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On Neutrosophic Topological Spaces in Šostak's Sense

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Abstract. Alexander P. Šostak has proposed a fundamental approach to the notion of fuzzy topological space, which depends on the generalization of classical (crisp) topology and Chang's fuzzy topology. Unlike previous approaches, not only the subsets were fuzzified, but also the conditions between them. In this paper, we present the notion of neutrosophic topological space in Šostak's sense by a simple way. Moreover, we investigate interesting properties of this topological space which is considered as a generalization of fuzzy and intuitionistic fuzzy topological spaces.

Keywords: Fuzzy sets; Atanassov's intuitionistic fuzzy sets; neutrosophic sets; topology;

1. Introduction

F. Smarandache [17] generalized the notions of fuzzy sets and Atanassov's intuitionistic fuzzy sets to the notion of neutrosophic sets (NSs). He introduced this notion to deal with imprecise and indeterminate data. NSs are defined by truth membership function (T), indeterminacy membership function (I) and falsity membership function (F). Many authors have studied and applied the notion of neutrosophic sets in several areas such as decision making problems (e.g. [21]), image processing (e.g. [25]), educational problem (e.g. [11]), conflict resolution (e.g. [15]), social problems (e.g. [10]), medical diagnosis (e.g. [22]), supply chain management (e.g. [1]). In particular, to exercise neutrosophic sets in real life applications suitably, Wang et al. [23] defined the notion of single valued neutrosophic sets (SVNSs) as a subclass of a neutrosophic sets, and investigated some of its properties. The studies, whether theoretical or applied on single valued neutrosophic set have been progressing rapidly. For instance, [2, 6, 8] and more others.

We study in this paper the notion of neutrosophic topological space in Šostak's sense as an important generalized fuzzy topological space. Also, several interesting properties on this structure are discussed.

The contents of the paper are organized as follows. In Section 2, we recall the necessary basic concepts and properties of neutrosophic sets, and some related notions that will be needed throughout this paper. In Section 3, we present the notion of neutrosophic topological space in Šostak's sense by a simple method. In Section 4, we study interesting properties of this topological space which is considered as a generalization of fuzzy and intuitionistic fuzzy topological spaces. Finally, we present some conclusions and discuss future research in Section 5.

2. Preliminaries

This section contains the basic definitions and properties of neutrosophic sets and some related notions that will be needed throughout this paper. The notion of fuzzy sets was first introduced by Zadeh [24].

Definition 2.1. [24] Let X be a nonempty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for any $x \in X$.

In 1983, Atanassov [3] proposed a generalization of Zadeh membership degree and introduced the notion of the intuitionistic fuzzy set.

Definition 2.2. [3] Let X be a nonempty set. An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\nu_A : X \to [0, 1]$ which satisfy the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \text{ for any } x \in X.$$

In 1998, Smarandache [17] defined the concept of a neutrosophic set as a generalization of Atanassov's intuitionistic fuzzy set. Also, he introduced neutrosophic logic, neutrosophic set and its applications in [18, 19]. In particular, Wang et al. [23] introduced the notion of a single valued neutrosophic set.

Definition 2.3. [18] Let X be a nonempty set. A neutrosophic set (NS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to]^-0, 1^+[$ and an indeterminacy function $\sigma_A : X \to]^-0, 1^+[$ and a non-membership function $\nu_A : X \to]^-0, 1^+[$ which satisfy the condition:

$$^-0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+, \text{ for any } x \in X.$$

Certainly, intuitionistic fuzzy sets are neutrosophic sets by setting $\sigma_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Next, we show the notion of single valued neutrosophic set as an instance of neutrosophic set which can be used in real scientific and engineering applications.

Definition 2.4. [23] Let X be a nonempty set. A single valued neutrosophic set (SVNS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a truth-membership function $\mu_A : X \to [0, 1]$, an indeterminacy-membership function $\sigma_A : X \to [0, 1]$ and a falsity-membership function $\nu_A : X \to [0, 1]$.

The class of single valued neutrosophic sets on X is denoted by $SVN(X)$.

For any two NSs A and B on a set X, several operations are defined (see, e.g., [20, 23]). Here we will present only those which are related to the present paper.

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\sigma_A(x) \leq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$,

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$ and $\sigma_A(x) = \sigma_B(x)$ and $\nu_A(x) = \nu_B(x)$, for all $x \in X$,

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$,

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$,

(v) $\overline{A} = \{\langle x, 1 - \nu_A(x), 1 - \sigma_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$,

(vi) $[A] = \{\langle x, \mu_A(x), \sigma_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$,

(vii) $\langle A\rangle = \{\langle x, 1 - \nu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$.

In the sequel, we need the following definition of level sets (which is also often called (α, β, γ)-cuts) of neutrosophic sets.

Definition 2.5. [2] Let A be a neutrosophic set on a set X. The (α, β, γ)-cut of A is a crisp subset

$$A_{\alpha,\beta,\gamma} = \{x \in X \mid \mu_A(x) \geq \alpha \text{ and } \sigma_A(x) \geq \beta \text{ and } \nu_A(x) \leq \gamma\},$$

where $\alpha, \beta, \gamma \in]0, 1]$.

Definition 2.6. [2] Let A be a neutrosophic set on a set X. The support of A is the crisp subset on X given by

$$Supp(A) = \{x \in X \mid \mu_A(x) \neq 0 \text{ and } \sigma_A(x) \neq 0 \text{ and } \nu_A(x) \neq 0\}.$$

3. Neutrosophic topological spaces

In this section, we provide the basic definitions of neutrosophic topological space and several properties of neutrosophic topological spaces in Šostak's sense.

3.1. *Definitions*

Definition 3.1. [16] Let X be non empty set and τ is a family of neutrosophic subsets in X satisfying the following axioms:

(i) $\emptyset, X \in \tau$, with $\emptyset = (0.0.1)$ and $X = (1.1.0)$.

(ii) For every $A, B \in \tau$, then $A \sqcap B \in \tau$.

(iii) For any $\{A_i, i \in I\} \sqsubseteq \tau$, then $\sqcup A_i \in \tau$.

In this case, the pair (X, τ) is called a neutrosophic topological space in Chang's sense, and any neutrosophic set in τ is known as neutrosophic open set in X.

Remark 3.2. To avoid any confusion or misunderstanding of some equation, we will use the symbols $(\sqsubset, \sqcup, \sqcup)$ to refer to the order, max, and min of neutrosophic sets and $(\leq, \wedge, \vee)$ to refer to the usual order, max, and min on the unit interval $[0, 1]$.

Now, we present the notion of neutrosophic topological space in Šostak's sense by a simple way.

Definition 3.3. Let X be non empty set and τ is a family of neutrosophic subsets in X satisfying the following axioms:

(i) $\mu_\tau(\emptyset) = \sigma_\tau(\emptyset) = 1$ and $\nu_\tau(\emptyset) = 0$,
$\mu_\tau(X) = \sigma_\tau(X) = 1$ and $\nu_\tau(X) = 0$.

(ii) For every $A_1, A_2 \in I^X$,
$\mu_\tau(A_1 \sqcap A_2) \geq \mu_\tau(A_1) \wedge \mu_\tau(A_2)$,
$\sigma_\tau(A_1 \sqcap A_2) \geq \sigma_\tau(A_1) \wedge \sigma_\tau(A_2)$ and
$\nu_\tau(A_1 \sqcap A_2) \leq \nu_\tau(A_1) \vee \nu_\tau(A_2)$.

(iii) For any $A_i \in I^X, i \in I$,
$\mu_\tau(\sqcup_{i \in I} A_i) \geq \wedge_{i \in I} \mu_\tau(A_i)$,
$\sigma_\tau(\sqcup_{i \in I} A_i) \geq \wedge_{i \in I} \sigma_\tau(A_i)$ and
$\nu_\tau(\sqcup_{i \in I} A_i) \leq \vee_{i \in I} \nu_\tau(A_i)$.

In this case the pair (X, τ) is called a neutrosophic topological space in Šostak's sense and any set in τ is known as neuterosophic open set in X. The functions μ_τ, σ_τ and ν_τ represent the degree of openness, the degree of neutral-openness, and the degree of non-openness, respectively.

Definition 3.4. Let X be non empty set and $\mathfrak{I}$ is a family of neutrosophic subsets in X satisfying the following axioms:

(i) $\mu_\mathfrak{I}(\emptyset) = \sigma_\mathfrak{I}(\emptyset) = 1$ and $\nu_\mathfrak{I}(\emptyset) = 0$,
$\mu_\mathfrak{I}(X) = \sigma_\mathfrak{I}(X) = 1$ and $\nu_\mathfrak{I}(X) = 0$.

(ii) For every $A_1, A_2 \in I^X$,
$\mu_{\mathfrak{I}}(A_1 \sqcup A_2) \geq \mu_{\mathfrak{I}}(A_1) \wedge \mu_{\mathfrak{I}}(A_2)$,
$\sigma_{\mathfrak{I}}(A_1 \sqcup A_2) \geq \sigma_{\mathfrak{I}}(A_1) \wedge \sigma_{\mathfrak{I}}(A_2)$ and
$\nu_{\mathfrak{I}}(A_1 \sqcup A_2) \leq \nu_{\mathfrak{I}}(A_1) \vee \nu_{\mathfrak{I}}(A_2)$.

(iii) For any $A_i \in I^X, i \in I$,
$\mu_{\mathfrak{I}}(\sqcap_{i\in I} A_i) \geq \wedge_{i\in I}\mu_{\mathfrak{I}}(A_i)$,
$\sigma_{\mathfrak{I}}(\sqcap_{i\in I} A_i) \geq \wedge_{i\in I}\sigma_{\mathfrak{I}}(A_i)$ and
$\nu_{\mathfrak{I}}(\sqcap_{i\in I} A_i) \leq \vee_{i\in I}\nu_{\mathfrak{I}}(A_i)$.

The pair $(X, \mathfrak{I})$ is called the closed set. The elements of $\mathfrak{I}$ are called closer neutrosophic sets.

4. **Properties of neutrosophic topological spaces in Šostak's sense**

In this section, we study interesting properties of neutrosophic topological spaces in Šostak's sense which is considered as a generalization of fuzzy and intuitionistic fuzzy topological spaces.

Proposition 4.1. *Let (X, τ) be a neutrosophic topological space in Šostak sense, and A be a neutrosophic subset on X. If the following statement hold:*

(i) $\mu_{\mathfrak{I}}(A) = \mu_\tau(A^c)$,

(ii) $\sigma_{\mathfrak{I}}(A) = \sigma_\tau(A^c)$,

(iii) $\nu_{\mathfrak{I}}(A) = \nu_\tau(A^c)$.

Then $\mathfrak{I}$ is a gradation of closeness.

Proof. (i) We show that $[\mu_{\mathfrak{I}}(\emptyset) = \sigma_{\mathfrak{I}}(\emptyset) = 1$ and $\nu_{\mathfrak{I}}(\emptyset) = 0]$ and $[\mu_{\mathfrak{I}}(X) = \sigma_{\mathfrak{I}}(X) = 1$ and $\nu_{\mathfrak{I}}(X) = 0]$. From $(i), (ii)$ and (iii) of Definition 3.3 we get that,
$\mu_{\mathfrak{I}}(\emptyset) = \mu_\tau(\emptyset^c) = \mu_\tau(X) = 1$,
$\sigma_{\mathfrak{I}}(\emptyset) = \sigma_\tau(\emptyset^c) = \sigma_\tau(X) = 1$ and
$\nu_{\mathfrak{I}}(\emptyset) = \nu_\tau(\emptyset^c) = \nu_\tau(X) = 0$.
By using the same methode, we obtain $[\mu_{\mathfrak{I}}(X) = \sigma_{\mathfrak{I}}(X) = 1$ and $\nu_{\mathfrak{I}}(X) = 0]$.

(ii) We show that $\mu_{\mathfrak{I}}(A_1 \sqcup A_2) \geq \mu_{\mathfrak{I}}(A_1) \wedge \mu_{\mathfrak{I}}(A_2)$, $\sigma_{\mathfrak{I}}(A_1 \sqcup A_2) \geq \sigma_{\mathfrak{I}}(A_1) \wedge \sigma_{\mathfrak{I}}(A_2)$ and $\nu_{\mathfrak{I}}(A_1 \sqcup A_2) \leq \nu_{\mathfrak{I}}(A_1) \vee \nu_{\mathfrak{I}}(A_2)$. By $(i), (ii)$ and (iii) of Definition 3.3 we get that, $\mu_{\mathfrak{I}}(A_1 \sqcup A_2) = \mu_\tau((A_1 \sqcup A_2)^c) = \mu_\tau(A_1^c \sqcap A_2^c)$. Also, by using the same Definition 3.3, we conclude that $\mu_\tau(A_1^c \sqcap A_2^c) \geq \mu_\tau(A_1^c) \wedge \mu_\tau(A_2^c) = \mu_{\mathfrak{I}}(A_1) \wedge \mu_{\mathfrak{I}}(A_2)$. Thus $\mu_{\mathfrak{I}}(A_1 \sqcup A_2) \geq \mu_{\mathfrak{I}}(A_1) \wedge \mu_{\mathfrak{I}}(A_2)$. In the same manner, we can show that $\sigma_{\mathfrak{I}}(A_1 \vee A_2) \geq \sigma_{\mathfrak{I}}(A_1) \sqcup \sigma_{\mathfrak{I}}(A_2)$ and $\nu_{\mathfrak{I}}(A_1 \sqcup A_2) \leq \nu_{\mathfrak{I}}(A_1) \vee \nu_{\mathfrak{I}}(A_2)$.

(iii) We show that $\mu_{\mathfrak{I}}(\sqcap_{i\in I} A_i) \geq \wedge_{i\in I}\mu_{\mathfrak{I}}(A_i)$, $\sigma_{\mathfrak{I}}(\sqcap_{i\in I} A_i) \geq \wedge_{i\in I}\sigma_{\mathfrak{I}}(A_i)$ and $\nu_{\mathfrak{I}}(\sqcap_{i\in I} A_i) \leq \vee_{i\in I}\nu_{\mathfrak{I}}(A_i)$. From $(i), (ii)$ and (iii) of Definition 3.3,we get that, $\mu_{\mathfrak{I}}(\sqcap_{i\in I} A_i) = \mu_\tau(\sqcap_{i\in I} A_i)^c = \mu_\tau(\sqcup_{i\in I}(A_i)^c)$. Also, by using the same Definition 3.3, we conclude that $\mu_\tau(\sqcup_{i\in I}(A_i)^c) \geq \wedge_{i\in I}\mu_\tau(A_i^c) = \wedge_{i\in I}\mu_{\Im}(A_i)$. Hence $\mu_{\Im}(\sqcap_{i\in I}A_i) \geq \wedge_{i\in I}\mu_{\Im}(A_i)$. In the same manner, we can show that $\sigma_{\Im}(\sqcap_{i\in I}A_i) \geq \wedge_{i\in I}\sigma_{\Im}(A_i)$ and $\nu_{\Im}(\sqcap_{i\in I}A_i) \leq \vee_{i\in I}\nu_{\Im}(A_i)$.

Therefore, $\Im$ satisfies the conditions of the gradation of closeness on X. □

Proposition 4.2. *Let $(X,\tau_i)_{i\in I}$ be a family of a neutrosophic topological spaces in Šostak's sense on X. Then their intersection $\curlywedge_{i\in I}(X,\tau_i)$ is a neutrosophic topological space in Šostak's sense on X.*

Proof. We put $\tau = \curlywedge_{k\in I}\tau_k$, and we will show that (X,τ) is a neutrosophic topological space in Šostak's sense on X.

(i) On the one hand $\mu_\tau(\emptyset) = \mu_{\curlywedge_{k\in I}}\tau_k(\emptyset) = \wedge_{k\in I}\mu_{\tau_k}(\emptyset) = \wedge_{k\in I}1 = 1$. In the same manner, we get that $\sigma_\tau(\emptyset) = 1$ and $\nu_\tau(\emptyset) = 0$. On the other hand $\mu_\tau(X) = \mu_{\curlywedge_{k\in I}}\tau_k(X) = \wedge_{k\in I}\mu_{\tau_k}(X) = \wedge_{k\in I}1 = 1$. By the same method, we get that $\sigma_\tau(X) = 1$ and $\nu_\tau(X) = 0$. Hence, $\mu_\tau(\emptyset) = \sigma_\tau(\emptyset) = 1$ and $\nu_\tau(\emptyset) = 0$, $\mu_\tau(X) = \sigma_\tau(X) = 1$ and $\nu_\tau(X) = 0$.

(ii) We show that $\mu_\tau(A_1 \sqcap A_2) \geq \mu_\tau(A_1) \wedge \mu_\tau(A_2), \sigma_\tau(A_1 \sqcap A_2) \geq \sigma_\tau(A_1) \wedge \sigma_\tau(A_2)$ and $\nu_\tau(A_1 \sqcap A_2) \leq \nu_\tau(A_1) \vee \nu_\tau(A_2)$, for every $A_1, A_2 \in I^X$.
$\mu_\tau(A_1 \sqcap A_2) = \mu_{\curlywedge_{k\in I}\tau_k}(A_1 \sqcap A_2) = \wedge_{k\in I}\mu_{\tau_k}(A_1 \sqcap A_2) \geq \wedge_{k\in I}(\mu_{\tau_k}(A_1) \wedge \mu_{\tau_k}(A_2) = (\wedge_{k\in I}\mu_{\tau_k}(A_1)) \wedge (\wedge_{k\in I}\mu_{\tau_k}(A_2)) = \mu_\tau(A_1) \wedge \mu_\tau(A_2)$, it follows that $\mu_\tau(A_1 \sqcap A_2) \geq \mu_\tau(A_1) \wedge \mu_\tau(A_2)$. In the same manner, we can show that $\sigma_\tau(A_1 \sqcap A_2) \geq \sigma_\tau(A_1) \wedge \sigma_\tau(A_2)$ and $\nu_\tau(A_1 \sqcap A_2) \leq \nu_\tau(A_1) \vee \nu_\tau(A_2)$.

(iii) We show that $\mu_\tau(\sqcup_{i\in I}A_i) \geq \wedge_{i\in I}\mu_\tau(A_i), \sigma_\tau(\sqcup_{i\in I}A_i) \geq \wedge_{i\in I}\sigma_\tau(A_i)$ and $\nu_\tau(\sqcup_{i\in I}A_i) \leq \vee_{i\in I}\nu_\tau(A_i)$. For any $A_i \in I^X$
$\mu_\tau(\sqcup_{i\in I}A_i) = \mu_{\curlywedge_{k\in I}\tau_k}(A_i) = \wedge_{k\in I}\mu_{\tau_k}(\sqcup_{i\in I}A_I) \geq \wedge_{k\in I}(\wedge_{i\in I}\mu_{\tau_k}(A_i))$. Since τ_k are a neutrosophic topologies in Šostak's sense, then it holds that $\wedge_{k\in I}(\wedge_{i\in I}\mu_{\tau_k}(A_i)) \geq \wedge_{i\in I}(\wedge_{k\in I}\mu_{\tau_k}(A_i)) = \wedge_{i\in I}\mu_\tau(A_i)$, it follows that $\mu_\tau(\sqcup_{i\in I}A_i) = \wedge_{i\in I}\mu_\tau(A_i)$, $\sigma_\tau(\sqcup_{i\in I}A_I) = \sigma_{\curlywedge_{k\in I}\tau_k}(\sqcap A_i) = \wedge_{k\in I}\sigma_{\tau_k}(\sqcup_{i\in I}A_I) \geq \wedge_{k\in I}(\wedge_{i\in I}\sigma_{\tau_k}(A_i)) \geq \wedge_{i\in I}(\wedge_{k\in I}\sigma_{\tau_k}(A_i)) = \wedge_{k\in I}\sigma_\tau(A_i)$, then it follows that $\sigma_\tau(\sqcup_{i\in I}A_i) = \wedge_{i\in I}\sigma_\tau(A_i)$ and $\nu_\tau(\sqcup_{i\in I}A_I) = \nu_{\curlywedge_{k\in I}\tau_k}(\sqcap A_i) = \wedge_{k\in I}\nu_{\tau_k}(\sqcup_{i\in I}A_I) \leq \wedge_{k\in I}(\wedge_{i\in I}\nu_{\tau_k}(A_i)) \leq \wedge_{i\in I}(\wedge_{k\in I}\nu_{\tau_k}(A_i)) = \wedge_{k\in I}\nu_\tau(A_i)$, then $\nu_\tau(\sqcup_{i\in I}A_i) \leq \vee_{i\in I}\nu_\tau(A_i)$. Hence $\mu_\tau(\sqcup_{i\in I}A_i) \geq \wedge_{i\in I}\mu_\tau(A_i)$, $\sigma_\tau(\sqcup_{i\in I}A_i) \geq \wedge_{i\in I}\sigma_\tau(A_i)$ and $\nu_\tau(\sqcup_{i\in I}A_i) \leq \vee_{i\in I}\nu_\tau(A_i)$.

Thus τ is a neutrosophic topological space in Šostak's sense. □

Next, we provide a characterization in neutrosophic topological space in Šostak's sense. First, we need the following notions.

Definition 4.3. Let $f : (X, \tau) \to (Y, \tau')$ be a mapping between two topologies. Then f is called :

(1) open if $\mu_\tau(A) \leq \mu_\tau(f(A)), \sigma_\tau(A) \leq \sigma_{\tau'}(f(A))$ and $\nu_\tau(A) \geq \nu_{\tau'}(f(A))$,
(2) closed if $\mu_\Im(A) \leq \mu_{\Im'}(f(A)), \sigma_\Im(A) \leq \sigma_{\Im'}(f(A))$ and $\nu_\Im(A) \geq \nu_{\Im'}(f(A))$.

Definition 4.4. Let (X, τ) and (Y, τ') are two neutrosophic topologies and $f : X \to Y$ is a mapping and is called a neutrosophic continuous if $\mu_\tau(f^{-1}(B)) \geq \mu_{\tau'}(B), \sigma_\tau(f^{-1}(B)) \geq \sigma_{\tau'}(B)$ and $\nu_\tau(f^{-1}(B)) \leq \nu_{\tau'}(B)$, for every $B \in I^Y$, where $f^{-1}[B]$ is defined by $f^{-1}[B](x) = B(f(x)), \forall x \in X$.

Theorem 4.5. *Let (X, τ) and $(X, \tau^{'})$ be two neutrosophic topological spaces in Šostak's sense and $f : X \longrightarrow Y$ be a bijective mapping. The following statement are equivalent*

(i) f^{-1} *is a neutrosophic continuous,*
(ii) f *is a neutrosophic open,*
(iii) f *is a neutrosophic closed.*

Proof. $(i) \Longrightarrow (ii)$
Suppose that f^{-1} is a neutrosophic continuous, we need to prove that f is neutrosophic open. Since f^{-1} is a continuous mapping, then it holds that $\mu_{\tau'}(f(B)) \geq \mu_\tau(B), \sigma_{\tau'}(f(B)) \geq \sigma_\tau(B)$ and $\nu_{\tau'}(f(B)) \leq \nu_\tau(B)$ for any $B \in I^X$. We get directly that f is a neutrosophic open.
$(ii) \Longrightarrow (iii)$
Suppose that f is a neutrosophic open i.e., $\tau(A) \leq \tau'(f(A))$, and we need to prove that f is a neutrosophic closed i.e., $\Im(A) \leq \Im'(f(A))$. We have that $\Im(A) = \tau(A^c)$. and $\mu_\Im(A) = \tau(A^c) \leq \tau'(f(A^c))$, because that f is a neutrosophic open $\mu_{\tau'}(f(A^c)) = \mu_{\tau'}[(f(A))^c] = \mu_{\Im'}(f(A))$, then it follows that $\mu_{\Im(A)} \leq \mu_{\Im'}(f(A))$. By using the same method, we can obtain that $\sigma_{\Im(A)} \leq \sigma_{\Im'}(f(A))$. Also, $\nu_\Im(A) = \tau(A^c) \geq \tau' f(A^c)$, since f is neutrosophic open $\nu_{\tau'} f(A^c) = \nu_{\tau'}[(f(A))^c] = \nu_{\Im'}(f(A))$, then it follows that $\nu_{\Im(A)} \geq \nu_{\Im'}(f(A))$. Hence, f is a neutrosophic closed.
$(iii) \Longrightarrow (i)$
Suppose that f is a neutrosophic closed i.e., $\Im(A) \leq \Im'(f(A))$, and we need to prove that f^{-1} is a neutrosophic continuous i.e., $\tau(A) \leq \tau'(f(A))$ for any $A \in I^X$, $\mu_\tau(A) = \mu_\Im(A^c) \leq \mu_{\Im'}(f(A^c))$, because that f is a neutrosophic closed $\mu_\Im(f(A^c)) = \mu_{\Im'}[(f(A))^c] = \mu_{\tau'} f(A)$, then it holds that $\mu_\tau(A) \leq \mu_{\tau'} f(A)$. By using the same method, we can obtain that $\sigma_\tau(A) \leq \mu_{\tau'} f(A)$. Also, $\nu_\tau(A) = \mu_\Im(A^c) \geq \nu_{\Im'}(f(A^c))$, from f is a neutrosophic closed $\nu_{\Im'}(f(A^c)) = \nu_{\Im'}[(f(A))^c] = \nu_{\tau'} f(A)$. Hence, $\nu_\tau(A) \geq \nu_{\tau'} f(A)$. Thus, f^{-1} is a neutrosophic continuous. □

5. Conclusion

In this work, we have studied the notion of neutrosophic topological spaces in Šostak's sense which is considered as a generalization of fuzzy and intuitionistic fuzzy topological spaces. Future work will be directed to study topological properties in this case such as completeness, compactness and other topological properties.

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