2$*, then*

$$\mathfrak{X}_{\mathfrak{K}_1(l)\otimes\mathfrak{K}_2(k)}\big((p,q_1)(p,q_2)\big)\, e^{iA_{\mathfrak{K}_1(l)\otimes\mathfrak{K}_2(k)}((p,q_1)(p,q_2))} = \big[\mathfrak{X}_{\mathfrak{S}_1(l)}(p) \wedge \mathfrak{X}_{\mathfrak{K}_2(k)}(q_1q_2)\big] \cdot e^{i\left(A_{\mathfrak{S}_1(l)}(p) \wedge A_{\mathfrak{K}_2(k)}(q_1q_2)\right)},$$

$$\mathfrak{Y}_{\mathfrak{K}_1(l)\otimes\mathfrak{K}_2(k)}\big((p,q_1)(p,q_2)\big)\, e^{iB_{\mathfrak{K}_1(l)\otimes\mathfrak{K}_2(k)}((p,q_1)(p,q_2))} = \big[\mathfrak{Y}_{\mathfrak{S}_1(l)}(p) \wedge \mathfrak{Y}_{\mathfrak{K}_2(k)}(q_1q_2)\big] \cdot e^{i\left(B_{\mathfrak{S}_1(l)}(p) \wedge B_{\mathfrak{K}_2(k)}(q_1q_2)\right)},$$

$$\mathfrak{Z}_{\mathfrak{K}_1(l)\otimes\mathfrak{K}_2(k)}\big((p,q_1)(p,q_2)\big)\, e^{iC_{\mathfrak{K}_1(l)\otimes\mathfrak{K}_2(k)}((p,q_1)(p,q_2))} = \big[\mathfrak{Z}_{\mathfrak{S}_1(l)}(p) \vee \mathfrak{Z}_{\mathfrak{K}_2(k)}(q_1q_2)\big] \cdot e^{i\left(C_{\mathfrak{S}_1(l)}(p) \vee C_{\mathfrak{K}_2(k)}(q_1q_2)\right)}.$$

(2) *If $(p_1, q)(p_2, q) \in E$, where $p_1p_2 \in E_1$ and $q \in V_2$, then*

$$\mathfrak{X}_{\mathfrak{K}_1(l)\otimes\mathfrak{K}_2(k)}\big((p_1,q)(p_2,q)\big)\, e^{iA_{\mathfrak{K}_1(l)\otimes\mathfrak{K}_2(k)}((p_1,q)(p_2,q))} = \big[\mathfrak{X}_{\mathfrak{K}_1(l)}(p_1p_2) \wedge \mathfrak{X}_{\mathfrak{S}_2(k)}(q)\big] \cdot e^{i\big(A_{\mathfrak{K}_1(l)}(p_1p_2)\wedge A_{\mathfrak{S}_2(k)}(q)\big)},$$

$$\mathfrak{Y}_{\mathfrak{K}_1(l)\otimes\mathfrak{K}_2(k)}\big((p_1,q)(p_2,q)\big)\, e^{iB_{\mathfrak{K}_1(l)\otimes\mathfrak{K}_2(k)}((p_1,q)(p_2,q))} = \big[\mathfrak{Y}_{\mathfrak{K}_1(l)}(p_1p_2) \wedge \mathfrak{Y}_{\mathfrak{S}_2(k)}(q)\big] \cdot e^{i\big(B_{\mathfrak{K}_1(l)}(p_1p_2)\wedge B_{\mathfrak{S}_2(k)}(q)\big)},$$

$$\mathfrak{Z}_{\mathfrak{K}_1(l)\otimes\mathfrak{K}_2(k)}\big((p_1,q)(p_2,q)\big)\, e^{iC_{\mathfrak{K}_1(l)\otimes\mathfrak{K}_2(k)}((p_1,q)(p_2,q))} = \big[\mathfrak{Z}_{\mathfrak{K}_1(l)}(p_1p_2) \vee \mathfrak{Z}_{\mathfrak{S}_2(k)}(q)\big] \cdot e^{i\big(C_{\mathfrak{K}_1(l)}(p_1p_2)\vee C_{\mathfrak{S}_2(k)}(q)\big)}.$$

Theorem 3.3. *Cartesian product $G_1 \bigotimes G_2 = (V_1 \bigotimes V_2, E_1 \bigotimes E_2)$ of two CNSGs, G_1 and G_2, is also the CNSG of $G_1 \bigotimes G_2$.*

Proof: *Two cases are examined. Case 1: For $p \in V_1$, $q_1q_2 \in E_2$*

$$\mathfrak{X}_{\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k)}((pq_1)(pq_2))e^{iA_{\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k)}((pq_1)(pq_2))} = [\mathfrak{X}_{\mathfrak{S}_1(l)}(p) \wedge \mathfrak{X}_{\mathfrak{K}_2(k)}(q_1q_2)]e^{i\{A_{\mathfrak{S}_1(l)}(p)\wedge A_{\mathfrak{K}_2(k)}(q_1q_2)\}}$$

$$\leq [\mathfrak{X}_{\mathfrak{S}_1(l)}(p) \wedge [\mathfrak{X}_{\mathfrak{S}_2(k)}(q_1) \wedge \mathfrak{X}_{\mathfrak{S}_2(k)}(q_2)]]e^{i\{A_{\mathfrak{S}_1(l)}(p)\wedge\{A_{\mathfrak{S}_2(k)}(q_1)\wedge A_{\mathfrak{S}_2(k)}(q_2)\}\}}$$

$$= [[\mathfrak{X}_{\mathfrak{S}_1(l)}(p) \wedge \mathfrak{X}_{\mathfrak{S}_2(k)}(q_1)] \wedge [\mathfrak{X}_{\mathfrak{S}_1(l)}(p) \wedge \mathfrak{X}_{\mathfrak{S}_2(k)}(q_2)]]e^{i\{A_{\mathfrak{S}_1(l)}(p)\wedge A_{\mathfrak{S}_2(k)}(q_1)\}\wedge\{(A_{\mathfrak{S}_1(l)}(p)\wedge A_{\mathfrak{S}_2(k)}(q_2)\}}$$

$$= [\mathfrak{X}_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(rq_1) \wedge \mathfrak{X}_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(rq_2)]e^{i\{A_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(rq_1)\wedge A_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(rq_2)\}}$$

$$\mathfrak{Y}_{\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k)}((pq_1)(pq_2))e^{iB_{\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k)}((pq_1)(pq_2))} = [\mathfrak{Y}_{\mathfrak{S}_1(l)}(p) \wedge \mathfrak{Y}_{\mathfrak{K}_2(k)}(q_1q_2)]e^{i\{B_{\mathfrak{S}_1(l)}(p)\wedge B_{\mathfrak{K}_2(k)}(q_1q_2)\}}$$

$$\leq [\mathfrak{Y}_{\mathfrak{S}_1(l)}(p) \wedge [\mathfrak{Y}_{\mathfrak{S}_2(k)}(q_1) \wedge \mathfrak{Y}_{\mathfrak{S}_2(k)}(q_2)]]e^{i\{B_{\mathfrak{S}_1(l)}(p)\wedge\{B_{\mathfrak{S}_2(k)}(q_1)\wedge B_{\mathfrak{S}_2(k)}(q_2)\}\}}$$

$$= [[\mathfrak{Y}_{\mathfrak{S}_1(l)}(p) \wedge \mathfrak{Y}_{\mathfrak{S}_2(k)}(q_1)] \wedge [\mathfrak{Y}_{\mathfrak{S}_1(l)}(p) \wedge \mathfrak{Y}_{\mathfrak{S}_2(k)}(q_2)]]e^{i\{B_{\mathfrak{S}_1(l)}(p)\wedge B_{\mathfrak{S}_2(k)}(q_1)\}\wedge\{(B_{\mathfrak{S}_1(l)}(p)\wedge B_{\mathfrak{S}_2(k)}(q_2)\}}$$

$$= [\mathfrak{Y}_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(rq_1) \wedge \mathfrak{Y}_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(rq_2)]e^{i\{B_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(rq_1)\wedge B_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(rq_2)\}}$$

$$\mathfrak{Z}_{\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k)}((pq_1)(pq_2))e^{iC_{\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k)}((pq_1)(pq_2))} = [\mathfrak{Z}_{\mathfrak{S}_1(l)}(p) \vee \mathfrak{Z}_{\mathfrak{K}_2(k)}(q_1q_2)]e^{i\{C_{\mathfrak{S}_1(l)}(p)\vee C_{\mathfrak{K}_2(k)}(q_1q_2)\}}$$

$$\leq [\mathfrak{Z}_{\mathfrak{S}_1(l)}(p) \vee [\mathfrak{Z}_{\mathfrak{S}_2(k)}(q_1) \vee \mathfrak{Z}_{\mathfrak{S}_2(k)}(q_2)]]e^{i\{C_{\mathfrak{S}_1(l)}(p)\vee\{C_{\mathfrak{S}_2(k)}(q_1)\vee C_{\mathfrak{S}_2(k)}(q_2)\}\}}$$

$$= [[\mathfrak{Z}_{\mathfrak{S}_1(l)}(p) \vee \mathfrak{Z}_{\mathfrak{S}_2(k)}(q_1)] \vee [\mathfrak{Z}_{\mathfrak{S}_1(l)}(p) \vee \mathfrak{Z}_{\mathfrak{S}_2(k)}(q_2)]]e^{i\{C_{\mathfrak{S}_1(l)}(p)\vee C_{\mathfrak{S}_2(k)}(q_1)\}\vee\{(C_{\mathfrak{S}_1(l)}(p)\vee C_{\mathfrak{S}_2(k)}(q_2)\}}$$

$$= [\mathfrak{Z}_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(rq_1) \vee \mathfrak{Z}_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(rq_2)]e^{i\{C_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(rq_1)\vee C_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(rq_2)\}}$$

for all $pq_1, pq_2 \in G_1 \bigotimes G_2$

Case 2: For $q \in V_2$, $p_1p_2 \in E_1$

$\mathfrak{X}_{\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k)}((p_1q)(p_2q))e^{iA_{\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k)}((p_1q)(p_2q))} = [\mathfrak{X}_{\mathfrak{S}_2(k)}(q)\wedge\mathfrak{X}_{\mathfrak{K}_1(l)}(p_1p_2)]e^{i\{A_{\mathfrak{S}_2(k)}(q)\wedge A_{\mathfrak{K}_1(l)}(p_1p_2)\}}$

$$\leq [\mathfrak{X}_{\mathfrak{S}_2(k)}(q)\wedge[\mathfrak{X}_{\mathfrak{S}_1(l)}(p_1)\wedge\mathfrak{X}_{\mathfrak{S}_1(l)}(p_2)]]e^{i\{A_{\mathfrak{S}_2(k)}(q)\wedge\{A_{\mathfrak{S}_1(l)}(p_1)\wedge A_{\mathfrak{S}_1(l)}(p_2)\}\}}$$
$$= [[\mathfrak{X}_{\mathfrak{S}_2(k)}(q)\wedge\mathfrak{X}_{\mathfrak{S}_1(l)}(p_1)]\wedge[\mathfrak{X}_{\mathfrak{S}_2(k)}(q)\wedge\mathfrak{X}_{\mathfrak{S}_1(l)}(p_2)]]e^{i\{A_{\mathfrak{S}_2(k)}(q)\wedge A_{\mathfrak{S}_1(l)}(p_1)\}\wedge\{A_{\mathfrak{S}_2(k)}(q)\wedge A_{\mathfrak{S}_1(l)}(p_2)\}}$$
$$= [\mathfrak{X}_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(p_1q)\wedge\mathfrak{X}_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(p_2q)]e^{i\{A_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(p_1q)\wedge A_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(p_2q)\}}$$

$\mathfrak{Y}_{\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k)}((p_1q)(p_2q))e^{iB_{\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k)}((p_1q)(p_2q))} = [\mathfrak{Y}_{\mathfrak{S}_2(k)}(q)\wedge\mathfrak{Y}_{\mathfrak{K}_1(l)}(p_1p_2)]e^{i\{B_{\mathfrak{S}_2(k)}(q)\wedge B_{\mathfrak{K}_1(l)}(p_1p_2)\}}$

$$\leq [\mathfrak{Y}_{\mathfrak{S}_2(k)}(q)\wedge[\mathfrak{Y}_{\mathfrak{S}_1(l)}(p_1)\wedge\mathfrak{Y}_{\mathfrak{S}_1(l)}(p_2)]]e^{i\{B_{\mathfrak{S}_2(k)}(q)\wedge\{B_{\mathfrak{S}_1(l)}(p_1)\wedge B_{\mathfrak{S}_1(l)}(p_2)\}\}}$$
$$= [[\mathfrak{Y}_{\mathfrak{S}_2(k)}(q)\wedge\mathfrak{Y}_{\mathfrak{S}_1(l)}(p_1)]\wedge[\mathfrak{Y}_{\mathfrak{S}_2(k)}(q)\wedge\mathfrak{Y}_{\mathfrak{S}_1(l)}(p_2)]]e^{i\{B_{\mathfrak{S}_2(k)}(q)\wedge B_{\mathfrak{S}_1(l)}(p_1)\}\wedge\{B_{\mathfrak{S}_2(k)}(q)\wedge B_{\mathfrak{S}_1(l)}(p_2)\}}$$
$$= [\mathfrak{Y}_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(p_1q)\wedge\mathfrak{Y}_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(p_2q)]e^{i\{B_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(p_1q)\wedge B_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(p_2q)\}}$$

$\mathfrak{Z}_{\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k)}((p_1q)(p_2q))e^{iC_{\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k)}((p_1q)(p_2q))} = [\mathfrak{Z}_{\mathfrak{S}_2(k)}(q)\vee\mathfrak{Z}_{\mathfrak{K}_1(l)}(p_1p_2)]e^{i\{C_{\mathfrak{S}_2(k)}(q)\vee C_{\mathfrak{K}_1(l)}(p_1p_2)\}}$

$$\leq [\mathfrak{Z}_{\mathfrak{S}_2(k)}(q)\vee[\mathfrak{Z}_{\mathfrak{S}_1(l)}(p_1)\vee\mathfrak{Z}_{\mathfrak{S}_1(l)}(p_2)]]e^{i\{C_{\mathfrak{S}_2(k)}(q)\vee\{C_{\mathfrak{S}_1(l)}(p_1)\vee C_{\mathfrak{S}_1(l)}(p_2)\}\}}$$
$$= [[\mathfrak{Z}_{\mathfrak{S}_2(k)}(q)\vee\mathfrak{Z}_{\mathfrak{S}_1(l)}(p_1)]\vee[\mathfrak{Z}_{\mathfrak{S}_2(k)}(q)\vee\mathfrak{Z}_{\mathfrak{S}_1(l)}(p_2)]]e^{i\{C_{\mathfrak{S}_2(k)}(q)\vee C_{\mathfrak{S}_1(l)}(p_1)\}\vee\{C_{\mathfrak{S}_2(k)}(q)\vee C_{\mathfrak{S}_1(l)}(p_2)\}}$$
$$= [\mathfrak{Z}_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(p_1q)\vee\mathfrak{Z}_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(p_2q)]e^{i\{C_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(p_1q)\vee C_{\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k)}(p_2q)\}}$$

for all $p_1q, p_2q \in G_1 \bigotimes G_2$.

Example 3.4. *Examine two CNSGs* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *of* $G = (V, E)$*, as shown in figure 1. From this, we obtain the cartesian product* $(G_1 \bigotimes G_2)$ *as shown in figure 2. Let* $(\mathfrak{S}, L)$ *be a CNSS over* V *and let* $L = (l)$ *be a set of parameters. The complex neutrosophic approximation function* $\mathfrak{S} : L \to L(V)$ *is defined by*

$$\mathfrak{S}(l_1) = \begin{cases} \{p_1, (.3e^{i\pi(1.6)}, .7e^{i\pi(1.8)}, .6e^{i\pi(1.4)})\} \\ \{p_2, (.2e^{i\pi(1.7)}, .6e^{i\pi(1.7)}, .7e^{i\pi(1.8)})\} \\ \{p_3, (.5e^{i\pi(1.4)}, .8e^{i\pi(1.6)}, .4e^{i\pi(1.6)})\} \end{cases}$$

Let $(\mathfrak{S}, K)$ *be a CNSS over* V *and let* $K = (k_1, k_2)$ *be a set of parameters. The complex neutrosophic approximation function* $\mathfrak{S} : K \to K(V)$ *is defined by*

$$\mathfrak{S}(l_1) = \begin{cases} \{q_1, (.4e^{i\pi(1.5)}, .5e^{i\pi(1.4)}, .6e^{i\pi(1.5)})\} \\ \{q_2, (.6e^{i\pi(1.3)}, .7e^{i\pi(1.5)}, .4e^{i\pi(1.8)})\} \end{cases}$$
$$\mathfrak{S}(l_2) = \begin{cases} \{q_1, (.3e^{i\pi(1.2)}, .7e^{i\pi(1.4)}, .6e^{i\pi(1.7)})\} \\ \{q_2, (.2e^{i\pi(1.6)}, .6e^{i\pi(1.5)}, .8e^{i\pi(1.5)})\} \end{cases}$$

Let $(\mathfrak{K}, L)$ be a CNSS over E and let $L = (l)$ be a set of parameters. The complex neutrosophic approximation function $\mathfrak{K} : L \to L(E)$ is defined by

$$\mathfrak{K}(l_1) = \begin{cases} \{p_1p_2, (.1e^{i\pi(1.5)}, .5e^{i\pi(1.6)}, .8e^{i\pi(1.9)})\} \\ \{p_2p_3, (.2e^{i\pi(1.3)}, .6e^{i\pi(1.5)}, .7e^{i\pi(1.8)})\} \end{cases}$$

Let $(\mathfrak{K}, K)$ be a CNSS over E and let $K = (k_1, k_2)$ be a set of parameters. The complex neutrosophic approximation function $\mathfrak{K} : K \to K(E)$ is defined by

$$\mathfrak{K}(k_1) = \Big\{ \{q_1q_2, (.3e^{i\pi(1.2)}, .4e^{i\pi(1.3)}, .7e^{i\pi(1.8)})\}$$
$$\mathfrak{K}(k_2) = \Big\{ \{q_1, q_2(.2e^{i\pi(1.2)}, .6e^{i\pi(1.4)}, .8e^{i\pi(1.7)})\}$$

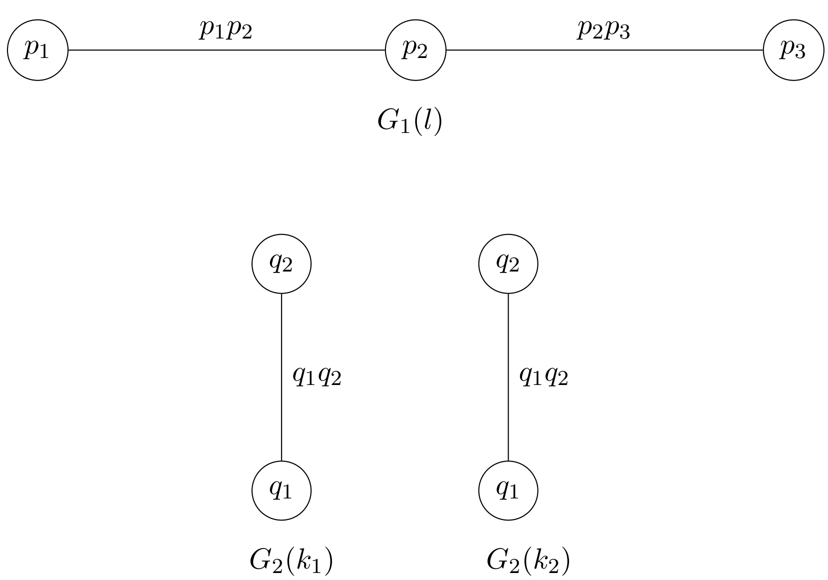

Figure 1: Complex Neutrosophic Soft Graphs.

Vertices of CNSG $G_1 \bigotimes G_2$

$$\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k_1)=\begin{cases}\{p_1q_1,(.3e^{i\pi(1.5)},.5e^{i\pi(1.4)},.6e^{i\pi(1.5)})\}\\\{p_2q_1,(.2e^{i\pi(1.5)},.5e^{i\pi(1.4)},.7e^{i\pi(1.8)})\}\\\{p_3q_1,(.4e^{i\pi(1.4)},.5e^{i\pi(1.4)},.6e^{i\pi(1.6)})\}\\\{p_1q_2,(.3e^{i\pi(1.3)},.7e^{i\pi(1.5)},.6e^{i\pi(1.8)})\}\\\{p_2q_2,(.2e^{i\pi(1.3)},.6e^{i\pi(1.5)},.7e^{i\pi(1.8)})\}\\\{p_3q_2,(.5e^{i\pi(1.3)},.7e^{i\pi(1.5)},.4e^{i\pi(1.8)})\}\end{cases}$$

$$\mathfrak{S}_1(l)\bigotimes\mathfrak{S}_2(k_2)=\begin{cases}\{p_1q_1,(.3e^{i\pi(1.2)},.7e^{i\pi(1.4)},.6e^{i\pi(1.7)})\}\\\{p_2q_1,(.2e^{i\pi(1.2)},.6e^{i\pi(1.4)},.7e^{i\pi(1.8)})\}\\\{p_3q_1,(.3e^{i\pi(1.2)},.7e^{i\pi(1.4)},.6e^{i\pi(1.7)})\}\\\{p_1q_2,(.2e^{i\pi(1.6)},.6e^{i\pi(1.5)},.8e^{i\pi(1.5)})\}\\\{p_2q_2,(.2e^{i\pi(1.6)},.6e^{i\pi(1.5)},.8e^{i\pi(1.8)})\}\\\{p_3q_2,(.2e^{i\pi(1.4)},.6e^{i\pi(1.5)},.8e^{i\pi(1.8)})\}\end{cases}$$

Edges of CNSG $G_1\bigotimes G_2$

$$\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k_1)=\begin{cases}\{p_1(q_1q_2),(.3e^{i\pi(1.2)},.4e^{i\pi(1.3)},.7e^{i\pi(1.8)})\}\\\{p_2(q_1q_2),(.2e^{i\pi(1.2)},.4e^{i\pi(1.3)},.7e^{i\pi(1.8)})\}\\\{p_3(q_1q_2),(.3e^{i\pi(1.2)},.4e^{i\pi(1.3)},.7e^{i\pi(1.8)})\}\\\{(p_1p_2)q_1,(.1e^{i\pi(1.5)},.5e^{i\pi(1.4)},.8e^{i\pi(1.9)})\}\\\{(p_1p_2)q_2,(.1e^{i\pi(1.3)},.6e^{i\pi(1.5)},.8e^{i\pi(1.9)})\}\\\{(p_2p_3)q_1,(.2e^{i\pi(1.3)},.5e^{i\pi(1.4)},.7e^{i\pi(1.8)})\}\\\{(p_2p_3)q_2,(.2e^{i\pi(1.3)},.6e^{i\pi(1.5)},.7e^{i\pi(1.8)})\}\end{cases}$$

$$\mathfrak{K}_1(l)\bigotimes\mathfrak{K}_2(k_2)=\begin{cases}\{p_1(q_1q_2),(.2e^{i\pi(1.2)},.6e^{i\pi(1.4)},.8e^{i\pi(1.7)})\}\\\{p_2(q_1q_2),(.2e^{i\pi(1.2)},.6e^{i\pi(1.4)},.8e^{i\pi(1.8)})\}\\\{p_3(q_1q_2),(.2e^{i\pi(1.2)},.6e^{i\pi(1.4)},.8e^{i\pi(1.7)})\}\\\{(p_1p_2)q_1,(.1e^{i\pi(1.2)},.5e^{i\pi(1.4)},.8e^{i\pi(1.9)})\}\\\{(p_1p_2)q_2,(.1e^{i\pi(1.5)},.5e^{i\pi(1.5)},.8e^{i\pi(1.9)})\}\\\{(p_2p_3)q_1,(.2e^{i\pi(1.2)},.6e^{i\pi(1.4)},.7e^{i\pi(1.8)})\}\\\{(p_2p_3)q_2,(.2e^{i\pi(1.3)},.6e^{i\pi(1.5)},.7e^{i\pi(1.8)})\}\end{cases}$$

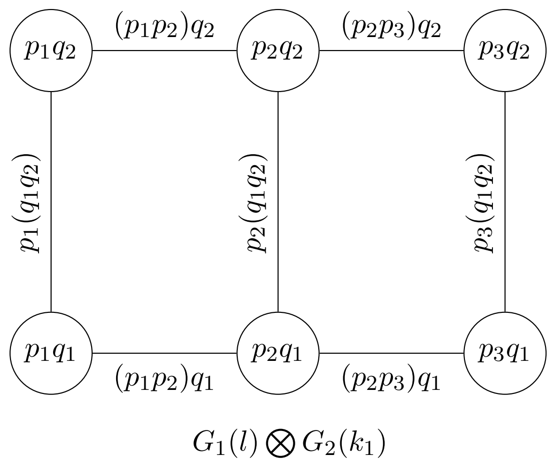

Figure 2: Cartesian product of CNSG.

4. Application: Decision-Making in Wireless Communication Networks using CNSG

A wireless sensor network (WSN) monitors system or environmental conditions by connecting several sensor nodes and exchanging data. However, uncertainties like noise, interference, and propagation delay, which result in phase changes in sent signals, often impede communication. Additionally, incomplete or missing data from particular sensors results in ambiguity, while transmission errors or packet losses lead to inaccurate information. Because of these features, it is difficult to assess communication link reliability using conventional deterministic or even fuzzy models. Implementing a more comprehensive framework that can simultaneously describe and analyze multiple types of uncertainty is therefore crucial. The Complex Neutrosophic soft Graph model generates such a structure by integrating truth, indeterminacy, and falsity degrees with complex (phase) and soft computing techniques. The goal of this technique is to determine the most reliable communication link between sensor nodes while accounting for all of the entangled uncertainty.

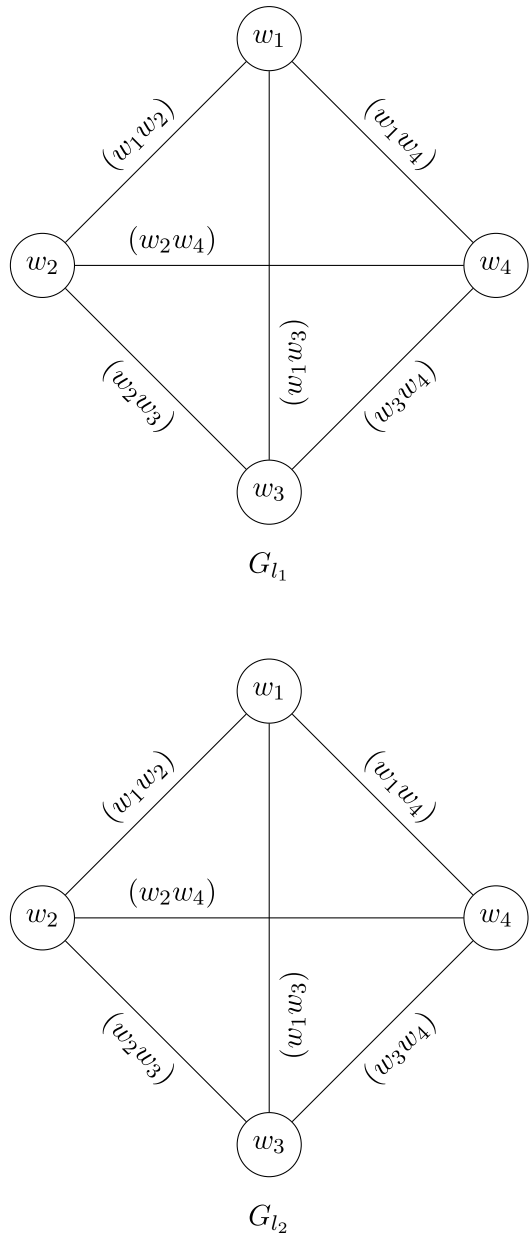

Figure 3: Wireless Communication Networks.

Sensor nodes:

$$\mathfrak{S}(l_1) = \begin{cases} \{u_1, (.5e^{i1.8\pi}, .4e^{i1.6\pi}, .4e^{i1.3\pi})\} \\ \{u_2, (.7e^{i1.6\pi}, .7e^{i1.3\pi}, .3e^{i1.5\pi})\} \\ \{u_3, (.5e^{i1.7\pi}, .6e^{i1.4\pi}, .5e^{i1.8\pi})\} \\ \{u_4, (.4e^{i1.9\pi}, .5e^{i1.2\pi}, .5e^{i1.7\pi})\} \end{cases}$$

$$\mathfrak{S}(l_2) = \begin{cases} \{u_1, (.6e^{i1.7\pi}, .3e^{i1.4\pi}, .3e^{i1.7\pi})\} \\ \{u_2, (.8e^{i1.6\pi}, .4e^{i1.3\pi}, .2e^{i1.5\pi})\} \\ \{u_3, (.6e^{i1.8\pi}, .6e^{i1.7\pi}, .4e^{i1.3\pi})\} \\ \{u_4, (.7e^{i1.9\pi}, .5e^{i1.8\pi}, .3e^{i1.4\pi})\} \end{cases}$$

Communication path:

$$\mathfrak{K}(l_1) = \begin{cases} \{u_1u_2, (.4e^{i1.6\pi}, .3e^{i1.2\pi}, .4e^{i1.5\pi})\} \\ \{u_2u_3, (.5e^{i1.5\pi}, .6e^{i1.3\pi}, .6e^{i1.6\pi})\} \\ \{u_3u_4, (.3e^{i1.7\pi}, .5e^{i1.2\pi}, .5e^{i1.8\pi})\} \\ \{u_1u_4, (.4e^{i1.6\pi}, .4e^{i1.1\pi}, .7e^{i1.6\pi})\} \\ \{u_1u_3, (.5e^{i1.6\pi}, .4e^{i1.3\pi}, .6e^{i1.7\pi})\} \\ \{u_2u_4, (.3e^{i1.5\pi}, .5e^{i1.2\pi}, .5e^{i1.6\pi})\} \end{cases}$$

$$\mathfrak{K}(l_2) = \begin{cases} \{u_1u_2, (.6e^{i1.5\pi}, .3e^{i1.2\pi}, .3e^{i1.6\pi})\} \\ \{u_2u_3, (.5e^{i1.4\pi}, .4e^{i1.3\pi}, .5e^{i1.4\pi})\} \\ \{u_3u_4, (.6e^{i1.7\pi}, .5e^{i1.6\pi}, .4e^{i1.3\pi})\} \\ \{u_1u_4, (.5e^{i1.6\pi}, .3e^{i1.5\pi}, .3e^{i1.2\pi})\} \\ \{u_1u_3, (.4e^{i1.6\pi}, .3e^{i1.4\pi}, .6e^{i1.5\pi})\} \\ \{u_2u_4, (.6e^{i1.5\pi}, .4e^{i1.2\pi}, .3e^{i1.4\pi})\} \end{cases}$$

Each edge in the Complex Neutrosophic Soft Graph is assigned a composite reliability score ($\mathfrak{R}$), which measures the overall reliability of each communication channel. This score takes into account the results of the complex neutrosophic representation's truth, indeterminacy, and falsity components. The equation for calculating the dependability of an edge is

$$\mathfrak{R}(w_iw_j) = \sum \left| \mathfrak{X}_{\mathfrak{K}(l)}(w_iw_j) - \frac{[\mathfrak{Y}_{\mathfrak{K}(l)}(w_iw_j) + \mathfrak{Z}_{\mathfrak{K}(l)}(w_iw_j)]}{2} \right| e^{i\sum\left|A_{\mathfrak{K}(l)}(w_iw_j) - \frac{[B_{\mathfrak{K}(l)}(w_iw_j)+C_{\mathfrak{K}(l)}(w_iw_j)]}{2}\right|}$$

Then,

$$\mathfrak{R}(w_1w_2) = .35e^{i0.35\pi}$$

$$\mathfrak{R}(w_2w_3) = .15e^{i0.10\pi}$$

$$\mathfrak{R}(w_3w_4) = .35e^{i0.55\pi}$$

$$\mathfrak{R}(w_1w_4) = .35e^{i0.50\pi}$$

$$\mathfrak{R}(w_1w_3) = .05e^{i0.25\pi}$$

$$\mathfrak{R}(w_2w_4) = .45e^{i0.30\pi}$$

The most reliable path is between w_2 and w_4 with $R(w_2w_4) = .45e^{i0.30\pi}$.

The least reliable path is $w_1 - w_3$ with $R(w_1w_3) = 0.05e^{i0.25\pi}$.

As a result, CNSG suggests the communication link (w_2w_4) as the most reliable path for data transfer.

5. Conclusion

This paper presented a formal definition and analysis of the Cartesian product of complex neutrosophic soft graphs. Vertex and edge membership functions were employed to investigate the fundamental structural properties of the resulting graphs. By establishing preservation results under the Cartesian product operation and examining its applications in CNSGs, it was shown that complex neutrosophic soft graphs can be systematically constructed from simpler ones. The proposed framework strengthens the theoretical foundations of complex neutrosophic soft graph theory and opens promising directions for further research and applications in complex network analysis and uncertainty modeling. It is also hoped that future studies will explore extensions based on Plithogenic Graphs [15, 16], HyperGraphs [17], and SuperHyperGraphs [18, 19, 23, 24].

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Use of Generative AI and AI-Assisted Tools

Generative AI and AI-assisted tools were used for tasks such as English grammar checking, and they were not used in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany this paper.

Disclaimer

The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

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The Randic Index in Neutrosophic Soft Graphs and its Applications

Suriyakumar G[1*], *V. J. Sudhakar*[2], *R. Dhivya*[3], *and Takaaki Fujita*[4]

[1] Research Scholar, Department of Mathematics, Islamiah College (Autonomous), Vaniyambadi, Affiliated to Thiruvalluvar University, Vellore, Tamil Nadu, India. gansuriyakumar@gmail.com

[2] Department of Mathematics, Islamiah College (Autonomous), Vaniyambadi, Affiliated to Thiruvalluvar University, Vellore, Tamil Nadu, India. vjsvec1@gmail.com

[3] Adhiparasakthi College of Engineering, Kalavai, Tamil Nadu, India dhivyarohi@gmail.com

[4] Independent Researcher, Tokyo, Japan. Takaaki.fujita060@gmail.com

*Correspondence: gansuriyakumar@gmail.com

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Abstract. The paper presents and studies the Randic Index for neutrosophic graphs and uncertain graphs. It provides an effective model for uncertainty and imprecision in edge and vertex relations. This study analyzes the Randic Index's properties for neutrosophic soft graphs and the impact of isomorphism between graphs. Additionally, the idea is expanded to directed neutrosophic soft graphs, providing a new method for analyzing directed structures in the presence of uncertainty. A real-world example shows how the Randic Index can be used to help choose the ideal location for a waste management plant. This technique underlines the Randic Index's practical usefulness in urban planning and infrastructure development, illustrating its greater influence across various disciplines.

Keywords: Randic Index; Neutrosophic soft graph; Randic Index in Neutrosophic soft graph.

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1. **Introduction**

Numerous practical problems in computer applications, systems analysis, computer networks, transportation, operations research, and economics may be resolved with the aid of graph theory. In essence, a graph is a relational model used to explain problems in the actual world that call for interactions between items. The items and their relationships are represented by the vertices and edges of the graph, accordingly. For a variety of causes, such as information loss, a lack of supporting evidence, faulty statistical data, and insufficient information, the information supplied in many optimization scenarios is erroneous or imprecise. Generally speaking, information concerning a real-life situation can be unclear.

Cantor proposed the fundamental concept of classical graph theory. Each vertex or edge in a classical graph can either belong to the graph or not. As a result, uncertain optimization scenarios cannot be represented by ordinary graphs. Since real-world issues are usually unforeseen, it is challenging to replicate them with traditional graphs. Fuzzy set theory was proposed by Zadeh [1]. Neutrosophic sets are a helpful mathematical tool for handling ambiguous, inconsistent, and incomplete data in the real world, according to Smarandache's theory. A truth-membership function (t), an indeterminacy-membership function (i), and a falsity-membership function (f) create neutrosophic sets independently in the real standard or nonstandard unit interval $[0, 1]$. As extensions of the Neutrosophic Set, concepts such as the Double-valued Neutrosophic Set [24] and the SuperHyperNeutrosophic Set [25] are also known. To facilitate the use of NS in real-world applications, Wang developed the notion of single-valued neutrosophic sets, a subclass of neutrosophic sets. The earlier study was motivated by the introduction of the notions of neutrosophic vague graphs and strong neutrosophic vague graphs by Satham Hussain [2]. Neutrosophic Graphs and their extended concepts, such as Neutrosophic HyperGraphs [18], Neutrosophic SuperHyperGraphs [19], Quadripartitioned neutrosophic graphs [22,23], and Plithogenic Graphs [20,21], have been studied in recent years.

In addition to discussing the characteristics of neighborly regular and irregular fuzzy graphs, Nagoor Gani [3, 4] defines regular and irregular fuzzy graphs. Then, Liangsong Huang [5] introduced a study of regular and irregular neutrosophic soft graphs, the two types of degree, d_m-regular, td_m-regular and m-highly irregular neutrosophic soft graphs, and some properties are shown.

The concept of complex neutrosophic soft graphs (CNSGs), which provide a sophisticated framework for simulating indeterminacy, ambiguity, and uncertainty in complex systems, is introduced and developed by Suriya [6–9]. High degrees of neutrosophic uncertainty and complex-valued memberships for both edges and vertices under soft set parameters define a strong complex neutrosophic soft graph (SCNSG). We present a novel measure that considers the cumulative strength and uncertainty of connections within graph characteristics, as well as the complements of a complex neutrosophic soft graph and a strong complex neutrosophic soft graph.

Yahya [10] introduces the features of ambiguous graphs and subgraphs as well as their Randic Index. This study uses isomorphic properties to investigate the upper and lower boundaries of the Randic Index of vague graphs. We introduce Randic indices for directed ambiguous graphs. Finally, an application of the Randic Index in construction was presented.

We introduce the neutrosophic graph features and the Randic Index. Using a variety of isomorphic features, this paper investigates a truth-membership function, indeterminacy-membership function, and falsity-membership function of the Randic Index in neutrosophic soft graphs. After introducing Randic indices for neutrosophic soft graphs and neutrosophic digraphs, theorems are explored through examples. The Randic Index is used in construction.

2. Preliminaries

This section provides the basic definitions and an example needed to support the main results. We begin with the definition of a fuzzy soft graph.

Definition 2.1. *([11]) A fuzzy soft graph $G = (G^*, M, N, R)$ is an ordered quadruple that meets the following conditions:*

- *$G^* = (V, E)$ is a simple graph,*
- *R is a non-empty set of parameters,*
- *(M, R) is a fuzzy set defined over the vertex set V,*
- *(N, R) is a fuzzy set defined over the edge set E,*
- *For each $a \in R$, the pair $(M(a), N(a))$ forms a fuzzy graph associated with G^*, such that:*

$$N(a)(xy) \leq \min\{M(a)(x), M(a)(y)\}$$

holds for all $x, y \in V$.

$\mathfrak{H}(a)$ represents the fuzzy graph $(M(a), N(a))$. A fuzzy soft graph can be thought of as a family of fuzzy graphs parameterized by R. $M(G^*)$ denotes the class of all fuzzy soft graphs over G^*.

Definition 2.2. *[12] An intuitionistic fuzzy soft graph $G = (G^*, F, K, Q)$ is an ordered four tuple if it satisfies the following conditions:*

- *$G^* = (V, E)$ is a simple graph,*
- *Q is a non-empty set of parameters,*
- *(F, Q) is an intuitionistic fuzzy set over V,*
- *(K, Q) is an intuitionistic fuzzy set over E,*
- *$(F(a), K(a))$ is an intuitionistic fuzzy graph of G^* for all $a \in Q$. That is*

$$\mathfrak{A}_{K(a)}(ab) \le \min\{\mathfrak{A}_{F(a)}(a), \mathfrak{A}_{F(a)}(b)\}$$

$$\mathfrak{C}_{K(a)}(ab) \ge \max\{\mathfrak{C}_{F(a)}(a), \mathfrak{C}_{F(a)}(b)\}$$

for all $a \in Q$ and $x, y \in V$. The intuitionistic fuzzy graph $(F(a), K(a))$ is denoted by $H(a)$ for convenience. An intuitionistic fuzzy soft graph is a parameterized family of intuitionistic fuzzy graphs.

Definition 2.3. *[13] A neutrosophic soft graph $G = (G^*, M, N, R)$ is defined as a quadruple satisfying the following conditions:*

- *$G^* = (V, E)$ represents a simple (i.e., undirected and without loops or multiple edges) graph,*
- *R is a non-empty collection of parameters,*
- *(M, R) denotes a neutrosophic set defined over the vertex set V,*
- *(N, R) denotes a neutrosophic set defined over the edge set E,*
- *For every $a \in R$, the pair $(M(a), N(a))$ forms a neutrosophic subgraph of G^*. Specifically, the following conditions hold:*

$$\mathfrak{A}_{N(a)}(xy) \le \min\{\mathfrak{A}_{M(a)}(x), \mathfrak{A}_{M(a)}(y)\},$$

$$\mathfrak{B}_{N(a)}(xy) \le \min\{\mathfrak{B}_{M(a)}(x), \mathfrak{B}_{M(a)}(y)\},$$

$$\mathfrak{C}_{N(a)}(xy) \ge \max\{\mathfrak{C}_{M(a)}(x), \mathfrak{C}_{M(a)}(y)\},$$

For ease of use, $\mathfrak{H}(a)$ is used to represent the neutrosophic fuzzy graph $(M(a), N(a))$. A neutrosophic fuzzy soft graph may therefore be thought of as a family of neutrosophic fuzzy graphs that are parameterized by R. $M(G^*)$ represents the class of all neutrosophic fuzzy soft graphs over G^*, for all $a \in R$ and $x, y \in V$.

Definition 2.4. *[10] The Randic Index of a vague graph $G = (S, K)$ is shown by $\mathfrak{RI}(G)$ and explained as follows:*

$$\mathfrak{RI}(G) = (\mathfrak{RI}_{\mathfrak{P}}(G), \mathfrak{RI}_{\mathfrak{Q}}(G), \mathfrak{RI}_{\mathfrak{R}}(G))$$

where,

$$\mathfrak{RI}_{\mathfrak{P}}(G) = \sum_{i \ne j, c_i c_j \in E} \left(\mathfrak{P}_S(c_i)\mathfrak{P}_S(c_j)\mathbb{D}_{\mathfrak{P}}(c_i)\mathbb{D}_{\mathfrak{P}}(c_j)\right)^{-\frac{1}{2}}$$

$$\mathfrak{RI}_{\mathfrak{R}}(G) = \sum_{i \ne j, c_i c_j \in E} \left(\mathfrak{R}_S(c_i)\mathfrak{R}_S(c_j)\mathbb{D}_{\mathfrak{R}}(c_i)\mathbb{D}_{\mathfrak{R}}(c_j)\right)^{-\frac{1}{2}}$$

where true and false parts of the node c_i degrees are $\mathbb{D}_{\mathfrak{P}}$ and $\mathbb{D}_{\mathfrak{R}}$, respectively.

Definition 2.5. *[8] The Randic Index of a neutrosophic graph $G = (S, K)$ is shown by $\mathfrak{RI}(G)$ and explained as follows:*

$$\mathfrak{R}(G) = (\mathfrak{RI}_{\mathfrak{P}}(G), \mathfrak{RI}_{\mathfrak{Q}}(G), \mathfrak{RI}_{\mathfrak{R}}(G))$$

where,

$$\mathfrak{R}_{\mathfrak{A}}(G) = \sum_{i \neq j, c_i c_j \in E} \frac{1}{\sqrt{(\mathfrak{A}_S(c_i)\mathfrak{P}_S(c_j)\mathbb{D}_{\mathfrak{P}}(c_i)\mathbb{D}_{\mathfrak{P}}(c_j))}}$$

$$\mathfrak{R}_{\mathfrak{B}}(G) = \sum_{i \neq j, c_i c_j \in E} \frac{1}{\sqrt{(\mathfrak{B}_S(c_i)\mathfrak{Q}_S(c_j)\mathbb{D}_{\mathfrak{Q}}(c_i)\mathbb{D}_{\mathfrak{Q}}(c_j))}}$$

$$\mathfrak{R}_{\mathfrak{C}}(G) = \sum_{i \neq j, c_i c_j \in E} \frac{1}{\sqrt{(\mathfrak{C}_S(c_i)\mathfrak{R}_S(c_j)\mathbb{D}_{\mathfrak{R}}(c_i)\mathbb{D}_{\mathfrak{R}}(c_j))}}$$

that the true, indeterminacy and false parts of the node c_i degrees are $\mathbb{D}_{\mathfrak{A}}$, $\mathbb{D}_{\mathfrak{B}}$ and $\mathbb{C}_{\mathfrak{R}}$, respectively.

3. The Randic Index in Neutrosophic Soft Graphs

In this section, we investigate the Randic index in neutrosophic soft graphs. The relevant definitions and related materials are presented below.

Definition 3.1. *The Randic Index of a neutrosophic soft graph $G = (S, K)$ is shown by $\mathfrak{R}(G)$ and explained as follows:*

$$\mathfrak{R}(G) = (\mathfrak{R}_{\mathfrak{A}(n)}(G), \mathfrak{R}_{\mathfrak{B}(n)}(G), \mathfrak{R}_{\mathfrak{C}(n)}(G))$$

where,

$$\mathfrak{R}_{\mathfrak{A}(n)}(G) = \sum_{n \in N} \sum_{i \neq j, c_i c_j \in E} \frac{1}{\sqrt{(\mathfrak{A}_{S(n)}(c_i)\mathfrak{A}_{S(n)}(c_j)\mathfrak{D}_{\mathfrak{A}(n)}(c_i)\mathfrak{D}_{\mathfrak{A}(n)}(c_j))}}$$

$$\mathfrak{R}_{\mathfrak{B}(n)}(G) = \sum_{n \in N} \sum_{i \neq j, c_i c_j \in E} \frac{1}{\sqrt{(\mathfrak{B}_{S(n)}(c_i)\mathfrak{B}_{S(n)}(c_j)\mathfrak{D}_{\mathfrak{B}(n)}(c_i)\mathfrak{D}_{\mathfrak{B}(n)}(c_j))}}$$

$$\mathfrak{R}_{\mathfrak{C}(n)}(G) = \sum_{n \in N} \sum_{i \neq j, c_i c_j \in E} \frac{1}{\sqrt{(\mathfrak{C}_{S(n)}(c_i)\mathfrak{C}_{S(n)}(c_j)\mathfrak{D}_{\mathfrak{C}(n)}(c_i)\mathfrak{D}_{\mathfrak{C}(n)}(c_j))}}$$

that the true, indeterminacy and false parts of the node c_i degrees are $\mathfrak{D}_{\mathfrak{A}(n)}$, $\mathfrak{D}_{\mathfrak{B}(n)}$ and $\mathfrak{D}_{\mathfrak{C}(n)}$, respectively. For each parameter $n \in N$, the pair $(S(n), K(n))$ forms a neutrosophic soft graph (NSG) on G^*.

Example 3.2. *Let G be the neutrosophic soft graph shown in Figure 1. Then*

$$\mathfrak{D}_{n_1}(c_1) = (0.6, 1.3, 2.0), \qquad \mathfrak{D}_{n_1}(c_2) = (0.5, 0.9, 1.4),$$

$$\mathfrak{D}_{n_1}(c_3) = (0.4, 1.7, 2.2), \qquad \mathfrak{D}_{n_1}(c_4) = (0.3, 1.1, 1.4),$$

and

$$\mathfrak{D}_{n_2}(c_1) = (0.4, 1.5, 2.4), \qquad \mathfrak{D}_{n_2}(c_2) = (0.4, 1.0, 1.6),$$

$$\mathfrak{D}_{n_2}(c_3) = (0.5, 1.3, 2.5), \qquad \mathfrak{D}_{n_2}(c_4) = (0.3, 1.0, 1.5).$$

Furthermore,

$$\sum_{n\in N}\sum_{\substack{i\neq j\\ c_ic_j\in E}}\frac{1}{\sqrt{\mathfrak{A}_{S(n)}(c_i)\mathfrak{A}_{S(n)}(c_j)\mathfrak{D}_{\mathfrak{A}(n)}(c_i)\mathfrak{D}_{\mathfrak{A}(n)}(c_j)}} = (5.27+7.91+11.79+7.86+8.33) + (10.21+9.13+10.21+11.18) = 81.89,$$

$$\sum_{n\in N}\sum_{\substack{i\neq j\\ c_ic_j\in E}}\frac{1}{\sqrt{\mathfrak{B}_{S(n)}(c_i)\mathfrak{B}_{S(n)}(c_j)\mathfrak{D}_{\mathfrak{B}(n)}(c_i)\mathfrak{D}_{\mathfrak{B}(n)}(c_j)}} = (1.69+1.17+0.91+1.32+1.06) + (1.09+1.48+1.60+1.18+1.13) = 12.63,$$

and

$$\sum_{n\in N}\sum_{\substack{i\neq j\\ c_ic_j\in E}}\frac{1}{\sqrt{\mathfrak{C}_{S(n)}(c_i)\mathfrak{C}_{S(n)}(c_j)\mathfrak{D}_{\mathfrak{C}(n)}(c_i)\mathfrak{D}_{\mathfrak{C}(n)}(c_j)}} = (1.09+0.96+0.96+1.09+0.74) + (0.79+0.72+0.82+0.89+0.55) = 8.61.$$

Hence,

$$\mathfrak{R}(G) = \big(\mathfrak{R}_{\mathfrak{A}(n)}(G), \mathfrak{R}_{\mathfrak{B}(n)}(G), \mathfrak{R}_{\mathfrak{C}(n)}(G)\big) = (81.89,\ 12.63,\ 8.61).$$

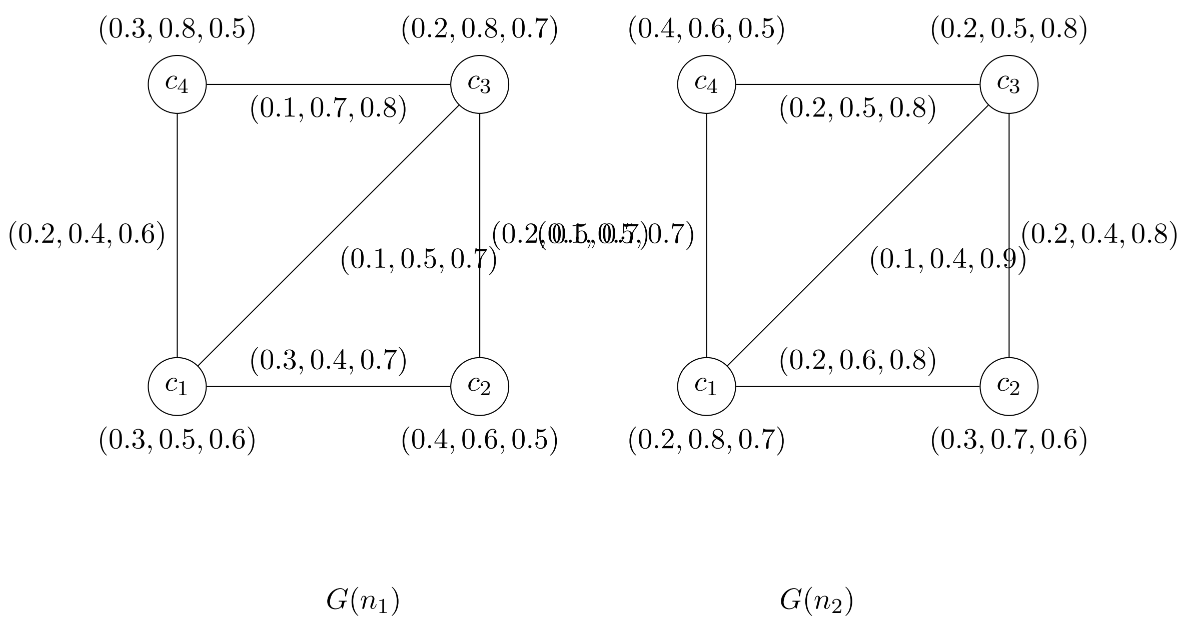


Figure 1. *Neutrosophic soft graph G.*

Theorem 3.3. *Let $G = (S, K)$ be a neutrosophic soft graph on the underlying graph*

$$G^* = (V, E),$$

and fix $n \in N$. Let

$$a := |V| \qquad \textit{and} \qquad b := |E|.$$

Assume that all terms appearing in $\mathfrak{R}_{\mathfrak{A}(n)}(G)$, $\mathfrak{R}_{\mathfrak{B}(n)}(G)$, and $\mathfrak{R}_{\mathfrak{C}(n)}(G)$ are well defined. Then

$$\mathfrak{R}_{\mathfrak{A}(n)}(G) \geq \frac{b}{a-1}, \qquad \mathfrak{R}_{\mathfrak{B}(n)}(G) \geq \frac{b}{a-1}, \qquad \mathfrak{R}_{\mathfrak{C}(n)}(G) \geq \frac{b}{a-1}.$$

Proof. Since G is a neutrosophic soft graph, for every $b_i, b_j \in V$ we have

$$0 \leq \mathfrak{A}_{S(n)}(b_i) \leq 1, \qquad 0 \leq \mathfrak{B}_{S(n)}(b_i) \leq 1, \qquad 0 \leq \mathfrak{C}_{S(n)}(b_i) \leq 1.$$

Hence,

$$\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j) \leq 1, \qquad \mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j) \leq 1, \qquad \mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j) \leq 1. \tag{1}$$

Since $|V| = a$, each vertex is adjacent to at most $a - 1$ vertices. Therefore,

$$\mathfrak{D}_{\mathfrak{A}(n)}(b_i) = \sum_{b_i b_j \in E} \mathfrak{A}_{K(n)}(b_i b_j) \leq a - 1,$$

$$\mathfrak{D}_{\mathfrak{B}(n)}(b_i) = \sum_{b_i b_j \in E} \mathfrak{B}_{K(n)}(b_i b_j) \leq a - 1,$$

$$\mathfrak{D}_{\mathfrak{C}(n)}(b_i) = \sum_{b_i b_j \in E} \mathfrak{C}_{K(n)}(b_i b_j) \leq a - 1$$

for all $b_i \in V$. Consequently,

$$\mathfrak{D}_{\mathfrak{A}(n)}(b_i)\mathfrak{D}_{\mathfrak{A}(n)}(b_j) \leq (a-1)^2,$$

$$\mathfrak{D}_{\mathfrak{B}(n)}(b_i)\mathfrak{D}_{\mathfrak{B}(n)}(b_j) \leq (a-1)^2,$$

$$\mathfrak{D}_{\mathfrak{C}(n)}(b_i)\mathfrak{D}_{\mathfrak{C}(n)}(b_j) \leq (a-1)^2$$

for all $b_i, b_j \in V$. Thus,

$$\begin{aligned}
\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{A}(n)}(b_i)\mathfrak{D}_{\mathfrak{A}(n)}(b_j) &\leq (a-1)^2,\\
\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{B}(n)}(b_i)\mathfrak{D}_{\mathfrak{B}(n)}(b_j) &\leq (a-1)^2,\\
\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{C}(n)}(b_i)\mathfrak{D}_{\mathfrak{C}(n)}(b_j) &\leq (a-1)^2.
\end{aligned} \tag{2}$$

By taking reciprocals and square roots, from (2) we obtain

$$\frac{1}{\sqrt{\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{A}(n)}(b_i)\mathfrak{D}_{\mathfrak{A}(n)}(b_j)}} \geq \frac{1}{a-1},$$

$$\frac{1}{\sqrt{\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{B}(n)}(b_i)\mathfrak{D}_{\mathfrak{B}(n)}(b_j)}} \geq \frac{1}{a-1},$$

$$\frac{1}{\sqrt{\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{C}(n)}(b_i)\mathfrak{D}_{\mathfrak{C}(n)}(b_j)}} \geq \frac{1}{a-1}$$

for every edge $b_ib_j \in E$. Summing these inequalities over all b edges yields

$$\mathfrak{R}_{\mathfrak{A}(n)}(G) \geq \frac{b}{a-1}, \qquad \mathfrak{R}_{\mathfrak{B}(n)}(G) \geq \frac{b}{a-1}, \qquad \mathfrak{R}_{\mathfrak{C}(n)}(G) \geq \frac{b}{a-1}.$$

This completes the proof. □

Example 3.4. *Let G be the neutrosophic soft graph given in Example 3.2. Then*

$$a = |V| = 4, \qquad b = |E| = 5,$$

and

$$\mathfrak{R}_{\mathfrak{A}(n)}(G) = 81.89, \qquad \mathfrak{R}_{\mathfrak{B}(n)}(G) = 12.63, \qquad \mathfrak{R}_{\mathfrak{C}(n)}(G) = 8.61.$$

Moreover,

$$\frac{b}{a-1} = \frac{5}{3} \approx 1.67.$$

Therefore,

$$\mathfrak{R}_{\mathfrak{A}(n)}(G) > \frac{b}{a-1}, \qquad \mathfrak{R}_{\mathfrak{B}(n)}(G) > \frac{b}{a-1}, \qquad \mathfrak{R}_{\mathfrak{C}(n)}(G) > \frac{b}{a-1}.$$

Theorem 3.5. *Let $G = (S, K)$ be a strong neutrosophic soft graph so that S is a constant function and $|V| = a$. Then, $\mathfrak{R}_{\mathfrak{A}(n)}(G) \leq \frac{a}{2v_{1(n)}^2}$, $\mathfrak{R}_{\mathfrak{B}(n)}(G) \leq \frac{a}{2v_{2(n)}^2}$ and $\mathfrak{R}_{\mathfrak{C}(n)}(G) \leq \frac{a}{2v_{3(n)}^2}$, where, $(v_{1(n)}, v_{2(n)}, v_{3(n)}) = (\mathfrak{A}_{S(n)}(b_i), \mathfrak{B}_{S(n)}(b_i), \mathfrak{C}_{S(n)}(b_i))$, $b_i \in V$.*

Proof: *Since a strong neutrosophic soft graph is generally a neutrosophic subgraph of the corresponding complete neutrosophic soft graph, it is simple to prove that $\mathfrak{R}_{\mathfrak{A}(n)}(G) \leq \frac{a}{2v_{1(n)}^2}$, $\mathfrak{R}_{\mathfrak{B}(n)}(G) \leq \frac{a}{2v_{2(n)}^2}$ and $\mathfrak{R}_{\mathfrak{C}(n)}(G) \leq \frac{a}{2v_{3(n)}^2}$.*

Theorem 3.6. *Let G be a complete neutrosophic soft graph so that S is a constant function. Then $\mathfrak{R}(G) = \left(\frac{a}{2v_{1(n)}^2}, \frac{a}{2v_{2(n)}^2}, \frac{a}{2v_{3(n)}^2}\right)$, where $a = |V|$ and $(v_{1(n)}, v_{2(n)}, v_{3(n)}) = (\mathfrak{A}_{S(n)}(b_i), \mathfrak{B}_{S(n)}(b_i), \mathfrak{C}_{S(n)}(b_i))$, $b_i \in V$.*

Proof: *Since S is a constant and $\mathfrak{A}_{S(n)}(b_i) = v_{1(n)}$, $\mathfrak{B}_{S(n)}(b_i) = v_{2(n)}$, $\mathfrak{C}_{S(n)}(b_i) = v_{3(n)}$, $b_i \in V$. Since G is complete neutrosophic soft graph, $\mathfrak{A}_{K(n)}(b_ib_j) = \{\mathfrak{A}_{S(n)}(b_i) \wedge \mathfrak{A}_{S(n)}(b_j)\}$, $\mathfrak{B}_{K(n)}(b_ib_j) = \{\mathfrak{B}_{S(n)}(b_i) \wedge \mathfrak{B}_{S(n)}(b_j)\}$ and $\mathfrak{C}_{K(n)}(b_ib_j) = \{\mathfrak{C}_{S(n)}(b_i) \wedge \mathfrak{C}_{S(n)}(b_j)\}$, $\forall b_ib_j \in E$.*

Since G is complete and $|V| = a$, there are $\frac{a(a-1)}{2}$ pairs of vertices and $\frac{a(a-1)}{2}$ edges. Additionally, each vertices in G is adjacent to $(a-1)$ vertices. Then,

$$\mathfrak{D}_{\mathfrak{A}(n)}(b_i) = \sum_{n \in N} \sum_{b_ib_j \in E} \mathfrak{A}_{K(n)}(b_ib_j) = v_{1(n)} \cdot v_{1(n)} \cdots v_{1(n)}(a-1 \ \ times) = (a-1)v_{1(n)}$$

$$\mathfrak{D}_{\mathfrak{B}(n)}(b_i) = \sum_{n \in N} \sum_{b_ib_j \in E} \mathfrak{B}_{K(n)}(b_ib_j) = v_{2(n)} \cdot v_{2(n)} \cdots v_{2(n)}(a-1 \ \ times) = (a-1)v_{2(n)}$$

$$\mathfrak{D}_{\mathfrak{C}(n)}(b_i) = \sum_{n\in N}\sum_{b_ib_j\in E}\mathfrak{C}_{K(n)}(b_ib_j) = v_{3(n)}\cdot v_{3(n)}\cdots v_{3(n)}(a-1\ \ times) = (a-1)v_{3(n)}$$

Therefore,

$$\begin{aligned}\mathfrak{R}_{\mathfrak{A}(n)}(G) &= \sum_{n\in N}\sum_{i\neq j, b_ib_j\in E}\frac{1}{\sqrt{\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{A}(n)}(b_i)\mathfrak{D}_{\mathfrak{A}(n)}(b_j)}}\\ &= \frac{a(a-1)}{2}\cdot\frac{1}{\sqrt{(v_{1(n)}v_{1(n)}(a-1)v_{1(n)}(a-1)v_{1(n)})}}\\ &= \frac{a}{2v_{1(n)}^2}\end{aligned}$$

$$\begin{aligned}\mathfrak{R}_{\mathfrak{B}(n)}(G) &= \sum_{n\in N}\sum_{i\neq j, b_ib_j\in E}\frac{1}{\sqrt{\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{B}(n)}(b_i)\mathfrak{D}_{\mathfrak{B}(n)}(b_j)}}\\ &= \frac{a(a-1)}{2}\cdot\frac{1}{\sqrt{(v_{2(n)}v_{2(n)}(a-1)v_{2(n)}(a-1)v_{2(n)})}}\\ &= \frac{a}{2v_{2(n)}^2}\end{aligned}$$

$$\begin{aligned}\mathfrak{R}_{\mathfrak{C}(n)}(G) &= \sum_{n\in N}\sum_{i\neq j, b_ib_j\in E}\frac{1}{\sqrt{\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{C}(n)}(b_i)\mathfrak{D}_{\mathfrak{C}(n)}(b_j)}}\\ &= \frac{a(a-1)}{2}\cdot\frac{1}{\sqrt{(v_{3(n)}v_{3(n)}(a-1)v_{3(n)}(a-1)v_{3(n)})}}\\ &= \frac{a}{2v_{3(n)}^2}\end{aligned}$$

Hence, $\mathfrak{R}(G) = \left(\frac{a}{2v_{1(n)}^2}, \frac{a}{2v_{2(n)}^2}, \frac{a}{2v_{3(n)}^2}\right)$.

Proposition 3.7. *If two neutrosophic soft graphs G and G' are isomorphic to each other, then,* $\mathfrak{R}(G) = \mathfrak{R}(G')$

Proof: *Let $G = (S, K)$ and $G' = (S', K')$ be two isomorphic neutrosophic soft graphs. Then , there is a bijection $l : G \to G'$ so that*

$$\mathfrak{A}_{S(n)}(b_i) = \mathfrak{A}_{S'(n)}(l(b_i)) = \mathfrak{A}_{S(n)}(b_i'),$$

$$\mathfrak{B}_{S(n)}(b_i) = \mathfrak{B}_{S'(n)}(l(b_i)) = \mathfrak{B}_{S(n)}(b_i'),$$

$$\mathfrak{C}_{S(n)}(b_i) = \mathfrak{C}_{S'(n)}(l(b_i)) = \mathfrak{C}_{S(n)}(b_i'),$$

$\forall\ b_i \in V$ *and*

$$\mathfrak{A}_{K(n)}(b_ib_j) = \mathfrak{A}_{K'(n)}(l(b_i)l(b_j)) = \mathfrak{A}_{K(n)}(b_i'b_j'),$$

$$\mathfrak{B}_{K(n)}(b_ib_j) = \mathfrak{B}_{K'(n)}(l(b_i)l(b_j)) = \mathfrak{B}_{K(n)}(b_i'b_j'),$$

$$\mathfrak{C}_{K(n)}(b_ib_j) = \mathfrak{C}_{K'(n)}(l(b_i)l(b_j)) = \mathfrak{C}_{K(n)}(b'_ib'_j),$$

$\forall\ b_ib_j \in E$.

So, for each $b_i \in V$ *there exists a vertex* $b'_i \in V'$ *so that* $\mathfrak{D}(b_i) = \mathfrak{D}(b'_i)$.

Thus, we have the proof.

Example 3.8. *Let* G *and* G' *be the neutrosophic soft graphs shown in Figure 2. Assume that there exists an isomorphism*

$$\ell : V(G) \to V(G')$$

such that

$$\mathfrak{A}_{S'(n)}\big(\ell(b_i)\big) = \mathfrak{A}_{S(n)}(a_i), \qquad \mathfrak{B}_{S'(n)}\big(\ell(b_i)\big) = \mathfrak{B}_{S(n)}(a_i), \qquad \mathfrak{C}_{S'(n)}\big(\ell(b_i)\big) = \mathfrak{C}_{S(n)}(a_i),$$

and

$$\mathfrak{A}_{K'(n)}\big(\ell(b_i)\ell(b_j)\big) = \mathfrak{A}_{K(n)}(a_ia_j),$$

$$\mathfrak{B}_{K'(n)}\big(\ell(b_i)\ell(b_j)\big) = \mathfrak{B}_{K(n)}(a_ia_j),$$

$$\mathfrak{C}_{K'(n)}\big(\ell(b_i)\ell(b_j)\big) = \mathfrak{C}_{K(n)}(a_ia_j),$$

for all $1 \leq i, j \leq 3$.

Then

$$\mathfrak{R}_{\mathfrak{A}(n)}(G) = 25.000 + 35.355 + 35.355 = 95.710 = \mathfrak{R}_{\mathfrak{A}(n)}(G'),$$

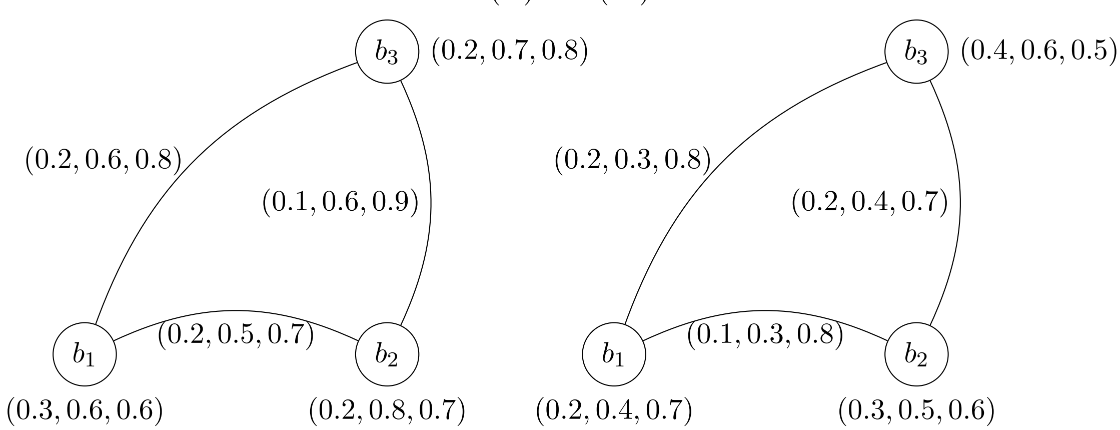
$$\mathfrak{R}_{\mathfrak{B}(n)}(G) = 1.207 + 1.304 + 1.102 = 3.613 = \mathfrak{R}_{\mathfrak{B}(n)}(G'),$$

and

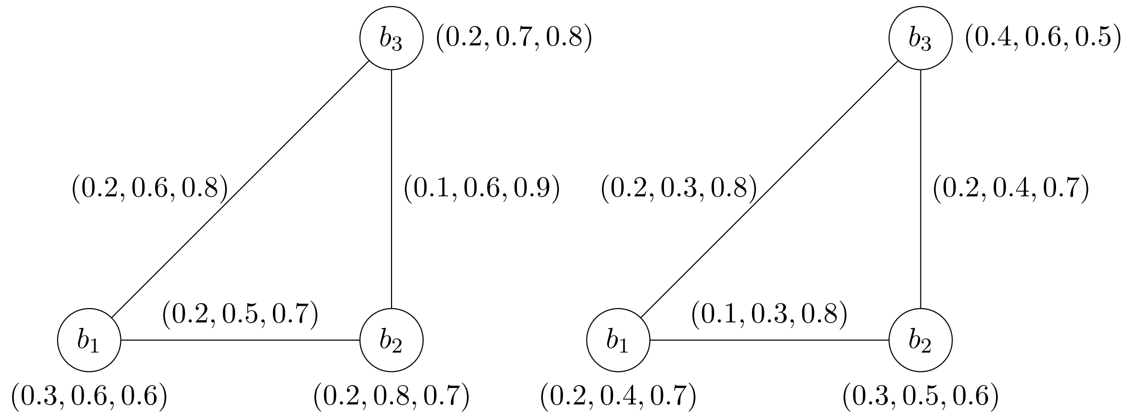
$$\mathfrak{R}_{\mathfrak{C}(n)}(G) = 1.181 + 1.056 + 1.398 = 3.635 = \mathfrak{R}_{\mathfrak{C}(n)}(G').$$

Hence,

$$\mathfrak{R}(G) = \mathfrak{R}(G').$$



$G(n_1)$ $G(n_2)$



$G'(n_1)$ $G'(n_2)$

Figure 2. *Two isomorphic neutrosophic soft graphs G and G'.*

Theorem 3.9. *Let $G = (S, K)$ be a neutrosophic soft graph, and fix $n \in N$. Assume that*

$$\mathfrak{A}_{S(n)}(b_i) > 0, \qquad \mathfrak{B}_{S(n)}(b_i) > 0, \qquad \mathfrak{C}_{S(n)}(b_i) > 0 \qquad (\forall\, b_i \in V),$$

and define

$$\mu_1 := \min_{b_i \in V} \mathfrak{D}_{\mathfrak{A}(n)}(b_i), \qquad \mu_2 := \min_{b_i \in V} \mathfrak{D}_{\mathfrak{B}(n)}(b_i), \qquad \mu_3 := \min_{b_i \in V} \mathfrak{D}_{\mathfrak{C}(n)}(b_i),$$
$$\eta_1 := \max_{b_i \in V} \mathfrak{D}_{\mathfrak{A}(n)}(b_i), \qquad \eta_2 := \max_{b_i \in V} \mathfrak{D}_{\mathfrak{B}(n)}(b_i), \qquad \eta_3 := \max_{b_i \in V} \mathfrak{D}_{\mathfrak{C}(n)}(b_i).$$

Suppose moreover that $\mu_1, \mu_2, \mu_3 > 0$. Define

$$f_1 := \sum_{b_i b_j \in E} \frac{1}{\sqrt{\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)}},$$

$$f_2 := \sum_{b_i b_j \in E} \frac{1}{\sqrt{\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)}},$$

$$f_3 := \sum_{b_i b_j \in E} \frac{1}{\sqrt{\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)}}.$$

Then

$$\frac{f_1}{\eta_1} \leq \mathfrak{R}_{\mathfrak{A}(n)}(G) \leq \frac{f_1}{\mu_1}, \qquad \frac{f_2}{\eta_2} \leq \mathfrak{R}_{\mathfrak{B}(n)}(G) \leq \frac{f_2}{\mu_2}, \qquad \frac{f_3}{\eta_3} \leq \mathfrak{R}_{\mathfrak{C}(n)}(G) \leq \frac{f_3}{\mu_3}.$$

Proof. For each $b_i \in V$, the $\mathfrak{A}$-, $\mathfrak{B}$-, and $\mathfrak{C}$-degrees are

$$\mathfrak{D}_{\mathfrak{A}(n)}(b_i) = \sum_{b_i b_j \in E} \mathfrak{A}_{K(n)}(b_i b_j), \qquad \mathfrak{D}_{\mathfrak{B}(n)}(b_i) = \sum_{b_i b_j \in E} \mathfrak{B}_{K(n)}(b_i b_j), \qquad \mathfrak{D}_{\mathfrak{C}(n)}(b_i) = \sum_{b_i b_j \in E} \mathfrak{C}_{K(n)}(b_i b_j).$$

By the definitions of μ_r and η_r, we have

$$\mu_1 \leq \mathfrak{D}_{\mathfrak{A}(n)}(b_i) \leq \eta_1, \qquad \mu_2 \leq \mathfrak{D}_{\mathfrak{B}(n)}(b_i) \leq \eta_2, \qquad \mu_3 \leq \mathfrak{D}_{\mathfrak{C}(n)}(b_i) \leq \eta_3$$

for all $b_i \in V$. Hence, for every edge $b_ib_j \in E$,

$$\mu_1^2 \leq \mathfrak{D}_{\mathfrak{A}(n)}(b_i)\mathfrak{D}_{\mathfrak{A}(n)}(b_j) \leq \eta_1^2,$$

$$\mu_2^2 \leq \mathfrak{D}_{\mathfrak{B}(n)}(b_i)\mathfrak{D}_{\mathfrak{B}(n)}(b_j) \leq \eta_2^2,$$

$$\mu_3^2 \leq \mathfrak{D}_{\mathfrak{C}(n)}(b_i)\mathfrak{D}_{\mathfrak{C}(n)}(b_j) \leq \eta_3^2.$$

Multiplying by the positive quantities $\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)$, $\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)$, and $\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)$, respectively, yields

$$\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)\mu_1^2 \leq \mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{A}(n)}(b_i)\mathfrak{D}_{\mathfrak{A}(n)}(b_j) \leq \mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)\eta_1^2,$$

$$\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)\mu_2^2 \leq \mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{B}(n)}(b_i)\mathfrak{D}_{\mathfrak{B}(n)}(b_j) \leq \mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)\eta_2^2,$$

$$\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)\mu_3^2 \leq \mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{C}(n)}(b_i)\mathfrak{D}_{\mathfrak{C}(n)}(b_j) \leq \mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)\eta_3^2.$$

Taking reciprocals and square roots reverses the inequalities, so for every edge $b_ib_j \in E$,

$$\frac{1}{\eta_1\sqrt{\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)}} \leq \frac{1}{\sqrt{\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{A}(n)}(b_i)\mathfrak{D}_{\mathfrak{A}(n)}(b_j)}} \leq \frac{1}{\mu_1\sqrt{\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)}},$$

$$\frac{1}{\eta_2\sqrt{\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)}} \leq \frac{1}{\sqrt{\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{B}(n)}(b_i)\mathfrak{D}_{\mathfrak{B}(n)}(b_j)}} \leq \frac{1}{\mu_2\sqrt{\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)}},$$

$$\frac{1}{\eta_3\sqrt{\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)}} \leq \frac{1}{\sqrt{\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{C}(n)}(b_i)\mathfrak{D}_{\mathfrak{C}(n)}(b_j)}} \leq \frac{1}{\mu_3\sqrt{\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)}}.$$

Summing over all edges gives the desired bounds. □

Theorem 3.10. *Let $G = (S, K)$ be a neutrosophic soft graph with $|V| = a$, and fix $n \in N$. Assume that*

$$\mathfrak{A}_{S(n)}(b_i) > 0, \qquad \mathfrak{B}_{S(n)}(b_i) > 0, \qquad \mathfrak{C}_{S(n)}(b_i) > 0 \qquad (\forall\, b_i \in V).$$

Then

$$\mathfrak{R}_{\mathfrak{A}(n)}(G) \geq \frac{1}{a-1}\sum_{b_ib_j\in E}\frac{1}{\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)},$$

$$\mathfrak{R}_{\mathfrak{B}(n)}(G) \geq \frac{1}{a-1}\sum_{b_ib_j\in E}\frac{1}{\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)},$$

$$\mathfrak{R}_{\mathfrak{C}(n)}(G) \geq \frac{1}{a-1}\sum_{b_ib_j\in E}\frac{1}{\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)}.$$

Proof. Since $|V| = a$, each vertex is incident with at most $a-1$ edges. Because G is a neutrosophic soft graph, for every edge $b_ib_j \in E$ we have

$$\mathfrak{A}_{K(n)}(b_ib_j) \le \min\{\mathfrak{A}_{S(n)}(b_i), \mathfrak{A}_{S(n)}(b_j)\},$$

$$\mathfrak{B}_{K(n)}(b_ib_j) \le \min\{\mathfrak{B}_{S(n)}(b_i), \mathfrak{B}_{S(n)}(b_j)\},$$

$$\mathfrak{C}_{K(n)}(b_ib_j) \le \min\{\mathfrak{C}_{S(n)}(b_i), \mathfrak{C}_{S(n)}(b_j)\}.$$

Hence, for each $b_i \in V$,

$$\mathfrak{D}_{\mathfrak{A}(n)}(b_i) = \sum_{b_ib_j \in E} \mathfrak{A}_{K(n)}(b_ib_j) \le (a-1)\mathfrak{A}_{S(n)}(b_i),$$

$$\mathfrak{D}_{\mathfrak{B}(n)}(b_i) = \sum_{b_ib_j \in E} \mathfrak{B}_{K(n)}(b_ib_j) \le (a-1)\mathfrak{B}_{S(n)}(b_i),$$

$$\mathfrak{D}_{\mathfrak{C}(n)}(b_i) = \sum_{b_ib_j \in E} \mathfrak{C}_{K(n)}(b_ib_j) \le (a-1)\mathfrak{C}_{S(n)}(b_i).$$

Therefore, for every edge $b_ib_j \in E$,

$$\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{A}(n)}(b_i)\mathfrak{D}_{\mathfrak{A}(n)}(b_j) \le (a-1)^2\mathfrak{A}_{S(n)}(b_i)^2\mathfrak{A}_{S(n)}(b_j)^2,$$

$$\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{B}(n)}(b_i)\mathfrak{D}_{\mathfrak{B}(n)}(b_j) \le (a-1)^2\mathfrak{B}_{S(n)}(b_i)^2\mathfrak{B}_{S(n)}(b_j)^2,$$

$$\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{C}(n)}(b_i)\mathfrak{D}_{\mathfrak{C}(n)}(b_j) \le (a-1)^2\mathfrak{C}_{S(n)}(b_i)^2\mathfrak{C}_{S(n)}(b_j)^2.$$

Taking reciprocals and square roots, we obtain

$$\frac{1}{\sqrt{\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{A}(n)}(b_i)\mathfrak{D}_{\mathfrak{A}(n)}(b_j)}} \ge \frac{1}{(a-1)\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)},$$

$$\frac{1}{\sqrt{\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{B}(n)}(b_i)\mathfrak{D}_{\mathfrak{B}(n)}(b_j)}} \ge \frac{1}{(a-1)\mathfrak{B}_{S(n)}(b_i)\mathfrak{B}_{S(n)}(b_j)},$$

$$\frac{1}{\sqrt{\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{C}(n)}(b_i)\mathfrak{D}_{\mathfrak{C}(n)}(b_j)}} \ge \frac{1}{(a-1)\mathfrak{C}_{S(n)}(b_i)\mathfrak{C}_{S(n)}(b_j)}.$$

Summing over all edges completes the proof. □

Theorem 3.11. *Let $G = (S, K)$ be a complete, regular, and totally regular neutrosophic soft graph on $|V| = a$ vertices, and fix $n \in N$. Assume that there exist positive constants*

$$x_{1(n)},\ x_{2(n)},\ x_{3(n)},\ h_{1(n)},\ h_{2(n)},\ h_{3(n)}$$

such that, for every $b_i \in V$,

$$\mathfrak{D}_{\mathfrak{A}(n)}(b_i) = x_{1(n)}, \qquad \mathfrak{D}_{\mathfrak{B}(n)}(b_i) = x_{2(n)}, \qquad \mathfrak{D}_{\mathfrak{C}(n)}(b_i) = x_{3(n)},$$

and

$$t\mathfrak{D}_{\mathfrak{A}(n)}(b_i) = h_{1(n)}, \qquad t\mathfrak{D}_{\mathfrak{B}(n)}(b_i) = h_{2(n)}, \qquad t\mathfrak{D}_{\mathfrak{C}(n)}(b_i) = h_{3(n)}.$$

Set

$$v_{1(n)} := h_{1(n)} - x_{1(n)}, \qquad v_{2(n)} := h_{2(n)} - x_{2(n)}, \qquad v_{3(n)} := h_{3(n)} - x_{3(n)}.$$

Then

$$\mathfrak{R}(G) = \big(\mathfrak{R}_{\mathfrak{A}(n)}(G), \mathfrak{R}_{\mathfrak{B}(n)}(G), \mathfrak{R}_{\mathfrak{C}(n)}(G)\big) = \binom{a}{2}\left(\frac{1}{v_{1(n)}x_{1(n)}}, \frac{1}{v_{2(n)}x_{2(n)}}, \frac{1}{v_{3(n)}x_{3(n)}}\right).$$

Proof. Since G is regular, the three degree functions are constant on V, namely,

$$\mathfrak{D}_{\mathfrak{A}(n)}(b_i) = x_{1(n)}, \qquad \mathfrak{D}_{\mathfrak{B}(n)}(b_i) = x_{2(n)}, \qquad \mathfrak{D}_{\mathfrak{C}(n)}(b_i) = x_{3(n)} \qquad (\forall\, b_i \in V).$$

Since G is totally regular, the total degrees are also constant:

$$t\mathfrak{D}_{\mathfrak{A}(n)}(b_i) = h_{1(n)}, \qquad t\mathfrak{D}_{\mathfrak{B}(n)}(b_i) = h_{2(n)}, \qquad t\mathfrak{D}_{\mathfrak{C}(n)}(b_i) = h_{3(n)} \qquad (\forall\, b_i \in V).$$

By definition of total degree,

$$t\mathfrak{D}_{\mathfrak{A}(n)}(b_i) = \mathfrak{D}_{\mathfrak{A}(n)}(b_i) + \mathfrak{A}_{S(n)}(b_i),$$

$$t\mathfrak{D}_{\mathfrak{B}(n)}(b_i) = \mathfrak{D}_{\mathfrak{B}(n)}(b_i) + \mathfrak{B}_{S(n)}(b_i),$$

$$t\mathfrak{D}_{\mathfrak{C}(n)}(b_i) = \mathfrak{D}_{\mathfrak{C}(n)}(b_i) + \mathfrak{C}_{S(n)}(b_i).$$

Therefore,

$$\mathfrak{A}_{S(n)}(b_i) = h_{1(n)} - x_{1(n)} = v_{1(n)},$$

$$\mathfrak{B}_{S(n)}(b_i) = h_{2(n)} - x_{2(n)} = v_{2(n)},$$

$$\mathfrak{C}_{S(n)}(b_i) = h_{3(n)} - x_{3(n)} = v_{3(n)}$$

for all $b_i \in V$. Thus the vertex-membership functions are constant.

Because G is complete on a vertices, the number of edges is

$$|E| = \binom{a}{2} = \frac{a(a-1)}{2}.$$

Moreover, every edge $b_ib_j \in E$ contributes the same quantity to each component of the Randic index. Hence

$$\begin{aligned}
\mathfrak{R}_{\mathfrak{A}(n)}(G) &= \sum_{b_ib_j \in E} \frac{1}{\sqrt{\mathfrak{A}_{S(n)}(b_i)\mathfrak{A}_{S(n)}(b_j)\mathfrak{D}_{\mathfrak{A}(n)}(b_i)\mathfrak{D}_{\mathfrak{A}(n)}(b_j)}} \\
&= \binom{a}{2}\frac{1}{\sqrt{v_{1(n)}v_{1(n)}x_{1(n)}x_{1(n)}}} = \binom{a}{2}\frac{1}{v_{1(n)}x_{1(n)}},
\end{aligned}$$

and similarly,

$$\mathfrak{R}_{\mathfrak{B}(n)}(G) = \binom{a}{2}\frac{1}{v_{2(n)}x_{2(n)}}, \qquad \mathfrak{R}_{\mathfrak{C}(n)}(G) = \binom{a}{2}\frac{1}{v_{3(n)}x_{3(n)}}.$$

Combining these equalities gives the result. □

4. **Application of the Randic Index in Construction**

A government organization plans to build a waste processing facility in a region that contains a number of medium-sized cities. The ideal site should minimize logistical and environmental issues, have existing infrastructure (partial facilities, land, staff), be easily accessible to nearby cities for effective waste transportation, and have administrative and public support.

The example shows how to use the Randic Index and a neutrosophic soft graph to choose the ideal city for a waste management plant. The selection process takes into account a variety of parameters, including public support, infrastructure availability, road connection and environmental concerns, which are represented as nodes (cities) and edges (roads) with neutrosophic weights signifying a truth, indeterminacy and falsity-membership function. Based on their population statistics, cities like C_1, C_2, C_3, and C_4 are evaluated using imprecise truth, indeterminacy, and falsity-membership ratings. The Randic Index, a graph-theoretic metric, is used to determine node centrality and significance.

Table 1: Weight of vertices in G

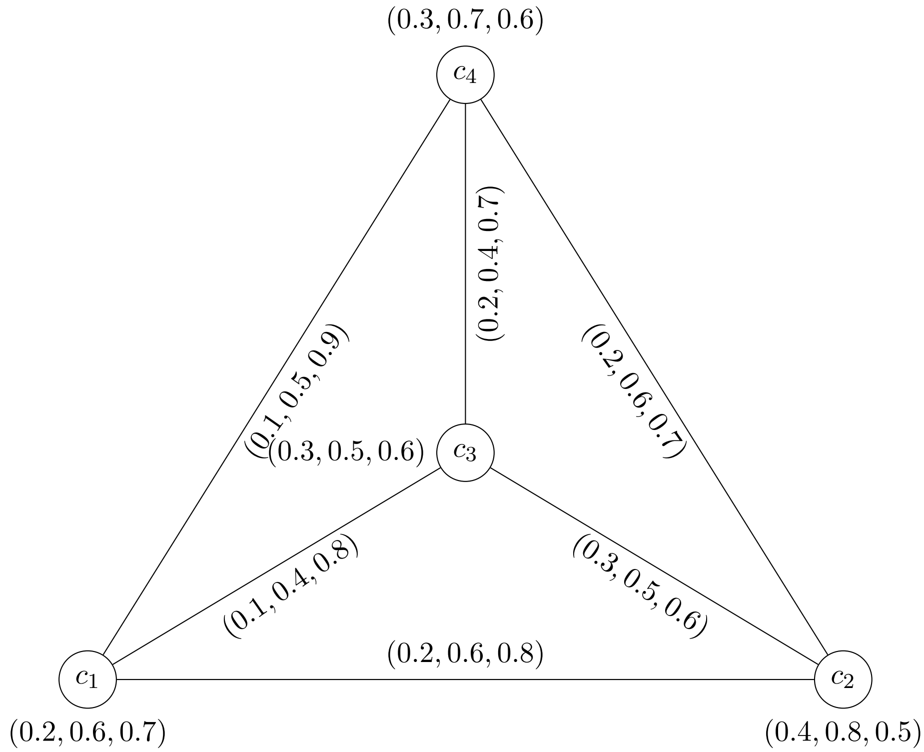
G	City 1	City 2	City 3	City 4
$(\mathfrak{A}_{S(1)}, \mathfrak{B}_{S(1)}, \mathfrak{C}_{S(1)})$ Road connection	(0.2,0.6,0.7)	(0.4,0.8,0.5)	(0.3,0.5,0.6)	(0.3,0.7,0.6)
$(\mathfrak{A}_{S(2)}, \mathfrak{B}_{S(2)}, \mathfrak{C}_{S(2)})$ Environment	(0.3,0.4,0.6)	(0.3,0.5,0.7)	(0.4,0.4,0.6)	(0.2,0.3,0.8)

Table 2: Weight of edges in G

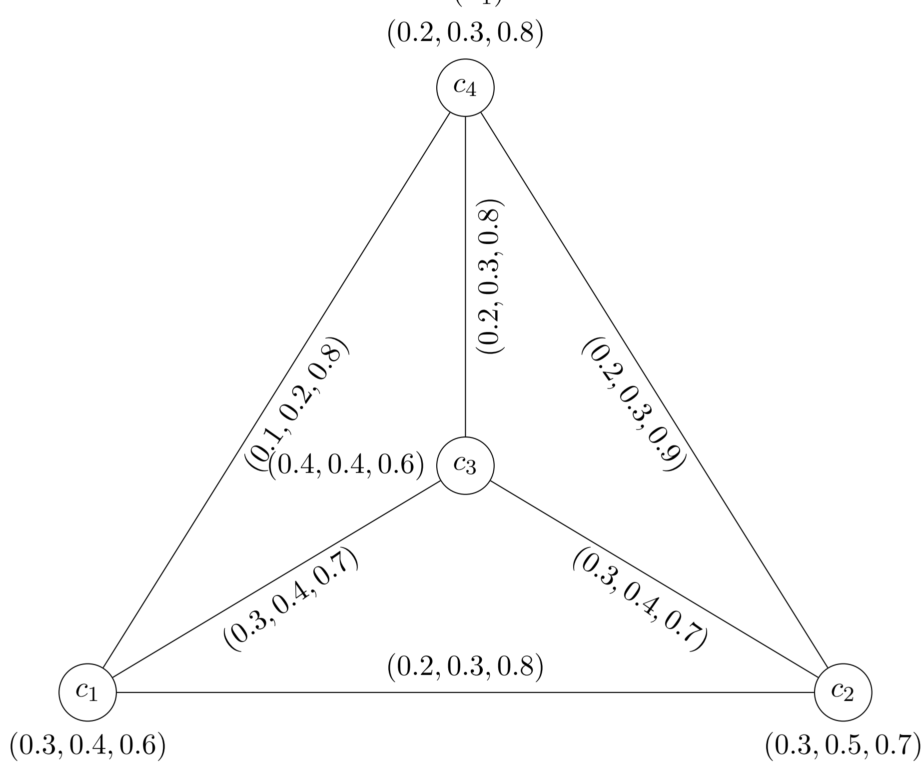
G	$C_1 - C_2$	$C_2 - C_3$	$C_3 - C_4$	$C_1 - C_4$	$C_1 - C_3$	$C_2 - C_4$
$(\mathfrak{A}_{K(1)}, \mathfrak{B}_{K(1)}, \mathfrak{C}_{K(1)})$	(0.2,0.6,0.8)	(0.3,0.5,0.6)	(0.2,0.4,0.7)	(0.1,0.5,0.9)	(0.1,0.4,0.8)	(0.2,0.6,0.7)
$(\mathfrak{A}_{K(2)}, \mathfrak{B}_{K(2)}, \mathfrak{C}_{K(2)})$	(0.2,0.3,0.8)	(0.3,0.4,0.7)	(0.2,0.3,0.8)	(0.1,0.2,0.8)	(0.3,0.4,0.7)	(0.2,0.3,0.9)

Table 3: The population of cities

G	City 1	City 2	City 3	City 4
Population	172567	92566	132890	159463



$\mathbf{G}(n_1)$



$\mathbf{G}(n_2)$

Figure 3: Neutrosophic soft graph G

Let

$$\mathfrak{D}_{n_1}(c_1) = (0.4, 1.5, 2.5), \quad \mathfrak{D}_{n_1}(c_2) = (0.7, 1.7, 2.1),$$

$$\mathfrak{D}_{n_1}(c_3) = (0.6, 1.3, 2.1), \quad \mathfrak{D}_{n_1}(c_4) = (0.5, 1.8, 2.2),$$

and

$$\mathfrak{D}_{n_2}(c_1) = (0.6, 0.9, 2.3), \quad \mathfrak{D}_{n_2}(c_2) = (0.7, 1.0, 2.4),$$

$$\mathfrak{D}_{n_2}(c_3) = (0.8, 1.1, 2.2), \quad \mathfrak{D}_{n_2}(c_4) = (0.5, 0.8, 2.5).$$

Then

$$\begin{aligned}\sum_{n\in N} \sum_{\substack{i\neq j\\ c_ic_j\in E}} \frac{1}{\sqrt{\mathfrak{A}_{S(n)}(c_i)\mathfrak{A}_{S(n)}(c_j)\mathfrak{D}_{\mathfrak{A}(n)}(c_i)\mathfrak{D}_{\mathfrak{A}(n)}(c_j)}} &= (6.68 + 6.30 + 6.09 + 9.13 + 8.33 + 4.88)\\ &\quad + (5.14 + 3.86 + 5.59 + 7.45 + 4.17 + 6.90)\\ &= 74.52,\end{aligned}$$

$$\begin{aligned}\sum_{n\in N} \sum_{\substack{i\neq j\\ c_ic_j\in E}} \frac{1}{\sqrt{\mathfrak{B}_{S(n)}(c_i)\mathfrak{B}_{S(n)}(c_j)\mathfrak{D}_{\mathfrak{B}(n)}(c_i)\mathfrak{D}_{\mathfrak{B}(n)}(c_j)}} &= (0.90 + 1.06 + 1.11 + 0.94 + 1.31 + 0.76)\\ &\quad + (2.36 + 2.13 + 3.08 + 3.40 + 2.51 + 2.89)\\ &= 22.45,\end{aligned}$$

and

$$\begin{aligned}\sum_{n\in N} \sum_{\substack{i\neq j\\ c_ic_j\in E}} \frac{1}{\sqrt{\mathfrak{C}_{S(n)}(c_i)\mathfrak{C}_{S(n)}(c_j)\mathfrak{D}_{\mathfrak{C}(n)}(c_i)\mathfrak{D}_{\mathfrak{C}(n)}(c_j)}} &= (0.74 + 0.87 + 0.78 + 0.66 + 0.73 + 0.85)\\ &\quad + (0.66 + 0.67 + 0.62 + 0.60 + 0.74 + 0.55)\\ &= 8.47.\end{aligned}$$

Hence,

$$\mathfrak{R}(G) = \big(\mathfrak{R}_{\mathfrak{A}(n)}(G), \mathfrak{R}_{\mathfrak{B}(n)}(G), \mathfrak{R}_{\mathfrak{C}(n)}(G)\big) = (74.52,\ 22.45,\ 8.47).$$

Further,

$$\mathfrak{R}_{\mathfrak{A}(n)}(G - c_1) = 74.52 - 5.14 - 7.45 - 4.17 - 6.68 - 9.13 - 8.33 = 33.62,$$

$$\mathfrak{R}_{\mathfrak{A}(n)}(G - c_2) = 74.52 - 6.68 - 6.30 - 4.88 - 5.14 - 3.86 - 6.90 = 40.76,$$

$$\mathfrak{R}_{\mathfrak{A}(n)}(G - c_3) = 74.52 - 6.30 - 6.09 - 8.33 - 3.86 - 5.59 - 4.17 = 40.18,$$

$$\mathfrak{R}_{\mathfrak{A}(n)}(G - c_4) = 74.52 - 6.09 - 9.13 - 4.88 - 5.59 - 7.45 - 6.90 = 34.48.$$

$$\Re_{\mathfrak{B}(n)}(G - c_1) = 22.45 - 0.90 - 0.94 - 1.31 - 2.36 - 3.40 - 2.51 = 11.03,$$
$$\Re_{\mathfrak{B}(n)}(G - c_2) = 22.45 - 0.90 - 1.06 - 0.76 - 2.36 - 2.13 - 2.89 = 12.35,$$
$$\Re_{\mathfrak{B}(n)}(G - c_3) = 22.45 - 1.06 - 1.11 - 1.31 - 2.13 - 3.08 - 2.51 = 11.25,$$
$$\Re_{\mathfrak{B}(n)}(G - c_4) = 22.45 - 1.11 - 0.94 - 0.76 - 3.08 - 3.40 - 2.89 = 10.27.$$

$$\Re_{\mathfrak{C}(n)}(G - c_1) = 8.47 - 0.74 - 0.66 - 0.73 - 0.66 - 0.60 - 0.74 = 4.34,$$
$$\Re_{\mathfrak{C}(n)}(G - c_2) = 8.47 - 0.74 - 0.87 - 0.85 - 0.66 - 0.67 - 0.55 = 4.13,$$
$$\Re_{\mathfrak{C}(n)}(G - c_3) = 8.47 - 0.87 - 0.78 - 0.73 - 0.67 - 0.62 - 0.74 = 4.06,$$
$$\Re_{\mathfrak{C}(n)}(G - c_4) = 8.47 - 0.78 - 0.66 - 0.85 - 0.62 - 0.60 - 0.55 = 4.41.$$

The calculation above indicates that avoiding nodes C_1, C_2, C_3, and C_4 is inappropriate for building a waste processing facility. City C_1 has the largest population and the greatest number of transportation routes, making it the most crucial and vital factor in maintaining a high Randic Index. City C_1 is the ideal location for the waste management facility.

5. **Conclusion**

Fuzzy graphs are less accurate, less adaptable, and less compatible than neutrosophic soft graphs. In social networks, neutrosophic soft graphs are frequently used to find the most productive members of a team or organization. The Connectivity Index has numerous uses in psychology, medical science, social groupings, and computer networks. This work presents Randic indices for neutrosophic soft graphs and subgraphs, along with their properties. Using certain isomorphic properties, a truth, indeterminacy, and falsity-membership function of the Randic Index of neutrosophic soft graphs is examined. Eccentricity index and Wiener index for neutrosophic incidence graphs are two things we intend to investigate further. It is also hoped that further research will advance extensions based on HyperGraphs [14, 15] and SuperHyperGraphs [16, 17, 26, 27].

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Use of Generative AI and AI-Assisted Tools

Generative AI and AI-assisted tools were used for tasks such as English grammar checking, and they were not used in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany this paper.

Disclaimer

The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

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A Proposed Mathematical Framework for Uncertain IT Service Management and Agile IT Service Management

Takaaki Fujita[1] *

[1] Independent Researcher, Tokyo, Japan. Takaaki.fujita060@gmail.com

*Correspondence: Takaaki.fujita060@gmail.com

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Abstract. Uncertainty is an essential feature of modern IT Service Management, because service quality, reliability, utility, process effectiveness, infrastructure status, and operational decisions are often evaluated under incomplete, imprecise, or indeterminate information. Motivated by this observation, this paper proposes a mathematical framework for Uncertain IT Service Management based on the notion of uncertain sets. After recalling basic uncertainty models, including fuzzy sets and neutrosophic sets, we introduce an M-Uncertain IT Service Management system, where M is a fixed uncertain model with degree domain $\mathrm{Dom}(M)$. The proposed framework represents IT services, infrastructure components, management processes, allocation relations, uncertainty-degree maps, quality, utility, reliability, aggregation schemes, and global evaluation functions in a unified mathematical form.

We prove that the overall uncertainty map induced by an M-Uncertain IT Service Management system is well-defined. Consequently, every such system naturally determines an uncertain set of type M on the set of IT services. We also show that fuzzy IT Service Management and neutrosophic IT Service Management are obtained as special cases by selecting the corresponding uncertainty model. Furthermore, we formulate Agile IT Service Management and Uncertain Agile IT Service Management in order to describe iterative service improvement through agile cycles, backlog items, and iteration-dependent evaluations. The proposed framework therefore provides a model-independent basis for studying IT Service Management and Agile IT Service Management under various forms of uncertainty, including fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and related settings.

Keywords: Uncertain set, fuzzy set, neutrosophic set, IT Service Management, Agile IT Service Management, uncertainty modeling, well-definedness.

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1. Preliminaries

This section introduces the basic notions and definitions used throughout this paper.

1.1. *Fuzzy Sets and Neutrosophic Sets*

A fuzzy set assigns to each element a membership degree in the interval $[0, 1]$. In this way, it represents uncertainty by allowing gradual membership rather than only binary membership or non-membership [1]. Related extensions, such as hyperfuzzy sets [2,3], hesitant fuzzy sets [4,5], bipolar fuzzy sets [6], and picture fuzzy sets [7,8], are also widely studied. We recall below the basic definitions needed for the present discussion.

Definition 1.1 (Fuzzy Set). *[1] Let Y be a nonempty universe. A* fuzzy set *τ on Y is a mapping*

$$\tau : Y \to [0, 1].$$

A fuzzy relation *on Y is a fuzzy subset δ of $Y \times Y$. If τ is a fuzzy set on Y and δ is a fuzzy relation on Y, then δ is called a* fuzzy relation on τ *if*

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Neutrosophic sets extend fuzzy sets by explicitly incorporating indeterminacy. This allows them to describe situations that cannot be adequately represented as simply true or false, or even as partially true in the usual fuzzy sense. Hence, neutrosophic sets provide a flexible framework for modeling uncertainty, ambiguity, and conflicting information [9–11]. Related concepts, such as neutrosophic graphs [12–15] are also known. Moreover, further extensions, including plithogenic sets [16–19], have also been studied. The basic definition is recalled as follows.

Definition 1.2 (Neutrosophic Set). *[10] Let X be a nonempty set. A* neutrosophic set *A on X is characterized by three membership functions*

$$T_A : X \to [0, 1], \qquad I_A : X \to [0, 1], \qquad F_A : X \to [0, 1],$$

where, for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

2. Uncertain Sets

An *uncertain set* assigns to each element a degree taken from a specified uncertainty model. This notion provides a unified way to treat fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and related set-theoretic frameworks [20]. We first recall the notion of an uncertain model, which specifies the admissible domain of uncertainty degrees.

Definition 2.1 (Uncertain Model). *[20] Let U denote the class of all uncertain models. Each $M \in U$ is specified by the following data:*

- *a nonempty set*

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

  of admissible degree tuples, for some integer $k \geq 1$;
- *model-specific algebraic or geometric constraints imposed on the elements of* $\mathrm{Dom}(M)$*;*
- *when M is used for graph structures, model-specific admissibility relations between vertex degrees and edge degrees.*

Typical examples include the following:

- ***Fuzzy model:***

$$\mathrm{Dom}(M) = [0,1].$$

- ***Intuitionistic fuzzy model:***

$$\mathrm{Dom}(M) = \{(\mu, \nu) \in [0,1]^2 \mid \mu + \nu \leq 1\}.$$

- ***Neutrosophic model:***

$$\mathrm{Dom}(M) = \{(T, I, F) \in [0,1]^3 \mid 0 \leq T + I + F \leq 3\}.$$

- ***Plithogenic model*** *and other related extensions.*

Definition 2.2 (Uncertain Set). *[20] Let X be a nonempty universe, and let M be a fixed uncertain model with degree domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

An uncertain set of type M*, or simply a* U-set*, on X is a pair*

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is called the uncertainty-degree function*, or the* membership map*, of $\mathcal{U}$.*

For each $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ represents the degree, or tuple of degrees, to which x belongs to the uncertain set according to the model M.

3. Results: Uncertain IT Service Management

IT Service Management systematically designs, delivers, operates, and improves IT services so that technology resources support organizational goals, business needs, users, and service quality [21–26]. In this section, we introduce a framework for Uncertain IT Service Management based on the notion of an uncertain set. The purpose of this framework is to provide a unified mathematical formulation that includes fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and other uncertainty-based IT management models as special cases.

Definition 3.1 (M-Admissible Aggregation Scheme)**.** *Let M be a fixed uncertain model, and let*

$$D_M = \mathrm{Dom}(M)$$

*be its degree domain. An M-*admissible aggregation scheme *on a finite set X is a family of maps*

$$\mathfrak{A}^X = \left\{ \mathfrak{A}^X_A : D_M^A \longrightarrow D_M \mid A \subseteq X \right\},$$

where

$$D_M^A = \{ f \mid f : A \to D_M \}.$$

Thus, for each subset $A \subseteq X$, the map $\mathfrak{A}^X_A$ aggregates an A-indexed family of uncertainty degrees into a single admissible uncertainty degree in D_M.
When $A = \emptyset$, the set $D_M^{\emptyset}$ consists of the unique empty function. In this case, $\mathfrak{A}^X_{\emptyset}$ assigns to it a prescribed default degree in D_M.

Definition 3.2 (Uncertain IT Service Management)**.** *Let M be a fixed uncertain model with degree domain*

$$D_M = \mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. An Uncertain IT Service Management system of type M*, briefly called an M-*UITSM system*, is a tuple*

$$\mathcal{UITSM}_M = \left(S, I, P, \alpha, \mu_S, \mu_I, \mu_P, Q_M, U_M, R_M, \mathfrak{A}^I, \mathfrak{A}^P, \Theta, C \right),$$

satisfying the following conditions.

(1) S is a finite nonempty set of IT services.
(2) I is a finite set of IT infrastructure components, such as servers, databases, storage devices, network devices, cloud resources, and security components.
(3) P is a finite set of IT management processes, such as Incident Management, Problem Management, Change Management, Release Management, Service Level Management, and Availability Management.

(4)

$$\alpha : S \longrightarrow 2^I \times 2^P$$

is an allocation map. If

$$\alpha(s) = (I_s, P_s),$$

then $I_s \subseteq I$ *is the set of infrastructure components supporting the service* s*, and* $P_s \subseteq P$ *is the set of management processes associated with* s*.*

(5)

$$\mu_S : S \longrightarrow D_M$$

is the service-level uncertainty map. The value $\mu_S(s)$ *represents the uncertain managerial status of the service* s*.*

(6)

$$\mu_I : I \longrightarrow D_M$$

is the infrastructure-level uncertainty map. The value $\mu_I(i)$ *represents the uncertain operational status, reliability, or adequacy of the infrastructure component* i*.*

(7)

$$\mu_P : P \longrightarrow D_M$$

is the process-level uncertainty map. The value $\mu_P(p)$ *represents the uncertain maturity, compliance, or effectiveness of the management process* p*.*

(8)

$$Q_M : S \longrightarrow D_M$$

is the uncertain quality map. The value $Q_M(s)$ *represents the uncertain quality assessment of the service* s*.*

(9)

$$U_M : S \longrightarrow D_M$$

is the uncertain utility map. The value $U_M(s)$ *represents the uncertain utility, business value, or economic evaluation of the service* s*.*

(10)

$$R_M : S \longrightarrow D_M$$

is the uncertain reliability map. The value $R_M(s)$ *represents the uncertain reliability or availability assessment of the service* s*.*

(11) $\mathfrak{A}^I$ *is an* M*-admissible aggregation scheme on* I*. It aggregates the uncertainty degrees of infrastructure components.*

(12) $\mathfrak{A}^P$ *is an* M*-admissible aggregation scheme on* P*. It aggregates the uncertainty degrees of management processes.*

(13)

$$\Theta : D_M^6 \longrightarrow D_M$$

is an M-admissible global evaluation function. It combines the service-level uncertainty, uncertain quality, uncertain utility, uncertain reliability, aggregated infrastructure uncertainty, and aggregated process uncertainty into a single final uncertainty degree.

(14) C is a collection of crisp cost parameters. For example, one may take

$$C = (c_I, c_P, B),$$

where

$$c_I : I \to \mathbb{R}_{\geq 0}, \qquad c_P : P \to \mathbb{R}_{\geq 0}, \qquad B \in \mathbb{R}_{\geq 0}.$$

Here, $c_I(i)$ is the operating cost of the infrastructure component i, $c_P(p)$ is the operating or administrative cost of the management process p, and B is a budget bound.

For each service $s \in S$, where $\alpha(s) = (I_s, P_s)$, define

$$\overline{\mu}_I(s) = \mathfrak{A}^I_{I_s}\big(\mu_I|_{I_s}\big)$$

and

$$\overline{\mu}_P(s) = \mathfrak{A}^P_{P_s}\big(\mu_P|_{P_s}\big).$$

The overall uncertain ITSM degree *of s is then defined by*

$$\mu_{\mathcal{UITSM}_M}(s) = \Theta\Big(\mu_S(s), Q_M(s), U_M(s), R_M(s), \overline{\mu}_I(s), \overline{\mu}_P(s)\Big).$$

The pair

$$\mathcal{U}_{\mathcal{UITSM}_M} = \big(S, \mu_{\mathcal{UITSM}_M}\big)$$

is called the uncertain set induced by the M-UITSM system*.*

Theorem 3.3 (Well-Definedness of Uncertain IT Service Management)**.** *Let*

$$\mathcal{UITSM}_M = \big(S, I, P, \alpha, \mu_S, \mu_I, \mu_P, Q_M, U_M, R_M, \mathfrak{A}^I, \mathfrak{A}^P, \Theta, C\big)$$

be an M-UITSM system. Then the map

$$\mu_{\mathcal{UITSM}_M} : S \longrightarrow D_M$$

is well-defined. Consequently,

$$\mathcal{U}_{\mathcal{UITSM}_M} = \big(S, \mu_{\mathcal{UITSM}_M}\big)$$

is an uncertain set of type M on S.

Proof. Let $s \in S$ be arbitrary. Since

$$\alpha : S \to 2^I \times 2^P$$

is a function, there exists a unique pair

$$\alpha(s) = (I_s, P_s)$$

such that $I_s \subseteq I$ and $P_s \subseteq P$.
Since

$$\mu_I : I \to D_M$$

is a function, its restriction

$$\mu_I|_{I_s} : I_s \to D_M$$

is also a function. Hence,

$$\mu_I|_{I_s} \in D_M^{I_s}.$$

Because $\mathfrak{A}^I$ is an M-admissible aggregation scheme on I, the map

$$\mathfrak{A}^I_{I_s} : D_M^{I_s} \to D_M$$

is defined. Therefore,

$$\overline{\mu}_I(s) = \mathfrak{A}^I_{I_s}\big(\mu_I|_{I_s}\big)$$

is a uniquely determined element of D_M.
Similarly, since

$$\mu_P : P \to D_M$$

is a function, its restriction

$$\mu_P|_{P_s} : P_s \to D_M$$

is an element of $D_M^{P_s}$. Since

$$\mathfrak{A}^P_{P_s} : D_M^{P_s} \to D_M$$

is defined, the value

$$\overline{\mu}_P(s) = \mathfrak{A}^P_{P_s}\big(\mu_P|_{P_s}\big)$$

is a uniquely determined element of D_M.
Moreover, by the definition of an M-UITSM system, all of the following values belong to D_M:

$$\mu_S(s), \quad Q_M(s), \quad U_M(s), \quad R_M(s), \quad \overline{\mu}_I(s), \quad \overline{\mu}_P(s).$$

Since

$$\Theta : D_M^6 \to D_M$$

is a function, the value

$$\Theta\Big(\mu_S(s), Q_M(s), U_M(s), R_M(s), \overline{\mu}_I(s), \overline{\mu}_P(s)\Big)$$

is uniquely determined and belongs to D_M. Hence,

$$\mu_{\mathcal{UITSM}_M}(s) \in D_M$$

is well-defined for every $s \in S$.

Therefore,

$$\mu_{\mathcal{UITSM}_M} : S \to D_M$$

is a well-defined function. Since $D_M = \mathrm{Dom}(M)$, the pair

$$(S, \mu_{\mathcal{UITSM}_M})$$

satisfies the definition of an uncertain set of type M on S. Thus,

$$\mathcal{U}_{\mathcal{UITSM}_M} = (S, \mu_{\mathcal{UITSM}_M})$$

is an uncertain set of type M. □

Proposition 3.4 (Fuzzy and Neutrosophic ITSM as Special Cases)**.** *The M-UITSM framework contains both Fuzzy IT Service Management and Neutrosophic IT Service Management as special cases.*

More precisely, the following statements hold:

(1) If

$$D_M = [0, 1],$$

then an M-UITSM system becomes a fuzzy ITSM framework.

(2) If

$$D_M = \{(T, I, F) \in [0, 1]^3 \mid 0 \le T + I + F \le 3\},$$

then an M-UITSM system becomes a neutrosophic ITSM framework.

Proof. First, suppose that

$$D_M = [0, 1].$$

Then every uncertainty map appearing in $\mathcal{UITSM}_M$ assigns to each object a single membership degree in $[0, 1]$. Hence, the service-level, infrastructure-level, process-level, quality, utility, and reliability evaluations are all fuzzy evaluations. Consequently, the induced uncertain set

$$(S, \mu_{\mathcal{UITSM}_M})$$

is a fuzzy set on S.

Next, suppose that

$$D_M = \{(T, I, F) \in [0, 1]^3 \mid 0 \le T + I + F \le 3\}.$$

Then every uncertainty degree is a neutrosophic triple consisting of truth, indeterminacy, and falsity degrees. Therefore, the service-level, infrastructure-level, process-level, quality, utility, and reliability evaluations are all neutrosophic evaluations. Consequently, the induced uncertain set

$$(S, \mu_{\mathcal{UITSM}_M})$$

is a neutrosophic set on S.

Thus, the proposed M-UITSM framework contains fuzzy ITSM and neutrosophic ITSM as particular instances obtained by selecting the corresponding uncertain model M. □

Remark 3.5. *The definition of an M-UITSM system is intentionally model-independent. Therefore, by changing the underlying uncertain model M, the same formal structure can represent intuitionistic fuzzy ITSM, neutrosophic ITSM, plithogenic ITSM, and other uncertainty-based IT management frameworks. This shows that Uncertain IT Service Management provides a unified mathematical setting for studying uncertainty-oriented approaches to IT service evaluation and management.*

4. Additional Results: Uncertain Agile IT Service Management

Agile IT Service Management organizes IT services, processes, infrastructure, backlogs, and iterations to improve quality, reliability, and value through continuous adaptive service delivery cycles iteratively [27–29]. Uncertain Agile IT Service Management extends agile ITSM by assigning uncertainty degrees to services, backlogs, iterations, quality, reliability, and utility under evolving information conditions dynamically.

Definition 4.1 (Agile IT Service Management)**.** *An* Agile IT Service Management system*, briefly an* Agile ITSM system*, is a tuple*

$$\mathcal{AITSM} = (S, I, P, K, B, \alpha, \beta, \gamma, Q, R, U, \Phi),$$

where:

- *S is a finite nonempty set of IT services;*
- *I is a finite set of IT infrastructure components;*
- *P is a finite set of IT management processes;*
- *K is a finite nonempty set of agile iterations, such as sprints;*
- *B is a finite set of backlog items;*
-
$$\alpha : S \to 2^I \times 2^P$$
is an allocation map assigning to each service its supporting infrastructure components and related IT management processes;

- $$\beta : K \to 2^B$$
  assigns to each iteration the backlog items selected for that iteration;
- $$\gamma : B \to 2^S \times 2^P$$
  associates each backlog item with the services and processes affected by it;
- $$Q : S \times K \to \mathbb{R}_{\geq 0}, \qquad R : S \times K \to [0,1], \qquad U : S \times K \to \mathbb{R}$$
  are, respectively, the quality, reliability, and utility evaluation maps for services during agile iterations;
- $$\Phi : S \times K \to \mathbb{R}^3$$
  is the global agile ITSM evaluation map, defined by
  $$\Phi(s,k) = \big(Q(s,k), R(s,k), U(s,k)\big).$$

Thus, an Agile ITSM system is a mathematical framework in which IT services, infrastructure, management processes, backlog items, and agile iterations are organized so that service quality, reliability, and utility can be evaluated iteratively.

Definition 4.2 (Uncertain Agile IT Service Management)**.** *Let M be a fixed uncertain model with degree domain*
$$D_M = \mathrm{Dom}(M) \subseteq [0,1]^m$$
for some integer $m \geq 1$. An Uncertain Agile IT Service Management system of type M, *briefly an M-*UAITSM system, *is a tuple*
$$\mathcal{UAITSM}_M = (S, I, P, K, B, \alpha, \beta, \gamma, \mu_S, \mu_I, \mu_P, \mu_B, Q_M, R_M, U_M, \Theta),$$
where:

- *$S, I, P, K, B, \alpha, \beta, \gamma$ are as in an Agile ITSM system;*
- $$\mu_S : S \to D_M, \qquad \mu_I : I \to D_M, \qquad \mu_P : P \to D_M, \qquad \mu_B : B \to D_M$$
  are uncertainty maps for services, infrastructure components, IT management processes, and backlog items, respectively;
- $$Q_M : S \times K \to D_M, \qquad R_M : S \times K \to D_M, \qquad U_M : S \times K \to D_M$$
  are uncertain quality, uncertain reliability, and uncertain utility maps;

- $$\Theta : D_M^4 \to D_M$$

  is an M-admissible global evaluation function.

For each service $s \in S$ and each iteration $k \in K$, the overall uncertain agile ITSM degree *is defined by*

$$\mu_{\mathcal{UAITSM}_M}(s, k) = \Theta\big(\mu_S(s), Q_M(s, k), R_M(s, k), U_M(s, k)\big).$$

The pair

$$\mathcal{U}_{\mathcal{UAITSM}_M} = \big(S \times K, \mu_{\mathcal{UAITSM}_M}\big)$$

is called the uncertain set induced by the M-UAITSM system*.*

Theorem 4.3 (Well-Definedness of Uncertain Agile ITSM)**.** *Let*

$$\mathcal{UAITSM}_M = (S, I, P, K, B, \alpha, \beta, \gamma, \mu_S, \mu_I, \mu_P, \mu_B, Q_M, R_M, U_M, \Theta)$$

be an M-UAITSM system. Then the map

$$\mu_{\mathcal{UAITSM}_M} : S \times K \to D_M$$

is well-defined. Consequently,

$$\mathcal{U}_{\mathcal{UAITSM}_M} = \big(S \times K, \mu_{\mathcal{UAITSM}_M}\big)$$

is an uncertain set of type M on $S \times K$.

Proof. Let $(s, k) \in S \times K$ be arbitrary. Since

$$\mu_S : S \to D_M, \qquad Q_M : S \times K \to D_M, \qquad R_M : S \times K \to D_M, \qquad U_M : S \times K \to D_M$$

are functions, the values

$$\mu_S(s), \quad Q_M(s, k), \quad R_M(s, k), \quad U_M(s, k)$$

are uniquely determined elements of D_M. Since

$$\Theta : D_M^4 \to D_M$$

is also a function, the value

$$\Theta\big(\mu_S(s), Q_M(s, k), R_M(s, k), U_M(s, k)\big)$$

is uniquely determined and belongs to D_M. Hence

$$\mu_{\mathcal{UAITSM}_M}(s, k) \in D_M$$

is well-defined for every $(s, k) \in S \times K$. Therefore,

$$\mu_{\mathcal{UAITSM}_M} : S \times K \to D_M$$

is a well-defined function. Since $D_M = \mathrm{Dom}(M)$, the pair

$$(S \times K, \mu_{\mathcal{UAITSM}_M})$$

satisfies the definition of an uncertain set of type M. Thus,

$$\mathcal{U}_{\mathcal{UAITSM}_M} = (S \times K, \mu_{\mathcal{UAITSM}_M})$$

is an uncertain set of type M on $S \times K$. □

Remark 4.4. *If $D_M = [0, 1]$, then an M-UAITSM system becomes a fuzzy agile ITSM framework. If*

$$D_M = \{(T, I, F) \in [0, 1]^3 \mid 0 \leq T + I + F \leq 3\},$$

then it becomes a neutrosophic agile ITSM framework. Hence, the proposed definition gives a unified model for agile IT service management under several types of uncertainty.

5. **Conclusion**

In this paper, we proposed a mathematical framework for Uncertain IT Service Management by using the notion of uncertain sets. The main idea was to represent IT services, infrastructure components, management processes, quality, reliability, utility, allocation relations, aggregation schemes, and global evaluation functions within a unified uncertainty-based structure. In particular, we defined an M-Uncertain IT Service Management system, where M is a fixed uncertain model with degree domain $D_M = \mathrm{Dom}(M)$.

We proved that the overall uncertainty map induced by an M-UITSM system is well-defined. Therefore, every such system naturally determines an uncertain set of type M on the set of IT services. This result provides a basic mathematical justification for treating uncertainty-aware IT service evaluation as an uncertain set-theoretic structure. We also showed that fuzzy IT Service Management and neutrosophic IT Service Management can be obtained as special cases by choosing appropriate degree domains. Hence, the proposed framework is not restricted to a single uncertainty model, but can be adapted to fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and related settings.

Furthermore, we introduced Agile IT Service Management and Uncertain Agile IT Service Management in order to describe service management processes that are updated iteratively through agile cycles, such as sprints and backlog-based improvements. In this setting, the evaluation of service quality, reliability, and utility may vary across iterations. The uncertain agile framework therefore offers a way to model time-dependent or iteration-dependent uncertainty in IT service improvement.

The present study is theoretical in nature. Future research may develop concrete examples, computational models, decision-making procedures, and empirical applications based on the proposed frameworks. In particular, it would be useful to investigate how uncertain ITSM and uncertain agile ITSM can be applied to incident management, change management, service reliability assessment, cloud service operations, cybersecurity operations, and multi-vendor service integration. Further studies may also examine suitable aggregation functions, ranking methods, optimization models, and software tools for practical implementation.

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Data Availability

This research is purely theoretical and involves no data collection or analysis. Future researchers are encouraged to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The author affirms that, to the best of the author's knowledge, this manuscript represents original research. It has not been previously published in any journal, nor is it currently being considered for publication elsewhere.

Disclaimer on Computational Tools

No mathematical software, symbolic computation systems, automated theorem provers, or proof assistants were employed in the development, analysis, or verification of the mathematical results contained in this paper. All derivations and proofs were conducted manually through analytical methods by the author.

Conflicts of Interest

The author confirms that there are no conflicts of interest related to this research or its publication.

Use of Generative AI and AI-Assisted Tools

The author used generative AI and AI-assisted tools only for English grammar checking, wording improvement, and typographical checks. These tools were not used to derive, prove, or verify the mathematical results.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials independently. The views and conclusions expressed here are the author's own.

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Article

The Neutral Set as a Re-Notation of the Single-Valued Neutrosophic Set

Florentin Smarandache[1,*]

[1] University on New Mexico, Campus de Gallup, États-Unis
* Correspondence: smarand@unm.edu

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Abstract: This short note examines the mathematical relationship between the recently introduced Neutral Set and the established Single-Valued Neutrosophic Set (SVNS). By means of a formal comparison of definitions, notation, and admissibility conditions, we show that the Neutral Set is mathematically equivalent to the SVNS, differing only in symbolic representation. Specifically, the membership functions denoted by truth, indeterminacy, and falsity in the Neutral Set correspond directly to the truth-membership, indeterminacy-membership, and falsity-membership functions of the SVNS under a simple notational transformation. Furthermore, the informational regimes identified for the Neutral Set—sub-ideal (incomplete information), intuitionistic (complete information), and paraconsistent (contradictory information)—coincide with the interpretative framework already established in neutrosophic set theory. We conclude that the Neutral Set should be regarded as a reformulation or particular representation of the Single-Valued Neutrosophic Set rather than a novel mathematical structure.

Keywords: Single-Valued Neutrosophic Set, Neutral Set, neutrosophic logic, neutrosophic set theory, mathematical equivalence, indeterminacy, truth membership, falsity membership, paraconsistent information, uncertainty modeling.

1. Introduction

The introduction of new mathematical structures in uncertainty modeling, fuzzy systems, and generalized set theories requires careful examination in order to distinguish genuinely novel concepts from reformulations of already established frameworks. Such distinctions are particularly important for preserving conceptual clarity, terminological consistency, and the cumulative development of scientific knowledge. In many cases, newly proposed structures may differ only in notation or terminology while remaining mathematically equivalent to previously defined models.

Recently, C. Ardil [1] introduced the concept of a *Neutral Set* as a framework intended to represent truth, indeterminacy, and falsity through three independent mappings taking values in the interval ([0,1]). The proposed structure identifies distinct informational regimes depending on the sum of these components, namely incomplete, intuitionistic, and paraconsistent information.

At the same time, the *Single-Valued Neutrosophic Set* (SVNS), originally developed within neutrosophic set theory [2], is defined through truth-membership, indeterminacy-membership, and falsity-membership functions satisfying the same admissibility condition. The SVNS framework already encompasses incomplete, complete, and contradictory information according to the total sum of its three components.

The objective of this short note is to examine the formal relationship between the Neutral Set and the Single-Valued Neutrosophic Set. By comparing their mathematical definitions and interpretative mechanisms, we demonstrate that the Neutral Set is mathematically equivalent to the Single-Valued Neutrosophic Set, differing primarily in symbolic notation. Furthermore, we show that the assumption of independent components in the Neutral Set corresponds to a particular case within the broader neutrosophic framework. The analysis presented here contributes to maintaining mathematical coherence in the taxonomy of uncertainty models and clarifies the position of the Neutral Set within neutrosophic set theory.

2. Definitions and Formal Comparison

In this section, we present the definitions of the *Neutral Set* and the *Single-Valued Neutrosophic Set* in order to establish a formal comparison between the two mathematical structures.

2.1. Neutral Set

According to Ardil [1], a *Neutral Set* (A) defined on a universe of discourse (X) is represented as:

$$\mathbf{A = \{\langle x, \mu(x), \eta(x), \nu(x)\rangle \mid x \in X\}}, \text{ where}$$

$$\boldsymbol{\mu, \eta, \nu : X \rightarrow [0,1]}$$

are independent mappings satisfying the condition

$$\mathbf{0 \le \mu(x) + \eta(x) + \nu(x) \le 3}$$

The quantities $\mathbf{\mu(x)}$, $\mathbf{\eta(x)}$, and $\mathbf{\nu(x)}$ denote the degrees of truth, indeterminacy, and falsity associated with the element $\mathbf{x}$, respectively. Furthermore, Ardil defines an informational density function:

$$\mathbf{S(x) = \mu(x) + \eta(x) + \nu(x)}$$

Based on the value of $\mathbf{S(x)}$, three informational regimes are distinguished:

- **Incomplete information**, when $\mathbf{S(x) < 1}$
- **Complete (intuitionistic) information**, when $\mathbf{S(x) = 1}$
- **Paraconsistent information**, when $\mathbf{S(x) > 1}$

2.2. Single-Valued Neutrosophic Set

The *Single-Valued Neutrosophic Set* (SVNS), introduced in neutrosophic set theory [2], is defined on a universe of discourse (U) as:

$$\mathbf{A = \{\langle x, T(x), I(x), F(x)\rangle \mid x \in U\}}$$

where

$$\mathbf{T, I, F : U \rightarrow [0,1]}$$

represent the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively, satisfying the condition

$$\mathbf{0 \le T(x) + I(x) + F(x) \le 3}$$

Within the neutrosophic framework, the functions $\mathbf{T(x)}$, $\mathbf{I(x)}$, and $\mathbf{F(x)}$ may be independent, partially independent, partially dependent, or dependent.

Similarly, the informational interpretation depends on the sum of the three components:

- **Incomplete information**, when $\mathbf{T(x) + I(x) + F(x) < 1}$
- **Complete information**, when $\mathbf{T(x) + I(x) + F(x) = 1}$
- **Contradictory (paraconsistent) information**, when $\mathbf{T(x) + I(x) + F(x) > 1}$

2.3. Preliminary Comparison

A direct comparison between the two definitions reveals a one-to-one correspondence between their components:

$$\mathbf{T(x) = \mu(x)}$$

$$\mathbf{I(x) = \eta(x)}$$

$$\mathbf{F(x) = \nu(x)}$$

Moreover, both structures satisfy the same admissibility condition:

$$0 \leq T(x) + I(x) + F(x) \leq 3$$

which is equivalent to

$$0 \leq \mu(x) + \eta(x) + \nu(x) \leq 3$$

Likewise, both frameworks adopt the same interpretation of informational regimes, namely incomplete, complete, and contradictory (paraconsistent) information according to the total sum of their respective components.

The only apparent distinction concerns the explicit requirement of independence among the mappings in the Neutral Set definition, whereas the Single-Valued Neutrosophic Set framework permits independent, partially dependent, or dependent components. Therefore, the Neutral Set may be viewed as a particular case within the broader neutrosophic framework.

The following section formally establishes the mathematical equivalence between the Neutral Set and the Single-Valued Neutrosophic Set.

3. Mathematical Equivalence Between the Neutral Set and the Single-Valued Neutrosophic Set

In this section, we formally establish the mathematical relationship between the *Neutral Set* and the *Single-Valued Neutrosophic Set* (SVNS).

Proposition 1

Let

$$A_N = \{\langle x, \mu(x), \eta(x), \nu(x)\rangle \mid x \in X\}$$

be a *Neutral Set* defined on a universe of discourse **X**, and let

$$A_{SVNS} = \{\langle x, T(x), I(x), F(x)\rangle \mid x \in U\}$$

be a *Single-Valued Neutrosophic Set* defined on a universe of discourse **U**.

If the following component-wise correspondence is established:

$$T(x) = \mu(x)$$
$$I(x) = \eta(x)$$
$$F(x) = \nu(x)$$

then the *Neutral Set* is mathematically equivalent to the *Single-Valued Neutrosophic Set*.

Proof

The definition of a Neutral Set requires that the mappings

$$\mu, \eta, \nu : X \to [0,1]$$

satisfy the admissibility condition

$$0 \leq \mu(x) + \eta(x) + \nu(x) \leq 3$$

Similarly, the definition of a Single-Valued Neutrosophic Set requires that

$$T, I, F : U \to [0,1]$$

satisfy

$$0 \leq T(x) + I(x) + F(x) \leq 3$$

Substituting the correspondence

$$T(x) = \mu(x)$$
$$I(x) = \eta(x)$$
$$F(x) = \nu(x)$$

into the SVNS condition yields:

$$0 \leq \mu(x) + \eta(x) + \nu(x) \leq 3$$

which is precisely the admissibility condition of the Neutral Set.

Moreover, the semantic interpretation of the three components coincides in both frameworks:

- $\mu(x)$ corresponds to $T(x)$ (truth degree),
- $\eta(x)$ corresponds to $I(x)$ (indeterminacy degree),
- $\nu(x)$ corresponds to $F(x)$ (falsity degree).

Likewise, the informational regimes are identical in both structures:

- **Incomplete information**, when $\mu(x) + \eta(x) + \nu(x) < 1$
  or equivalently $T(x) + I(x) + F(x) < 1$

- **Complete information**, when $\mu(x) + \eta(x) + \nu(x) = 1$
  or equivalently $T(x) + I(x) + F(x) = 1$
- **Contradictory (paraconsistent) information**, when $\mu(x) + \eta(x) + \nu(x) > 1$
  or equivalently $T(x) + I(x) + F(x) > 1$

Hence, the Neutral Set and the Single-Valued Neutrosophic Set possess identical mathematical structures, identical admissibility constraints, and identical interpretative mechanisms.

The only distinction lies in notation and in the explicit assumption that the three mappings are independent in the Neutral Set framework. Since the Single-Valued Neutrosophic Set already permits independent components as a valid case, the Neutral Set constitutes a particular instance of the more general neutrosophic framework.

Therefore, the so-called *Neutral Set* is mathematically equivalent to the *Single-Valued Neutrosophic Set*. □

4. Discussion

The result established in Proposition 1 demonstrates that the *Neutral Set* does not introduce a new mathematical structure distinct from the *Single-Valued Neutrosophic Set* (SVNS). Rather, the comparison reveals a direct correspondence between the components of the two frameworks, namely:

$$T(x) = \mu(x),\ I(x) = \eta(x),\ F(x) = \nu(x)$$

together with the same admissibility condition:

$$0 \le T(x) + I(x) + F(x) \le 3$$

and identical interpretations of informational regimes.

From a mathematical perspective, two set-theoretical models that possess the same functional structure, the same constraints, and the same semantic interpretation should be regarded as equivalent representations of the same concept rather than distinct theories. The introduction of alternative notation, by itself, does not constitute mathematical novelty unless accompanied by new operators, new algebraic properties, generalized domains, or genuinely different semantics.

A possible distinction may be identified in the explicit requirement of independence among the mappings $\mu(x)$, $\eta(x)$, and $\nu(x)$ in the Neutral Set framework. However, this distinction does not invalidate the equivalence established above. In fact, the Single-Valued Neutrosophic Set already includes independent components among its admissible configurations, while also allowing partial dependence or full dependence. Consequently, the Neutral Set can be interpreted as a restricted or particular case of the broader neutrosophic framework rather than an independent mathematical construction.

The present analysis also highlights a broader methodological issue in uncertainty modeling and generalized set theories: the proliferation of terminologies for mathematically equivalent structures may create conceptual fragmentation and unnecessary ambiguity in the literature. For scientific coherence, newly introduced frameworks should be carefully examined in relation to pre-existing theories in order to determine whether they represent genuine mathematical innovations or alternative reformulations of known concepts.

Therefore, based on the formal correspondence established in this paper, the Neutral Set should be understood as a notational reformulation of the Single-Valued Neutrosophic Set and, more specifically, as a particular case included within neutrosophic set theory.

5. Conclusion

This short note examined the mathematical relationship between the recently introduced *Neutral Set* and the established *Single-Valued Neutrosophic Set* (SVNS). Through a formal comparison of definitions, admissibility conditions, and semantic interpretations, we demonstrated that the two frameworks are mathematically equivalent under a direct component-wise correspondence between their respective functions of truth, indeterminacy, and falsity.

The analysis showed that both structures employ identical mappings into the interval **[0,1]**, satisfy the same condition

$$\mathbf{0 \le T(x) + I(x) + F(x) \le 3}$$

and admit the same informational regimes associated with incomplete, complete, and contradictory (paraconsistent) information. The only distinction identified concerns the explicit assumption of independence among the mappings in the Neutral Set framework, which constitutes a particular case already included within the broader Single-Valued Neutrosophic Set formalism.

Consequently, the Neutral Set should be regarded not as a new mathematical structure, but as a reformulation of the Single-Valued Neutrosophic Set under alternative notation. More generally, this study emphasizes the importance of evaluating newly proposed uncertainty models against established frameworks in order to preserve conceptual consistency and avoid unnecessary terminological proliferation in the scientific literature.

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Article

New Generalized Aggregation Operator in Single Valued Neutrosophic Set Environment and Its Application to Crop Land Selection Problem

Shyamal Dalapati[1], Supriya Mukherjee[2], Florentin Smarandache[3], Abhijit Mukherjee[4,5], Animesh Mahata[*5] and Banamali Roy[6]

[1]Department of Mathematics, Dum Dum Motijheel Rabindra Mahavidyalaya, Dum Dum Road, Chatkal, Dum Dum, Kolkata, Pin-700074, West Bengal India. Email: dalapatishyamal30@gmail.com

[2]Department of Mathematics, Gurudas College,1/1 Suren Sarkar Road, Kolkata, pin-700054, West Bengal India. Email: supriyaskbu2013@gmail.com

[3]Mathematics, physics and Natural science Division, University of New Mexico, Gallup Campus, NM 87301, USA. Email: smarand@unm.edu

[4]Department of Mathematics, Swami Vivekananda University, Telinipara, Barasat-Barrackpore RD Bara Kanthalia, West Bengal-700121, West Bengal, India. Email: abhijitmukherjee470@gmail.com

[5]Department of Mathematics, Sri Ramkrishna Sarada Vidyamahapitha, Kamarpukur-712612, West Bengal, India. Email: animeshmahata.2023@srsvidyamahapitha.org

[6]Department of Mathematics, Bangabasi Evening, 19, Rajkumar Chakraborty Sarani, Kolkata-700009, West Bengal, India.

* Correspondence: animeshmahata8@gmail.com

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Abstract: In real-life situation, making the right decision can significantly impact One's lifespan. Cropland selection for agriculture is a popular Multi Criteria Decision Making (MCDM)problem. Therefore, many researchers are interested in conducting research in the agricultural field. The main aim of farmer optimizing land and water resources and enhancing agricultural system. However, making the right decision in crop land selection is a challenging task due to the uncertainty of available information. Neutrosophic sets have the potential to deal with uncertain or incomplete data due to the independent components structure. In this paper, we have defined a gene5ralised aggregation operator in single valued Neutrosophic set (SVNS)context and its basic properties. We have designed a general structure for MCDM problem, followed by some steps. A real-life example of Crop land selection is also demonstrated to highlight effectiveness of the proposed operator and

MCDM method. The effectiveness of the proposed method was examined by comparing it with some existing methods.

Keywords: Single valued Neutrosophic set; Generalized Aggregation operator; Crop land selection; Multi Criteria Decision making.

1.Introduction

The invention of the neutrosophic set (NS) by F. Smarandache is a significant exploration in research. Neutrosophic set is the generalization of intuionistic fuzzy set (IFS). IFSs contain only two independent components: truth and falsity, whereas NSs contain three independent components: truth, falsity, and indeterminacy which are held in the $]^{-}0,1^{+}[$. In 2010 Wang et. al introduced Single Values Neutrosophic Set (SVNS) on the basis of standard interval [0,1]. Due to its three components, any incomplete and uncertain information can be easily handled by the SVNSs. The attractive structure of SVNSs has gained significant attention from many researchers, who are now focusing their work in the SVNS environment.

1.1 *Motivation of the study*

Senapati [2] in 2024 develop Aczel-Alsina aggregation operator in SVNSs and proved its necessary properties and construct a strategy for MADM issues in SVNS environment. Ye [3] introduced two aggregation operators for SVNSs, applying them to solve MCDM problems. Nancy and Garg [4] proposed some operations for SVNNs under Frank norm operations and developed averaging and geometric aggregation operators for decision-making problems. Liu et al. [5] proposed a MAGDM strategy based on power average operators in SVNSs and grey relational analysis (GRA) for the Ranking according to Compromise Solution (MARCOS) method. Liu et al. [6] integrated SVNS with Dombi extended power aggregation operators, proposing two weighted operators and applying them to an intelligent transportation system. Kara et al. [7] proposed the SVNS-ARLON method for service provider ranking, demonstrating its applicability in real-world scenarios. Garai et al. [8] proposed an SVNS-softmax aggregation approach, validating it through application to an MCDM problem. Agriculture is one of the fundamental resources of the global economy. Agricultural technology advancements are leading to higher crop yields, increased food production, and more efficient land use, revolutionizing farming practices. Sustainable farming methods play a vital role in guaranteeing food security and reducing ecological footprint simultaneously. The agricultural sector is crucial to India's economy, focusing on promoting equitable growth, boosting rural incomes, and strengthening food security measures. MCDM is a method that evaluates alternatives holistically, considering multiple conflicting criteria, and has been effectively used to assess various aspects of the agriculture sector. The presence of uncertainty significantly impacts on the assessment of agriculture land selection. Due to the advantages of TMF, FMF and IMF, SVNS can cope with incomplete information for selection agricultural decision-making problem. An SVN-distance measure was proposed by Mishra et al. [9] to determine the dissimilarity between SVNSs. They also proposed a hybrid ranking strategy in SVNS to assess the agro-climatic regions of India. An MCDM strategy was formulated by Garai & Garg [10] to tackle water resource management (WRM) issues in the Puruliya district of West Bengal, under a neutrosophic environment. In 2021 Zeng et. Al [11] defined SVN-Graph and applied this graph in Food and Agriculture Organization.

Inspired by these innovative ideas. We developed a generalized aggregation operator in an SVNS environment and proposed an MCDM technique based on the aggregation operator for agricultural land selection.

1.2 *Contribution of the study*

The purpose of this study is to design a method that recommends the best alternative from a collection of feasible options. For this decision-making purpose, at first, we develop a generalized aggregation operator and proves its related impotent properties. We also investigate the sensitivity analysis for this operator. A multi-step MCDM approach was developed and demonstrated through a numerical example focusing on crop land location selection based on key criteria. The main contributions of this study are that:

- We defined a generalized SVNS aggregation operator.
- We proved its basics properties.
- We design a MCDM framework for decision making based on proposed operator.
- we solve a real decision-making task to check validity and practicability of MCDM method.
- A comparative study with existing methods is conducted to verify the proposed approach and demonstrate its strengths.

*The results are outlined and discussed to assess the method's reliability and effectiveness.

Structure of this study

This manuscript is arranged into the following sections. Section 2 provides a brief overview of fundamental concepts related to SVNSs and their operations. In section 3 introduces a novel neutrosophic power mean aggregation operator, along with a discussion of its fundamental properties. In Sect. 4, we defined new power mean operator based MCDM decision making method. Section 5 provides a numerical illustration of land selection for agriculture. In section 6, we provide a comparative analysis. Finally, in Sect. 7 the paper is concluded.

2.**Basic Preliminaries**

Neutrosophic sets(N.S): [1] Let $\breve{X}$ be a universe. A neutrosophic set $\breve{A}$ over $\breve{X}$ is defined by $\breve{A}$ ={< $\breve{x}, \left(T_{\breve{A}}(\breve{x}), I_{\breve{A}}(\breve{x}), F_{\breve{A}}(\breve{x})\right) >: \breve{x} \in \breve{X}\}$ where $T_{\breve{A}}(\breve{x}), I_{\breve{A}}(\breve{x})$ and $F_{\breve{A}}(\breve{x})$ are known as truth part, indeterminacy parts and falsity part, in function form respectively. These are in function form as :

$T_{\breve{A}}$: $\breve{X} \rightarrow]-_0, 1^+[$,$I_{\breve{A}}$: $\breve{X} \rightarrow]-_0, 1^+[$, $F_{\breve{A}}$: $\breve{X} \rightarrow]-_0, 1^+[$ such that $0^- \leq T_{\breve{A}}(\breve{x}) + I_{\breve{A}}(\breve{x}) + F_{\breve{A}}(\breve{x}) \leq 3^+$

Single valued Neutrosophic set (SVNS):[14] Let $\breve{X}$ be a universe. A single valued neutrosophic (SVN)set $\breve{A}$ over $\breve{X}$ is represent as

$T_{\breve{A}}$: $\breve{X} \rightarrow [0,1] I_{\breve{A}}$: $\breve{X} \rightarrow [0,1] F_{\breve{A}}$: $\breve{X} \rightarrow [0,1]$ such that $0 \leq T_{\breve{A}}(\breve{x}) + I_{\breve{A}}(\breve{x}) + F_{\breve{A}}(\breve{x}) \leq 3$. For simplicity, a SVNN can be written as $\breve{A} = (T_{\breve{A}}, I_{\breve{A}}, F_{\breve{A}}) T_{\breve{A}} \in [0,1]$,$I_{\breve{A}} \in [0,1], F_{\breve{A}} \in [0,1]$

and $0 \leq T_{\breve{A}} + I_{\breve{A}} + F_{\breve{A}} \leq 3$

2.1 *Some Operations [14]*

Let $\breve{A}$={<$\breve{x}, \left(T_{\breve{A}}(\breve{x}), I_{\breve{A}}(\breve{x}), F_{\breve{A}}(\breve{x})\right) >: \breve{x} \in \breve{X}\}$ and $\breve{B}$={<$\breve{x}, \left(T_{\breve{B}}(\breve{x}), I_{\breve{B}}(\breve{x}), F_{\breve{B}}(\breve{x})\right) >: \breve{x} \in \breve{X}\}$ be two SVNSs in $\breve{X}$,then operations between them defined by Wang *et al.* [14]as follows:

1.$\breve{A} \subseteq \breve{B} \leftrightarrow T_{\breve{A}}(\breve{x}) \leq T_{\breve{B}}(\breve{x})$,$I_{\breve{A}}(\breve{x}) \geq I_{\breve{B}}(\breve{x})$, ,$F_{\breve{A}}(\breve{x}) \geq F_{\breve{B}}(\breve{x})$,$\forall \breve{x} \in \breve{X}$

2.$\breve{A} = \breve{B}$ if and only if $\breve{A} \subseteq \breve{B}, \breve{B} \subseteq \breve{A}$,$\forall \breve{x} \in \breve{X}$

3.$\breve{A}^c$={<$\breve{x}, (F_{\breve{A}} 1 - I_{\breve{A}}, T_{\breve{A}}) >: \breve{x} \in \breve{X}\}$,$\forall \breve{x} \in \breve{X}$

However, the likely correction would be $\breve{A}^c$ = {< $\breve{x}, (F_{\breve{A}} 1 - I_{\breve{A}}, T_{\breve{A}}) >: \breve{x} \in \breve{X}\}$ should be $\breve{A}^c$={<$\breve{x}, \left(F_{\breve{A}}(\breve{x}), 1 - I_{\breve{A}}(\breve{x}), T_{\breve{A}}(\breve{x})\right) >: \breve{x} \in \breve{X}\}$

4. $\breve{A} \cup \breve{B} = \breve{A}$={<$\breve{x}, \left(\max\left(T_{\breve{A}}(\breve{x}), T_{\breve{B}}(\breve{x})\right), \min(I_{\breve{A}}(\breve{x}), I_{\breve{B}}(\breve{x})), \min(F_{\breve{A}}(\breve{x}), F_{\breve{B}}(\breve{x})\right) >: \breve{x} \in \breve{X}\}$,$\forall \breve{x} \in \breve{X}$

5. $\breve{A} \cap \breve{B} = \breve{A}$={<$\breve{x}, \left(\min\left(T_{\breve{A}}(\breve{x}), T_{\breve{B}}(\breve{x})\right), \max(I_{\breve{A}}(\breve{x}), I_{\breve{B}}(\breve{x})), \max(F_{\breve{A}}(\breve{x}), F_{\breve{B}}(\breve{x})\right) >: \breve{x} \in \breve{X}\}$,$\forall \breve{x} \in \breve{X}$

3.**New Neutrosophic Power mean operator**

Let,$\breve{N} = \breve{A}_1, \breve{A}_2, \breve{A}_3, \breve{A}_4, \dots \dots \dots \dots \dots . \breve{A}_P$ be the set of P neutrosophic sets. The neutrosophic power mean operator $f_q: \breve{N}^P \rightarrow \breve{N}$is expressed as

$$f_q(\breve{x}) = (\frac{1}{p}\sum_{i=1}^{-p}\left(T_{\breve{A}_i}(\breve{x})\right)^q)^{\frac{1}{q}}, (\frac{1}{p}\sum_{i=1}^{-p}\left(I_{\breve{A}_i}(\breve{x})\right)^q)^{\frac{1}{q}}, (\frac{1}{p}\sum_{i=1}^{-p}\left(F_{\breve{A}_i}(\breve{x})\right)^q)^{\frac{1}{q}} \quad (1)$$

Where q>0, areal number and p is a positive number

Special Cases:

Case 1: when q=1, the Eq (1) reduces to the following form:

$$f_1(\breve{x})=(\frac{1}{p}\sum_{i=1}^{p} T_{\breve{A}_i}(\breve{x}), \frac{1}{p}\sum_{i=1}^{p} I_{\breve{A}_i}(\breve{x}), \frac{1}{p}\sum_{i=1}^{p} F_{\breve{A}_i}(\breve{x}),)$$

Case 2: when q=0, the Eq (1) reduces to the following form:

$$f_0(\breve{x}) = (\sqrt[p]{\prod_{i=1}^{p} T_{\breve{A}_i}(\breve{x}) \prod_{i=1}^{p} I_{\breve{A}_i}(\breve{x}), \prod_{i=1}^{p} F_{\breve{A}_i}(\breve{x}),},)$$

Case 3: Similarly, when q=2, 3,……… the Eq. (1) reduces to quadratic mean transformation function, cubic mean transformations functions and so on.

Case 4: When q→ ∞

$$f_\infty(\breve{x}) = (\max\{T_{\breve{A}_i}(\breve{x}),\}, \max\{I_{\breve{A}_i}(\breve{x})\}, \max\{F_{\breve{A}_i}(\breve{x})\})$$

3.1 ***Properties of power mean operator***

Property 1: Idempotency

If all $T_{\breve{A}_i}(i = 1,2,3,\ldots\ldots.) I_{\breve{A}_i}(i = 1,2,3,\ldots\ldots.) and F_{\breve{A}_i}(i = 1,2,3,\ldots\ldots.)$ are equal, i,e, $T_{\breve{A}_1} = T_{\breve{A}_2} = T_{\breve{A}_3} = T_{\breve{A}_4} = T_{\breve{A}_5} = T_{\breve{A}_6} = \cdots\ldots\ldots..T_{\breve{A}}$, $I_{\breve{A}_1} = I_{\breve{A}_2} = I_{\breve{A}_3} = I_{\breve{A}_4} = I_{\breve{A}_5} = I_{\breve{A}_6} = \cdots\ldots\ldots..I_{\breve{A}}, F_{\breve{A}_1} = F_{\breve{A}_2} = F_{\breve{A}_3} = F_{\breve{A}_4} = F_{\breve{A}_5} = F_{\breve{A}_6} = \cdots\ldots\ldots..F_{\breve{A}}$

Then $f_q(\breve{x}) = (\frac{1}{p}\sum_{i=1}^{-p}\left(T_{\breve{A}_i}(\breve{x})\right)^q)^{\frac{1}{q}}, (\frac{1}{p}\sum_{i=1}^{-p}\left(I_{\breve{A}_i}(\breve{x})\right)^q)^{\frac{1}{q}}, (\frac{1}{p}\sum_{i=1}^{-p}\left(F_{\breve{A}_i}(\breve{x})\right)^q)^{\frac{1}{q}}$

$$=((\frac{1}{p} \times p \times (T_{\breve{A}})^q)^{\frac{1}{q}}, (\frac{1}{p} \times p \times (I_{\breve{A}})^q)^{\frac{1}{q}}, (\frac{1}{p} \times p \times (F_{\breve{A}})^q)^{\frac{1}{q}}) = <T_{\breve{A}}, I_{\breve{A}}, F_{\breve{A}}>$$

Property 2: Monotonically

Let set of neutrosophic numbers $\breve{A}_i \leq \breve{B}_i$

i.e, $T_{\overline{A_\iota}}(\breve{x}) \leq T_{\overline{B_\iota}}(\breve{x})$,$I_{\overline{A_\iota}}(\breve{x}) \geq I_{\overline{B_\iota}}(\breve{x})$, ,$F_{\overline{A_\iota}}(\breve{x}) \geq F_{\overline{B_\iota}}(\breve{x})$,$\forall \breve{x} \in \breve{X}$

$f_q A(\breve{x})$={$\left(\frac{1}{p}\sum_{i=1}^{p}(T_{\breve{A}_i}(\breve{x}))^q\right)^{\frac{1}{q}}, \left(\frac{1}{p}\sum_{i=1}^{p}(I_{\breve{A}_i}(\breve{x}))^q\right)^{\frac{1}{q}}, \left(\frac{1}{p}\sum_{i=1}^{p}(F_{\breve{A}_i}(\breve{x}))^q\right)^{\frac{1}{q}}$} ≤{

$\left(\frac{1}{p}\sum_{i=1}^{p}(T_{\breve{B}_i}(\breve{x}))^q\right)^{\frac{1}{q}}, \left(\frac{1}{p}\sum_{i=1}^{p}((I_{\breve{B}_i}(\breve{x}))^q)^q\right)^{\frac{1}{q}}, \left(\frac{1}{p}\sum_{i=1}^{p}(F_{\breve{B}_i}(\breve{x}))^q\right)^{\frac{1}{q}}$}=$f_q B(\breve{x})$

Proof: since $T_{\overline{A_\iota}}(\breve{x}) \leq T_{\overline{B_\iota}}(\breve{x})$for every i

Then we have $\left(T_{\overline{A_\iota}}(\breve{x})\right)^q \leq \left(T_{\overline{B_\iota}}(\breve{x})\right)^q \Rightarrow \sum_{i=1}^{p}\left(T_{\overline{A_\iota}}(\breve{x})\right)^q \leq \sum_{i=1}^{p}\left(T_{\overline{B_\iota}}(\breve{x})\right)^q$

$$\Rightarrow \frac{1}{p}\sum_{i=1}^{p}\left(T_{\overline{A_\iota}}(\breve{x})\right)^q \leq \frac{1}{p}\sum_{i=1}^{p}\left(T_{\overline{B_\iota}}(\breve{x})\right)^q$$

$$\Rightarrow \left(\frac{1}{p}\sum_{i=1}^{p}\left(T_{\overline{A_\iota}}(\breve{x})\right)^q\right)^{\frac{1}{q}} \leq \left(\frac{1}{p}\sum_{i=1}^{p}\left(T_{\overline{B_\iota}}(\breve{x})\right)^q\right)^{\frac{1}{q}} \qquad (5)$$

Further $I_{\overline{A_\iota}}(\breve{x}) \geq I_{\overline{B_\iota}}(\breve{x})$

$$\Rightarrow \left(I_{\overline{A_\iota}}(\breve{x})\right)^q \geq \left(I_{\overline{B_\iota}}(\breve{x})\right)^q$$

$$\Rightarrow \sum_{i=1}^{p}\left(\mathrm{I}_{\overline{A_\iota}}(\breve{x})\right)^q \geq \sum_{i=1}^{p}\left(I_{\overline{B_\iota}}(\breve{x})\right)^q$$

$$\Rightarrow \frac{1}{p}\sum_{i=1}^{p}\left(I_{\overline{A_\iota}}(\breve{x})\right)^q \geq \frac{1}{p}\sum_{i=1}^{p}\left(I_{\overline{B_\iota}}(\breve{x})\right)^q$$

$$\Rightarrow \left(\frac{1}{p}\sum_{i=1}^{p}\left(I_{\overline{A_\iota}}(\breve{x})\right)^q\right)^{\frac{1}{q}} \geq \left(\frac{1}{p}\sum_{i=1}^{p}\left(I_{\overline{B_\iota}}(\breve{x})\right)^q\right)^{\frac{1}{q}} \qquad (6)$$

And

$$F_{\overline{A_\iota}}(\breve{x}) \geq F_{\overline{B_\iota}}(\breve{x})$$

$$\Rightarrow \left(F_{\overline{A_\iota}}(\breve{x})\right)^q \geq \left(F_{\overline{B_\iota}}(\breve{x})\right)^q$$

$$\Rightarrow \sum_{i=1}^{p} \left(F_{\overline{A_i}}(\breve{x}) \right)^q \geq \sum_{i=1}^{p} \left(F_{\overline{B_i}}(\breve{x}) \right)^q$$

$$\Rightarrow \frac{1}{p}\sum_{i=1}^{p} \left(F_{\overline{A_i}}(\breve{x}) \right)^q \geq \frac{1}{p}\sum_{i=1}^{p} \left(F_{\overline{B_i}}(\breve{x}) \right)^q$$

$$\Rightarrow \left(\tfrac{1}{p}\textstyle\sum_{i=1}^{p} \left(F_{\overline{A_i}}(\breve{x}) \right)^q \right)^{\frac{1}{q}} \geq \left(\tfrac{1}{p}\textstyle\sum_{i=1}^{p} \left(F_{\overline{B_i}}(\breve{x}) \right)^q \right)^{\frac{1}{q}} \qquad (7)$$

From Eq(5), Eq(6), Eq(7) we obtain

$$f_q A(\breve{x}) =< (\frac{1}{p}\sum_{i=1}^{p} \left(T_{\breve{A}_i}(\breve{x}) \right)^q)^{\frac{1}{q}}, (\frac{1}{p}\sum_{i=1}^{p} \left(I_{\breve{A}_i}(\breve{x}) \right)^q)^{\frac{1}{q}} (\frac{1}{p}\sum_{i=1}^{p} \left(F_{\breve{A}_i}(\breve{x}) \right)^q)^{\frac{1}{q}} >\leq$$

$$< (\frac{1}{p}\sum_{i=1}^{p} \left(T_{\breve{B}_i}(\breve{x}) \right)^q)^{\frac{1}{q}}, (\frac{1}{p}\sum_{i=1}^{p} \left(I_{\breve{B}_i}(\breve{x}) \right)^q)^{\frac{1}{q}} (\frac{1}{p}\sum_{i=1}^{p} \left(F_{\breve{B}_i}(\breve{x}) \right)^q)^{\frac{1}{q}} = f_q B(\breve{x})$$

Property 3: Boundness

Let $\breve{N} = N(\breve{x})$ be collection of Neutrosophic sets in$((\breve{X})$, and $\breve{N}^+ =<$ $\max_i\{T_{\breve{A}_i}(\breve{x})\}$,$\min_i\{I_{\breve{A}_i}(\breve{x})\}$ $\min_i\{F_{\breve{A}_i}(\breve{x})\}$>,$\breve{N}^- =<$ $\min_i\{T_{\breve{A}_i}(\breve{x})\}$,$\max_i\{I_{\breve{A}_i}(\breve{x})\}$ $\max_i\{F_{\breve{A}_i}(\breve{x})\}$>, then $\breve{N}^- \leq f_q(\breve{x}) \leq \breve{N}^+$

3.2 Score and accuracy function

Here, I adopted [12] score and accuracy function for comparing two single valued Neutrosophic number as follows:

Let $\breve{A} = (T_{\breve{A}}, I_{\breve{A}}, F_{\breve{A}})$,be a SVNN, then

i). S($\breve{A}$)=$\frac{1+(T_{\breve{A}}-F_{\breve{A}})-2I_{\breve{A}}}{2}$, where S($\breve{A}$) $\in [0,1]$

ii)$\delta(\breve{A}) = T_{\breve{A}} - I_{\breve{A}}(1 - T_{\breve{A}}) - F_{\breve{A}}(1 - I_{\breve{A}})$ where $\delta(\breve{A}) \in [-1,1]$

In [12], there is a order relation for two SVNNs, which is given as follows:

Let $\breve{P} = (T_{\breve{P}}, I_{\breve{P}}, F_{\breve{P}})$ and let $\breve{Q} = (T_{\breve{Q}}, I_{\breve{Q}}, F_{\breve{Q}})$ are two SVNNs.

If S($\breve{P}$)>S($\breve{Q}$)$\Longrightarrow$ $\breve{P}$>$\breve{Q}$,again if S($\breve{P}$)=S($\breve{Q}$), then If $\delta(\breve{P})$>$\delta(\breve{Q})$$\Longrightarrow$ $\breve{P}$>$\breve{Q}$

4 An MCDM technique constructed by employing the proposed aggregation operator in the SVNS setting

This part presents a systematic framework for assessing and prioritizing various options under neutrosophic sets, facilitating effective decision analysis. Let us assume that $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \dots \dots \mathcal{L}_n\}$

represents n alternatives or options, which are ranked based on the attributes $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \dots \dots \mathcal{C}_n\}$

The decision-maker expresses opinions for alternatives based on attributes in terms of single-valued neutrosophic numbers (SVNs). Now, we describe the MCDM framework using the following steps based on the proposed operators:

Step 1: Formation of alternatives versus attributes decision matrix

The interrelationship of the alternatives $\mathcal{L}_i (i = 1,2,3 \dots, n)$ and their corresponding criteria$\mathcal{C}_j (j = 1,2,3 \dots, n)$ is captured in terms of SVNs and can be illustrated through a matrix representation M^1: M^1: Alternatives versus attributes decision matrix

$$\begin{array}{c|cccc} & \mathcal{C}_1 & \mathcal{C}_2 & \dots\dots\dots & \mathcal{C}_n \\ \hline \mathcal{L}_1 & (\breve{T}_{11}, \breve{I}_{11}, \breve{F}_{11}) & (\breve{T}_{12}, \breve{I}_{12}, \breve{F}_{12}) & & (\breve{T}_{1m}, \breve{I}_{1m}, \breve{F}_{1m}) \\ \mathcal{L}_2 & (\breve{T}_{21}, \breve{I}_{21}, \breve{F}_{21}) & (\breve{T}_{22}, \breve{I}_{22}, \breve{F}_{22}) & & (\breve{T}_{2m}, \breve{I}_{2m}, \breve{F}_{2m}) \\ \dots\dots\dots & \dots\dots\dots & \dots\dots & \dots\dots\dots & \\ \mathcal{L}_n & (\breve{T}_{n1}, \breve{I}_{n1}, \breve{F}_{n1}) & (\breve{T}_{n2}, \breve{I}_{n2}, \breve{F}_{n2}) & & (\breve{T}_{nm}, \breve{I}_{nm}, \breve{F}_{nm}) \end{array}$$

Step 2: Aggregate the decision matrix using proposed operator

Utilizing the proposed operator (Eq. 1), we aggregate the original decision matrix (M1) and derive a new decision matrix (M^2) that synthesizes the information in a structured form.

M^2: Aggregated decision matrix

$$\begin{array}{c|c}
 & \check{\mathcal{C}}_{AGG} \\
\hline
\mathcal{L}_1 & (T_{11}^{\breve{A}gg}, I_{11}^{\breve{A}gg}, F_{11}^{\breve{A}gg}) \\
\mathcal{L}_2 & \left(T_{21}^{\breve{A}gg}, I_{21}^{\breve{A}gg}, F_{21}^{\breve{A}gg}\right) \\
 & \ldots\ldots. \\
\mathcal{L}_n & (T_{n1}^{\breve{A}gg}, I_{n1}^{\breve{A}gg}, F_{n1}^{\breve{A}gg})
\end{array}$$

Step 3: Calculate Score Values from Aggregate decision matrix using proposed Score function

Eq. (8) is employed to compute the score values of alternatives from matrix M2, resulting in real-valued scores that quantify their performance.

Step 4: Calculate Accuracy Values from Aggregate decision matrix using proposed Score function

Eq. (9) is employed to compute the score values of alternatives from matrix M2, resulting in real-valued scores that quantify their performance.

Step 5: Prepare Ranking order based on score, accuracy and values

Alternatives are ranked according to the obtained score and accuracy measures. Alternatives with higher score values are considered the most preferable. When two alternatives have identical score values, their accuracy values are then used to differentiate them.

Step 6: End

5 Crop-land selection by the multi criteria decision making approach through SVNSs

Agricultural systems are vulnerable to climate change, with notable effects on water resources and crop output. Climate change poses serious threats to agricultural production, potentially escalating the likelihood of food scarcity and famine. The growing population's demands and natural resource degradation pose significant threats to sustainable agriculture. Advancements in agricultural technologies boost productivity, yet they gradually.

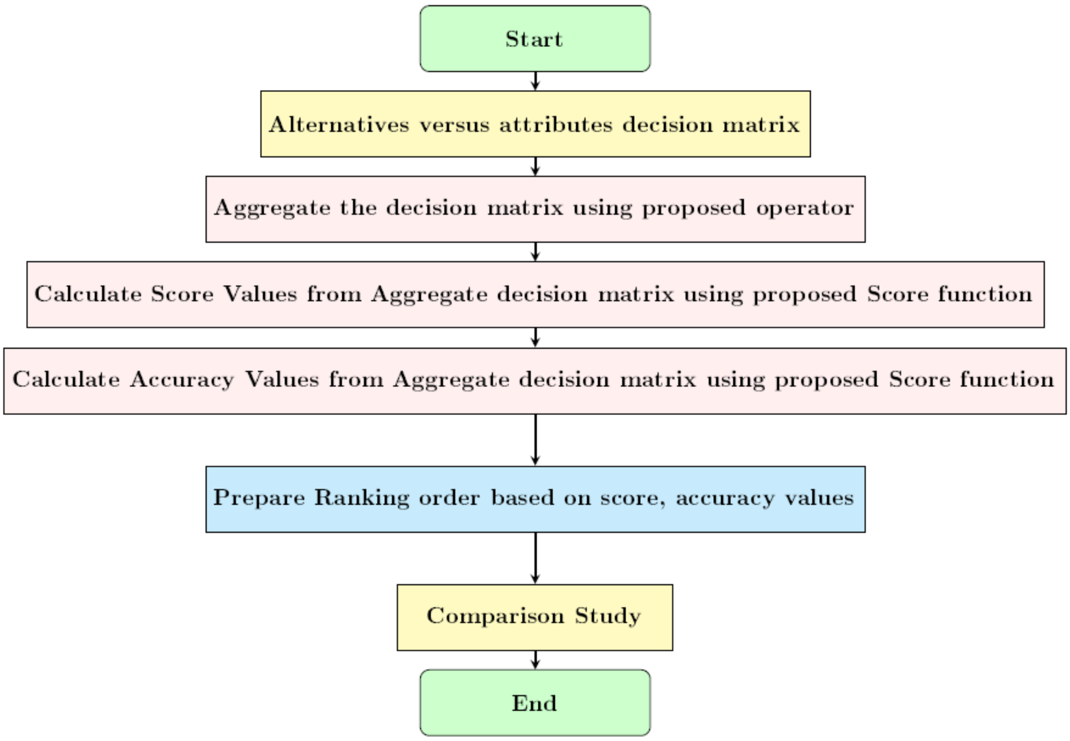

Figure 1: Flow chart of the decision-making process using power aggregation operator.

exhaust natural resources. Introducing technology in farming requires careful consideration of its ecological consequences. A plant needs rain for specific time frame, and the soil must contain moisture substance etc. Therefore, for selecting appropriate agricultural crop land is a challenging thing. Knowledgeable producers select farmland by examining soil conditions, water availability, and local climate to ensure optimal crop growth and limited environmental effects.

Properly chosen agricultural land supports efficient production, reducing input use and mitigating negative effects on the environment.

Determining the appropriate soil and climate conditions is a necessary prerequisite to starting agricultural production that reduce the farmer loss and increase the productivity. Here, I adapted [13] following criteria for choosing appropriate Crop land for planting.

($\breve{C}_1$) Climate
($\breve{C}_2$) Soil quality
($\breve{C}_3$) Water availability
($\breve{C}_4$) Transportation
($\breve{C}_5$) Energy access
($\breve{C}_6$) Biodiversity

Step 1: Formation of Land versus Criteria decision matrix

The decision maker provides the rating values of criteria in terms of neutrosophic number with respect to the land, and represents these values in matrix form for easy calculation and understanding. The following matrix $M^2$is presented below:

M^2: Land versus criteria decision matrix

	$\check{C}_1$	$\check{C}_2$	$\check{C}_3$	$\check{C}_4$	$\check{C}_5$	$\check{C}_6$
$\mathcal{L}_1$	(0.6,0.4,0.3)	(0.7,0.3,0.4)	(0.8,0.4,0.6)	(0.6,0.2,0.4)	(0.8,0.3,0.6)	(0.9,0.5,0.6)
$\mathcal{L}_2$	(0.3,0.1,0.4)	(0.5,0.2,0.2)	(0.8,0.3,0.4)	(0.6,0.3,0.5)	(0.7,0.4,0.3)	(0.4,0.5,0.6)
$\mathcal{L}_3$	(0.6,0.3,0.5)	(0.7,0.3,0.5)	(0.8,0.3,0.5)	(0.4,0.3,0.2)	(0.6,0.4,0.6)	(0.8,0.4,0.3)
$\mathcal{L}_4$	(0.7,0.3,0.3)	(0.3,0.4,0.3)	(0.7,0.4,0.5)	(0.8,0.3,0.4)	(0.7,0.3,0.5)	(0.5,0.3,0.6)

Step 2: Aggregated decision matrix using proposed operator

By using proposed operator (Eq. 1), the aggregate decision matrix of (M^2) and new decision matrix are obtained, synthesizing the information in a structured form.

Aggregated decision matrix

When q=1 the aggregate decision matrix is:

	$\check{C}_{Agg}$
$\mathcal{L}_1$	(0.73,0.35,0.48)
$\mathcal{L}_2$	(0.55,0.30,0.42)
$\mathcal{L}_3$	(0.65,0.35,0.40)
$\mathcal{L}_4$	(0.62,0.33,0.42)

When q=2 the aggregate decision matrix is:

	$\check{C}_{Agg}$
$\mathcal{L}_1$	(0.74,0.36,0.46)
$\mathcal{L}_2$	(0.58,0.34,0.43)
$\mathcal{L}_3$	(0.67,0.28,0.37)
$\mathcal{L}_4$	(0.61,0.27,0.43)

When q=0 the aggregate decision matrix is:

	$\check{C}_{Agg}$
$\mathcal{L}_1$	(0.72,0.34,0.47)
$\mathcal{L}_2$	(0.52,0.27,0.40)
$\mathcal{L}_3$	(0.63,0.35,0.38)
$\mathcal{L}_4$	(0.59,0.33,0.40)

When q→∞the aggregate decision matrix is:

	$\check{C}_{Agg}$
$\mathcal{L}_1$	(0.90,0.50,0.60)
$\mathcal{L}_2$	(0.80,0.50,0.60)
$\mathcal{L}_3$	(0.80,0.40,0.50)
$\mathcal{L}_4$	(0.80,0.40,0.60)

Step 3: Score values of Aggregated decision matrix

Eq (8) is employed to compute the score values of alternatives from aggregated decision matrix, resulting in real –valued score that quantify their performance.

When q=1,

S($\mathcal{L}_1$)=0.285, S($\mathcal{L}_2$)=0.265, S($\mathcal{L}_3$)=0.265, S($\mathcal{L}_4$)=0.27

When q=2,

S($\mathcal{L}_1$)=0.30, S($\mathcal{L}_2$)=0.235, S($\mathcal{L}_3$)=0.37, S($\mathcal{L}_4$)=0.32

When q=0

S($\mathcal{L}_1$)=0.285, S($\mathcal{L}_2$)=0.290, S($\mathcal{L}_3$)=0.275, S($\mathcal{L}_4$)=0.265

When q→∞, S($\mathcal{L}_1$)=0.15, S($\mathcal{L}_2$)=0.10, S($\mathcal{L}_3$)=0.20, S($\mathcal{L}_4$)=0.20

Step 4: Accuracy values of Aggregated decision matrix

Eq (9) is employed to compute the accuracy values of alternatives from aggregated decision matrix, resulting in real –valued score that quantify their performance

When q=1,

$\delta(\mathcal{L}_1)$=0.3505,δ $(\mathcal{L}_2)$=0.121, δ $(\mathcal{L}_3)$=0.268,δ $(\mathcal{L}_4)$=0.213.

When q=2,

$\delta(\mathcal{L}_1)$=0.348,δ $(\mathcal{L}_2)$=0.153, δ $(\mathcal{L}_3)$=0.311,δ $(\mathcal{L}_4)$=0.191.

When q=0,

$\delta(\mathcal{L}_1)$=0.315,δ $(\mathcal{L}_2)$=0.099, δ $(\mathcal{L}_3)$=0.254,δ $(\mathcal{L}_4)$=0.187.

When q→∞,

$\delta(\mathcal{L}_1)$=0.55,δ $(\mathcal{L}_2)$=0.40, δ $(\mathcal{L}_3)$=0.60,δ $(\mathcal{L}_4)$=0.36

Step 4: Ranking order based on score, accuracy values

When q=0,

Score ranking:

0.290>0.285>0.275>0.265⇒ $\mathcal{L}_2 > \mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_4$

Accuracy ranking:

0.3146>0.2535>0.1867>0.0984 ⇒ $\mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_4 > \mathcal{L}_2$

When q=`1,

Score ranking:

0.285>0.270>0.265(TWO)⇒ $\mathcal{L}_1 > \mathcal{L}_4 > \mathcal{L}_2 = \mathcal{L}_3$

Accuracy ranking:

0.3505>0.268>0.2132>0.121 ⇒ $\mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_4 > \mathcal{L}_2$

When q=2,

Score ranking:

0.370>0.320>0.30>0.235⇒ $\mathcal{L}_3 > \mathcal{L}_4 > \mathcal{L}_1 > \mathcal{L}_2$

Accuracy ranking:

0.348>0.3112>0.1908>0.1534 ⇒ $\mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_4 > \mathcal{L}_2$.

When q→∞

Score ranking:

0.20(two)>0.15>0.10⇒ $\mathcal{L}_3 = \mathcal{L}_4 > \mathcal{L}_1 > \mathcal{L}_2$
Accuracy ranking:
0.60>0.55>0.40>0.36 $\Rightarrow \mathcal{L}_3 > \mathcal{L}_1 > \mathcal{L}_2 > \mathcal{L}_4$
Step6: End

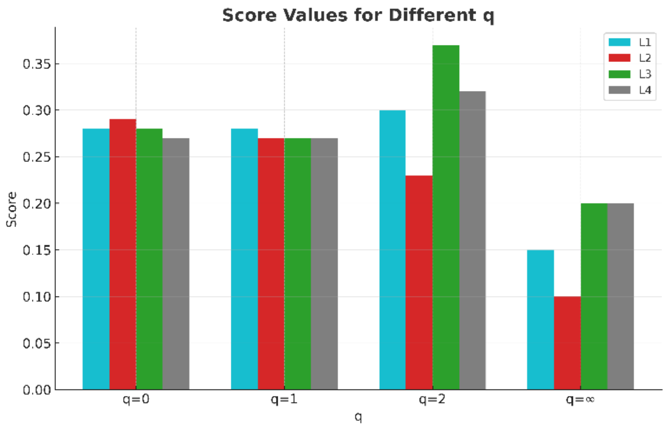

Figure 2: Score values for different $q$parameters for alternatives $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$,and $\mathcal{L}_4$. This plot shows how scores vary with changing $q$values in our proposed method.

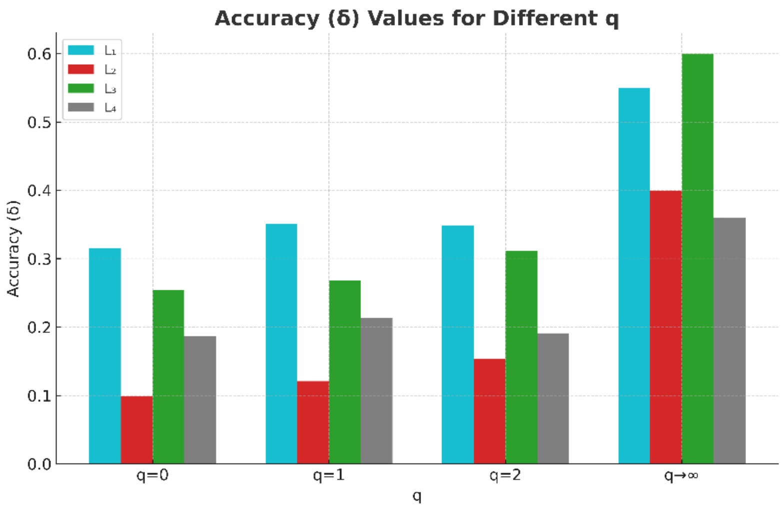

Figure 3: Accuracy (δ) values for different q parameters for alternatives $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$and $\mathcal{L}_4$. This plot shows the variation in accuracy with changing q values in the proposed method.

Comparative study:

A comparative analysis is performed to evaluate the efficacy of the proposed ranking method against existing techniques, including [2], [4]

Operator	Ranking order alternatives	Best Alternatives
Senapati(2024)	$\mathcal{L}_1 > \mathcal{L}_2 > \mathcal{L}_3 > \mathcal{L}_4(\epsilon = 1)$	$\mathcal{L}_1$
Senapati(2024)	$\mathcal{L}_1 > \mathcal{L}_2 > \mathcal{L}_4 > \mathcal{L}_3(\epsilon = 2)$	$\mathcal{L}_1$
Nancy and Garg(2016)	$\mathcal{L}_3 > \mathcal{L}_4 > \mathcal{L}_2 > \mathcal{L}_1$	$\mathcal{L}_3$
Our Method	$\mathcal{L}_2 > \mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_4(q = 0)$	$\mathcal{L}_2$
Our Method	$\mathcal{L}_1 > \mathcal{L}_4 > \mathcal{L}_2 = \mathcal{L}_3(q = 1)$	$\mathcal{L}_1$
Our Method	$\mathcal{L}_3 > \mathcal{L}_4 > \mathcal{L}_1 > \mathcal{L}_2(q = 2)$	$\mathcal{L}_3$
Our Method	$\mathcal{L}_3 = \mathcal{L}_4 > \mathcal{L}_1 > \mathcal{L}_2(q \to \infty)$	$\mathcal{L}_3$
…….	$\mathcal{L}_3 > \mathcal{L}_1 > \mathcal{L}_2 > \mathcal{L}_4(AccuracyRanking)$	$\mathcal{L}_3$

Table 1: Comparison of ranking results using different operators

In Table-1, we have presented a comparative analysis. By senapati [2] (2024) method, the best alternative is $\mathcal{L}_1$ and the worst one is $\mathcal{L}_3$. According to the Nancy and Garg (2016) method, the best alternative is $\mathcal{L}_2$, and the worst one is L1. By our proposed method, when q=1, the best alternative is $\mathcal{L}_1$ and the worst one is $\mathcal{L}$ 3. when q=2, the best alternative is $\mathcal{L}_3$and the worst one is $\mathcal{L}_2$. when q →∞, the best alternative is $\mathcal{L}_3$ and the worst one is $\mathcal{L}_4$. Based on the above comparative study, it is evident that the proposed method achieves better results more quickly than the existing methods.

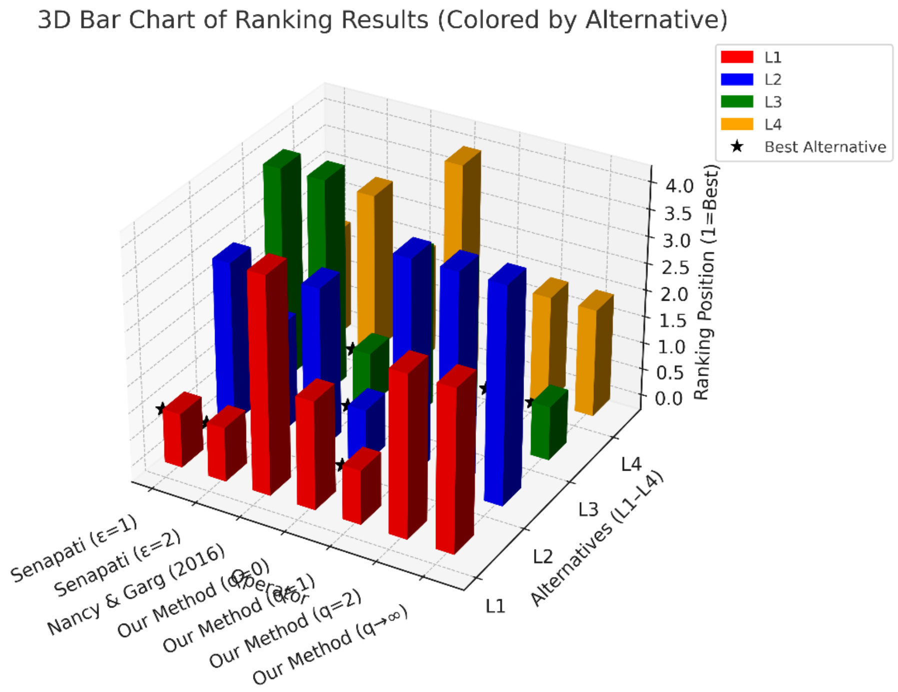

Figure 4: 3D bar chart showing ranking results of alternatives using different operators. The bars are colored by alternative, and stars indicate the best-performing alternatives.

Conclusions

In recent times, aggregation operators have emerged as a prominent area of research within decision-making contexts. Selecting suitable cropland is a crucial decision-making issue in agriculture. The decision-making method with an aggregation operator, which has been successfully applied in the agricultural field, is an interesting topic of research.In this study, we have proposed a new aggregation operator based on SVNSs and proved its necessary properties. We structured a general MCDM method to select cropland for agriculture. The introduced approach is tested on a cropland selection example based on standard criteria. We analyzed the ranking results of our proposed method with the help of a significant comparative study with existing methods. The results obtained for different values of q indicate that the proposed operator works stably and reliably. For q=1, it reflects the best alternative, which is the same as the best alternative obtained from the Senapati (2024) operator result and For q=2 and as q $\to\infty$, the best alternative is the same as the result obtained by Nancy and Garg (2016). This proves that the operator can efficiently handle the complexity and uncertainty of real problems. The proposed MCDM technique requires that each alternative be evaluated against the same set of criteria. This is a necessary condition because the aggregation operators used in this technique are designed to handle a uniform number of criteria across all alternatives enabling the calculation of overall aggregation values. It also has application potential in practical decision-making, especially in teachers' election processes, school management decisions, weather forecasting, and investment evaluation. Future research will concentrate on expanding this operator to other neutrosophic sets and applying it across diverse social and technological domains.

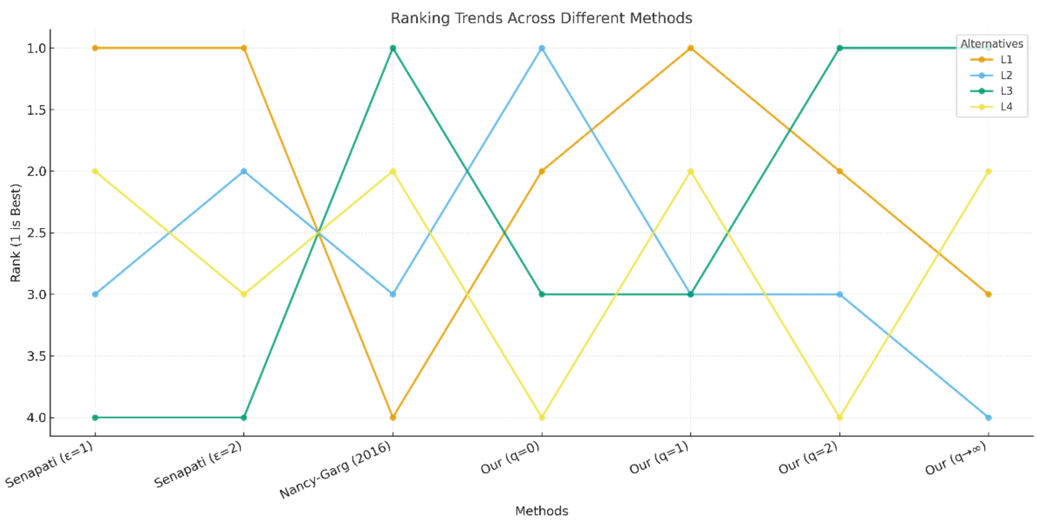

Figure 5: Line graph showing the ranking order of alternatives $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ using differ-ent decision-making approaches. The methods compared include Senapati's operators for ϵ = 1, 2, Nancy-Garg (2016), and the proposed method for q = 0, 1, 2, ∞. A lower rank indicates a better performing alternative (Rank 1 being the best).

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Conflicts of Interest: The authors declare no conflict of interest.

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Article

Applications of Neutrosophic Logic in Psychology and Human Behavior: A Systematic Review

Hanane El ghouat *[1]

[1] Qualified lecturer; Educational sciences; philosophy; and psychology
Affiliation ; Regional Center for Education and Training Professions; Kenitra; Morocco. Email: animeshmahata8@gmail.com

* Correspondence: animeshmahata8@gmail.com

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Abstract: This systematic review examines how neutrosophic logic has been applied in psychology and behavioral science, with particular attention to the representation of psychological indeterminacy through the Truth, Indeterminacy, and Falsity triad. Following a PRISMA-based search strategy, records were identified from Scopus, Web of Science, PubMed, PsycINFO, Google Scholar, ResearchGate, and specialized neutrosophic sources. After screening and eligibility assessment, 30 studies were included in the qualitative synthesis. The reviewed literature indicates that neutrosophic approaches have been used mainly in psychological assessment, behavioral decision-making, clinical diagnostic support, sentiment/perception analysis, and theoretical modeling of ambivalence. The findings suggest that neutrosophic tools may complement classical and fuzzy approaches by explicitly modeling neutrality, hesitation, contradiction, and incomplete information. However, the current evidence remains uneven, with many studies being conceptual, computational, or preliminary rather than clinically validated. Future research should prioritize empirical validation, transparent reporting, reproducible tools, and accessible software for researchers and practitioners.

Keywords: Neutrosophic Logic; Neutropsyche; Human Behavior; Indeterminacy; Systematic Review; Psychometrics; Cognitive Dissonance; Decision-Making; PRISMA; TIF Triad

1. Introduction

1.1. Limitations of Classical Psychological Approaches in Explaining Human Behavior

Traditional psychological approaches often rely on Boolean logic or classical crisp sets to differentiate between psychological states, for example rational and irrational thinking. However, such differences are complex and cannot always be captured through a simple true/false distinction or by a strict membership/non-membership decision. Fuzzy logic improves on Boolean logic by allowing degrees of truth or membership, yet it remains limited when a psychological state contains uncertainty, neutrality, hesitation, or conflicting evidence. These dimensions are particularly important in decision-making situations involving stress, incomplete information, and subjective ambivalence.

Current psychometric models frequently treat ambiguous or inconsistent responses as measurement error. Acciarini et al. (2021) compare traditional psychometric models with stochastic descriptions of human behavior and show that decision processes may include contradictions. Berthet (2022) similarly argues that psychological measurement often contains cognitive noise or ambivalence. A more comprehensive framework should therefore be able to represent contradictory, neutral, and uncertain information without reducing it to error alone.

1.2. Theoretical Foundations of Neutrosophic Logic and Its Interdisciplinary Applications

Florentin Smarandache introduced Neutrosophic Logic as an extension of classical and fuzzy approaches. In this framework, a proposition is evaluated through three independent components: Truth (T), Indeterminacy (I), and Falsity (F). These components form the foundation of neutrosophic modeling and have been applied in several disciplines, including psychology, where the emerging concept of Neutropsyche attempts to describe human tendencies and behavior through a triadic structure.

Recent publications have explored neutrosophic concepts in cultural psychology, cognitive science, psychological ambivalence, and the modeling of human tendencies as dynamic systems. Examples include works by Christianto and Smarandache (2024), Wang and Smarandache (2025), and Smarandache (2026). These studies provide the theoretical background for examining whether neutrosophic logic can contribute to the analysis of complex behavioral data.

1.3. Research Objectives and Research Questions of the Systematic Review

This systematic review synthesizes the growing literature at the intersection of neutrosophic logic and behavioral science. By following PRISMA-oriented procedures, the study evaluates how neutrosophic tools have been used to model human behavior and psychological indeterminacy.

The primary research objectives are:

- To identify the limitations of classical and fuzzy logic in selected psychological literature.
- To analyze applications of neutrosophic sets in personality, emotions, decision-making, and assessment.
- To determine the theoretical contribution of Neutropsyche to contemporary psychological modeling.

This review is guided by the following research questions:

1. How does neutrosophic logic represent psychological indeterminacy compared with classical and fuzzy models?
2. In what ways have neutrosophic Likert scales and related assessment tools been used to improve behavioral data extraction?
3. What gaps remain in the integration of neutrosophic logic into clinical and social psychology?

2. Materials and Methods (PRISMA)

This systematic review was conducted in accordance with the Preferred Reporting Items for Systematic Reviews and Meta-Analyses (PRISMA) statement. The methodology was designed to support a transparent and reproducible synthesis of the literature concerning applications of neutrosophic logic in psychological and behavioral sciences.

2.1. Literature Search Strategy and Databases Used

A comprehensive search was performed in major international databases to identify relevant studies published between 2014 and 2026. The databases searched included Scopus, Web of Science, PubMed, and PsycINFO. Additional searches were conducted through Google Scholar, ResearchGate, and specialized neutrosophic sources to identify recent or field-specific publications. The archives of Neutrosophic Sets and Systems were also manually consulted to include foundational works by Smarandache and collaborators.

The search strategy used Boolean operators AND/OR with terms appearing in titles, abstracts, and/or keywords, including: neutrosophic logic, Neutropsyche, indeterminacy, neutrosophy, human behavior, decision-making, psychological assessment, and neutrosophic sets.

2.2. Inclusion and Exclusion Criteria for Study Selection

Inclusion criteria were defined as follows:

- Studies applying neutrosophic logic, neutrosophic sets, or neutrosophic probability structures to psychological theories, behavioral models, social science diagnostics, or decision-making.
- Peer-reviewed journal articles, conference papers, and book chapters presenting a clear mathematical or conceptual framework.
- English-language publications accessible to the international community of neutrosophic and behavioral-science researchers.
- Studies explicitly addressing one or more T, I, and F components in relation to human cognition, behavior, or assessment.

Exclusion criteria were defined as follows:

- Purely mathematical works on neutrosophic topology, abstract algebra, or geometry without application to human behavior, psychology, or decision-making.
- Editorials, blog-style material, raw unpublished data, and unpublished student theses.
- Duplicate records, repeated versions of the same study, or preliminary records superseded by a later publication.

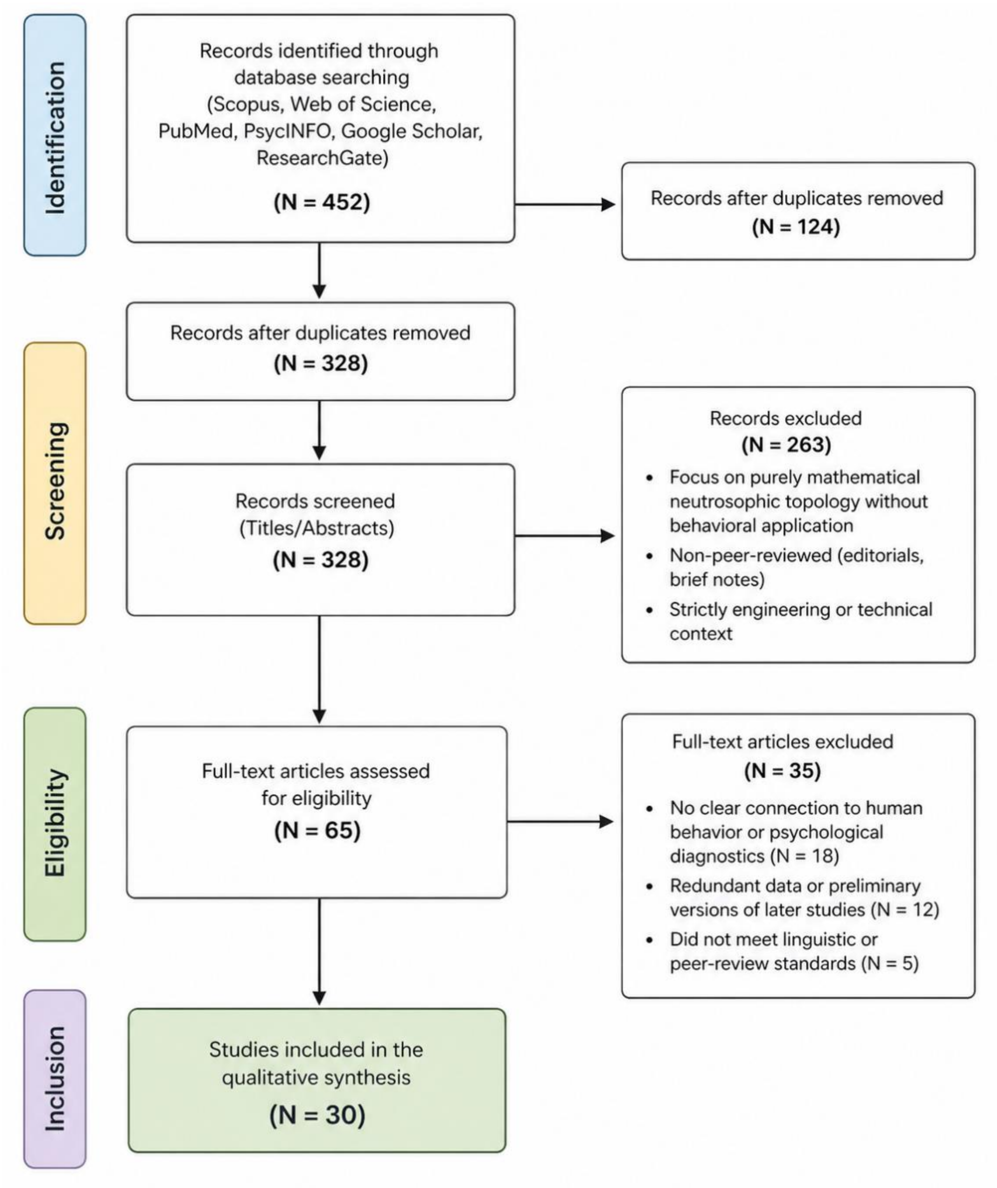
2.3. Study Selection Process and Data Extraction

The selection process consisted of four stages: identification, screening, eligibility assessment, and inclusion.

1. Identification: records were identified through database searches and supplementary searches, then imported into reference management software.
2. Screening: titles and abstracts were screened for relevance to neutrosophic logic, behavioral science, and psychological applications.
3. Eligibility: full-text articles were assessed against the inclusion and exclusion criteria. Particular attention was given to studies addressing behavioral decision-making, psychological assessment, and diagnostic ambiguity.
4. Inclusion: 30 studies met the criteria and were included in the qualitative synthesis.

For each included study, data were extracted using a structured form covering authorship, year of publication, neutrosophic method used, psychological domain, and contribution to the modeling of human behavior or cognition. Extracted methods included single-valued neutrosophic sets, interval neutrosophic sets, neutrosophic Likert scales, and neutrosophic multi-criteria decision-making approaches.

Figure 1. Study selection process (PRISMA).



3. Results

3.1. General Characteristics of the Selected Studies

The final synthesis included 30 papers addressing non-binary or neutrosophic approaches to behavioral and psychological modeling. The publication pattern suggests increased attention after 2020. Earlier papers were largely theoretical and focused on the concept of Neutropsyche and its implications, while more recent studies have increasingly addressed empirical applications, decision-making frameworks, and software-assisted neutrosophic data analysis.

The reviewed studies came from different research contexts and included journal articles, conference papers, and specialized book chapters. The collection reflects an interdisciplinary literature rather than a mature clinical evidence base. Therefore, the findings should be interpreted as a structured synthesis of current applications and theoretical directions rather than as proof of clinical superiority.

3.2. Applications of Neutrosophic Logic in Psychological and Behavioral Studies

Three main areas of application were identified in the reviewed literature:

- Psychological assessment and measurement: neutrosophic scales, particularly Neutrosophic Likert Scales, have been proposed to allow respondents to express agreement, disagreement, neutrality, and uncertainty more explicitly than in classical Likert formats.
- Clinical diagnosis and decision-making: neutrosophic multi-criteria decision-making methods have been applied to problems involving overlapping symptoms and incomplete information. For example, Gomez and Ludeña (2023) applied neutrosophic reasoning to anxiety diagnosis, while Abdel-Basset et al. (2019) used neutrosophic MCDM in professional selection contexts relevant to psychology.
- Organizational and social behavior: studies such as Nabeeh (2023) and Al Awadh (2022) used single-valued neutrosophic sets to model decision-maker behavior under uncertainty in professional and organizational contexts.

3.3. Main Findings on the Contribution of Neutrosophic Logic to Understanding Human Behavior

Across the 30 reviewed studies, neutrosophic logic appears to contribute mainly by formalizing indeterminacy rather than treating it only as random error. The following findings summarize the main contributions identified in the selected studies.

1. Ambivalence can be represented as a measurable component. Classical models often treat ambivalence as an exception or outlier, whereas neutrosophic approaches model it through the I component within a triadic structure.
2. Neutrosophic sets may refine sentiment and perception analysis by separating neutral or indeterminate responses from positive and negative polarities. Reported accuracy gains should be interpreted cautiously because they depend on datasets, models, and validation procedures.
3. The Neutropsyche and related personality models describe psychological traits as dynamic tendencies rather than fixed binary categories.
4. Neutrosophic models may be useful in difficult or crisis situations where information is incomplete, contradictory, or uncertain, because they explicitly encode the unknown component instead of forcing a binary classification.

Table 1. Summary of representative studies and their contributions to behavioral science.

Author(s) and year	Neutrosophic tool/model	Psychological domain	Key contribution to psychology
Smarandache (2016, 2026)	Neutropsyche; neutrosophic personality	Theoretical psychology	Introduced and developed a triadic structure (T, I, F) for modeling human tendencies as dynamic systems.
Patro and Smarandache (2024)	Neutrosophic Likert Scale	Psychometrics	Proposed a scale format that allows respondents to express uncertainty or neutral feelings during assessment.
Gomez and Ludeña (2023)	Neutrosophic MCDM/TOPSIS	Clinical diagnosis	Applied indeterminacy to the interpretation of overlapping symptoms in anxiety-related diagnostic support.

Wang and Smarandache (2025)	Single-valued neutrosophic sets	Cognitive science	Formalized the representation of psychological ambivalence and states of uncertainty.
Nabeeh (2023)	Integrated SVNS decision-making	Organizational behavior	Modeled decision strategies under professional uncertainty and multi-criteria constraints.
Dhamodharan (2024)	Neutrosophic sentiment analysis	Social psychology	Separated neutral sentiment from positive and negative polarities in perception analysis.
Șahin (2022)	Neutrosophic uncertainty modeling	Cognitive psychology	Discussed limitations of binary logic in explaining gray areas of behavior and risk perception.
Duran et al. (2021)	Confirmatory neutrosophic analysis	Well-being and life satisfaction	Converted the Satisfaction with Life Scale into a neutrosophic form and compared confirmatory analysis results.
Berthet (2022)	Cognitive-bias framework	Behavioral economics	Interpreted cognitive biases as responses to uncertain or incomplete information rather than only as errors.
Heredia et al. (2024)	AHP, Delphi, and neutrosophic logic	Crisis management	Integrated expert judgment and neutrosophic logic for decision-making in complex environments.

4. Discussion

4.1. Interpretation of Findings in the Context of Contemporary Psychological Theories

This systematic review indicates that neutrosophic logic can function as a bridge between mathematical formalization and the complexity of human cognition. In classical interpretations of cognitive dissonance, two conflicting beliefs or facts may produce psychological tension. A neutrosophic interpretation adds an explicit indeterminate component, allowing the model to represent hesitation, uncertainty, or the subjective feeling of not knowing how to decide.

The analysis of personality traits also suggests that rigid boundaries between traits may be insufficient for describing actual psychological variability. In this context, the neutrosophic component of indeterminacy can represent ambiguous or unstable responses without immediately classifying them as error. This perspective is consistent with recent discussions of cognitive biases, where some responses may reflect processing under uncertainty rather than simple irrationality.

4.2. Theoretical and Conceptual Contributions of Neutrosophic Logic to Psychology

Smarandache (2016, 2026) proposed the Neutropsyche model to represent human tendencies and behaviors through a triadic structure. The theoretical value of this framework lies in moving from binary opposition toward a model that can include truth, falsity, and indeterminacy simultaneously. This is particularly relevant in psychological contexts where respondents may agree with an item, disagree with it, and still retain some degree of uncertainty or neutrality.

For psychological assessment, neutrosophic scales may provide a more detailed structure for collecting data on perceptions, attitudes, experiences, and evaluations. In clinical or diagnostic contexts, neutrosophic multi-criteria decision-making can support the interpretation of overlapping symptoms when available information is incomplete or uncertain. These contributions remain promising, but they require stronger empirical testing before broad practical adoption.

4.3. Limitations of the Current Literature and Future Research Directions

Despite growing interest, the literature remains limited. Many studies are still theoretical, computational, or based on preliminary applications. There is a lack of large longitudinal clinical studies comparing traditional psychological diagnostic methods with neutrosophic frameworks under controlled conditions.

A second limitation concerns usability. Neutrosophic methods may require mathematical knowledge that is not common among behavioral researchers and clinicians. Future work should therefore focus on transparent validation studies, open datasets when possible, user-friendly software, and reporting standards that make neutrosophic models easier to evaluate and reproduce.

5. Conclusions

This systematic review examined how neutrosophic logic can be used to model aspects of human behavior that are difficult to capture through binary or fuzzy approaches alone. The reviewed studies show that the Truth-Indeterminacy-Falsity triad may be useful for representing ambivalence, uncertainty, and neutrality in psychological assessment, behavioral decision-making, and theoretical personality modeling.

The use of Neutrosophic Likert Scales and neutrosophic multi-criteria decision-making methods suggests a possible direction for instruments that preserve uncertain or neutral information rather than eliminating it during analysis. Nevertheless, the field requires stronger empirical evidence, clearer methodological standards, and accessible computational tools.

Future research should prioritize empirical validation in clinical and social psychology, comparative studies against existing psychometric models, and software that allows researchers to apply neutrosophic methods without excessive computational barriers.

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Conflicts of Interest: The author(s) declare no conflict of interest.

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Article

Neutral/Neutrality are parts of the generic group called Indeterminacy

Florentin Smarandache[1,*]

[1] University on New Mexico, Campus de Gallup, United States
* Correspondence: smarand@unm.edu

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Abstract: In this short note we show that C. Ardil's assertion "neutrality is frequently interpreted as a manifestation of the indeterminacy component" is false. We have showed from the beginning in 1998 that "Neutrosophy is a new branch of philosophy, which studies the origin, nature, and scope of **neutralities**, as well as their interactions with different ideational spectra" [1].
His Score-Based Ranking for Neutrosophic Logic is wrong [3] and his Neutrosophic Conjunction and Neutrosophic Disjunction operators are superficial (reduced only to min/max operations), while the Neutrosophic Logic Operators are based on T-norm and T-conorm and have a general form.

Neutral Set/Logic, that uses only neutrality, is a subclass of the Neutrosophic Set/Logic, that uses both neutrality and all types of indeterminacies.

Keywords: Neutrosophy, Neutrality, Single-Valued Neutrosophic Set, Neutral Set, neutrosophic logic, neutrosophic set theory, mathematical equivalence, indeterminacy, truth membership, falsity membership, paraconsistent information, uncertainty modeling.

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1. Introduction

In [3] C. Ardil asserts: "Neutrosophic Logic and related neutrosophic set theories have emerged as influential frameworks for representing uncertainty, inconsistency, incompleteness, and indeterminacy. Within this paradigm, neutrality is frequently interpreted as a manifestation of the indeterminacy component. This paper re-examines that assumption from philosophical, ontological, and mathematical perspectives."

2. Etymology and Definition of Neutrosophy ([1], p. 16, 1998)

A) Etymology: Neutro-sophy [French neutre < Latin neuter, neutral, and Greek sophia, skill/wisdom] means knowledge of neutral thought.
B) Definition: Neutrosophy is a new branch of philosophy, which studies the origin, nature, and scope of **neutralities**, as well as their interactions with different ideational spectra.
We have clearly explained in 2021 on the paper [2] that:

"Indeterminacy" is just a generic name for everything that exists in between the opposites
<A > and <antiA>,
it is not strictly (literary) "indeterminacy" (= unclearness).

3. Type of Indeterminacies and Neutralities [2]

"Indeterminacy" is a whole group or class of concepts in between the opposites <A> and <antiA>, and the type of "indeterminacy" depends on each application's space.

For examples (see again the link https://fs.unm.edu/Indeterminacy.pdf):

(Friend, **Neutral**, Enemy), Indeterminacy = Neutral (in this case)

(Proton, **Neutron**, Electron), Indeterminacy = Neuron

(Win, **Tie-game**, Lose), Indeterminacy = Tie-game

(Matter, **Unmatter**, Antimatter), Indeterminacy = Unmatter

(Male, **Transgender**, Female), Indeterminacy = Transgender

(Small, **Medium**, Tall), Indeterminacy = Medium

(White, **Red**, Black), in this case Indeterminacy = Red

(True, **Partially-true & Partially-false**, False), in this case
Indeterminacy = Partially-true & Partially-false

"The most general definition, the classification, and many real examples of Indeterminacies from our everyday life, utilized in the neutrosophic theories and their applications, are presented below in an understandable manner. **"Indeterminacy" should not be taken into the narrow sense of a lexical dictionary, but as something that is in between the opposites**.

Because of dealing with various types of indeterminacies (vague, unclear, uncertain, conflicting, incomplete, hesitancy, **neutrality**, unknown, etc.) related to the data or to the procedures employed in our real world, we may extend by neutrosophication any classical scientific or cultural crisp concept from any field of knowledge to a corresponding neutrosophic (un-crisp) concept, since in our world more things are indeterminate or partially indeterminate than completely determinate.

Firstly, let's define the neutrosophic triplets. Let be an item (concept, notion, idea, sentence, theory etc.) <A> and its opposite <antiA>. In between the opposites <A> and <antiA>, there is a **neutral** (or indeterminacy) part, denoted by **<neutA>**.
The <neutA > is neither <A> nor <antiA>,
or sometimes the <neutA> is a mixture of partial <A> and partial <antiA>. [2]

Of course, we consider the neutrosophic triplets (T, I, F) that make sense in the world, and there are plenty of such triplets in our everyday life [1].

4. Score Function for Neutrosophic Logic [4]

It is defined as:

$$s(T,I,F) = [T+(1-I)+(1-F)]/3 = [2+T-I-F]/3$$

and represents the average of positiveness of the single-valued neutrosophic components T, I, F, where T is considered of positive quality, while I and F of negative quality which implies that 1 – I and 1- F are of positive quality.

Ardil's (actually AI's) Scor Function, defined as

$$s(T, I, F) = T - F$$

is wrong, because the component I is completely ignored.

And, by the way, only the Score Function is not enough in order to get a total order on the set of triplets (T, I, F), two more functions are needed:
Accuracy Function and Certainty Function [see 4].

5. Neutrosophic Conjunction and Neutrosophic Disjunction

C. Ardil uses only min/max for these operators, which is superficial, since the Neutrosophic Conjunction and Disjunction are based more generally on T-norm and T-conorm.
About applying min/max/max or min/min/max (and similarly for max/min/min or max/max/min) it depends to where is indeterminacy/neutrality I close to, either much close to F, or respectively much closer to T.

Neutral Set/Logic, that uses only neutrality, is a subclass of the Neutrosophic Set/Logic, that uses both neutrality and all types of indeterminacies.

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DOUBLE-VALUED NEUTROSOPHIC HYPERGRAPHS

Suriyakumar G[1], *V. J. Sudhakar*[2*], *and Takaaki Fujita*[3]

[1] Research Scholar, Department of Mathematics, Islamiah College (Autonomous), Vaniyambadi, Affiliated to Thiruvalluvar University, Vellore, Tamil Nadu, India. gansuriyakumar@gmail.com

[2] Department of Mathematics, Islamiah College (Autonomous), Vaniyambadi, Affiliated to Thiruvalluvar University, Vellore, Tamil Nadu, India. vjsvec1@gmail.com

[3] Independent Researcher, Tokyo, Japan. Takaaki.fujita060@gmail.com

[*]Correspondence: vjsvec1@gmail.com

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Abstract. Double-valued neutrosophic sets provide an effective framework for representing uncertainty, indeterminacy, and inconsistency by incorporating truth, falsity, and two distinct indeterminacy components. Hypergraphs generalize classical graphs by allowing a hyperedge to connect more than two vertices, making them suitable for modeling higher-order relationships in complex systems.

In this paper, we extend the concept of double-valued neutrosophic sets to hypergraph structures and introduce the notion of a double-valued neutrosophic hypergraph (DVNHG). We develop the fundamental theory of DVNHGs by defining their height, level hypergraphs, lower and upper truncations, transition levels, and core hypergraph structures. Furthermore, we investigate ordered and sectionally elementary DVNHGs and study properties of double-valued neutrosophic transversals and minimal transversals.

In addition, we introduce the concept of T-related double-valued neutrosophic hypergraphs and establish several fundamental results concerning their transversal structures. The proposed framework generalizes existing neutrosophic hypergraph models and provides a richer mathematical tool for representing complex uncertainty in higher-order relational systems.

Keywords: Neutrosophic hypergraph, Single-valued neutrosophic hypergraph, Double-valued neutrosophic hypergraph.

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1. **Introduction**

Fuzzy set theory was introduced by Zadeh [1] as a mathematical framework for representing vague and imprecise information. Later, Smarandache introduced neutrosophic sets (NSs) as a more general tool for handling ambiguous, incomplete, indeterminate, and inconsistent information [2, 3]. A neutrosophic set is characterized by three independent components: a truth-membership function (T), an indeterminacy-membership function (I), and a falsity-membership function (F), whose values are usually taken from the standard or nonstandard real unit interval. To facilitate the application of neutrosophic sets in practical problems, Wang et al. [4] proposed the notion of a single-valued neutrosophic set (SVNS), in which the three membership degrees are restricted to the standard interval $[0, 1]$.

Neutrosophic sets generalize fuzzy sets and intuitionistic fuzzy sets by treating truth, indeterminacy, and falsity as independent information components. This feature makes neutrosophic models more flexible for describing uncertainty than classical, fuzzy, or intuitionistic fuzzy models. In graph theory, neutrosophic information has been incorporated into vertices and edges, giving rise to several types of neutrosophic graphs (cf. [35,36]). Broumi et al. [5] studied various classes of single-valued neutrosophic graphs, including complete, constant, and strong single-valued neutrosophic graphs. Akram et al. [6, 7] further investigated operations and structural properties of complex and single-valued neutrosophic graphs. Moreover, plithogenic sets are also known as an extended concept of neutrosophic sets [26, 27].

Double-valued neutrosophic sets, together with their applications to minimal spanning trees and clustering techniques, were first proposed by Ilanthenral Kandasamy [8]. Double-valued neutrosophic graphs were subsequently introduced by Takaaki Fujita [9]. In the double-valued neutrosophic framework, the indeterminacy component is divided into two distinct parts: indeterminacy leaning toward truth and indeterminacy leaning toward falsity. This separation enables a more refined representation of ambiguous and mixed information than the single-valued neutrosophic setting. More recently, triple-valued neutrosophic sets were proposed as an extension of double-valued neutrosophic sets [10, 11, 25].

Hypergraphs provide a natural extension of classical graphs by allowing a hyperedge to connect more than two vertices. In uncertain environments, this higher-order structure has been combined with fuzzy and neutrosophic frameworks. Kaufmann introduced the notion of fuzzy hypergraphs, and Chen [12] later introduced interval-valued fuzzy hypergraphs. Lee-Kwang [13] generalized the concept of fuzzy hypergraphs and reformulated it for the fuzzy partition of a system. Akram and Dudek [14] investigated several properties of intuitionistic fuzzy hypergraphs and presented their applications. Akram [15] further introduced various concepts related to single-valued neutrosophic hypergraphs, including line graphs, dual hypergraphs, and transversal single-valued neutrosophic hypergraphs. Because of the importance of studying neutrosophic hypergraphs, neutrosophic hypergraphs have been investigated in works such as [31–34]. Table 1 provides a concise comparison between single-valued neutrosophic hypergraphs and double-valued neutrosophic hypergraphs.

Many related concepts have also been developed in the theory of neutrosophic graphs. For example, neutrosophic soft graphs combine soft-set parameters with neutrosophic graph structures, thereby modeling uncertain relationships through truth, indeterminacy, and falsity degrees assigned to vertices and edges [21–24]. In addition, complex neutrosophic soft graphs (CNSGs) have been developed by SuriyaKumar et al. [16–20] as an advanced framework for representing ambiguity, uncertainty, and indeterminacy in complex systems. In particular, strong complex neutrosophic soft graphs (SCNSGs) incorporate soft-set parameters together with complex-valued membership degrees for both vertices and edges, providing a flexible representation of neutrosophic uncertainty. Related studies have investigated the complements of complex neutrosophic soft graphs and strong complex neutrosophic soft graphs, proposed measures reflecting the cumulative strength and uncertainty of graph relations, and investigated structural properties of strong double-valued complex neutrosophic graphs, including their subgraphs and complements.

The advantages of DVNHG over SVNHG are DVNHGs provide a more detailed representation of uncertainty by separating indeterminacy into truth-oriented and falsity-oriented components. DVNHGs can effectively handle inconsistent, incomplete, and conflicting information. DVNHGs possess greater descriptive power for modelling complex systems. DVNHGs offer more accurate representations of real-world scenarios involving multiple information sources. DVNHGs enhance decision-making processes under uncertain environments. The relationship between SVNHG and DVNHG is an SVNHG can be regarded as a special case of a DVNHG when the truth-oriented and falsity-oriented indeterminacy components are combined into a single indeterminacy degree. Therefore,

$$\text{SVNHG} \subseteq \text{DVNHG}.$$

Hence, DVNHGs constitute a more generalized and expressive framework for modelling uncertainty in hypergraph structures.

Table 1. Comparison between SVNHGs and DVNHGs

Feature	**SVNHG**	**DVNHG**
Basic Structure	Based on Single-Valued Neutrosophic Sets (SVNSs)	Based on Double-Valued Neutrosophic Sets (DVNSs)
Membership Components	Truth ($\mathfrak{t}$), Indeterminacy ($\mathfrak{i}$), Falsity ($\mathfrak{f}$)	Truth ($\mathfrak{t}$), Truth-oriented Indeterminacy ($\mathfrak{i}^{\mathfrak{t}}$), Falsity-oriented Indeterminacy ($\mathfrak{i}^{\mathfrak{f}}$), Falsity ($\mathfrak{f}$)
Vertex Representation	Each vertex has three degrees: $(\mathfrak{t}_S, \mathfrak{i}_S, \mathfrak{f}_S) \in [0,1]^3$.	Each vertex has four degrees: $(\mathfrak{t}_S, \mathfrak{i}^{\mathfrak{t}}_S, \mathfrak{i}^{\mathfrak{f}}_S, \mathfrak{f}_S) \in [0,1]^4$.
Hyperedge Representation	Each hyperedge has three degrees: $(\mathfrak{t}_K, \mathfrak{i}_K, \mathfrak{f}_K) \in [0,1]^3$.	Each hyperedge has four degrees: $(\mathfrak{t}_K, \mathfrak{i}^{\mathfrak{t}}_K, \mathfrak{i}^{\mathfrak{f}}_K, \mathfrak{f}_K) \in [0,1]^4$.
Constraint	$0 \le \mathfrak{t} + \mathfrak{i} + \mathfrak{f} \le 3$	$0 \le \mathfrak{t} + \mathfrak{i}^{\mathfrak{t}} + \mathfrak{i}^{\mathfrak{f}} + \mathfrak{f} \le 4$
Height of Hypergraph	$(\max \mathfrak{t}, \max \mathfrak{i}, \min \mathfrak{f})$	$(\max \mathfrak{t}, \max \mathfrak{i}^{\mathfrak{t}}, \max \mathfrak{i}^{\mathfrak{f}}, \min \mathfrak{f})$
Treatment of Indeterminacy	Single indeterminacy degree	Two independent indeterminacy degrees
Information Richness	Moderate	High
Ability to Model Uncertainty	Good	Better
Ability to Capture Conflicting Information	Limited	More effective
Expressiveness	Lower	Higher
Computational Complexity	Lower	Higher
Main advantage	Simpler and easier to handle.	More expressive for ambiguous and mixed evidence.

Motivated by these developments, this paper introduces double-valued neutrosophic hypergraphs. Specifically, we define double-valued neutrosophic hypergraphs, level hypergraphs, lower truncations, upper truncations, and transition levels. Furthermore, we introduce T-related double-valued neutrosophic hypergraphs and discuss several properties of their minimal transversals.

2. Preliminaries

In this section, we recall the basic definitions and notation needed in the sequel. In particular, we review single-valued neutrosophic sets, single-valued neutrosophic hypergraphs, single-valued neutrosophic graphs, and double-valued neutrosophic graphs. These notions serve as the foundation for the construction of double-valued neutrosophic hypergraphs developed in the next section.

Definition 2.1. *The support of a single-valued neutrosophic set* $A = \{(x, \mathfrak{t}_A(x), \mathfrak{i}_A(x), \mathbf{f}_A(x)) : x \in X\}$ *is denoted by supp(A), defined by* $supp(A) = \{x \mid \mathfrak{t}_A(x) \neq 0; \mathfrak{i}_A(x) \neq 0; \mathfrak{f}_A(x) \neq 0\}$. *The support of a single-valued neutrosophic set is a crisp set.*

Definition 2.2. *The height of a single-valued neutrosophic set* $A = \{(x, \mathfrak{t}_A(x), \mathfrak{i}_A(x), \mathbf{f}_A(x)) : x \in X\}$ *is defined as* $h(A) = (sup_{x\in X}\mathfrak{t}_A(x); sup_{x\in X}\mathfrak{i}_A(x); inf_{x\in X}\mathfrak{f}_A(x))$. *A single-valued neutrosophic set A is called normal if there exists at least one element* $x \in X$ *such that* $\mathfrak{t}_A(x) = 1, \mathfrak{i}_A(x) = 1, \mathbf{f}_A(x) = 0$.

Definition 2.3. *Let* $A = \{(x, \mathfrak{t}_A(x), \mathfrak{i}_A(x), \mathbf{f}_A(x)) : x \in X\}$ *be a single-valued neutrosophic set on X and let* $a, b, c \in [0, 1]$ *such that* $a + b + c \leq 3$. *Then the set* $A_{(a,b,c)} = \{(x; \mathfrak{t}_A(x) \geq a, \mathfrak{i}_A(x) \geq b, \mathbf{f}_A(x) \leq c)\}$ *is called* (a, b, c) *-level subset of A.* (a, b, c) *-level set is a crisp set.*

Definition 2.4. *The height of a single-valued neutrosophic hypergraph* $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$*, is denoted by* $h(\mathfrak{H})$*, is defined by* $h(\mathfrak{H}) = \bigvee_i \{h(\mathfrak{E}_i) \mid \mathfrak{E}_i \in \mathfrak{E}\}$

Definition 2.5. *Let* $G^* = (V, E)$ *be a graph. A SVNG on* G^* *or an SVNG where* $S = (\mathfrak{t}_S, \mathfrak{i}_S, \mathfrak{f}_S)$ *is a single-valued neutrosophic set on V and* $K = (\mathfrak{t}_K, \mathfrak{i}_K, \mathfrak{f}_K)$ *is a single-valued neutrosophic set on E*

1. $V = \{u_1, u_2, ..., u_n\}$ *such that* $\mathfrak{t}_S : V \to [0, 1]$, $\mathfrak{i}_S : V \to [0, 1]$ *and* $\mathfrak{f}_S : V \to [0, 1]$, *indicates the degree of truth, indeterminacy and falsity membership function of the element* $u_i \in V$

$$0 \leq \mathfrak{t}_S(u_i) + \mathfrak{i}_S(u_i) + \mathfrak{f}_S(u_i) \leq 3$$

2. $E \subseteq V \times V$ *where* $\mathfrak{t}_K : V \times V \to [0, 1]$, $\mathfrak{i}_K : V \times V \to [0, 1]$ *and* $\mathfrak{f}_K : V \times V \to [0, 1]$. *indicates the degree of truth, indeterminacy and falsity membership of the element* $u_iu_j \in E$, *respectively and such that,*

$$0 \leq \mathfrak{t}_K(u_iu_j) + \mathfrak{i}_K(u_iu_j) + \mathfrak{f}_K(u_iu_j) \leq 3$$

such that

$$\mathfrak{t}_K(u_iu_j) \leq \min\{\mathfrak{t}_S(u_i), \mathfrak{t}_S(u_j)\}$$

$$\mathfrak{i}_K(u_iu_j) \leq \min\{\mathfrak{i}_S(u_i), \mathfrak{i}_S(u_j)\}$$

$$\mathfrak{f}_K(u_iu_j) \geq \max\{\mathfrak{f}_S(u_i), \mathfrak{f}_S(u_j)\}$$

Definition 2.6. *Let us examine the graph $G^* = (V, E)$. A double-valued neutrosophic graph (DVNG) on G^* or a DVNG where $S = (\mathfrak{t}_S, \mathfrak{i}^{\mathfrak{t}}_S, \mathfrak{i}^{\mathfrak{f}}_S, \mathfrak{f}_S)$ is a double-valued neutrosophic set on V and $K = (\mathfrak{t}_K, \mathfrak{i}^{\mathfrak{t}}_K, \mathfrak{i}^{\mathfrak{f}}_K, \mathfrak{f}_K)$ is a double-valued neutrosophic set $E \subseteq V \times V$ where,*

1. $V = \{u_1, u_2, ..., u_n\}$ such that $\mathfrak{t}_S : V \to [0,1]$, $\mathfrak{i}^{\mathfrak{t}}_S : V \to [0,1]$, $\mathfrak{i}^{\mathfrak{f}}_S : V \to [0,1]$ and $\mathfrak{f}_S : V \to [0,1]$, indicates the degree of truth membership function, indeterminacy leaning towards truth, indeterminacy leaning towards falsity and falsity membership function of the element $u_i \in V$

$$0 \leq \mathfrak{t}_S(u_i) + \mathfrak{i}^{\mathfrak{t}}_S(u_i) + \mathfrak{i}^{\mathfrak{f}}_S(u_i) + \mathfrak{f}_S(u_i) \leq 4$$

2. $E \subseteq V \times V$ where $\mathfrak{t}_K : V \times V \to [0,1]$, $\mathfrak{i}^{\mathfrak{t}}_K : V \times V \to [0,1]$, $\mathfrak{i}^{\mathfrak{f}}_K : V \times V \to [0,1]$ and $\mathfrak{f}_K : V \times V \to [0,1]$. indicates the degree of truth membership function, indeterminacy membership and falsehood membership of the element $u_i u_j \in E$, respectively and such that,

$$0 \leq \mathfrak{t}_K(u_i u_j) + \mathfrak{i}^{\mathfrak{t}}_K(u_i u_j) + \mathfrak{i}^{\mathfrak{f}}_K(u_i u_j) + \mathfrak{f}_K(u_i u_j) \leq 4$$

such that

$$\mathfrak{t}_K(u_i u_j) \leq \min\{\mathfrak{t}_S(u_i), \mathfrak{t}_S(u_j)\}$$
$$\mathfrak{i}^{\mathfrak{t}}_K(u_i u_j) \leq \min\{\mathfrak{i}^{\mathfrak{t}}_S(u_i), \mathfrak{i}^{\mathfrak{t}}_S(u_j)\}$$
$$\mathfrak{i}^{\mathfrak{f}}_K(u_i u_j) \leq \min\{\mathfrak{i}^{\mathfrak{f}}_S(u_i), \mathfrak{i}^{\mathfrak{f}}_S(u_j)\}$$
$$\mathfrak{f}_K(u_i u_j) \geq \max\{\mathfrak{f}_S(u_i), \mathfrak{f}_S(u_j)\}$$

Definition 2.7. *Let S be a finite nonempty set of vertices. A fuzzy hypergraph is an ordered pair $G = (S, K)$, where $K = \{K_1, K_2, \cdots, K_m\}$ is a family of fuzzy subsets of S. Each hyperedge S_i is defined by a membership function $\mathfrak{t}_{K_i} : S \to [0,1]$, where $\mathfrak{t}_{K_i}(u)$ represents the degree of membership of the vertex u in the hyperedge K_i. Hence, a fuzzy hypergraph may be written as*

$$G = (S, \{\mathfrak{t}_{K_1}, \mathfrak{t}_{K_2}, \cdots, \mathfrak{t}_{K_m}\}).$$

Definition 2.8. *Let S be a finite nonempty set of vertices. An intuitionistic fuzzy hypergraph is an ordered pair $G = (S, K)$, where $K = \{K_1, K_2, \cdots, K_m\}$ is a family of intuitionistic fuzzy subsets of S. For each hyperedge K_i, there exist two functions $\mathfrak{t}_{K_i} : S \to [0,1]$ and $\mathfrak{f}_{K_i} : S \to [0,1]$, representing the membership degree and non-membership degree of a vertex in K_i, respectively, satisfying*

$$0 \leq \mathfrak{t}_{K_i}(u) + \mathfrak{f}_{K_i}(u) \leq 1, \qquad \forall u \in S.$$

The hesitation degree is defined by

$$\rho_{K_i}(u) = 1 - \mathfrak{t}_{K_i}(u) - \mathfrak{f}_{K_i}(u).$$

Therefore, an intuitionistic fuzzy hypergraph can be represented as

$$G = (S, \{(\mathfrak{t}_{K_1}, \mathfrak{f}_{K_1}), (\mathfrak{t}_{K_2}, \mathfrak{f}_{K_2}), \cdots, (\mathfrak{t}_{K_m}, \mathfrak{f}_{K_m})\}).$$

Definition 2.9. *Let S be a finite nonempty set of vertices. A neutrosophic hypergraph is an ordered pair $G = (S, K)$, where $K = \{K_1, K_2, \cdots, K_m\}$ is a family of neutrosophic subsets of S. For each hyperedge K_i, there exist three functions $\mathfrak{t}_{K_i} : S \to [0,1]$, $\quad \mathfrak{i}_{K_i} : S \to [0,1]$, $\quad \mathfrak{f}_{K_i} : S \to [0,1]$, where $\mathfrak{t}_{K_i}(u)$, $\mathfrak{i}_{K_i}(u)$, and $\mathfrak{f}_{K_i}(u)$ denote the truth-membership, indeterminacy-membership, and falsity-membership degrees of the vertex u in the hyperedge K_i, respectively. These functions satisfy*

$$0 \le \mathfrak{t}_{K_i}(u) + \mathfrak{i}_{K_i}(u) + \mathfrak{f}_{K_i}(u) \le 3, \qquad \forall u \in S.$$

Therefore, a neutrosophic hypergraph can be represented as

$$G = \left(S, \{(\mathfrak{t}_{K_1}, \mathfrak{i}_{K_1}, \mathfrak{f}_{K_1}), (\mathfrak{t}_{K_2}, \mathfrak{i}_{K_2}, \mathfrak{f}_{K_2}), \cdots, (\mathfrak{t}_{K_m}, \mathfrak{i}_{K_m}, \mathfrak{f}_{K_m})\}\right).$$

Definition 2.10. *A Neutrosophic Super-Hypergraph is a generalization of a neutrosophic hypergraph in which both vertices and hyperedges may themselves be neutrosophic sets. This model is capable of representing higher-order and hierarchical relationships under uncertainty. Let $S = \{S_1, S_2, \cdots, S_n\}$ be a finite family of neutrosophic sets called super-vertices. A* Neutrosophic Super-Hypergraph *is an ordered pair $\mathcal{G} = (S, K)$, where $K = \{K_1, K_2, \cdots, K_m\}$ is a family of neutrosophic super-hyperedges joining one or more super-vertices. Each super-vertex S_i is associated with three membership functions $\mathfrak{t}_S(S_i)$, $\quad \mathfrak{i}_S(S_i)$, $\quad \mathfrak{f}_S(S_i) : S \to [0,1]$, representing the truth-membership, indeterminacy-membership, and falsity-membership degrees, respectively. Similarly, each super-hyperedge K_j is characterized by $\mathfrak{t}_K(K_j)$, $\quad \mathfrak{i}_K(K_j)$, $\quad \mathfrak{f}_K(K_j) : K \to [0,1]$, such that*

$$0 \le \mathfrak{t}_S(S_i) + \mathfrak{i}_S(S_i) + \mathfrak{f}_S(S_i) \le 3,$$

and

$$0 \le \mathfrak{t}_K(K_j) + \mathfrak{i}_K(K_j) + \mathfrak{f}_K(K_j) \le 3.$$

Therefore, a Neutrosophic Super-Hypergraph is represented as

$$\mathcal{G} = \left(S, K, \mathfrak{t}_S, \mathfrak{i}_S, \mathfrak{f}_S, \mathfrak{t}_K, \mathfrak{i}_K, \mathfrak{f}_K\right).$$

3. DOUBLE-VALUED NEUTROSOPHIC HYPERGRAPHS (DVNHGs)

Double-valued neutrosophic hypergraphs model higher-order relations by assigning truth, falsity, and two indeterminacy degrees to vertices and hyperedges under compatibility constraints, enabling richer information representation.

Definition 3.1. *A double-valued neutrosophic hypergraph on $\mathfrak{G}$ is an ordered pair*

$$\mathfrak{H} = (\mathfrak{V}, \mathfrak{E}),$$

where

$$\mathfrak{V} = \{V_1, V_2, \ldots, V_n\}$$

is a finite family of double-valued neutrosophic sets on $\mathfrak{G}$, and $\mathfrak{E}$ is a double-valued neutrosophic relation on $\mathfrak{V}$ satisfying the following conditions.

(1) For every crisp hyperedge

$$e = \{u_1, u_2, \ldots, u_r\} \subseteq \mathfrak{G},$$

the following inequalities hold:

$$\mathfrak{t}_{\mathfrak{E}}(e) \leq \min\{\mathfrak{t}_{V_i}(u_1), \mathfrak{t}_{V_i}(u_2), \ldots, \mathfrak{t}_{V_i}(u_r)\},$$

$$\mathfrak{i}^{\mathfrak{t}}_{\mathfrak{E}}(e) \leq \min\{\mathfrak{i}^{\mathfrak{t}}_{V_i}(u_1), \mathfrak{i}^{\mathfrak{t}}_{V_i}(u_2), \ldots, \mathfrak{i}^{\mathfrak{t}}_{V_i}(u_r)\},$$

$$\mathfrak{i}^{\mathfrak{f}}_{\mathfrak{E}}(e) \leq \min\{\mathfrak{i}^{\mathfrak{f}}_{V_i}(u_1), \mathfrak{i}^{\mathfrak{f}}_{V_i}(u_2), \ldots, \mathfrak{i}^{\mathfrak{f}}_{V_i}(u_r)\},$$

and

$$\mathfrak{f}_{\mathfrak{E}}(e) \geq \max\{\mathfrak{f}_{V_i}(u_1), \mathfrak{f}_{V_i}(u_2), \ldots, \mathfrak{f}_{V_i}(u_r)\}.$$

Moreover,

$$0 \leq \mathfrak{t}_{\mathfrak{E}}(e) + \mathfrak{i}^{\mathfrak{t}}_{\mathfrak{E}}(e) + \mathfrak{i}^{\mathfrak{f}}_{\mathfrak{E}}(e) + \mathfrak{f}_{\mathfrak{E}}(e) \leq 4.$$

(2) The supports of the double-valued neutrosophic sets in $\mathfrak{V}$ cover $\mathfrak{G}$, that is,

$$\bigcup_{i=1}^{n} \operatorname{supp}(V_i) = \mathfrak{G}.$$

The set $e = \{u_1, u_2, \ldots, u_r\}$ is called a crisp hyperedge of $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$.

Definition 3.2. *Assume that $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is a DVNHG on $\mathfrak{G}$. The height of $\mathfrak{H}$, denoted by $h(\mathfrak{H})$, is defined by*

$$h(\mathfrak{H}) = \big(h_{\mathfrak{t}}(\mathfrak{H}), h_{\mathfrak{i}^{\mathfrak{t}}}(\mathfrak{H}), h_{\mathfrak{i}^{\mathfrak{f}}}(\mathfrak{H}), h_{\mathfrak{f}}(\mathfrak{H})\big),$$

where

$$h_{\mathfrak{t}}(\mathfrak{H}) = \max_{\delta_i \in \mathfrak{E},\, v \in \mathfrak{G}} \mathfrak{t}_{\delta_i}(v),$$

$$h_{\mathfrak{i}^{\mathfrak{t}}}(\mathfrak{H}) = \max_{\delta_i \in \mathfrak{E},\, v \in \mathfrak{G}} \mathfrak{i}^{\mathfrak{t}}_{\delta_i}(v),$$

$$h_{\mathfrak{i}^{\mathfrak{f}}}(\mathfrak{H}) = \max_{\delta_i \in \mathfrak{E},\, v \in \mathfrak{G}} \mathfrak{i}^{\mathfrak{f}}_{\delta_i}(v),$$

and

$$h_{\mathfrak{f}}(\mathfrak{H}) = \min_{\delta_i \in \mathfrak{E},\, v \in \mathfrak{G}} \mathfrak{f}_{\delta_i}(v).$$

Here, $\mathfrak{t}_{\delta_i}(v)$, $\mathfrak{i}^{\mathfrak{t}}_{\delta_i}(v)$, $\mathfrak{i}^{\mathfrak{f}}_{\delta_i}(v)$, *and* $\mathfrak{f}_{\delta_i}(v)$ *denote, respectively, the truth-membership degree, the truth-oriented indeterminacy degree, the falsity-oriented indeterminacy degree, and the falsity-membership degree of the vertex* v *with respect to the hyperedge* δ_i.

Definition 3.3. *Assume that* $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ *is a DVNHG on* $\mathfrak{G}$*. Let* $A, B^A, B^C, C \in [0, 1]$ *satisfy*

$$0 \le A + B^A + B^C + C \le 4.$$

The (A, B^A, B^C, C)*-level hypergraph of* $\mathfrak{H}$ *is the crisp hypergraph*

$$\mathfrak{H}^{(A,B^A,B^C,C)} = \big(\mathfrak{V}^{(A,B^A,B^C,C)}, \mathfrak{E}^{(A,B^A,B^C,C)}\big),$$

where, for each $\delta_i \in \mathfrak{E}$,

$$\mathfrak{E}_i^{(A,B^A,B^C,C)} = \left\{ x \in \mathfrak{G} \;\middle|\; \mathfrak{t}_{\delta_i}(x) \ge A,\ \mathfrak{i}^{\mathfrak{t}}_{\delta_i}(x) \ge B^A,\ \mathfrak{i}^{\mathfrak{f}}_{\delta_i}(x) \ge B^C,\ \mathfrak{f}_{\delta_i}(x) \le C \right\}.$$

The hyperedge set is defined by

$$\mathfrak{E}^{(A,B^A,B^C,C)} = \left\{ \mathfrak{E}_i^{(A,B^A,B^C,C)} \;\middle|\; \delta_i \in \mathfrak{E},\ \mathfrak{E}_i^{(A,B^A,B^C,C)} \ne \emptyset \right\},$$

and the vertex set is defined by

$$\mathfrak{V}^{(A,B^A,B^C,C)} = \bigcup_{\delta_i \in \mathfrak{E}} \mathfrak{E}_i^{(A,B^A,B^C,C)}.$$

Thus, $\mathfrak{H}^{(A,B^A,B^C,C)}$ *is a crisp hypergraph.*

Definition 3.4. *Assume that* $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ *is a DVNHG on* $\mathfrak{G}$*, and write*

$$h(\mathfrak{H}) = \big(h_{\mathfrak{t}}, h_{\mathfrak{i}^{\mathfrak{t}}}, h_{\mathfrak{i}^{\mathfrak{f}}}, h_{\mathfrak{f}}\big).$$

For parameters satisfying

$$0 < A \le h_{\mathfrak{t}}, \qquad 0 < B^A \le h_{\mathfrak{i}^{\mathfrak{t}}}, \qquad 0 < B^C \le h_{\mathfrak{i}^{\mathfrak{f}}}, \qquad h_{\mathfrak{f}} \le C \le 1,$$

let

$$\mathfrak{H}^{(A,B^A,B^C,C)} = \big(\mathfrak{V}^{(A,B^A,B^C,C)}, \mathfrak{E}^{(A,B^A,B^C,C)}\big)$$

be the (A, B^A, B^C, C)*-level hypergraph of* $\mathfrak{H}$*.*
A finite sequence

$$\big\{(A_1, B_1^A, B_1^C, C_1), (A_2, B_2^A, B_2^C, C_2), \ldots, (A_m, B_m^A, B_m^C, C_m)\big\}$$

is called a fundamental sequence of $\mathfrak{H}$ *if the following conditions hold:*

$$h_{\mathfrak{t}} = A_1 > A_2 > \cdots > A_m > 0,$$

$$h_{\mathfrak{i}^{\mathfrak{t}}} = B_1^A > B_2^A > \cdots > B_m^A > 0,$$

$$h_{\mathfrak{if}} = B_1^C > B_2^C > \cdots > B_m^C > 0,$$

and

$$h_{\mathfrak{f}} = C_1 < C_2 < \cdots < C_m \leq 1.$$

Moreover, for each $k \in \{1, 2, \ldots, m-1\}$, the following two conditions are satisfied:

(i) If

$$A_{k+1} < A' \leq A_k, \qquad B_{k+1}^A < (B^A)' \leq B_k^A, \qquad B_{k+1}^C < (B^C)' \leq B_k^C,$$

and

$$C_k \leq C' < C_{k+1},$$

then

$$\mathfrak{E}^{(A',(B^A)',(B^C)',C')} = \mathfrak{E}^{(A_k,B_k^A,B_k^C,C_k)}.$$

(ii) The level hyperedge sets are strictly nested as

$$\mathfrak{E}^{(A_k,B_k^A,B_k^C,C_k)} \subset \mathfrak{E}^{(A_{k+1},B_{k+1}^A,B_{k+1}^C,C_{k+1})}.$$

The fundamental sequence of $\mathfrak{H}$ is denoted by $\mathfrak{F}_s(\mathfrak{H})$.

The set of level hypergraphs

$$\left\{\mathfrak{H}^{(A_1,B_1^A,B_1^C,C_1)}, \mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)}, \ldots, \mathfrak{H}^{(A_m,B_m^A,B_m^C,C_m)}\right\}$$

is called the set of core hypergraphs, or the core set, of $\mathfrak{H}$ and is denoted by $c(\mathfrak{H})$.

Example 3.5. *Assume a DVNHG $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ on*

$$\mathfrak{G} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$$

, where $u_1 = (0.8, 0.6, 0.7, 0.5)$, $u_2 = (0.5, 0.5, 0.4, 0.3)$, $u_3 = (0.6, 0.4, 0.5, 0.4)$, $u_4 = (0.2, 0.3, 0.2, 0.1)$, $u_5 = (0.7, 0.6, 0.8, 0.5)$, $u_6 = (0.4, 0.5, 0.4, 0.3)$. The DVNR $\mathfrak{E}$ is given as, $\mathfrak{E}(\{u_1, u_2, u_3\}) = (0.5, 0.4, 0.4, 0.3)$, $\mathfrak{E}(\{u_4, u_5, u_6\}) = (0.2, 0.3, 0.2, 0.1)$, $\mathfrak{E}(\{u_1, u_6\}) = (0.4, 0.5, 0.4, 0.4)$, $\mathfrak{E}(\{u_2, u_5\}) = (0.5, 0.5, 0.4, 0.4)$ and $\mathfrak{E}(\{u_3, u_4\}) = (0.2, 0.3, 0.2, 0.1)$. The corresponding DVNHG is given in Figure 1.

$$(A_1, B_1^A, B_1^C, C_1) = (0.8, 0.6, 0.7, 0.5)$$

$$(A_2, B_2^A, B_2^C, C_2) = (0.5, 0.4, 0.4, 0.3)$$

$$(A_3, B_3^A, B_3^C, C_3) = (0.4, 0.4, 0.4, 0.3)$$

$$(A_4, B_4^A, B_4^C, C_4) = (0.2, 0.3, 0.2, 0.1)$$

Note that the sequence

$$\{(A_1, B_1^A, B_1^C, C_1), (A_2, B_2^A, B_2^C, C_2), (A_3, B_3^A, B_3^C, C_3), (A_4, B_4^A, B_4^C, C_4)\}$$

satisfies all the conditions of definition 3.4. Thus, it is a fundamental sequence of $\mathfrak{H}$. The corresponding (A_j, B_j^A, B_j^C, C_j) -level hypergraphs are shown in Figures 2-5.

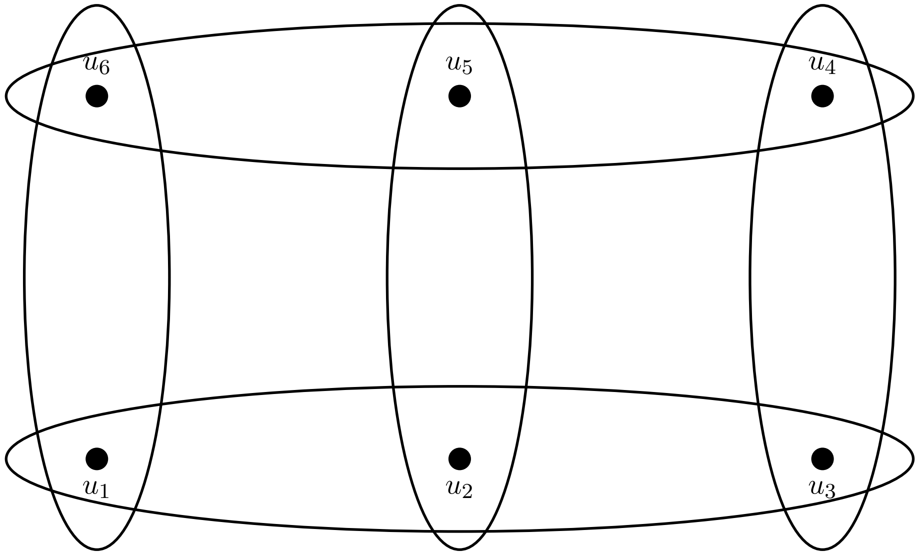

FIGURE 1. double-valued neutrosophic hypergraph $\mathfrak{H}$

FIGURE 2. $\mathfrak{H}^{(A_1,B_1^A,B_1^C,C_1)}$ -level hypergraph

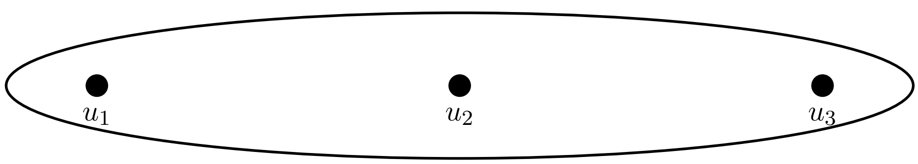

FIGURE 3. $\mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)}$ -level hypergraph

Definition 3.6. *A DVNHG $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is called ordered if its core set $c(\mathfrak{H})$ is ordered. More precisely, if*

$$c(\mathfrak{H}) = \left\{\mathfrak{H}^{(A_1,B_1^A,B_1^C,C_1)}, \mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)}, \dots, \mathfrak{H}^{(A_m,B_m^A,B_m^C,C_m)}\right\},$$

then

$$\mathfrak{H}^{(A_1,B_1^A,B_1^C,C_1)} \subseteq \mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)} \subseteq \cdots \subseteq \mathfrak{H}^{(A_m,B_m^A,B_m^C,C_m)}.$$

A DVNHG $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is called simply ordered if $c(\mathfrak{H})$ is simply ordered. That is, writing

$$\mathfrak{H}^{(A_i,B_i^A,B_i^C,C_i)} = (\mathfrak{G}_i, \mathfrak{E}_i), \qquad i = 1, 2, \dots, m,$$

if

$$e \in \mathfrak{E}_{i+1} \setminus \mathfrak{E}_i,$$

then

$$e \nsubseteq \mathfrak{G}_i.$$

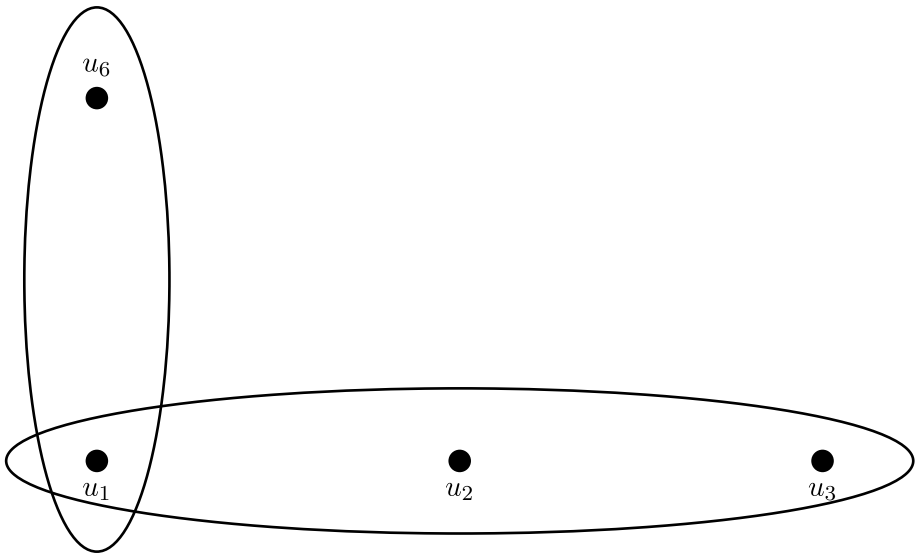

FIGURE 4. $\mathfrak{H}^{(A_3,B_3^A,B_3^C,C_3)}$ -level hypergraph

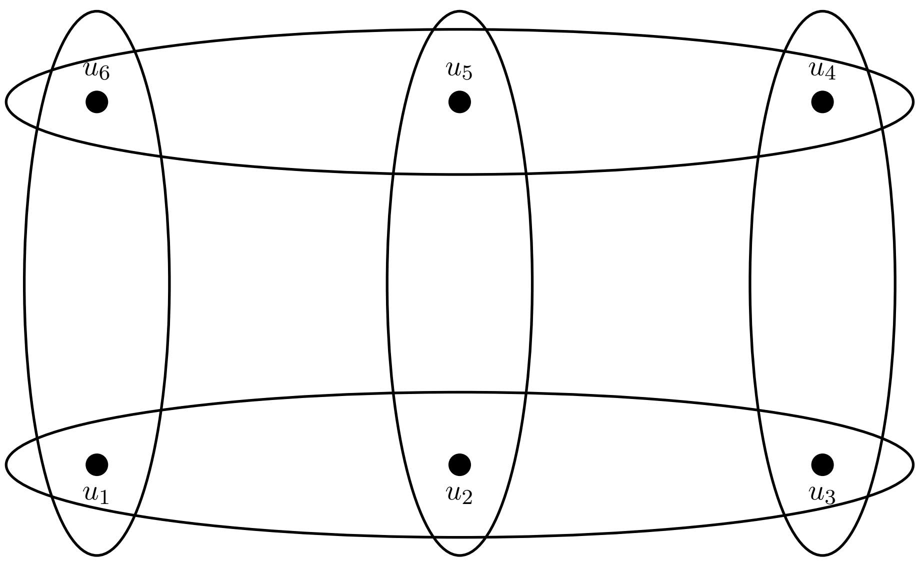

FIGURE 5. $\mathfrak{H}^{(A_4,B_4^A,B_4^C,C_4)}$ -level hypergraph

Example 3.7. *Assume that $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is a DVNHG as shown in Figure 1. The set of core hypergraphs is given by*

$$c(\mathfrak{H}) = \left\{\mathfrak{H}^{(A_1,B_1^A,B_1^C,C_1)}, \mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)}, \mathfrak{H}^{(A_3,B_3^A,B_3^C,C_3)}, \mathfrak{H}^{(A_4,B_4^A,B_4^C,C_4)}\right\},$$

where

$$\mathfrak{H}^{(A_1,B_1^A,B_1^C,C_1)} = (\mathfrak{G}_1, \mathfrak{E}_1), \qquad \mathfrak{G}_1 = \{u_1\}, \qquad \mathfrak{E}_1 = \emptyset,$$

$$\mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)} = (\mathfrak{G}_2, \mathfrak{E}_2), \qquad \mathfrak{G}_2 = \{u_1, u_2, u_3\}, \qquad \mathfrak{E}_2 = \{\{u_1, u_2, u_3\}\},$$

$$\mathfrak{H}^{(A_3,B_3^A,B_3^C,C_3)} = (\mathfrak{G}_3, \mathfrak{E}_3), \qquad \mathfrak{G}_3 = \{u_1, u_2, u_3, u_6\},$$

$$\mathfrak{E}_3 = \{\{u_1, u_2, u_3\}, \{u_1, u_6\}\},$$

and

$$\mathfrak{H}^{(A_4,B_4^A,B_4^C,C_4)} = (\mathfrak{G}_4, \mathfrak{E}_4), \qquad \mathfrak{G}_4 = \{u_1, u_2, u_3, u_4, u_5, u_6\},$$

$$\mathfrak{E}_4 = \{\{u_1, u_2, u_3\}, \{u_1, u_6\}, \{u_4, u_5, u_6\}, \{u_2, u_5\}, \{u_3, u_4\}\}.$$

Note that

$$\mathfrak{H}^{(A_1,B_1^A,B_1^C,C_1)} \subseteq \mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)} \subseteq \mathfrak{H}^{(A_3,B_3^A,B_3^C,C_3)} \subseteq \mathfrak{H}^{(A_4,B_4^A,B_4^C,C_4)}.$$

Hence, $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ *is an ordered DVNHG. Also,* $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ *is simply ordered.*

Definition 3.8. *Let* $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ *be a DVNHG with*

$$\mathfrak{I}_s(\mathfrak{H}) = \left\{(A_1, B_1^A, B_1^C, C_1), (A_2, B_2^A, B_2^C, C_2), \ldots, (A_m, B_m^A, B_m^C, C_m)\right\}.$$

Then $\mathfrak{H}$ *is called sectionally elementary if, for every* $k \in \{1, 2, \ldots, m-1\}$*, the equality*

$$\mathfrak{E}^{(A',(B^A)',(B^C)',C')} = \mathfrak{E}^{(A_k,B_k^A,B_k^C,C_k)}$$

holds for all parameters satisfying

$$A_{k+1} < A' \leq A_k, \qquad B_{k+1}^A < (B^A)' \leq B_k^A, \qquad B_{k+1}^C < (B^C)' \leq B_k^C,$$

and

$$C_k \leq C' < C_{k+1}.$$

Definition 3.9. *Assume that* V *is a DVNS on* $\mathfrak{I}$*. For* $0 < A, B^A, B^C, C < 1$*, define the level set*

$$V^{(A,B^A,B^C,C)} = \left\{u \in \mathfrak{I} \;\middle|\; \mathfrak{t}_V(u) \geq A,\ \mathfrak{i}_V^{\mathfrak{t}}(u) \geq B^A,\ \mathfrak{i}_V^{\mathfrak{f}}(u) \geq B^C,\ \mathfrak{f}_V(u) \leq C\right\}.$$

The lower truncation of V *at level* (A, B^A, B^C, C) *is the DVNS* $V_{[(A,B^A,B^C,C)]}$ *defined by*

$$\mathfrak{t}_{V_{[(A,B^A,B^C,C)]}}(u) = \begin{cases} \mathfrak{t}_V(u), & \text{if } u \in V^{(A,B^A,B^C,C)}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathfrak{i}^{\mathfrak{t}}_{V_{[(A,B^A,B^C,C)]}}(u) = \begin{cases} \mathfrak{i}^{\mathfrak{t}}_V(u), & \text{if } u \in V^{(A,B^A,B^C,C)}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathfrak{i}^{\mathfrak{f}}_{V_{[(A,B^A,B^C,C)]}}(u) = \begin{cases} \mathfrak{i}^{\mathfrak{f}}_V(u), & \text{if } u \in V^{(A,B^A,B^C,C)}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\mathfrak{f}_{V_{[(A,B^A,B^C,C)]}}(u) = \begin{cases} \mathfrak{f}_V(u), & \text{if } u \in V^{(A,B^A,B^C,C)}, \\ 0, & \text{otherwise}. \end{cases}$$

Definition 3.10. *Assume that V is a DVNS on $\mathfrak{J}$. For $0 < A, B^A, B^C, C < 1$, define*

$$V^{(A,B^A,B^C,C)} = \left\{ u \in \mathfrak{J} \;\middle|\; \mathfrak{t}_V(u) \geq A,\ \mathfrak{i}_V^{\mathfrak{t}}(u) \geq B^A,\ \mathfrak{i}_V^{\mathfrak{f}}(u) \geq B^C,\ \mathfrak{f}_V(u) \leq C \right\}.$$

The upper truncation of V at level (A, B^A, B^C, C) is the DVNS $V^{[(A,B^A,B^C,C)]}$ defined by

$$\mathfrak{t}_{V^{[(A,B^A,B^C,C)]}}(u) = \begin{cases} A, & \text{if } u \in V^{(A,B^A,B^C,C)}, \\ \mathfrak{t}_V(u), & \text{otherwise}, \end{cases}$$

$$\mathfrak{i}^{\mathfrak{t}}_{V^{[(A,B^A,B^C,C)]}}(u) = \begin{cases} B^A, & \text{if } u \in V^{(A,B^A,B^C,C)}, \\ \mathfrak{i}^{\mathfrak{t}}_V(u), & \text{otherwise}, \end{cases}$$

$$\mathfrak{i}^{\mathfrak{f}}_{V^{[(A,B^A,B^C,C)]}}(u) = \begin{cases} B^C, & \text{if } u \in V^{(A,B^A,B^C,C)}, \\ \mathfrak{i}^{\mathfrak{f}}_V(u), & \text{otherwise}, \end{cases}$$

and

$$\mathfrak{f}_{V^{[(A,B^A,B^C,C)]}}(u) = \begin{cases} C, & \text{if } u \in V^{(A,B^A,B^C,C)}, \\ \mathfrak{f}_V(u), & \text{otherwise}. \end{cases}$$

Definition 3.11. *Assume that $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is a DVNHG. The lower truncation of $\mathfrak{H}$ at level (A, B^A, B^C, C) is denoted by*

$$\mathfrak{H}_{[(A,B^A,B^C,C)]} = \left(\mathfrak{V}_{[(A,B^A,B^C,C)]}, \mathfrak{E}_{[(A,B^A,B^C,C)]} \right),$$

where

$$\mathfrak{V}_{[(A,B^A,B^C,C)]} = \left\{ V_{[(A,B^A,B^C,C)]} \;\middle|\; V \in \mathfrak{V} \right\},$$

and $\mathfrak{E}_{[(A,B^A,B^C,C)]}$ is obtained by applying the same lower truncation to each member of $\mathfrak{E}$. The upper truncation of $\mathfrak{H}$ at level (A, B^A, B^C, C) is denoted by

$$\mathfrak{H}^{[(A,B^A,B^C,C)]} = \left(\mathfrak{V}^{[(A,B^A,B^C,C)]}, \mathfrak{E}^{[(A,B^A,B^C,C)]} \right),$$

where

$$\mathfrak{V}^{[(A,B^A,B^C,C)]} = \left\{ V^{[(A,B^A,B^C,C)]} \;\middle|\; V \in \mathfrak{V} \right\},$$

and $\mathfrak{E}^{[(A,B^A,B^C,C)]}$ is obtained by applying the same upper truncation to each member of $\mathfrak{E}$.

Definition 3.12. *Assume that V is a DVNS on $\mathfrak{G}$. A quadruple (A, B^A, B^C, C) is called a transition level of V if*

$$A \in (0, \mathfrak{t}(h(V))), \qquad B^A \in (0, \mathfrak{i}^{\mathfrak{t}}(h(V))), \qquad B^C \in (0, \mathfrak{i}^{\mathfrak{f}}(h(V))),$$

and

$$C \in (\mathfrak{f}(h(V)), 1],$$

and the level cut of V changes at (A, B^A, B^C, C).

Example 3.13. *Assume that $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is a DVNHG as shown in Figure 1. The $(0.4, 0.4, 0.4, 0.3)$-level hypergraph of $\mathfrak{H}$ is shown in Figure 4. Then the lower truncation*

$$\mathfrak{H}_{[(0.4,0.4,0.4,0.3)]} = \left(\mathfrak{V}_{[(0.4,0.4,0.4,0.3)]}, \mathfrak{E}_{[(0.4,0.4,0.4,0.3)]}\right)$$

of $\mathfrak{H}$ is a DVNHG on

$$\mathfrak{G}_1 = \{u_1, u_2, u_3, u_6\}$$

as given in Figure 6.

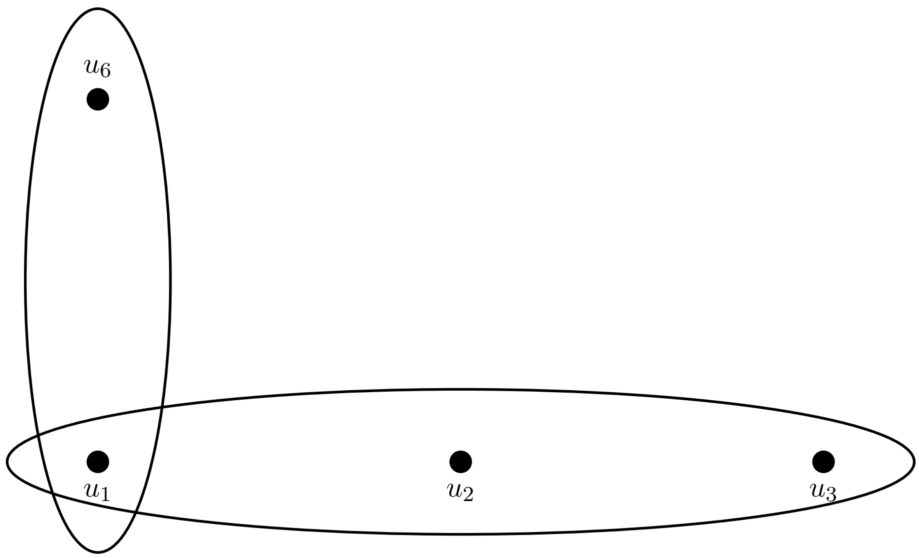

Figure 6. Lower truncation of $\mathfrak{H}$

Note that

$$\mathfrak{G}_1 = \bigcup_{V \in \mathfrak{V}} \text{supp}\left(V_{[(0.4,0.4,0.4,0.3)]}\right).$$

The upper truncation

$$\mathfrak{H}^{[(0.4,0.4,0.4,0.3)]} = \left(\mathfrak{V}^{[(0.4,0.4,0.4,0.3)]}, \mathfrak{E}^{[(0.4,0.4,0.4,0.3)]}\right)$$

of $\mathfrak{H}$ is a DVNHG on

$$\mathfrak{G} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$$

as given in Figure 7.

Definition 3.14. *Assume that $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is a DVNHG. A double-valued neutrosophic transversal (DVNT) ρ is a DVNS of $\mathfrak{G}$ satisfying the condition $\delta^{h(\delta)} \cap \rho^{h(\delta)} \neq \emptyset$, for all $\delta \in E$, where $h(\delta)$, is the height of δ.*
A minimal double-valued neutrosophic transversal ρ_1 is the DVNT of $\mathfrak{H}$ with the property that if $\rho \subset \rho_1$, then ρ is not a DVNT of $\mathfrak{H}$.
Let us denote the family of minimal DVNTs of $\mathfrak{H}$ by $T_r(\mathfrak{H})$.

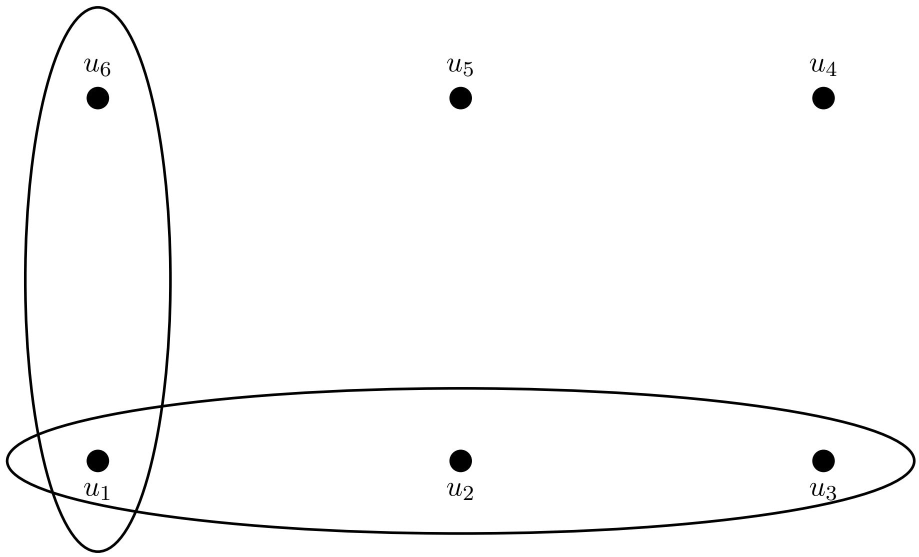

FIGURE 7. Upper truncation of $\mathfrak{H}$

Definition 3.15. *A DVNT ρ with the property that $\rho^{(A,B^A,B^C,C)} \in T_r(\mathfrak{H}^{(A,B^A,B^C,C)})$, for all$(A, B^A, B^C, C) \in [0,1]$, is called the locally minimal DVNT of $\mathfrak{H}$. The collection of all locally minimal DVNTs of $\mathfrak{H}$ is represented by $T_r^*(\mathfrak{H})$.*
Note that $T_r^*(\mathfrak{H}) \subseteq T_r(\mathfrak{H})$, but the converse is not generally true.

Definition 3.16. *Assume that V is a DVNS on $\mathfrak{G}$. Then, the basic sequence of V determined by V, denoted by $D_s(V)$, is defined as $\{(A_1, B_1^A, B_1^C, C_1)^V, (A_2, B_2^A, B_2^C, C_2)^V, \cdots, (A_n, B_n^A, B_n^C, C_n)^V\}$, where*

- $A_1 > A_2 > \cdots > A_n, B_1^A > B_2^A > \cdots > B_n^A$, $B_1^C > B_2^C > \cdots > B_n^C$, $C_1 < C_2 < \cdots < C_n$
- $(A_1, B_1^A, B_1^C, C_1) = h(V)$
- $\{(A_2, B_2^A, B_2^C, C_2)^V, \cdots, (A_n, B_n^A, B_n^C, C_n)^V\}$ *are the transition levels of V*

Definition 3.17. *Assume that $D_s(V) = \{(A_1, B_1^A, B_1^C, C_1)^V, (A_2, B_2^A, B_2^C, C_2)^V, \cdots, (A_n, B_n^A, B_n^C, C_n)^V\}$ is the basic sequence of V. Then the set of basic cuts $D_s(V)$ is defined as,*

$$D_c(V) = \{V^{(A,B^A,B^C,C)} | (A, B^A, B^C, C) \in D_s(V)\}$$

.

Lemma 3.18. *Assume that $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is a DVNHG with*
$\mathfrak{I}_s(\mathfrak{H}) = \{(A_1, B_1^A, B_1^C, C_1), (A_2, B_2^A, B_2^C, C_2), \cdots, (A_n, B_n^A, B_n^C, C_n)\}$. Then

- *If (A, B^A, B^C, C) is a transition level of $\rho \in T_r(\mathfrak{H})$, then there exists an $\epsilon > 0$ such that for all $A_1 \in (A, A+\epsilon]$, $B_1^A \in (B^A, B^A+\epsilon]$, $B_1^C \in (B^C, B^C+\epsilon]$, $C_1 \in (C, C+\epsilon]$ is* *minimal* $\mathfrak{H}^{(A,B^A,B^C,C)}$ *transversal extension of* $\rho^{(A,B^A,B^C,C)}$, *i.e.,* $\rho^{(A_1,B_1^A,B_1^C,C_1)} \subseteq J \subset \rho^{(A,B^A,B^C,C)}$, *then* J *is not a transversal of* $\mathfrak{H}^{(A,B^A,B^C,C)}$.

- $T_r(\mathfrak{H})$, *i.e., the collection of minimal transversals of* $\mathfrak{H}$ *is sectionally elementary.*
- $\mathfrak{F}_s(T_r(\mathfrak{H}))$ *is properly contained in* $\mathfrak{F}_s(\mathfrak{H})$.
- $\rho^{(A,B^A,B^C,C)} \in T_r(\mathfrak{H}^{(A,B^A,B^C,C)})$, *for all* $\rho \in T_r(\mathfrak{H})$ *and for every* $A_2 < A \leq A_1$, $B_2^A < B^A \leq B_1^A$, $B_2^C < B^C \leq B_1^C$, $C_2 > C \geq C_1$.

4. T-Related DVNHGs

T-related DVNHGs are double-valued neutrosophic hypergraphs whose consecutive core hypergraphs preserve minimal transversals compatibly, ensuring global and locally minimal transversal structures coincide across transition levels.

Definition 4.1. *A DVNHG* $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ *is N-tempered DVNHG of* $\mathbb{H} = (\mathbb{V}, \mathbb{E})$ *if there exists* $\mathbb{H} = (\mathbb{V}, \mathbb{E})$, *a crisp hypergraph and a DVNS* N *such that* $\mathfrak{E} = \{\xi_e | e \in \mathbb{E}\}$, *where*

$$\mathfrak{t}_\xi(x) = \begin{cases} \min\{\mathfrak{t}_N(x) | x \in e\}, & if \;\; x \in e \\ 0, & otherwise. \end{cases}$$

$$\mathfrak{i^t}_\xi(x) = \begin{cases} \min\{\mathfrak{i^t}_N(x) | x \in e\}, & if \;\; x \in e \\ 0, & otherwise. \end{cases}$$

$$\mathfrak{i^f}_\xi(x) = \begin{cases} \min\{\mathfrak{i^f}_N(x) | x \in e\}, & if \;\; x \in e \\ 0, & otherwise. \end{cases}$$

$$\mathfrak{f}_\xi(x) = \begin{cases} \max\{\mathfrak{f}_N(x) | x \in e\}, & if \;\; x \in e \\ 0, & otherwise. \end{cases}$$

An N-tempered DVNHG $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ *determined by* $\mathbb{H}$ *and DVNS* N *is denoted by* $N \otimes \mathbb{H}$.

Definition 4.2. *A pair* $(\mathfrak{G}, \mathfrak{J})$ *of crisp hypergraphs is T-related if whenever* g *is a minimal transversal of* $\mathfrak{G}$, k *is any transversal of* $\mathfrak{J}$ *and* $g \subseteq k$, *then there exists a minimal transversal* l *of* $\mathfrak{J}$ *such that* $g \subseteq l \subseteq k$.

Definition 4.3. *Assume that* $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ *is a DVNHG with* $\mathfrak{F}_s(\mathfrak{H}) = \{(A_1, B_1^A, B_1^C, C_1), (A_2, B_2^A, B_2^C, C_2), \cdots, (A_n, B_n^A, B_n^C, C_n)\}$. *Then,* $\mathfrak{H}$ *is T-related if from the core set*

$$c(\mathfrak{H}) = \{\mathfrak{H}^{(A_1,B_1^A,B_1^C,C_1)}, \mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)}, \cdots, \mathfrak{H}^{(A_m,B_m^A,B_m^C,C_m)}\}$$

of $\mathfrak{H}$, *every successive ordered pair* $(\mathfrak{H}^{(A_j,B_j^A,B_j^C,C_j)}, \mathfrak{H}^{(A_{j-1},B_{j-1}^A,B_{j-1}^C,C_{j-1})})$ *is T-related.* *If* $\mathfrak{F}_s(\mathfrak{H})$ *contains only one element,* $\mathfrak{H}$ *is considered to be trivially T-related.*

Theorem 4.4. *If $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is a T-related DVNHG, then*

$$T_r(\mathfrak{H}) = T_r^*(\mathfrak{H}).$$

Proof. Assume that $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is a T-related DVNHG with

$$\mathfrak{F}_s(\mathfrak{H}) = \left\{(A_1, B_1^A, B_1^C, C_1), (A_2, B_2^A, B_2^C, C_2), \ldots, (A_n, B_n^A, B_n^C, C_n)\right\}.$$

We prove that every minimal DVNT of $\mathfrak{H}$ is locally minimal. Since

$$T_r^*(\mathfrak{H}) \subseteq T_r(\mathfrak{H}),$$

it is enough to show that

$$T_r(\mathfrak{H}) \subseteq T_r^*(\mathfrak{H}).$$

Let $\delta \in T_r(\mathfrak{H})$. We show that, for every admissible level (A, B^A, B^C, C),

$$\delta^{(A,B^A,B^C,C)} \in T_r\big(\mathfrak{H}^{(A,B^A,B^C,C)}\big).$$

First, suppose that $\mathfrak{F}_s(\mathfrak{H})$ consists of only one element, say

$$\mathfrak{F}_s(\mathfrak{H}) = \left\{(A_1, B_1^A, B_1^C, C_1)\right\}.$$

Then Lemma 3.18 implies that, for each $\delta \in T_r(\mathfrak{H})$,

$$\delta^{(A,B^A,B^C,C)} \in T_r\big(\mathfrak{H}^{(A,B^A,B^C,C)}\big)$$

for all admissible levels satisfying

$$0 < A \leq \mathfrak{t}\big(h(\mathfrak{H})\big), \qquad 0 < B^A \leq \mathfrak{i}^{\mathfrak{t}}\big(h(\mathfrak{H})\big),$$

$$0 < B^C \leq \mathfrak{i}^{\mathfrak{f}}\big(h(\mathfrak{H})\big), \qquad 0 < \mathfrak{f}\big(h(\mathfrak{H})\big) \leq C.$$

Hence,

$$\delta \in T_r^*(\mathfrak{H}).$$

Therefore,

$$T_r(\mathfrak{H}) = T_r^*(\mathfrak{H})$$

in this case.

Next, suppose that $\mathfrak{F}_s(\mathfrak{H})$ contains at least two elements. Let

$$\delta^{(A_1,B_1^A,B_1^C,C_1)} \subseteq \delta^{(A_2,B_2^A,B_2^C,C_2)} \subseteq \cdots \subseteq \delta^{(A_n,B_n^A,B_n^C,C_n)}$$

be the corresponding sequence of level cuts of δ.

We first prove that

$$\delta^{(A_2,B_2^A,B_2^C,C_2)} \in T_r\big(\mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)}\big).$$

Suppose, to the contrary, that

$$\delta^{(A_2,B_2^A,B_2^C,C_2)} \notin T_r\big(\mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)}\big).$$

Since

$$\delta^{(A_1,B_1^A,B_1^C,C_1)} \in T_r\big(\mathfrak{H}^{(A_1,B_1^A,B_1^C,C_1)}\big)$$

and the ordered pair

$$\Big(\mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)}, \mathfrak{H}^{(A_1,B_1^A,B_1^C,C_1)}\Big)$$

is T-related, there exists a minimal transversal ρ_2 of $\mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)}$ such that

$$\delta^{(A_1,B_1^A,B_1^C,C_1)} \subseteq \rho_2 \subset \delta^{(A_2,B_2^A,B_2^C,C_2)}.$$

Let

$$\rho_1 = \delta^{(A_1,B_1^A,B_1^C,C_1)}.$$

Construct a DVNS ξ from δ by replacing the cut $\delta^{(A_2,B_2^A,B_2^C,C_2)}$ with ρ_2, while preserving the remaining lower-level structure. Equivalently, one may write

$$\xi = \delta^{(A_3,B_3^A,B_3^C,C_3)} \cup (\tau_2 \cap \tau_1),$$

where τ_k is an elementary DVNS with support ρ_k and height

$$(A_k, B_k^A, B_k^C, C_k), \qquad k = 1, 2.$$

Then ξ is a DVNT of $\mathfrak{H}$ and

$$\xi \subset \delta,$$

which contradicts the minimality of δ in $T_r(\mathfrak{H})$. Therefore,

$$\delta^{(A_2,B_2^A,B_2^C,C_2)} \in T_r\big(\mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)}\big).$$

Repeating the same argument successively for all adjacent pairs in the fundamental sequence, we obtain

$$\delta^{(A_k,B_k^A,B_k^C,C_k)} \in T_r\big(\mathfrak{H}^{(A_k,B_k^A,B_k^C,C_k)}\big)$$

for every $k = 1, 2, \ldots, n$. By Lemma 3.18, this also holds for every admissible level lying between two consecutive fundamental levels. Hence

$$\delta^{(A,B^A,B^C,C)} \in T_r\big(\mathfrak{H}^{(A,B^A,B^C,C)}\big)$$

for every admissible level (A, B^A, B^C, C). Thus,

$$\delta \in T_r^*(\mathfrak{H}).$$

Since $\delta \in T_r(\mathfrak{H})$ was arbitrary, we obtain

$$T_r(\mathfrak{H}) \subseteq T_r^*(\mathfrak{H}).$$

Together with $T_r^*(\mathfrak{H}) \subseteq T_r(\mathfrak{H})$, this proves

$$T_r(\mathfrak{H}) = T_r^*(\mathfrak{H}).$$

□

Example 4.5. *Assume that $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is a DVNHG represented by the incidence matrix in Table 2. Note that*

$$\mathfrak{E}^{(0.8,0.8,0.8,0.8)} = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_1, u_3\}\},$$
$$\mathfrak{E}^{(0.4,0.4,0.4,0.4)} = \{\{u_1, u_2, u_4\}, \{u_2, u_3, u_5\}, \{u_1, u_3, u_4\}\},$$
$$\mathfrak{E}^{(0.2,0.2,0.2,0.2)} = \{\{u_1, u_2, u_4, u_5\}, \{u_2, u_3, u_4, u_5\}, \{u_1, u_3, u_4, u_5\}\}.$$

Clearly,

$$\mathfrak{F}_s(\mathfrak{H}) = \{(0.8, 0.8, 0.8, 0.8), (0.4, 0.4, 0.4, 0.4), (0.2, 0.2, 0.2, 0.2)\}.$$

Also,

$$T_r(\mathfrak{H}) = \{\rho_1, \rho_2, \rho_3\} = T_r^*(\mathfrak{H}),$$

where

$$\rho_1 = \{(u_1, 0.8, 0.8, 0.8, 0.8), (u_2, 0.8, 0.8, 0.8, 0.8)\},$$

$$\rho_2 = \{(u_2, 0.8, 0.8, 0.8, 0.8), (u_3, 0.8, 0.8, 0.8, 0.8)\},$$

and

$$\rho_3 = \{(u_1, 0.8, 0.8, 0.8, 0.8), (u_3, 0.8, 0.8, 0.8, 0.8)\}.$$

Table 2. Incidence matrix of the DVNHG $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$

	$\mathfrak{E}_1$	$\mathfrak{E}_2$	$\mathfrak{E}_3$
u_1	$(0.8, 0.8, 0.8, 0.8)$	$(0, 0, 0, 1)$	$(0.8, 0.8, 0.8, 0.8)$
u_2	$(0.8, 0.8, 0.8, 0.8)$	$(0.8, 0.8, 0.8, 0.8)$	$(0, 0, 0, 1)$
u_3	$(0, 0, 0, 1)$	$(0.8, 0.8, 0.8, 0.8)$	$(0.8, 0.8, 0.8, 0.8)$
u_4	$(0.4, 0.4, 0.4, 0.4)$	$(0.4, 0.4, 0.4, 0.4)$	$(0.2, 0.2, 0.2, 0.2)$
u_5	$(0.2, 0.2, 0.2, 0.2)$	$(0.2, 0.2, 0.2, 0.2)$	$(0.4, 0.4, 0.4, 0.4)$

Since

$$\{u_4, u_5\} \in T_r\big(\mathfrak{H}^{(0.4,0.4,0.4,0.4)}\big)$$

and

$$\{u_4\} \in T_r\big(\mathfrak{H}^{(0.2,0.2,0.2,0.2)}\big),$$

no minimal transversal of $\mathfrak{H}^{(0.2,0.2,0.2,0.2)}$ contains $\{u_4, u_5\}$. Hence the ordered pair

$$\Big(\mathfrak{H}^{(0.4,0.4,0.4,0.4)}, \mathfrak{H}^{(0.2,0.2,0.2,0.2)}\Big)$$

is not T-related. Therefore, $\mathfrak{H}$ is not T-related.

Theorem 4.6. *Assume that $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is an ordered DVNHG. Then*

$$T_r(\mathfrak{H}) = T_r^*(\mathfrak{H}) \iff \mathfrak{H} \text{ is T-related.}$$

Proof. By Theorem 4.4, if $\mathfrak{H}$ is T-related, then

$$T_r(\mathfrak{H}) = T_r^*(\mathfrak{H}).$$

It remains to prove the converse.
Assume that

$$T_r(\mathfrak{H}) = T_r^*(\mathfrak{H})$$

and suppose, for a contradiction, that $\mathfrak{H}$ is not T-related. Let

$$\mathfrak{F}_s(\mathfrak{H}) = \left\{(A_1, B_1^A, B_1^C, C_1), (A_2, B_2^A, B_2^C, C_2), \ldots, (A_n, B_n^A, B_n^C, C_n)\right\},$$

and let

$$c(\mathfrak{H}) = \left\{\mathfrak{H}^{(A_1,B_1^A,B_1^C,C_1)}, \mathfrak{H}^{(A_2,B_2^A,B_2^C,C_2)}, \ldots, \mathfrak{H}^{(A_n,B_n^A,B_n^C,C_n)}\right\}.$$

Since $\mathfrak{H}$ is not T-related, there exists an index $j \in \{1, 2, \ldots, n-1\}$ such that the ordered pair

$$\left(\mathfrak{H}^{(A_j,B_j^A,B_j^C,C_j)}, \mathfrak{H}^{(A_{j+1},B_{j+1}^A,B_{j+1}^C,C_{j+1})}\right)$$

is not T-related.
By the definition of T-relatedness, there exist a minimal transversal

$$\rho_j \in T_r\left(\mathfrak{H}^{(A_j,B_j^A,B_j^C,C_j)}\right)$$

and a transversal ρ_{j+1} of

$$\mathfrak{H}^{(A_{j+1},B_{j+1}^A,B_{j+1}^C,C_{j+1})}$$

such that

$$\rho_j \subseteq \rho_{j+1},$$

but no minimal transversal N of $\mathfrak{H}^{(A_{j+1},B_{j+1}^A,B_{j+1}^C,C_{j+1})}$ satisfies

$$\rho_j \subseteq N \subseteq \rho_{j+1}.$$

Since $\mathfrak{H}$ is ordered, we have

$$\mathfrak{H}^{(A_j,B_j^A,B_j^C,C_j)} \subseteq \mathfrak{H}^{(A_{j+1},B_{j+1}^A,B_{j+1}^C,C_{j+1})}.$$

Therefore, ρ_j is not a transversal of $\mathfrak{H}^{(A_{j+1},B_{j+1}^A,B_{j+1}^C,C_{j+1})}$; otherwise, it would contradict the above choice of ρ_j and ρ_{j+1}.
Choose a minimal transversal ξ of $\mathfrak{H}^{(A_{j+1},B_{j+1}^A,B_{j+1}^C,C_{j+1})}$ such that

$$\xi \subseteq \rho_{j+1}.$$

By the non-T-relatedness assumption, ξ cannot contain ρ_j. Hence

$$\rho_j \nsubseteq \xi.$$

Using ρ_j as the cut at level (A_j, B_j^A, B_j^C, C_j) and ξ as the cut at level $(A_{j+1}, B_{j+1}^A, B_{j+1}^C, C_{j+1})$, one can construct a minimal DVNT δ of $\mathfrak{H}$ such that

$$\delta \in T_r(\mathfrak{H})$$

but

$$\delta^{(A_{j+1}, B_{j+1}^A, B_{j+1}^C, C_{j+1})} \notin T_r\big(\mathfrak{H}^{(A_{j+1}, B_{j+1}^A, B_{j+1}^C, C_{j+1})}\big).$$

Thus,

$$\delta \notin T_r^*(\mathfrak{H}).$$

This gives

$$\delta \in T_r(\mathfrak{H}) \setminus T_r^*(\mathfrak{H}),$$

which contradicts the assumption

$$T_r(\mathfrak{H}) = T_r^*(\mathfrak{H}).$$

Therefore, $\mathfrak{H}$ must be T-related. □

Corollary 4.7. *Assume that $\mathfrak{H} = (\mathfrak{V}, \mathfrak{E})$ is an ordered DVNHG with*

$$\mathfrak{F}_s(\mathfrak{H}) = \big\{(A_1, B_1^A, B_1^C, C_1), (A_2, B_2^A, B_2^C, C_2), \ldots, (A_n, B_n^A, B_n^C, C_n)\big\}$$

and

$$c(\mathfrak{H}) = \left\{\mathfrak{H}^{(A_1, B_1^A, B_1^C, C_1)}, \mathfrak{H}^{(A_2, B_2^A, B_2^C, C_2)}, \ldots, \mathfrak{H}^{(A_n, B_n^A, B_n^C, C_n)}\right\}.$$

If an ordered pair

$$\left(\mathfrak{H}^{(A_j, B_j^A, B_j^C, C_j)}, \mathfrak{H}^{(A_{j+1}, B_{j+1}^A, B_{j+1}^C, C_{j+1})}\right)$$

is not T-related, then:

- $$(A_{j+1}, B_{j+1}^A, B_{j+1}^C, C_{j+1}) \in \mathfrak{F}_s\big(T_r(\mathfrak{H})\big).$$

- $$(A_{j+1}, B_{j+1}^A, B_{j+1}^C, C_{j+1})$$

  is a transition level for some

$$\delta \in T_r(\mathfrak{H}) \setminus T_r^*(\mathfrak{H}).$$

5. Conclusions

In this study, we integrated the concept of double-valued neutrosophic sets into hypergraph theory in order to handle uncertainty and indeterminacy situations more effectively. The proposed double-valued neutrosophic hypergraph model provides a flexible framework for representing uncertainty through truth, indeterminacy, and falsity membership degrees. We investigated several fundamental properties of double-valued neutrosophic hypergraphs, including transition levels, lower truncations, and upper truncations. Moreover, we studied the properties of minimal transversals and their relationship with T-related double-valued neutrosophic hypergraphs. The proposed framework shows its potential for describing double-valued uncertain systems and identifying hidden relationships among interconnected structures. Future research may extend this framework to interval-valued (hyperfuzzy) [39,40], bipolar [37,38], and other generalized neutrosophic hypergraphs. We also hope that further extensions using HyperSoft graphs [28], SuperHyperGraphs [29,30], and related higher-order structures will be investigated in future studies.

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