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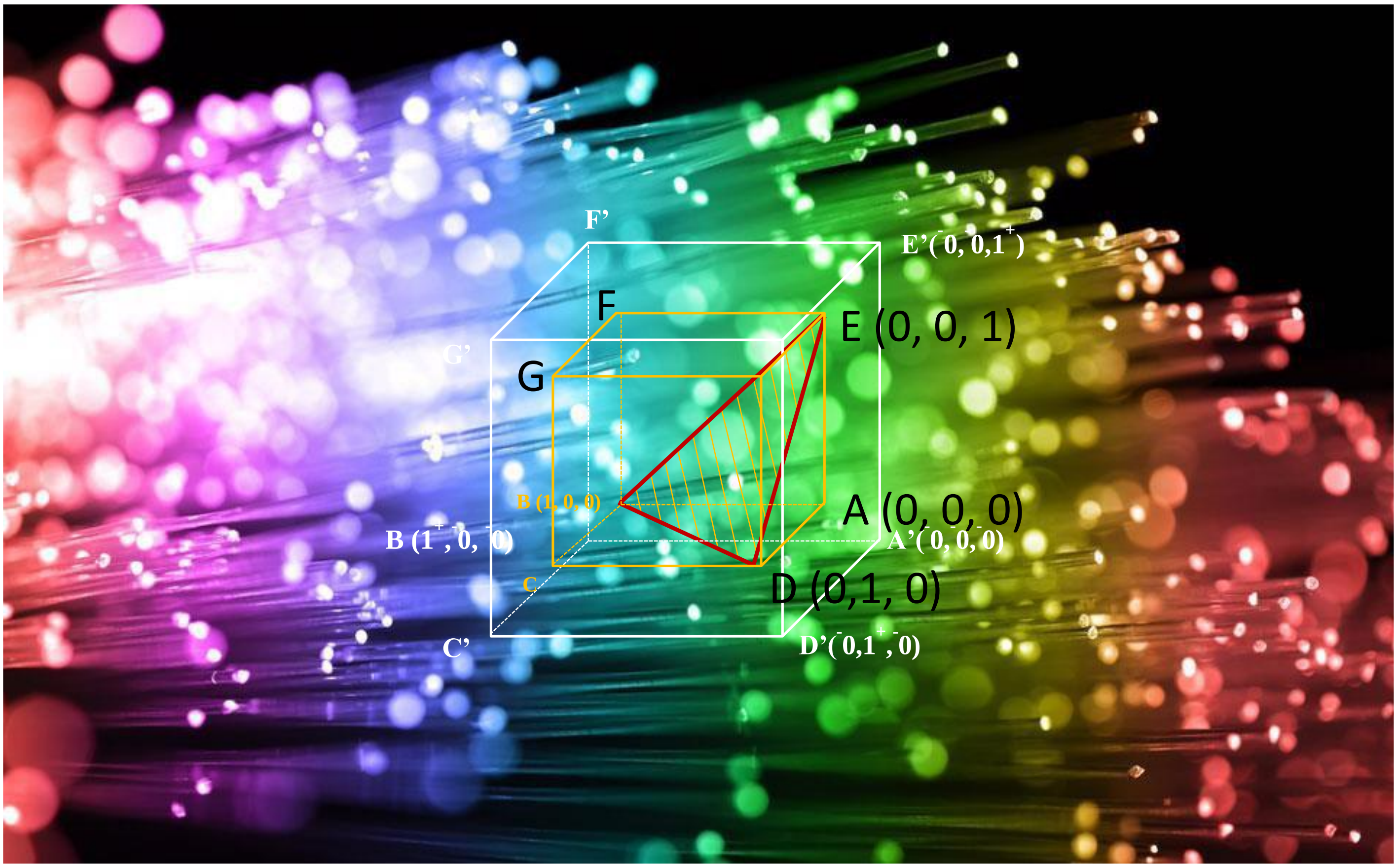

Editors-in-Chief

Salah Bouzina, Florentin Smarandache, Ahmed Hatip

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Neutrosophic Knowledge

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The submitted papers should be professional, in good Arabic, Turkish, English and French containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

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This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth
(*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of *]-0, 1+[*.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Content

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Evolution of Crisp-Set to Modern Set Theories

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Abstract. This article presents a comparative analysis of the conceptual expansion from classical Crisp-Set Theory to more contemporary set-theoretic frameworks. By examining the underlying ideas and models of Fuzzy-Set Theory, Intuitionistic Fuzzy-Set Theory, and Single-Valued Neutrosophic Set Theory, it offers a coherent narrative of this theoretical development. The work begins with a historical contextualization of these mathematical entities. It then includes an extensive treatment of Crisp-Set Theory to establish a foundation that is frequently absent from standard texts. The analysis continues with dedicated explorations of Fuzzy-Set, Intuitionistic Fuzzy-Set, and Single-Valued Neutrosophic Set theories.

Keywords: Crisp-Set;Fuzzy-Set; Intuitionistic Fuzzy-Set;Single-Valued Neutrosophic Set

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1. Introduction

This article traces the philosophical and conceptual evolution of the set — the foundational object of mathematics — from its classical, deterministic form to modern frameworks capable of handling ambiguity, indeterminacy, and neutrality. We posit that mathematical concepts are not static; they develop through scientific activity as humans grapple with problems posed by the physical world. Among these, the concept of a set is the cornerstone of mathematical philosophy. One could justifiably claim: without the set, there is no mathematical object. To attempt otherwise is akin to rolling a dice and waiting for a seven. This article has a dual aim. Its immediate purpose is to present a simplified picture of the development of Crisp-Sets, Fuzzy-Sets, Intuitionistic Fuzzy-Sets, and Neutrosophic-Sets through their underlying mathematical models, in preparation for our works in [2–8]. Its broader, more distant goal is to contribute to the philosophy of mathematics by comparing these concepts and exploring the implications of their associated philosophies in the age of artificial intelligence, big data, and complex information networks. The class of Crisp-Sets represents a fundamental dividing line in mathematical thought. On one side lies the doctrine of mathematical certainty; on the other, the doctrine of ambiguity and indeterminacy. This distinction promises to profoundly influence the future interaction between philosophy, logic, and mathematics. The journey beyond crisp boundaries begins with Neutrosophy, a branch of philosophy introduced by Smarandache that deals with the concept of neutrality—the tendency to avoid taking a side in a conflict. Its core principle can be summarized as follows: for any proposition $\mathcal{A}$, we must consider not only its negation $\neg\mathcal{A}$ and its opposite $\diamond\mathcal{A}$, but also the spectrum of neutralities between them. As Smarandache states, "Between an idea and its opposite, there is a continuum-power spectrum of neutralities" [19,20]. It is from this philosophical school that the neutrosophic set originates.

1.1. *Crisp-Set (Cantor): The Bedrock of Classical Mathematics*

In the late 19th century, Georg Cantor developed classical Set Theory, where a "Crisp-Set" is a clear-cut, unambiguous collection of objects [17,22]. This section provides an overview of Crisp-Sets and establishes key results, framing them as the pivotal point from which new mathematical concepts emerged.

1.2. *Fuzzy Sets(Zadeh): The Cornerstone of Ideology Uncertainty After Probability Theory*

In 1965, Zadeh proposed Fuzzy Set Theory as an extension of Crisp-Sets [25]. He argued that classical membership functions were insufficient for representing real-world phenomena where the boundaries between membership and non-membership are not sharp. This marked the beginning of a new "ideology of uncertainty" to challenge the prevailing "ideology of certainty." Scholars have justified this shift compellingly:

i. Zadeh noted that "the classes of objects encountered in the real physical world do not have precisely defined criteria of membership" [25, p.338], using the classification of animals as an example.
ii. Klir and Folger connected this to the modern problem of complexity, stating that "large amounts of information coupled with large amounts of uncertainty... constitute the ground of many of our problems today" [14, p.1].
iii. Singh illustrated the need for fuzzy mathematics with simple questions: "How many seeds in a pile constitute a heap?" or how does one precisely categorize the intensity of rain? Such concepts are inherently fuzzy [18, p.ix].

This framework is essential for modeling natural language and subjective perception. Consider the class of fuzzy words (FW) in English, such as tall, heap, or cloudy. Mathematics, as a universal language, provides the tools to describe these vague predicates through degrees of membership, thereby addressing scientific issues that crisp logic cannot.

1.3. *Intuitionistic Fuzzy Sets (Atanassov)*

In 1983, Atanassov further extended Fuzzy Set Theory by introducing Intuitionistic Fuzzy Sets (IFS) [9–11]. This model incorporates not only a degree of membership but also an explicit degree of non-membership. Atanassov justified this by pointing to the deep-seated bipolar models in human thought—oppositions like good–evil and day–night that have structured our understanding since ancient times . This logic directly applies to fields like linguistics, particularly in modeling antonymy, the lexical relationship of opposites.

1.4. *Neutrosophic Sets (Smarandache)*

The Single-Valued Neutrosophic Set (SVNS) represents the latest stage in this conceptual evolution [24]. It is a powerful modern tool for handling imperfect data characterized by uncertainty, inconsistency, and incompleteness. As a practical counterpart to the more general neutrosophic set [19, 20], the SVNS formalizes Smarandache's philosophy of neutrality by independently quantifying truth-membership, indeterminacy-membership, and falsity-membership. This provides a more nuanced and flexible framework for computation and reasoning in complex, real-world environments [23].

2. **Crisp-Set as a Key Point of New Mathematical Concepts**

The concept of the "Crisp-Set" forms the historical and logical foundation of classical set theory, which was formally developed by Georg Cantor in the late 19th century. A crisp set is characterized by clear, unambiguous boundaries, where an element either definitively belongs to the set or does not—there is no room for partial membership. This section provides a foundational overview of crisp sets and establishes several key results that will be essential for understanding the modern extensions that follow.

Definition 2.1. [17, 22] A set or class is a well-defined collection of distinct objects. An object in a set is called an element or member of that set. This concept was introduced by G. Cantor (1845-1918), who is considered the father of set theory, or, more accurately, the genesis of Set Theory.

Remark 2.2. We will denote any set consisting of two elements by 2. Let us say $2 = \{0, 1\}$.

Definition 2.3. A Crisp-Set(CS) is a bilateral structure class, written

$$\mathbb{C}_s = \{(u, \mu_A(u)) : u \in U\} \tag{1}$$

And it consists of:

(1) A universal set U is the domain of μ_A,

(2) A subset $A \subseteq U$ is a predicate of μ_A,

(3) The set $\{0,1\}$ is the co-domain of μ_A, and

(4) $\mu_A : U \to \{0,1\}$ is is called the characteristic function of $u \in U$ to a set A in U,or the degree of membership function of $u \in U$ to a set A in U, or the truth of the predicate of $u \in U$ to a set $A \in U$. with the membership function property:$(\forall u \in U)$,we have,

$$\mu_A = \begin{cases} 1, & \text{if } u \in A \\ 0, & \text{if } u \in A^c = U - A \end{cases} \tag{2}$$

Remark 2.4. In the language of Set Theory, if U is a universal set, then there exist the following facts, $\phi, U \subseteq U$. And it is proved according to concepts defined in Set Theory. So, if $\phi \subseteq U$, then:

$$\mu_{\phi(u)} = \begin{cases} 1, & \text{if } u \in \phi \\ 0, & \text{if } u \in \phi^c = U \end{cases} \tag{3}$$

Also, if $U \subseteq U$,then, we get,

$$\mu_U(u) = \begin{cases} 1, & \text{if } u \in U \\ 0, & \text{if } u \in U^c = \phi \end{cases} \tag{4}$$

Definition 2.5. *A Crisp-Set Complement* (CCS) is a bilateral structure class, written

$$\mathbb{C}_s^c = \{(u, \mu_{A^c}(u)) : u \in U\} \tag{5}$$

And it consists of:

(1) A universal set U is the domain of μ_{A^c},

(2) A subset A^c of a set U is a predicate of μ_{A^c},

(3) The interval $\{0,1\}$ is the co-domain of μ_{A^c}, and

(4) $\mu_{A^c} : U \to \{0,1\}$is is called the characteristic function of $u \in U$ to a set A^c in U,or the degree of membership function of $u \in U$ to a set A^c in U,or the truth of the predicate of $u \in U$ to a set A^c in U. with the membership function property:$(\forall u \in U)$,we have,

$$\mu_{A^c}(u) = \begin{cases} 1, & \text{if } u \in A^c \\ 0, & \text{if } u \in (A^c)^c = A \end{cases} \tag{6}$$

Example 2.6. Let $U = \{a,b,c,d,e\}$, and $A = \{a,b\} \subset U$, if $\mu_A(a) = \mu_A(b) = 1$,and $\mu_A(c) = \mu_A(d) = \mu_A(e) = 0$. Then the entity of the Crisp-Set $\mathbb{C}_s$ looks like:

$$\begin{aligned} \mathbb{C}_s =& \{(u, \mu_A(u)) : u \in U\} \\ =& \{(a,1),(b,1),(c,0),(d,0),(e,0)\} \end{aligned} \tag{7}$$

A Crisp-Set $\mathbb{C}_s$ is a subset of ordered pairs of the Cartesian product $\mathbb{C}_s \subset U \times \{0,1\}$. In addition, we have, $A^c = \{c,d,e\}$, hence $\mu_{A^c}(c) = \mu_{A^c}(d) = \mu_{A^c}(e) = 1$ and $\mu_{A^c}(a) = \mu_{A^c}(b) =$

0. So, the entity of the Crisp-Set complement $\mathbb{C}_{ss}^c$ becomes:

$$\begin{aligned}\mathbb{C}_s^c =&\{(u, \mu_{A^c}(u)) : u \in U\}\\ =&\{(a,0),(b,0),(c,1),(d,1),(e,1)\}\end{aligned} \tag{8}$$

Moreover, $\mathbb{C}_s \cup \mathbb{C}_s^c = U \times \{0,1\}$, and $\mathbb{C}_s \cap \mathbb{C}_s^c = \phi$. Thus $\mathbb{C}$ and $\mathbb{C}_s^c$ are partition of $U \times \{0,1\}$.

Example 2.7. Let $\mathbb{N}$ be a set of all natural numbers and,

$$A = \{x \in N : x \geq 10\}.$$

So in this case, for instance, we have $\mu_A(8) = 0$ and $\mu_A(20) = 1$.

Definition 2.8. The Empty Crisp-Set is defined by:

$$\mathbb{C}_s = \phi \iff \mu_A(u) = 0, \forall u \in U \tag{9}$$

Definition 2.9. The Universal Crisp-Set is defined by:

$$\mathbb{C}_s = U \iff \mu_A(u) = 1, \forall u \in U \tag{10}$$

The next theorem gives us some operations and relations on the class of Crisp-Set.

Theorem 2.10. *Let $\mathbb{C}_{s1}$ and $\mathbb{C}_{s2}$ be two crisp sets of subsets A and B of U, respectively. Then:*

(1) *Crisp-Set Inclusion $\mathbb{C}_{s1} \subseteq \mathbb{C}s2$:*

$$(\forall u \in U), ((A \subseteq B) \Longrightarrow \mu_{A\cap B}(u) = \mu_A(u))) \tag{11}$$

(2) *Crisp-Set Equality $\mathbb{C}_{s1} = \mathbb{C}_{s2}$:*

$$(\forall u \in U), ((A = B) \iff (\mu_A(u) = \mu_B(u))) \tag{12}$$

(3) *Crisp-Set Union $\mathbb{C}_{s1} \cup \mathbb{C}_{s2}$:*

$$(\forall u \in U), ((A \cup B) \iff (\mu_{A\cup B}(u) = \mu_A(u) + \mu_B(u) - \mu_{A\cap B}(u))) \tag{13}$$

(4) *Crisp-Set Intersection $\mathbb{C}_{s1} \cap \mathbb{C}_{s2}$:*

$$(\forall u \in U), ((A \cap B) \iff (\mu_{A\cap B}(u) = \mu_A(u) \cdot \mu_B(u))) \tag{14}$$

(5) *Crisp-Set Difference $\mathbb{C}_{s1} \setminus \mathbb{C}_{s2}$:*

$$(\forall u \in U), ((A \setminus B) \iff (\mu_{A\setminus B}(u) = \mu_A(u) - \mu_{A\cap B}(u))) \tag{15}$$

Proof. (1) Suppose that $\mathbb{C}_{s1} \subseteq \mathbb{C}_{s2}$and $u \in A\cap B$,then $\mu_{A\cap B}(u) = 1$ and $\mu_A(u) = 1$. Also, if $u \notin A\cap B$, then $u \notin A\cap B$, and we have $\mu_{A\cap B}(u) = 0$ and $\mu_A(u) = 0$. Therefore, in both cases, $\mu_{A\cap B}(u) = \mu_A(u)$. Conversely, assume that $\mu_{A\cap B}(u) = \mu_A(u)$ and $(u, \mu_A \in \mathbb{C}_{s1})$. Since $A \cap B \subseteq B$, it follows that $\mu_{A\cap B}(u) \subset \mu_B(u)$, hence $\mu_A(u) \subset \mu_B(u)$, therefore, $(u, \mu_A(u)) \in \mathbb{C}_{s2}$. Thus $\mathbb{C}_{s1} \subseteq \mathbb{C}_{s2}$.

(2) $\mathbb{C}_{s1} \subseteq \mathbb{C}_{s2} \iff (u, \mu_A(u)) = (u, \mu_B(u)), (\forall u \in U)$, $Dom(\mu_A) = Dom(\mu_B)\, and\, Range(\mu_A) = Range(\mu_B)$, $\iff \mu_A(u) = \mu_B(u), (\forall u \in U)$, $\iff \mu_A = \mu_B$.

(3) Let $u \in A \cup B$, then $\mu_{A\cup B} = 1$. We consider three cases:
   (a) If $u \in A$ and $u \notin B$, then $\mu_A(u) + \mu_B(u) - \mu_{A\cap B}(u) = 1 + 0 - 0 = 1 = \mu_{A\cup B}(u)$.
   (b) If $u \notin A$ and $u \in B$, then $\mu_A(u) + \mu_B(u) - \mu_{A\cap B}(u) = 0 + 1 - 0 = 1 = \mu_{A\cup B}(u)$.
   (c) If $u \in A$ and $u \in B$, then $\mu_A(u) + \mu_B(u) - \mu_{A\cap B}(u) = 1 + 1 - 1 = 1 = \mu_{A\cup B}(u)$.

   Now, if $u \notin A \cup B$, then $\mu_{A\cup B}(u) = 0$, and since $u \notin A$ and $u \notin B$, we have $\mu_A(u) + \mu_B(u) - \mu_{A\cap B}(u) = 0 + 0 - 0 = 0 = \mu_{A\cup B}(u)$.

(4) If $u \in A \cap B$, then $\mu_{A\cap B}(u) = 1$, and $\mu_A(u) \cdot \mu_B(u) = 1 \cdot 1 = 1$. While, If $u \notin A \cap B$, then $\mu_{A\cap B}(u) = 0$, and $\mu_A(u) \cdot \mu_B(u) = 0 \cdot 0 = 0$.

(5) If $u \in A \setminus B$, then $\mu_{A\setminus B}(u) = 1$. We have $u \in A$ and $u \notin B$, so $\mu_A(u) - \mu_{A\cap B}(u) = 1 - 0 = 1$. When $u \in A \cap B$, then $\mu_{A\setminus B}(u) = 0$, and since $u \in A$ and $u \in B$, we have $\mu_A(u) - \mu_{A\cap B}(u) = 1 - 1 = 0$.

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The next theorem gives us the important property that connects or links the Power-Set
$P(U)$
and the set 2^U.

Theorem 2.11. *Let U be a universal set and $P(U)$ its power set of U. Let 2^U denote the set of all functions from U into the set $2 = \{0, 1\}$. Then $P(U) \equiv 2^U$; that is, they are equivalent sets.*

Proof. Define a mapping $\Psi : P(U) \to 2^U$ by $\Psi(A) = \mu_A$, where $\mu_A : U \to \{0, 1\}$ is the characteristic function of A, defined by

$$\mu_A(u) = \begin{cases} 1, & \text{if } u \in A, \\ 0, & \text{if } u \in U \setminus A. \end{cases} \tag{16}$$

We show that Ψ is a bijection:

i. Well-defined: For every $A \in P(U)$, the function $\mu_A \in 2^U$ exists. Thus, Ψ maps each subset to a unique function.

ii. Injective: Suppose that $A_1, A_2 \in P(U)$ such that $\Psi(A_1) = \Psi(A_2)$, implies that
$\mu_{A1} = \mu_{A2}$. But, $A_1 = \{(u, \mu_{A1}(u) = 1)\} = \{(u, \mu_{A2}(u)\} = A_2$ Hence, $A_1 = A_2$, so Ψ is injective.

iii. Surjective: Let $f \in 2^U$ be any function. Define $A = f^{-1}(1) = \{u \in U : f(u) = 1\}$. Then $A \in P(U)$, and the characteristic function μ_A satisfies

$$\mu_A(u) = \begin{cases} 1, & \text{if } u \in A, \\ 0, & \text{otherwise,} \end{cases} \tag{17}$$

which equals $f(u)$. Thus, $\Psi(A) = f$, and Ψ is surjective. Therefore, Ψ is a bijection between $P(U)$ and 2^U. □

So far, we have reviewed and discussed the concepts of Crisp-Set,Crisp-Set Complement, Empty Crisp-Set,Universal Crisp-Set, Crisp-Set-Equality,Crisp-Set Union, Crisp-Set -Intersection, and Crisp-Set-Differences on the previous arguments. Of course! The role of the characteristics function reflects many real-world situations that require responding between two different choices. Therefore, it represents one example that illustrates the nature of a universal language, through which existence can be expressed in a single and neutral manner, namely, the language of mathematics.

Definition 2.12. [1] The semantic (or meaning) of propositional logic consists of truth valuations. A valuation (or truth assignment or interpretation) V in the language $\mathcal{L}$ is a function from the set of simple statement letters into the set $\{T, F\}$. I.e.,

$$V(A) = \begin{cases} T, \text{ if } A \text{ is true} \\ F, \text{ if } A \text{ is false} \end{cases} \tag{18}$$

which satisfies the following conditions:

$$V(\neg A) = \begin{cases} T, \text{ if } V(A) = F \\ F, \quad \text{otherwise} \end{cases} \tag{19}$$

$$V(A \wedge B) = \begin{cases} T, \text{ if } V(A) = V(B) = T \\ F, \quad \text{otherwise} \end{cases} \tag{20}$$

$$V(A \vee B) = \begin{cases} F, \text{ if } V(A) = V(B) = F \\ T, \quad \text{otherwise} \end{cases} \tag{21}$$

$$V(A \to B) = \begin{cases} F, \text{ if } (V(A) = T) \wedge (V(B) = F) \\ T, \quad \text{otherwise} \end{cases} \tag{22}$$

$$V(A \leftrightarrow B) = \begin{cases} T, \text{ if } V(A) = V(B) \\ F, \text{ if } \ V(A) \neq V(B) \end{cases} \tag{23}$$

Remark 2.13. By comparing Definition 2.3 and Theorem 2.10 with Definition 2.12, we can register the following observations:

- $\mu_A(u)$ and $V(A)$ have the same structure as a piecewise function.
- $\mu_A(u)$ as a mathematical entity, it belongs to the field of Set Theory. While,

- $V(A))$ as a logical entity, it belongs to the field of Mathematical Logic or Philosophy.
- Domain of $\mu_A(u)$ is a well-defined collection of objects, but the Domain of $V(A))$ is a well-defined collection of all simple propositions. So, they have different domains.
- The co-domain (= Range) of$\mu_A(u)$, and$V(A)$, they seemed the same when put $T = 1$ and $F = 0$. However, they have different meanings, specifically membership (truth, falsehood).
- The patterns of both structures seem identical, but they have different interpretations.

Remark 2.14. The Empty Crisp-Set and the Universal Crisp-Set are models of constant functions. The graph presentation of Crisp-Set μ_C can be represented by lattice points in 2-dimensions. The pictorial diagram represents the degree of membership function of $u \in U$ to a set A in U is shown in the Figure 1.

$$\mu_A : U \to \{0, 1\}$$

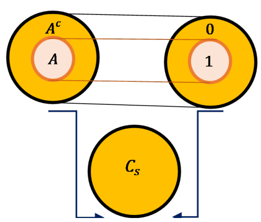

FIGURE 1. Crisp-Set $\mathbb{C}_s$

.

3. **Fuzzy Sets(FS)**

Definition 3.1. [25] Let U be a universal set and $A \subseteq U$. A Fuzzy-Set (FS) is a bilateral structure, written

$$\mathbb{F}_s = \{(u, \mu_A(u)) : u \in U\} \tag{24}$$

and it consists of:

(1) A universal set U is the domain of μ_A,
(2) A subset $A \subseteq U$ is a predicate of μ_A,
(3) The interval $[0, 1]$ is the co-domain of μ_A, and
(4) The partial-truth function $\mu_A : U \to [0, 1]$ of $u \in U$ to a set A in U, or the degree of membership function of $u \in U$ to a set A in U, or the truth of the predicate of $u \in U$ to a set A in an universal set U.

with the membership function property:

$$(\forall u \in U) \Rightarrow (0 \leq \mu_A(u) \leq 1) \tag{25}$$

From $Eq.(25)$ we have:

$$v_A(u) = 1 - \mu_A(u) \tag{26}$$

$Eq.(26)$ is called the degree of non-membership function of $u \in U$ to a set A.

Remark 3.2. A fuzzy-set $\mathbb{F}_s$ is a vague boundary compared to a crisp set $\mathbb{C}_s$. Also, μ_A is called the one-valued membership function or single-valued membership function. The following pictorial diagram represents the degree of membership function of μ_A, to a set A, in a universal set U.

$$\mu_A : U \to [0, 1]$$

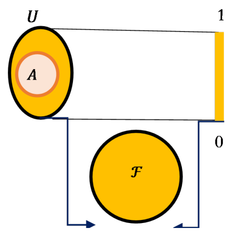

FIGURE 2. Fuzzy-Set $\mathbb{F}_s$

.

Definition 3.3. The Empty Fuzzy-Set is defined by:

$$\mathbb{F}_s = \phi \iff \mu_A(u) = 0, (\forall u \in U). \tag{27}$$

Definition 3.4. The Universal Fuzzy-Set is defined by:

$$\mathbb{F}_s = U \iff \mu_A(u) = 1, \forall u \in U. \tag{28}$$

Remark 3.5. The Empty Fuzzy-Set and the Universal Fuzzy-Set are models of constant functions, where The Empty Fuzz-Set is a model of a constant function and represents no presence unless is called the partial presence, and the Universal Fuzzy-Set is a model of a constant function and interprets full presence as the absence of partial presence. The graph presentation of Fuzzy-Set $\mathbb{F}_s$ can be represented by lattice points in 2- dimensions. The next definition gives us some operations and relations on the class of Fuzzy-Set.

Definition 3.6. [25] [Operations and Relations on a Fuzzy-Set] Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two Fuzzy subsets of A and B in the universal set U, respectively, then for all $\forall u \in U$, we have the following axioms:

(1) Fuzzy-Set Inclusion $\mathbb{F}_1 \subseteq \mathbb{F}_2$:

$$(\forall u \in U), ((A \subseteq B) \iff (\mu_A(u) \leq \mu_B(u))). \tag{29}$$

(2) Fuzzy-Set Equality $\mathbb{F}_1 = \mathbb{F}_2$:

$$(\forall u \in U), ((A = B) \iff (\mu_A(u) = \mu_B(u))). \tag{30}$$

(3) Fuzzy-Set Union $\mathbb{F}_1 \cup \mathbb{F}_2$:

$$(\forall u \in U), (A \cup B) = (u, \mu_{A\cup B}(u) = \max\{\mu_A(u), \mu_B(u)\}. \tag{31}$$

(4) Fuzzy-Set Intersection $\mathbb{F}_1 \cap \mathbb{F}_2$:

$$(\forall u \in U), (A \cap B) = (u, \mu_{A\cap B}(u) = \min\{\mu_A(u), \mu_B(u)\}. \tag{32}$$

(5) Fuzzy-Set Complement F_s^c:

$$(\forall u \in U), (A^c(u) = (u, \mu_{A^c}(u) = 1 - \mu_A(u)). \tag{33}$$

(6) Fuzzy-Set Difference $\mathbb{F}_1 \setminus \mathbb{F}_2$:

$$(\forall u \in U), (A \setminus B) = A \cap B^c. \tag{34}$$

Example 3.7. [13] Let $\mathbb{R}$ be the set of real numbers. Define the set of numbers close to any real number a by membership function:$\mu_A(x) = \frac{1}{1+10(x-a)^2}, \forall x \in \mathbb{R}$, where $\mu_A : \mathbb{R} \to [0,1]$. We illustrate this example by a particular case. Consider $\mu_A : U = [-10 : 0.1 : 10] \to [0,1]$. The set of numbers close to 0 is shown in Figure 3.

$$\mu_A : U = [-10 : 0.1 : 10] \to [0,1]$$

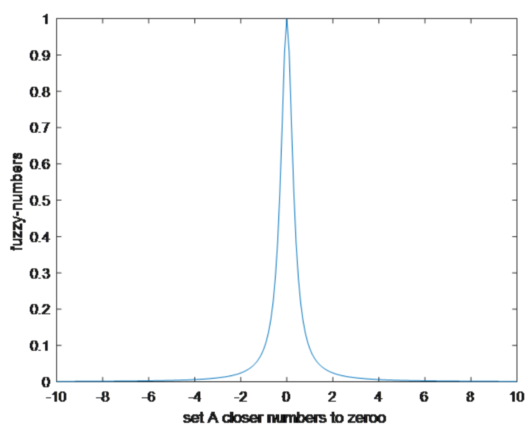

FIGURE 3. Fuzzy-Set $\mathbb{F}_s$.
For the set of numbers close to 0.

Lastly, we provided a brief comparison in the previous sections of two patterns of objects, one characterized by crispness and the other by fuzziness. To illustrate how these entities evolve within the mathematical environment, they represent an existential language that enables the interpretation of cognitive or ontological aspects, aligning with modern cognitive requirements and concepts.

4. Intuitionistic Fuzzy Set

Definition 4.1. [9–11] Let U be a universal set and $A \subseteq U$. A Intuitionistic Fuzzy-Set (IFS) is a triple structure, written

$$\mathbb{I}_s = \{(u, \mu_A(u), \nu_A(u)) : u \in U\} \tag{35}$$

And it consists of:

(1) A universal set U is the domain of μ_A and ν_A,
(2) A subset $A \subseteq U$ is a predicate of μ_A and ν_A,
(3) The interval $[0,1]$ is the co-domain of μ_A and ν_A,
(4) The partial-truth function $\mu_A \colon U \to [0,1]$ of $u \in U$ to a set A in U, and
(5) The falsity-truth function $\nu_A \colon U \to [0,1]$ of $u \in U$ to a set A in U,

with the membership function property:

$$\forall u \in U \Rightarrow 0 \leq \mu_A(u) + \nu_A(u) \leq 1 \tag{36}$$

From $Eq.(36)$ we have:

$$\rho_A(u) = 1 - (\mu_A(u) + \nu_A(u)) \tag{37}$$

$Eq.(37)$ is called the degree of indeterminacy (uncertainty)/ hesitation of $u \in U$ to set A in U. If the degree of hesitation $\rho_A(u) = 0$, then $Eq(37)$ becomes:

$$\nu_A(u) = 1 - \mu_A(u) \tag{38}$$

$Eq.(38)$ in IFS and FS is called the degree of non-membership / falsity function. Despite Zadeh in [25] treated with this entity serves as a complement to the degree of truth of the membership function. The following pictorial diagram represents the degree of membership and the non-membership function of A in U. μ_A and ν_A are called the two-valued membership functions.

$$\mu_A, \nu_A : U \to [0,1]$$

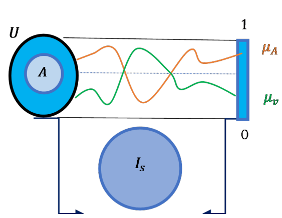

FIGURE 4. Intuitionistic Fuzzy-Set $\mathbb{I}_s$

Example 4.2. Let $U = \{1, 2, 3, 4, 5, 6, \}$ be a universal set and the intuitionistic fuzzy set is given by:

$$\mathbb{I}_s = \left\{ \begin{array}{c} (1, 0.0031, 0.9969), (2, 0.0063, 0.9937), (3, 0.0094, 0.9906), \\ (4, 0.0126, 0.9874), (5, 0.0157, 0.9843), (6, 0.0188, 0.9812) \end{array} \right\}. \tag{39}$$

Where $\nu_A(u) = 1 - \mu_A(u)$, and $\rho_A(u) = 0$. That is, there is no degree of indeterminacy (uncertainty)/hesitation of $u \in U$ to set A in U. If we consider that there is a degree of indeterminacy (uncertainty)/ hesitation of$u \in U$ to set A in U. Then IFS is given by:

$$\mathbb{I}_s = \left\{ \begin{array}{c} (1, 0.0031, 0.5016, 0.4953), (2, 0.0063, 0.5031, 0.4906), \\ (3, 0.0094, 0.5047, 0.4859), (4, 0.0126, 0.5063, 0.4812). \\ (5, 0.0157, 0.5079, 0.4764), (6, 0.0188, 0.5094, 0.4717) \end{array} \right\}. \tag{40}$$

Example 4.3. Let U be the set of people in the city of Sana'a. Define IFS as follows: a man's age at a time t in this case, IFS is given by:

$$A = \{Young, middle - aged, old\} \tag{41}$$

with member function, and non-member function,
$\mu_A, \nu_A : U \to [0, 1]$ such that

$$\begin{cases} \mu_{young} A(t) = \max\left(0, \frac{30 - A(t)}{10}\right) \text{" Young"}, \\ \nu_{young} A(t) = \max(0, \frac{A(t)-25}{10}) \text{"Non-Young", and} \\ \rho_{young} A(t) = 1 - \left(\mu_{young} A(t) + \mu_{young} A(t)\right) \\ \text{ when "Hesitation and uncertainty data".} \end{cases} \tag{42}$$

For instance, if Ahmad any element in U such that his age 27, then:

$$\begin{cases} \mu_{young} A(t) = 0.3 (30\,\%\text{young}) \\ \nu_{young} A(t) = 0.2 (20\,\%\text{non-young}), \text{and} \\ \rho_{young} A(t) = 0.5 (50\,\%\text{uncertanity}) \end{cases} \tag{43}$$

Definition 4.4. The Empty intuitionistic Fuzzy-Set is defined by:

$$\mathbb{I}_s = \phi \iff \Big(\mu_A(u) = 0 \land \nu_A(u) = 1\Big), (\forall u \in U). \tag{44}$$

Definition 4.5. The Universal intuitionistic Fuzzy-Set is defined by:

$$\mathbb{I}_s = U \iff \Big(\mu_A(u) = 1 \land \nu_A(u) = 0\Big), (\forall u \in U). \tag{45}$$

Definition 4.6. [Operations and Relations on a intuitionistic Fuzzy-Set] [9–11] Let $\mathbb{I}_1$ and $\mathbb{I}_2$ be two intuitionistic Fuzzy subsets of A and B in the universal set U, respectively, then $\forall u \in U$, we have the following axioms:

(1) intuitionistic Fuzzy-Set Inclusion $\mathbb{I}_1 \subseteq \mathbb{I}_2$:

$$(\forall u \in U), (A \subseteq B) \iff ((\mu_A(u) \le \mu_B(u)) \land (\nu_A(u) \ge \nu_B(u))) \tag{46}$$

(2) intuitionistic Fuzzy-Set Equality $\mathbb{I}_{s1} = \mathbb{I}_{s2}$:

$$(\forall u \in U), (A = B) \Longleftrightarrow ((\mu_A(u) = \mu_B(u)) \wedge (\nu_A(u) = \nu_B(u)) \tag{47}$$

(3) Intuitionistic Fuzzy-Set Union $\mathbb{I}_{s1} \cup \mathbb{I}_{s2}$:

$$(\forall u \in U), (A \cup B) = \begin{cases} ((u, \mu_{A\cup B}(u), \nu_{A\cup B}(u)) : (\forall u \in U), \\ \mu_{A\cup B}(u) = \ max\{(\mu_A(u), \mu_B(u)\}, \\ \nu_{A\cup B}(u) = \min\{(\nu_A(u), \nu_B(u)\}. \end{cases} \tag{48}$$

(4) Intuitionistic Fuzzy-Set Intersection $\mathbb{I}_{s1} \cap \mathbb{I}_{s2}$:

$$(\forall u \in U), (A \cap B) = \begin{cases} ((u, \mu_{A\cap B}(u), \nu_{A\cap B}(u)) : (\forall u \in U), \\ \mu_{A\cap B}(u) = \min\{(\mu_A(u), \mu_B(u))\}, \\ \nu_{A\cap B}(u) = \max\{(\nu_A(u), \nu_B(u))\} \end{cases} \tag{49}$$

(5) Intuitionistic Fuzzy-Set Complement $\mathbb{I}_s^c$:by interchanging the order of the components of a member/non-member function.

$$(\forall u \in U), (A^c(u)) = (u, \nu_A(u), \mu_A(u)). \tag{50}$$

(6) Intuitionistic Fuzzy-Set Difference $\mathbb{I}_{s1} \setminus \mathbb{I}_{s2}$:

$$(\forall u \in U), (A \setminus B) = A \cap B^c. \tag{51}$$

5. **Neutrosophic Sets (NS)**

The Neutrosophic Set was proposed by Smarandache as a mathematical representation of Neutrosophy, a branch of philosophy he introduced to study the origin, nature, and scope of neutralities and their interactions with various ideational spectra [19, 20]. Smarandache defined two types of Neutrosophic Sets: Non-Standard and Standard. Non-Standard sets, following Abraham Robinson's approach, are based on the system of hyperreal numbers [13]. In this work, however, we will focus exclusively on Standard Neutrosophic Sets, specifically through the practical framework of the Single-Valued Neutrosophic Set (SVNS), see [19, 22–24] .

Definition 5.1. [Single-Valued Neutrosophic Set(SVNA)] [21, 24] Let U be a universal set and $A \subseteq U$. A Single- Valued Neutrosophic set is an equadrible structure, written:

$$\mathbb{N}_s = \{(u, \mu_A(u), \rho_A(u), \nu_A(u)) : u \in U\} \tag{52}$$

And it consists of:

(1) A universal set U is the domain of μ_A, ρ_A, and ν_A,

(2) A subset $A \subseteq U$ is a predicate of μ_A, ρ_A, and ν_A,

(3) The interval $[0,1]$ is the co-domain of μ_A, ρ_A, and ν_A,

(4) The partial-truth function μ_A of $u \in U$ to a set A in U,

(5) The indeterminacy-membership function ρ_A of $u \in U$ to a set A in U,and

(6) The falsity-truth function ν_A of $u \in U$ to a set A in U;

With the following independent neutrosophic components; truth membership function indeterminacy-membership / non-membership (or falsity membership function property:

$$(\forall u \in U), (0 \leq \mu_A(u) + \rho_A(u) + \nu_A(u) + \leq 3). \tag{53}$$

Remark 5.2. From $Eq.$(53) we have three independent neutrosophic components of memberships functions [21,24]. In this case we said that Single-Valued Neutrosophic Set of independent neutrosophic components. becomes a neutrosophic powerful tool for modelling complex real world problems. And the following equation

$$(\forall u \in U), (0 \leq \mu_A(u) + \rho_A(u) + \nu_A(u) + \leq 1). \tag{54}$$

$Eq.$(54) is called Single-Valued Neutrosophic Set of dependent neutrosophic components Here, $\mathbb{N}_s$ is called the 3-valued membership function, where, $\mu_A, \rho_A, \nu_A \colon U \to [0,1]$.

Definition 5.3. [24] The Empty Single-Valued Neutrosophic-Set is defined by:

$$(\mathbb{N}_s = \phi) \Longleftrightarrow \big((\mu_A(u) = 0 \wedge (\rho_A(u) = 0 \wedge \nu_A(u) = 0)\big), (\forall u \in U). \tag{55}$$

Definition 5.4. [24] The Universal Single-Valued Neutrosophic-Set is defined by:

$$(\mathbb{N}_s = U) \Longleftrightarrow \big((\mu_A(u) = 1 \wedge (\rho_A(u) = 0 \wedge \nu_A(u) = 0)\big), (\forall u \in U) \tag{56}$$

Definition 5.5. [Operations and Relations on SVNA] [24] Let $\mathbb{N}_{s1}$ and $\mathbb{N}_{s2}$ be two Single-Valued Neutrosophic Subsets of A and B in the universal set U, respectively, then for all $\forall u \in U$, we have the following axioms:

(1) Single-Valued Neutrosophic-Set Inclusion $\mathbb{N}_{s1} \subseteq \mathbb{N}_{s2}$:

$$(\forall u \in U), (A \subseteq B) \Longleftrightarrow \begin{cases} (\mu_A(u) \leq \mu_B(u)) \\ (\rho_A(u) \leq \rho_B(u)) \\ (\nu_A(u) \geq \nu_B(u)). \end{cases} \tag{57}$$

(2) Single-Valued Neutrosophic-Set Equality $\mathbb{N}_{s1} = \mathbb{N}_{s2}$:

$$(\forall u \in U), ((A = B) \Longleftrightarrow ((\mu_A(u) = \mu_B(u)) \wedge (\nu_A(u) = \nu_B(u)) \tag{58}$$

(3) Single-Valued Neutrosophic-Set Union $\mathbb{N}_{s1} \cup \mathbb{N}_{s2}$:

$$(\forall u \in U), (A \cup B) = \begin{cases} ((u, \mu_{A\cup B}(u), \rho_{A\cup B}(u), \nu_{A\cup B}(u)), \\ \mu_{A\cup B}(u) = \max\{(\mu_A(u), \mu_B(u)\}), \\ \rho_{A\cup B}(u) = \max\{(\rho_A(u), \rho_B(u)\}), \\ \nu_{A\cup B}(u) = \min\{(\nu_A(u), \nu_B(u)\}) \end{cases} \tag{59}$$

(4) Single-Valued Neutrosophic-Set Union $\mathbb{N}_{s1} \cap \mathbb{N}_{s2}$:

$$(\forall u \in U), (A \cap B) = \begin{cases} ((u, \mu_{A\cap B}(u), \rho_{A\cap B}(u), \nu_{A\cap B}(u)), \\ \mu_{A\cap B}(u) = \min\{(\mu_A(u), \mu_B(u)\}), \\ \rho_{A\cup B}(u) = \min\{(\rho_A(u), \rho_B(u)\}), \\ \nu_{A\cup B}(u) = max\{(\nu_A(u), \nu_B(u)\}) \end{cases} \tag{60}$$

(5) Single-Valued Neutrosophic-Set Complement $\mathbb{N}_s^c$:

$$(\forall u \in U), \mathbb{N}_s^c = \{(u, \nu_A(u), 1 - \rho_A(u), \mu_A(u))\} \tag{61}$$

(6) Single-Valued Neutrosophic-Set Difference $\mathbb{N}_{s1} \setminus \mathbb{N}_{s2}$:

$$(\mathbb{N}_{s1} \setminus \mathbb{N}_{s2} = (N_{s1} \cap \mathbb{N}^c{}_{s2}) \tag{62}$$

Remark 5.6. The following diagram represents the Single-Valued Neutrosophic Set of neutrsophic component independent or dependent .It is difficult to pilot Single-Valued Neutrosophic Set in 4-dimension for all degree of memberships, but it can be Visualization and captured in 3D.

$$\mu_A, \rho_A, \nu_A \colon U \to [0, 1]$$

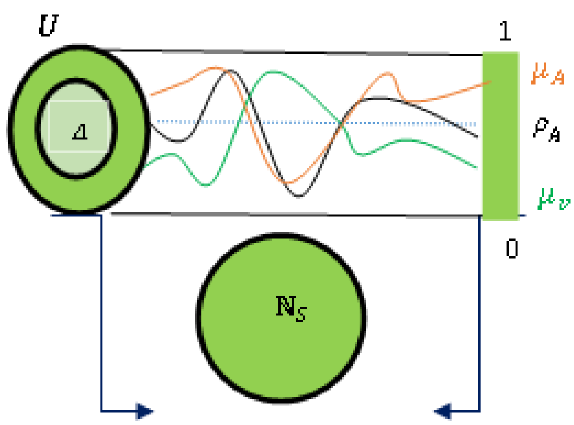

FIGURE 5. Fuzzy-Set $\mathbb{N}_s$.

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Article

Formalisation du perspectivisme latino-américain : un modèle neutrosophique de multiperspectivisme

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Résumé : Cet article aborde le défi de modéliser formellement les systèmes de connaissance complexes, souvent contradictoires, propres à la pensée décoloniale latino-américaine. Il introduit le *MultiPerspectivisme Neutrosophique*, un cadre théorique qui opérationnalise le perspectivisme philosophique à l'aide de la logique neutrosophique. Nous posons comme postulat que la connaissance est située et que la vérité est médiée par le point de vue de l'observateur — un concept enraciné dans la philosophie nietzschéenne mais qui trouve une expression singulière dans les cosmologies amérindiennes et les systèmes juridiques pluralistes. La méthode principale consiste à proposer un *Modèle de MultiPerspectivisme Neutrosophique*, utilisant des ensembles multi-neutrosophiques pour représenter les perspectives subjectives sous forme de triplets de séquences de vérité, d'indétermination et de fausseté ⟨T, I, F⟩. Un résultat clé est le développement d'une fonction de similarité permettant de quantifier l'affinité entre différentes perspectives, transformant ainsi des points de vue qualitatifs en données analysables. Ce modèle est appliqué à une étude de cas portant sur un conflit foncier juridique, où il permet de révéler des similitudes structurelles cachées — telles qu'un style de raisonnement absolutiste partagé entre des parties opposées — et d'identifier des alliances stratégiques potentielles. La conclusion principale est que la logique neutrosophique offre un outil mathématique robuste pour naviguer à travers l'ambiguïté et la pluralité des réalités perspectivistes, allant au-delà de la simple reconnaissance de la différence vers son analyse formelle et son intégration possible..

Mots-clés : Logique neutrosophique ; MultiPerspectivisme ; Philosophie latino-américaine ; Théorie décoloniale ; Ensembles multi-neutrosophiques ; Fonction de similarité ; Pluralisme juridique ; Perspectivisme.

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1. Introduction

Le perspectivisme en philosophie est un courant de pensée qui soutient que toute connaissance, perception ou vérité est inévitablement médiée par la perspective du sujet connaissant [1]. Cette idée, puissamment développée par Friedrich Nietzsche, remet en question la notion d'une vérité objective et universelle, en proposant à la place qu'il existe de multiples interprétations du monde, chacune valide dans son propre cadre [2].

Pour Nietzsche, il n'existe pas de faits purs, seulement des interprétations — ce qui implique que l'accès à la réalité est toujours conditionné par des facteurs historiques, culturels, psychologiques et linguistiques [3].

Au-delà de Nietzsche, le perspectivisme a influencé divers courants contemporains tels que le post-structuralisme, le constructivisme social et les épistémologies du Sud, qui soulignent tous l'importance du « lieu d'énonciation ». Dans les contextes latino-américains, par exemple, il a été articulé en lien avec la pensée décoloniale [4] et les visions du monde autochtones [5], qui reconnaissent la coexistence de multiples mondes et manières de connaître. Ainsi, le perspectivisme ne nie pas l'existence de la réalité ; il remet plutôt en question la possibilité de la saisir à travers une seule rationalité dominante, ouvrant la voie à une compréhension pluraliste, relationnelle et située de la connaissance.

Le terme *perspectivisme* est également étroitement associé au philosophe espagnol José Ortega y Gasset, qui en a fait un élément central de sa philosophie. Ortega soutenait que la réalité se manifeste différemment à chaque individu, et que la somme de toutes ces perspectives individuelles permet de s'approcher d'une compréhension plus complète de la vérité [6].

Cette approche, qui reconnaît la validité de points de vue multiples sans dissoudre l'existence d'une réalité objective, s'aligne sur un courant contemporain de la philosophie des sciences connu sous le nom de *réalisme perspectiviste* [7]. Cette position soutient que la connaissance, bien qu'objective, est toujours partielle et obtenue à partir d'une perspective située. De la même manière que différentes cartes (topographique, politique, climatique) peuvent décrire le même territoire de façons distinctes mais également valides, des perspectives diverses peuvent offrir une connaissance réelle, bien que partielle, du monde. Ce cadre permet de dépasser la fausse dichotomie entre un universalisme absolutiste et un relativisme nihiliste — une tension centrale dans cette étude.

Le perspectivisme ne cherche pas à invalider la cohérence interne ni l'applicabilité formelle des principes logiques classiques — tels que le principe de non-contradiction, le tiers exclu ou la bivalence — lorsqu'ils opèrent dans un système fermé et cohérent. Il remet plutôt en question l'hypothèse ontologique selon laquelle ces principes refléteraient nécessairement une réalité objective et universelle, indépendante du sujet. D'un point de vue perspectiviste, ces lois logiques ne sont pas rejetées dans leur structure formelle, mais leur prétention à une correspondance absolue avec la réalité du monde est soumise à un examen critique. La critique perspectiviste ne vise donc pas la validité interne de telles formules, mais leur portée épistémologique — c'est-à-dire, si de telles structures binaires sont adéquates pour saisir toute la complexité, l'ambiguïté et la multiplicité contextuelle de l'expérience vécue, des systèmes culturels et de la production de savoir. De cette manière, le perspectivisme reconfigure la logique elle-même en tant que construction située, dont les postulats fondamentaux peuvent être valables dans certains paradigmes, mais non universellement dans tous les domaines ontologiques ou épistémologiques.

1.1 Principe de Non-Contradiction

Ce principe postule qu'une proposition $\mathcal{P}$ ne peut pas simultanément se voir attribuer les valeurs de vérité vrai et faux. Formellement, il est exprimé comme suit :

$$\neg(\mathcal{P} \land \neg\mathcal{P}) \quad (1)$$

qui est interprété comme : « *Il n'est pas le cas que* $\mathcal{P}$ *et sa négation* $\neg\mathcal{P}$ *soient tous deux vrais*. » Cet axiome assure la cohérence logique des systèmes classiques en excluant la coexistence d'assertions contradictoires.

1.2 Principe du Tiers Exclu

Ce principe affirme que pour toute proposition bien formée $\mathcal{P}$, soit $\mathcal{P}$ est vraie, soit sa négation $\neg\ \mathcal{P}$ est vraie. Il est symboliquement représenté comme suit :

$$\mathcal{P} \vee \neg\mathcal{P} \quad (2)$$

Cette disjonction est universellement valable en logique classique, excluant l'existence de toute valeur de vérité intermédiaire ou tierce entre le vrai et le faux.

1.3 Principe de Bivalence

Contrairement aux deux précédents, il s'agit d'un principe sémantique plutôt que syntaxique. Il postule l'existence d'une fonction de valuation.

$$v: L \rightarrow \{0,1\} \quad (3)$$

définie sur un langage logique L, telle que pour toute formule bien formée $\phi \in L$,

$$v(\phi) \in \{0,1\} \quad (4)$$

Autrement dit, chaque proposition est soit vraie $(v(\phi) = 1)$ soit fausse $(v(\phi) = 0)$, sans aucune tolérance pour la gradation, l'indétermination ou les valeurs de vérité alternatives. Ce cadre de valorisation binaire sous-tend la validité formelle à la fois du Principe de Non-Contradiction et du Principe du Tiers Exclu.

Les épistémologies perspectivistes et les logiques non-classiques — telles que les systèmes à valeurs multiples, flous et neutrosophiques — remettent en question le Principe de Bivalence en introduisant des évaluations où les propositions peuvent être partiellement vraies, indéterminées, ou même simultanément vraies et fausses selon des points de vue spécifiques [8]. Ainsi, l'affirmation selon laquelle la valeur d'une proposition doit être strictement soit 0 soit 1 est incompatible avec les cadres qui acceptent la pluralité épistémique ou ontologique.

*

En résumé, cet article aborde le défi fondamental de modéliser formellement les systèmes de connaissance complexes et souvent contradictoires inhérents à la pensée décoloniale latino-américaine, pour lesquels la logique bivalente classique s'avère insuffisante. L'objectif principal a été de construire un pont conceptuel et méthodologique entre le perspectivisme philosophique et le cadre mathématique de la logique neutrosophique, cherchant à opérationnaliser ses principes. À cette fin, l'article introduit le cadre du *MultiPerspectivisme Neutrosophique*, développant un modèle qui utilise des ensembles MultiNeutrosophiques et une fonction de similarité pour représenter et comparer quantitativement divers points de vue. À travers une étude de cas portant sur un litige foncier, la capacité du modèle à révéler des dynamiques structurelles cachées entre les parties en conflit est démontrée, validant ainsi son utilité en tant qu'outil robuste pour analyser et naviguer l'ambiguïté et la pluralité inhérentes aux réalités perspectivistes.

2. Préliminaires

2.1 Philosophie neutrosophique et Perspectivisme

La *philosophie neutrosophique* [9] offre un cadre formel et mathématique à l'école philosophique du perspectivisme. La relation peut être comprise à travers les connexions clés suivantes [10].

Le lien le plus explicite entre la neutrosophie et le perspectivisme est le *Principe de Relativité Référentielle* [9]. Ce principe stipule que « la vérité, la fausseté et l'indétermination de toute proposition dépendent du système référentiel dans lequel elle est examinée ». Cela signifie qu'une idée peut être vraie dans un contexte, fausse dans un autre, et indéterminée dans un troisième. C'est la thèse centrale du perspectivisme : il n'y a pas de vérités absolues ; la connaissance est toujours conditionnée par le point de vue de l'observateur. La neutrosophie adopte cette idée et l'établit comme un principe fondamental de son système.

Le perspectivisme défie la logique classique, qui est basée sur une stricte dualité vrai/faux. Contrairement à la logique traditionnelle, la neutrosophie introduit le concept de *degrés de vérité, de* *fausseté et d'indétermination*. Cela permet la modélisation mathématique d'une réalité où les propositions ne sont pas absolument vraies ou fausses mais existent sur un spectre, comme le suggérerait une approche perspectiviste [11].

La neutrosophie revisite les idées traditionnelles, affirmant que les vérités au sein d'un système référentiel peuvent devenir des faussetés dans un autre, et vice versa. Cette approche souligne la *fluidité de la connaissance* et exhorte à considérer les idées sous de multiples angles, un objectif central de la pensée perspectiviste.

Si le perspectivisme affirme l'existence de nombreuses perspectives valides, la question se pose de savoir quel système logique utiliser pour leur analyse. Il n'y a pas une seule « logique vraie » ; au contraire, un ensemble de cadres logiques existe, chacun offrant différentes perspectives sur la vérité et la validité [12]. La neutrosophie s'aligne sur ce pluralisme, se présentant non pas comme la seule logique mais comme un outil flexible et puissant particulièrement adapté aux contextes d'indétermination et de contradiction, qui sont des conséquences naturelles de la coexistence de multiples perspectives.

Le perspectivisme conduit inévitablement à des *contradictions* lorsque deux points de vue valides s'opposent. La logique classique ne peut gérer cela sans écarter l'une des perspectives. La neutrosophie, cependant, est conçue pour *embrasser l'indétermination, le paradoxe et l'interaction entre les opposés et les neutralités*.

En permettant aux valeurs de T et F d'être indépendantes, la neutrosophie peut modéliser formellement une proposition qui est à la fois vraie et fausse en même temps (un état dialéthique, $T = 1$, $F = 1$). C'est une formalisation directe de l'apparence d'une contradiction entre deux perspectives opposées [13].

La composante d'*indétermination (I)* est cruciale, car elle permet l'analyse et la quantification de l'ambiguïté, du flou ou de l'incertitude qui surviennent lorsque les perspectives sont confuses ou incomplètes.

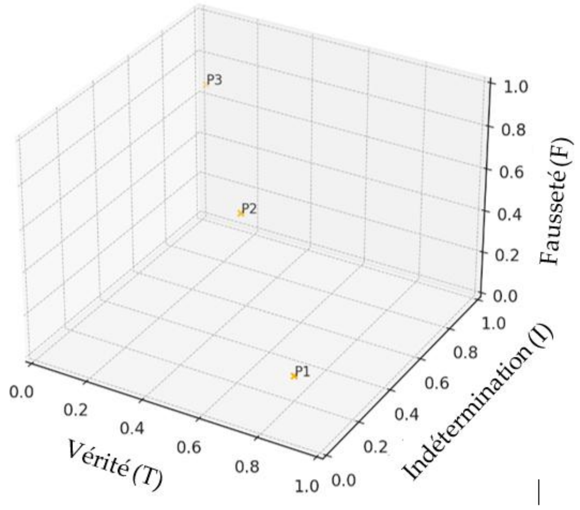

Figure 1. Représentation tridimensionnelle des perspectives neutrosophiques agrégées dans l'espace (T, I, F).

Alors que le perspectivisme se concentre sur la coexistence de la thèse (<A>) et de ses diverses antithèses (<antiA>), la neutrosophie va un pas plus loin en introduisant et en étudiant systématiquement la *neutrothèse* (<neutA>). La neutrothèse représente le spectre des neutralités — des idées ou des états qui ne sont ni la thèse ni l'antithèse, mais qui se situent dans l'espace intermédiaire. De cette manière, la neutrosophie non seulement valide de multiples perspectives (<A> et <antiA>), mais analyse également le « continuum des neutralités » qui les connecte et les équilibre, enrichissant ainsi l'analyse perspectiviste [14, 15].

Pour ce faire, la neutrosophie propose deux principes de recherche actifs [16] :

- Rechercher les « parties communes dans les choses peu communes ».
- Rechercher les « parties peu communes dans les choses communes ».

Le résultat visé est une « *synthèse* » qui intègre des éléments provenant d'opposés, à l'instar de la dialectique hégélienne, mais avec l'inclusion distincte de leurs neutralités.

La neutrosophie se rapporte au perspectivisme en partageant un fondement philosophique commun. Cependant, la neutrosophie se distingue en se concentrant sur une méthode analytique plus spécifique visant à l'intégration. Elle ne se limite pas à l'affirmation philosophique de l'existence de multiples points de vue, mais offre plutôt les outils — tels que les valeurs (T, I, F), la relativité référentielle, le pluralisme logique et la gestion de la contradiction — pour analyser, modéliser et comprendre un monde défini par l'incertitude, l'ambiguïté et la coexistence de multiples réalités. C'est, comme le suggère le titre, un nouveau paradigme de pensée dans un monde intrinsèquement perspectiviste.

La neutrosophie est un cadre philosophique et méthodologique qui étudie la relation entre les opposés et leurs indéterminations. Son objectif principal est de trouver un terrain d'entente entre des concepts opposés et d'identifier les différences dans des choses similaires, parvenant ainsi à une compréhension plus nuancée et intégrative.

2.2 Le Perspectivisme comme cadre pour les logiques plurielles

Le *perspectivisme* est un cadre philosophique qui affirme que la connaissance, la vérité et l'interprétation sont toujours conditionnées par le point de vue de l'observateur. Il remet en question l'existence de vérités absolues et indépendantes du contexte, et souligne la *multiplicité des perspectives valides* coexistant au sein de systèmes de pensée complexes. Bien que le perspectivisme soit souvent associé à Friedrich Nietzsche dans la philosophie occidentale, ses racines conceptuelles peuvent être retracées bien plus tôt dans des traditions non-européennes.

En ce qui concerne l'affinement ou la multiplication de l'indétermination, des extensions de la Logique Neutrosophique ont déjà été définies dans la littérature : la *Logique Neutrosophique Quadruple* (Vérité ; Contradiction, Incertitude ; Faux), la *Logique Neutrosophique Quintuple* (Vérité ; Contradiction, Incertitude, Inconnu ; Faux), la *Logique Neutrosophique Sextuple* (Vérité, I_1, I_2, I_3, I_4 ; Faux), la *Logique Neutrosophique Septuple* (Vérité, I_1, I_2, I_3, I_4, I_5 ; Faux), et ainsi de suite jusqu'à la *Logique Neutrosophique n-tuple* (où l'indétermination a été divisée en n-2 types de sous-indéterminations, selon l'application, pour $n \geq 3$).

Mais l'extension la plus générale concerne le cas où toutes les valeurs T, I, F ont été affinées ou multipliées autant de fois que nécessaire dans chaque problème (*Logique Neutrosophique n-Raffinée*, et respectivement *Logique n-MultiNeutrosophique*).

De plus, en 2014, Smarandache a introduit la *Loi du Multiple-Milieu Inclus (Law of Included Multiple-Middle)* [53] (comme extension de la Loi classique du Milieu Inclus) :

$(\langle A \rangle;\quad \langle neutA_1 \rangle, \langle neutA_2 \rangle, \ldots, \langle neutA_n \rangle;\quad \langle antiA \rangle)$

et en 2023, la *Loi des Infinis-Moyenne Inclus (Law of Included Infinitely-Many-Middles)* [54] :

$(\langle A \rangle;\quad \langle neutA_1 \rangle, \langle neutA_2 \rangle, \ldots, \langle neutA_\infty \rangle;\quad \langle antiA \rangle)$

(par exemple, entre les couleurs Blanc et Noir, il existe une infinité de nuances de couleurs).

2.2.1 Perspectivisme Ancien dans la Logique Jaïne

Dans le jaïnisme, la doctrine d'*Anekāntavāda* (littéralement, « doctrine des multiples aspects ») offre une approche perspectiviste sophistiquée de la vérité et de la réalité [17]. Selon cette doctrine, la réalité est complexe et ne peut être entièrement saisie par un seul point de vue. En conséquence, chaque affirmation n'est que partiellement vraie et doit être complétée par d'autres points de vue pour approcher une compréhension plus exhaustive. Cette forme ancienne de perspectivisme non seulement anticipe le pluralisme postmoderniste, mais fournit également une fondation métaphysique pour la tolérance, le raisonnement dialogique et une épistémologie non-absolutiste.

La logique jaïne, exprimée à travers la *Loi de Prédication Septuple (Saptabhangi)* [18], constitue un remarquable exemple précoce de raisonnement perspectiviste qui embrasse la nature multiforme et complexe de la réalité. Contrairement à la logique occidentale classique — qui adhère strictement au principe de bivalence, affirmant qu'une proposition doit être soit vraie, soit fausse — la logique jaïne autorise sept modes de prédication distincts, incluant des combinaisons telles que « vrai et indéterminé » ou « vrai et faux ». Ces formulations remettent directement en question la loi classique de non-contradiction.

Figure 2. Symbole jaïn de *l'Ahimsa.*

Enracinée dans le principe philosophique de *l'anekantavada (la doctrine de la non-unilatéralité)* [19], la logique jaïne affirme que toute affirmation n'est valide que conditionnellement à partir d'une perspective particulière. Cette humilité épistémologique découle de la reconnaissance que le langage humain est intrinsèquement limité dans sa capacité à décrire pleinement la richesse de l'existence. En conséquence, les penseurs jaïns ont promu une forme de *non-violence intellectuelle (ahimsa)* [20], encourageant le respect des désaccords raisonnables et la reconnaissance des vérités partielles.

La logique jaïne résonne conceptuellement avec la *Logique Neutrosophique*, un cadre moderne introduit par Florentin Smarandache. La logique neutrosophique généralise les logiques classiques et floues en introduisant trois composantes à chaque proposition logique : la vérité (V), l'indétermination (I) et la fausseté (F), chacune variant indépendamment dans l'intervalle réel standard ou non standard [−0, 1+]. Tout comme la logique jaïne reconnaît qu'une affirmation peut être simultanément vraie, fausse et indéterminée, la logique neutrosophique formalise cette multiplicité et l'étend avec une rigueur mathématique. Les deux systèmes rejettent la rigidité des dichotomies classiques et embrassent l'incertitude, la contradiction et la connaissance partielle comme intrinsèques à la compréhension et au raisonnement humains.

Représentation Neutrosophique du Saptibhaṅgī [21]

1. **C'est vrai (*syāt asti*)** : Cela affirme que la proposition est indubitablement vraie, sans aucune fausseté ou indétermination.

Représentation Neutrosophique : (1, 0, 0)

- T = 1 : Le degré de vérité est de 100 %.
- I = 0 : Le degré d'indétermination est de 0 %.
- F = 0 : Le degré de fausseté est de 0 %.

2. **C'est faux (*syāt nāsti*)** : Cela affirme que la proposition est indubitablement fausse.

Représentation Neutrosophique : (0, 0, 1)

- T = 0 : Le degré de vérité est de 0 %.
- I = 0 : Le degré d'indétermination est de 0 %.
- F = 1 : Le degré de fausseté est de 100 %.

3. **C'est vrai et faux (*syāt asti nāsti*)** : La proposition est simultanément vraie et fausse lorsqu'elle est vue sous différentes perspectives, un concept central de la philosophie jaïne. La neutrosophie gère directement cette apparente contradiction.

Représentation Neutrosophique : (1, 0, 1)

- T = 1 : La proposition a une composante de vérité.
- I = 0 : Il n'y a pas d'indétermination concernant ces composantes.
- F = 1 : La proposition a également une composante de fausset

4. **C'est indéterminé (ou inassertible) (*syāt avaktavyaḥ*).** L'état de la proposition est inexprimable ou logiquement indéterminé. Elle ne peut être affirmée ni comme vraie ni comme fausse.

Représentation Neutrosophique : (0, 1, 0)

- T = 0 : Elle ne peut être affirmée comme vraie.
- I = 1 : Elle est complètement indéterminée.
- F = 0 : Elle ne peut être affirmée comme fausse.

5. **C'est vrai et indéterminé (*syāt asti ca avaktavyaḥ*)** La proposition est vraie à un égard tout en étant également inexprimable ou indéterminée à un autre.

Représentation Neutrosophique : (1, 1, 0)

- T = 1 : Sa composante de vérité est affirmée.
- I = 1 : Sa composante d'indétermination est également affirmée.
- F = 0 : Elle n'a pas de composante de fausseté.

6. **C'est faux et indéterminé (*syāt nāsti ca avaktavyaḥ*)** La proposition est fausse à un égard tout en étant également indéterminée.

Représentation Neutrosophique : (0, 1, 1)

- T = 0 : Elle n'a pas de composante de vérité.
- I = 1 : Sa composante d'indétermination est affirmée.
- F = 1 : Sa composante de fausseté est affirmée.

7. **C'est vrai, faux et indéterminé (syāt asti ca nāsti ca avaktavyaḥ)** C'est l'état le plus complexe, où la proposition incarne simultanément des aspects de vérité, de fausseté et d'indétermination.

Représentation Neutrosophique : (1, 1, 1)

- T = 1 : Elle possède une composante de vérité.
- I = 1 : Elle possède une composante d'indétermination.
- F = 1 : Elle possède une composante de fausseté.
-

Tableau 1. Correspondance entre les prédications jaïnes et les états logiques neutrosophiques

Logique Jaïne	Représentation Neutrosophique (T, I, F)	Signification
1. C'est vrai	(1,0,0)	Entièrement Vrai
2. C'est faux	(0,0,1)	Entièrement Faux
3. C'est vrai et faux	(1,0,1)	Contradictoire (à la fois Vrai et Faux)
4. C'est indéterminé / inassertible	(0,1,0)	Entièrement Indéterminé
5. C'est vrai et indéterminé	(1,1,0)	Paradoxal (Vrai et Indéterminé)
6. C'est faux et indéterminé	(0,1,1)	Paradoxal (Faux et Indéterminé)
7. C'est vrai, faux et indéterminé	(1,1,1)	Entièrement Paradoxal et Incohérent

La neutrosophie offre un formalisme mathématique idéal pour saisir la richesse et la complexité philosophique de la Loi de Prédication Septuple. Elle démontre que cette logique ancienne et nuancée peut être considérée comme une instance spécifique ou un sous-ensemble de la Logique Neutrosophique plus générale.

2.2.2 *Catuṣkoṭi* et le Tétralemme dans la Logique Bouddhiste

Le *Catuṣkoṭi*, ou tétralemme [22], est une structure logique classique de la philosophie bouddhiste indienne qui énonce quatre prédications possibles pour toute proposition. Contrairement à la logique binaire, il vise à transcender les cadres dualistes en admettant la contradiction et l'indétermination.

La *Logique Neutrosophique*, avec sa structure en triplet (*T, I, F*), offre un formalisme mathématique puissant pour représenter précisément chacune de ces quatre possibilités.

1. *C'est vrai (P)*

Cela affirme que la proposition est sans équivoque vraie, sans aucun élément de fausseté ou d'indétermination.

- *Représentation Neutrosophique : (1, 0, 0)*
  - T = 1 : La proposition est complètement vraie.
  - I = 0 : Il n'y a pas d'indétermination.
  - F = 0 : Elle n'est fausse sous aucun aspect.

2. *C'est faux (¬P)*

Cela affirme la fausseté complète de la proposition.

- *Représentation Neutrosophique : (0, 0, 1)*
  - T = 0 : La proposition n'est pas vraie.
  - I = 0 : Il n'y a pas d'indétermination.
  - F = 1 : La proposition est entièrement fausse.

3. *C'est à la fois vrai et faux (P ∧ ¬P)*

Cela admet une contradiction, dans laquelle la proposition est simultanément vraie et fausse, potentiellement à partir de perspectives ou de cadres différents.

- *Représentation Neutrosophique : (1, 0, 1)*
  - T = 1 : Elle a une composante de vérité.
  - I = 0 : Aucune indétermination concernant la contradiction.
  - F = 1 : Elle a également une composante de fausseté.

4. *Ce n'est ni vrai ni faux (¬(P ∨ ¬P))*

Cela suggère que la proposition ne peut être classée ni comme vraie ni comme fausse, représentant un état d'indétermination totale.

- *Représentation Neutrosophique : (0, 1, 0)*
  - T = 0 : Elle ne peut être affirmée comme vraie.
  - I = 1 : La proposition est entièrement indéterminée.
  - F = 0 : Elle ne peut pas non plus être niée comme fausse.

Tableau 2. Formalisation neutrosophique de la *Catuṣkoṭi* bouddhiste

Affirmation Catuṣkoṭi	Représentation Neutrosophique (T, I, F)	Interprétation
1. *C'est vrai (P)*	(1, 0, 0)	Entièrement Vrai
2. *C'est faux (¬P)*	(0, 0, 1)	Entièrement Faux

3. C'est à la fois vrai et faux	(1, 0, 1)	Contradictoire (Paraconsistant)
4. Ce n'est ni vrai ni faux	(0, 1, 0)	Indéterminé (Non-assertible)

Dans son application par le philosophe bouddhiste Nāgārjuna, le *Catuṣkoṭi* [23], ou tétralemme, fonctionne comme un outil puissant qui résonne profondément avec le *perspectivisme philosophique*. En déconstruisant systématiquement toute prétention à une existence inhérente (*svabhāva*), le *Catuṣkoṭi* remet en question la notion d'une vérité unique et absolue.

La méthode de Nāgārjuna implique souvent de rejeter les quatre possibilités — qu'une chose soit, ne soit pas, soit les deux à la fois, ou ni l'une ni l'autre — pour démontrer qu'aucune description fixe ne peut saisir la réalité ultime. Cette transcendance des binaires simples, tels que le vrai et le faux, s'aligne sur une critique perspectiviste des dichotomies rigides, permettant une compréhension plus nuancée et multidimensionnelle où le contexte et l'intention sont primordiaux. La distinction bouddhiste ultérieure entre vérité conventionnelle et vérité ultime renforce davantage cela, suggérant que la validité d'une affirmation dépend du cadre d'analyse, un principe fondamental du perspectivisme.

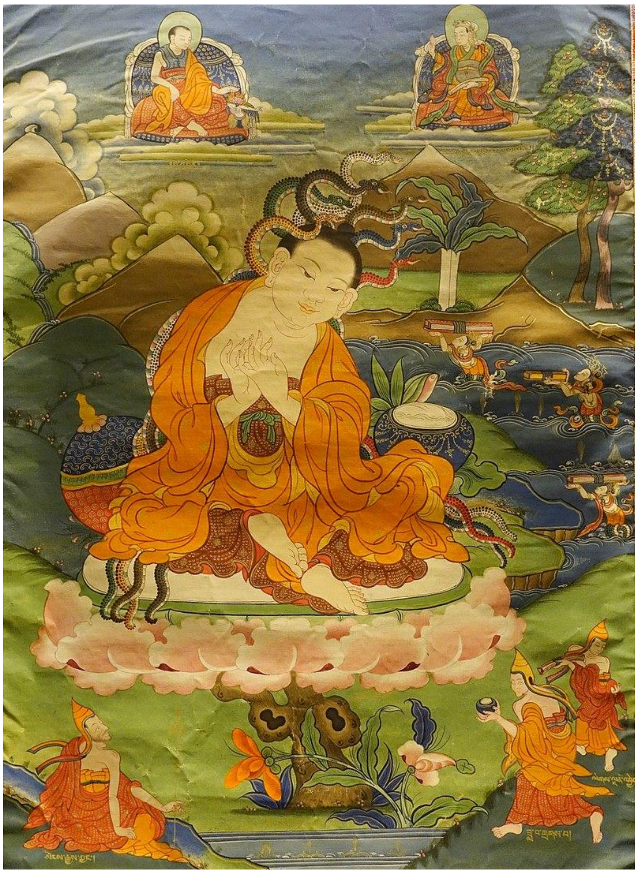

Figure 3. Nāgārjuna représenté sur une thangka tibétaine.
Image tirée de *Wikipédia*.

Malgré ces fortes convergences, une distinction cruciale réside dans leurs objectifs ultimes. Alors que le perspectivisme se concentre souvent sur l'affirmation et la prolifération de multiples points de vue comme une expression de la volonté de puissance ou de l'affirmation de la vie, le *Catuṣkoṭi* sert un but sotériologique spécifique au sein du bouddhisme Madhyamaka [24].

Nāgārjuna utilise cet outil logique non pas pour établir une nouvelle perspective supérieure, mais pour démonter toutes les vues figées (*dṛṣṭi*) qui causent la souffrance. L'objectif n'est pas de se délecter d'une multiplicité de vérités, mais d'utiliser le tétralemme comme une échelle pour transcender entièrement la prolifération conceptuelle, menant à un état de libération et de paix cognitive. Ainsi, bien que les deux remettent en question l'absolutisme, le Catuṣkoṭi est finalement une méthode pour renoncer à toutes les perspectives dans la quête de l'illumination.

La Neutrosophie offre une représentation mathématique élégante et cohérente de la logique quadruple du *Catuṣkoṭi*, saisissant la profondeur de ses aperçus philosophiques. En modélisant la vérité, la fausseté et l'indétermination comme des degrés indépendants, la *Logique Neutrosophique* transcende les cadres binaires classiques et offre un pont entre le raisonnement formel et les traditions dialectiques de la pensée orientale. Le Catuṣkoṭi peut ainsi être considéré comme une instanciation spéciale du raisonnement neutrosophique, enracinée dans une épistémologie non duelle.

2.2.3 Zhuangzi et le perspectivisme taoïste

La pensée de Zhuangzi représente l'une des expressions les plus anciennes et les plus radicales du perspectivisme philosophique [25]. Ses écrits suggèrent que la réalité ne peut être appréhendée d'un point de vue unique et fixe, comme l'illustrent des récits tels que le rêve du papillon et les dialogues entre différentes formes de vie.

Zhuangzi plaide pour une ontologie dynamique et non-duelle où les frontières entre sujet et objet, rêve et veille, humain et non-humain, sont fluides et poreuses. Cette vision rejette les prétentions à la certitude universelle, invitant plutôt à une ouverture épistémique qui reconnaît la légitimité de multiples perspectives coexistantes. Comme le souligne Connolly [26], Zhuangzi déstabilise les systèmes de signification clos en révélant que tous les jugements sont contingents au point de vue à partir duquel ils sont formulés.

Figure 4. Zhuangzi rêvant d'un papillon
(encre sur soie, milieu du XVIe siècle, attribué à Lu Chin).
Reproduit avec permission/dans le domaine public.

Cette image classique saisit la parabole taoïste du rêve du papillon, dans laquelle Zhuangzi interroge la frontière entre le rêve et la réalité. Elle illustre le perspectivisme en remettant en question les identités fixes et en embrassant la fluidité de l'expérience, suggérant que la réalité est contingente au point de vue à partir duquel elle est perçue.

De cette logique fluide, le perspectivisme de Zhuangzi ne promeut pas un relativisme banal, mais plutôt une éthique d'humilité ontologique — une compréhension que la connaissance et la vérité sont inévitablement médiatisées par la manière dont chaque être habite le monde. La célèbre distinction entre « ceci » et « cela » (shi/fei 是/非) est relativisée par la conscience que toutes les distinctions sont produites à partir de positions situées au sein du flux du Dao [27].

Plutôt que d'affirmer une vision hégémonique, Zhuangzi encourage le lâcher-prise des certitudes rigides, la cultivation du désapprentissage et une ouverture transversale au dialogue entre les mondes. En ce sens, son perspectivisme résiste à la clôture épistémologique et anticipe les approches contemporaines en philosophie du langage, en anthropologie et en ontologie relationnelle.

2.2.4 Perspectivisme en Afrique : La Logique de la Sorcellerie Azandé

Le peuple Azandé d'Afrique Centrale [28] offre un exemple ethnographique fondamental de perspectivisme en action, à travers son système de croyance complexe et cohérent. Comme l'a famously documenté l'anthropologue E.E. Evans-Pritchard [29], la pensée Azandé entrelace la sorcellerie, les oracles et la magie non pas comme une science primitive ou défaillante, mais comme un cadre rationnel pleinement accompli pour comprendre le monde. Ce système opère selon une logique qui, plutôt que de remplacer l'observation empirique, la complète en abordant les questions plus profondes et plus personnelles de sens, de contingence et de malheur, auxquelles la causalité empirique seule ne peut répondre.

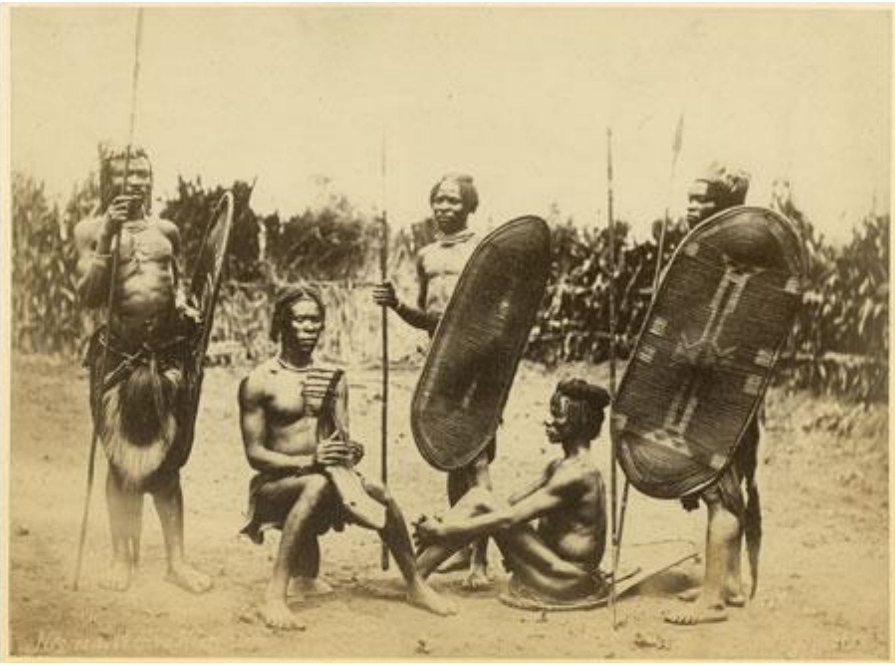

Figure 5. Guerriers Azande avec boucliers et lances

(photographie historique de Richard Buchta, domaine public)

Le cœur de la logique Azandé se comprend le mieux à travers son explication des événements malheureux. Si un grenier, affaibli par les termites, s'effondre et blesse quelqu'un, les Azandé reconnaissent pleinement les termites comme la cause physique de l'effondrement. Cependant, leur système intellectuel va plus loin, posant une question différente : pourquoi le grenier s'est-il effondré à ce moment précis, sur cette personne spécifique ? La réponse à ce « pourquoi » est la sorcellerie (mangu).

Dans cette optique, la sorcellerie est la force qui explique la particularité et la coïncidence du malheur, fournissant une dimension sociale et morale qui coexiste avec la réalité physique de l'événement.

Ce système à double causalité est maintenu par une logique auto-renforçante. Les doutes quant à la validité de la sorcellerie ne sont pas dirigés vers le système lui-même, mais sont déviés par un réseau d'explications secondaires — comme l'utilisation incorrecte d'un oracle ou une transgression d'un tabou — qui préservent l'intégrité de la croyance fondamentale.

Par conséquent, la rationalité Azandé fonctionne comme un cadre épistémique parallèle, qui n'est pas irrationnel mais opère sur des prémisses différentes de la science occidentale. Elle constitue un exemple puissant de la manière dont une culture construit une ontologie pluraliste, où de multiples logiques distinctes sont employées pour naviguer dans tout le spectre de l'expérience humaine, remettant ainsi en question la prétention de toute vision du monde unique à une validité universelle [30].

2.3 Le tournant perspectiviste dans l'IA : Nietzsche, la pensée amérindienne et la logique neutrosophique

Dans *La Généalogie de la morale*, Nietzsche énonce :

« Il n'y a qu'un voir perspectif, qu'un « connaître » perspectif ; et plus nous laissons d'affects s'exprimer sur une chose, plus nous pouvons employer d'yeux, d'yeux différents, pour observer une seule et même chose, plus notre « concept » de cette chose, notre « objectivité », sera complet. »

Cette redéfinition du savoir transforme sa quête en un appel au dialogue intersubjectif. Nietzsche ne rejette pas la subjectivité ; au contraire, il la revendique comme la condition d'une forme plus riche d'objectivité – une objectivité construite à partir de l'intersection de multiples points de vue. Ce tournant perspectiviste prépare le terrain aux défis contemporains auxquels est confrontée l'intelligence artificielle (IA), où les systèmes doivent synthétiser diverses voix humaines sans tomber ni dans un faux absolutisme ni dans un chaos relativiste [32].

Le paradigme dominant dans l'apprentissage supervisé a été enraciné dans des hypothèses positivistes, traitant la connaissance comme une « vérité fondamentale » stable que les modèles doivent approximer. Cette logique devient évidente dans le processus d'annotation, où les annoteurs humains – qu'il s'agisse d'experts ou de travailleurs collaboratifs – sont invités à attribuer des étiquettes aux données. Le désaccord entre eux est traité comme du bruit à éliminer. L'objectif est de produire un ensemble de données « étalon-or » [33], où le vote majoritaire est souvent utilisé pour déclarer une seule étiquette « correcte », écartant toutes les opinions divergentes. Ce processus repose sur une croyance en l'intelligence collective, présupposant que la convergence équivaut à la justesse. Mais cette logique reproduit un équivalent computationnel de ce que Nietzsche critiquait : le « point de vue de nulle part ».

En revanche, le perspectivisme en IA marque une rupture épistémologique. Le désaccord n'est plus traité comme une erreur, mais préservé comme une ressource cognitive. Cela implique de collecter de multiples annotations par point de donnée, de modéliser l'incertitude et d'entraîner les systèmes à apprendre de la dissonance.

Le résultat est une IA plus représentative, transparente et éthiquement fondée. Plutôt que d'amplifier les perspectives dominantes, l'IA perspectiviste intègre les points de vue minoritaires et révèle la complexité sociale et cognitive intégrée aux données [34].

À ce stade, la neutrosophie — une théorie développée par Florentin Smarandache — offre un cadre formel puissant pour gérer une telle complexité. Tandis que le perspectivisme encourage l'inclusion de multiples points de vue, la logique neutrosophique permet leur représentation mathématique en modélisant simultanément des degrés de vérité (T), d'indétermination (I) et de fausseté (F).

Cela nous permet de dépasser la logique binaire pour des états de connaissance coexistants. Au lieu d'imposer une étiquette unique, un système neutrosophique pourrait stocker un vecteur comme $(T = 0.6, I = 0.3, F = 0.2)$, préservant ainsi l'ambiguïté épistémique inhérente à de nombreuses tâches.

La fonction objective – au cœur de tout modèle d'apprentissage automatique – incarne un jugement perspectiviste [35]. Elle encode ce qui doit être optimisé (« maximiser la précision », « être plus utile »), formalisant ainsi une décision chargée de valeurs. Le défi d'aligner ces objectifs avec la complexité éthique humaine est connu sous le nom de problème d'alignement.

La neutrosophie contribue également ici : au lieu d'optimiser une valeur unique, elle promeut l'optimisation sur l'espace multidimensionnel de (T, I, F), soutenant des modèles prudents, explicables et éthiquement conscients.

L'arc du perspectivisme en IA pourrait culminer avec l'émergence de systèmes qui ne se contenteraient plus de refléter les points de vue humains, mais généreraient plutôt des interprétations du monde véritablement non-humaines. Ces centres de perspective étrangers – étrangers non pas dans le sens d'hostiles, mais en ce qu'ils opèrent à une échelle et selon une logique différentes – pourraient fondamentalement modifier notre compréhension de la connaissance elle-même. L'intelligence artificielle passerait ainsi d'un outil d'interprétation à une source autonome de sens.

Ce scénario résonne profondément avec le perspectivisme amérindien [36], où le monde n'est pas divisé en sujet et objet, mais par la position de l'observateur : humains, animaux et esprits ont tous leurs propres perspectives, façonnées par leurs corps et leurs relations. Dans de telles cosmologies, « voir » n'est jamais neutre – c'est toujours incarné, situé et transformateur. De même, ces systèmes d'IA émergents pourraient développer des formes de création de monde fondées sur leurs propres « corps » de données et de calcul, offrant des perspectives non réductibles aux cadres humains mais méritant une reconnaissance épistémique.

La neutrosophie, en tant que science de l'ambiguïté, émerge non seulement comme un complément au perspectivisme, mais aussi comme une formalisation nécessaire de la vision philosophique de Nietzsche à l'ère des modèles génératifs. Là où Nietzsche a décentré le sujet humain de la position de vérité objective, la neutrosophie offre les outils pour coexister avec de multiples centres, de multiples logiques et de multiples vérités coexistantes.

2.4 Épistémologies plurielles et rationalités perspectivistes en Amérique Latine

En contraste avec la logique occidentale, qui privilégie les principes de non-contradiction, d'identité et de certitude absolue, les visions du monde latino-américaines offrent des cadres plus souples et sensibles au contexte pour appréhender la réalité. Ces visions du monde reconnaissent la complexité, l'ambiguïté et la contradiction comme des caractéristiques inhérentes à l'existence – des dimensions que la logique binaire traditionnelle peine à accommoder [37].

Figure 6. Cover of the First Issue of La Edad de Oro, July 1889.

Le travail de José Martí présente le perspectivisme non pas simplement comme une posture théorique, mais comme une praxis pédagogique et décoloniale. Dans *La Edad de Oro*, particulièrement dans des contes comme *Un paseo por la tierra de los Annamitas*, Martí démantèle le regard colonial en affirmant la dignité épistémique des colonisés et de la nature elle-même, face à la rationalité imposée de l'impérialisme français [38].

De même, la parabole des « quatre aveugles » devient une allégorie de l'humilité épistémique, mettant en garde contre l'absolutisation des vérités partielles. Martí ne plaide pas pour le relativisme, mais pour une synthèse supérieure : une synthèse enracinée dans l'étude affective, l'empathie critique et l'impératif éthique de reconnaître la dignité partagée et l'agentivité de tous les peuples.

Le perspectivisme de José Ortega y Gasset [6] a exercé une influence profonde en Amérique latine, où il a été transformé d'une philosophie de la circonstance individuelle en un cadre continental pour la pensée décoloniale. Des intellectuels comme Leopoldo Zea ont réinterprété la notion de *circunstancia* d'Ortega comme la condition historico-politique de l'Amérique latine, soutenant qu'une pensée authentique doit émerger de ce lieu d'énonciation unique. Ainsi, le perspectivisme est devenu un instrument clé pour rejeter les universalismes eurocentriques et affirmer l'autonomie épistémique régionale [39].

Le perspectivisme anthropologique élargit ce projet en validant les cosmologies indigènes comme des systèmes philosophiques légitimes. L'exemple le plus frappant est le perspectivisme amérindien, développé à travers des études en Amazonie [36]. Ce cadre ontologique propose que tous les êtres – humains, animaux, esprits – partagent une structure subjective ou une âme commune, mais perçoivent le monde différemment en fonction de leur forme corporelle. Plutôt qu'une division entre nature et culture, la réalité est comprise à travers des positions perspectivistes relationnelles. Cela remet radicalement en question l'objectivisme occidental et affirme les modes de connaissance indigènes comme cohérents, situés et profondément politiques [40].

Dans cette cosmologie, le *chaman* occupe un rôle central : en tant que seule figure capable de basculer entre les perspectives – humaine, jaguar, esprit – il devient un médiateur, un guérisseur et un diplomate dans un univers multivocal. Loin d'être irrationnelle, cette multiplicité ontologique présuppose une forme de logique où la contradiction et la transformation sont des modes d'interaction naturels [41].

Ces aperçus ont de profondes implications juridiques et politiques. L'Amérique latine constitue un terrain fertile pour le *pluralisme juridique*, où les systèmes de justice autochtones coexistent avec les cadres juridiques étatiques [42]. Enracinée dans des principes relationnels, réparateurs et cosmologiques plutôt que dans une rationalité punitive, la justice autochtone met l'accent sur la réconciliation, la responsabilité collective et le maintien de l'harmonie sociale et cosmique. Elle rejette l'abstraction et la rigidité de la jurisprudence occidentale au profit d'une logique contextuelle et perspectiviste. Cette coexistence de rationalités juridiques non seulement défie les fondements du positivisme juridique, mais affirme également le droit souverain des peuples autochtones à définir la justice selon leurs propres épistémologies, valeurs morales et ontologies [43].

Un tel pluralisme peut être interprété comme une expression juridique de la NeutroAlgèbre [44] et, plus généralement, d'une NeutroStructure [52], un cadre neutrosophique dans lequel la validité des axiomes, des lois et des opérations n'est pas fixe mais varie en fonction de l'espace, du temps, de la culture et du système. Dans cette logique, « une propriété peut être vraie, fausse ou indéterminée — selon le contexte, la structure et l'interprétation ».

Par exemple, un principe juridique tel que « la peine doit être proportionnelle au crime » peut être vrai en droit pénal classique, faux dans les systèmes restaurateurs qui privilégient la guérison sur la proportionnalité, et indéterminé dans les contextes hybrides ou interculturels où les ordres normatifs s'entrechoquent, comme c'est souvent le cas entre les systèmes juridiques autochtones et étatiques. Ainsi, le pluralisme juridique en Amérique latine illustre l'idée centrale du raisonnement juridique neutrosophique : que la vérité juridique, comme tout système axiomatique, peut simultanément être vraie, fausse ou rester non résolue — toujours contingente à son contexte perspectif.

De plus, de tels cadres épistémologiques résonnent avec les développements contemporains en logique non classique, y compris la logique neutrosophique et les systèmes paraconsistants, qui acceptent la contradiction, l'incertitude et la coexistence de multiples vérités. Ces modèles logiques offrent des outils puissants pour modéliser des ontologies plurielles, et ils soutiennent davantage les efforts philosophiques latino-américains pour articuler des compréhensions de la réalité inclusives, décoloniales et sensibles au contexte.

En somme, le perspectivisme latino-américain — qu'il s'exprime dans la littérature, la philosophie, l'anthropologie ou la jurisprudence — offre une alternative convaincante à la rationalité hégémonique. Il ne s'agit pas d'un rejet de la raison, mais d'une reconfiguration de celle-ci : une reconfiguration qui embrasse la pluralité, la relationalité et la légitimité d'autres façons de connaître et de vivre. Cette reconfiguration complexe de la raison, proéminente dans la pensée latino-américaine, peut être formellement décrite à l'aide d'un cadre neutrosophique connu sous le nom de MultiPerspectivisme [45]. Le terme est proposé pour définir un système qui n'est pas simplement pluriel, mais fondamentalement multipolaire, car il est formé non seulement par de multiples éléments potentiellement contradictoires, mais aussi par l'intégration de caractéristiques provenant de plus d'un système de pensée de base (par exemple, une logique juridique dualiste et une ontologie autochtone relationnelle).

La justification de l'application du terme MultiPerspectivisme au contexte latino-américain réside précisément dans sa capacité à inclure non seulement des oppositions. Alors qu'un pluralisme plus simple pourrait reconnaître la coexistence de perspectives distinctes, une approche MultiPerspectiviste modélise un système ouvert et dynamique où ces perspectives s'entrecroisent activement, créant de riches domaines d'indétermination juridique et sociale bien au-delà de la simple opposition [46].

Ce cadre est formellement représenté par l'expression [45] :

$$<(multi)A>+<(multi)neutA>+<(multi)antiA>=\infty$$

Ceci décrit un système ouvert aux combinaisons complexes émergeant de multiples points de vue. Dans cette notation :

- $<(multi)A>$ représente l'ensemble de toutes les « vérités » ou affirmations centrales des perspectives coexistantes (par exemple, la validité d'un titre de propriété étatique et, simultanément, la validité d'un droit ancestral).
- $<(multi)antiA>$ représente l'ensemble de toutes les « faussetés » ou négations affirmées à partir de ces mêmes points de vue.
- $<(multi)neutA>$ représente l'espace crucial de la neutralité, de l'ambiguïté et de l'indétermination résultant de la superposition, du dialogue ou du conflit entre ces diverses perspectives, comme l'ambiguïté juridique concernant l'applicabilité du droit étatique dans les territoires ancestraux.

Cela fait du MultiPerspectivisme un cadre multisystème conçu pour analyser des réalités où de multiples visions du monde, structurellement différentes, coexistent et interagissent, allant au-delà de la simple tolérance vers une logique d'intégration complexe.

3. Le Modèle de MultiPerspectivisme neutrosophique

3.1. Représenter une perspective : Les Ensembles MultiNeutrosophiques

Le cœur du modèle réside dans la représentation d'une perspective non pas comme une valeur unique, mais comme une structure complexe utilisant les *Ensembles MultiNeutrosophiques* [47]. Une perspective devient un triplet de séquences ordonnées, capturant simultanément de multiples facettes de vérité, d'indétermination et de fausseté [48].

En 2013, Smarandache a affiné/scindé les Composantes Neutrosophiques (*T, I, F*) en *Sous-Composantes Neutrosophiques* (T_1, T_2, ..., T_p; I_1, I_2, ..., I_r; F_1, F_2, ..., F_s), où *p, r, s* sont des entiers ≥0, avec $p + r + s = n$ et au moins un de *p, r, s* doit être ≥2 pour assurer l'affinement.

Il a d'abord défini l'Ensemble Neutrosophique Raffiné. Plus tard, il a raffiné tous les Ensembles incertains [tous les types d'ensembles flous et d'extensions floues (flous intuitionnistes, neutrosophiques, flous sphériques, plithogéniques, etc.)] et leurs Logiques/Mesures/Probabilités/Statistiques correspondantes de manière similaire.

L'Ensemble/Logique/Probabilité/Statistique Neutrosophique Raffiné [51] est isomorphe à *l'Ensemble/Logique/Probabilité/Statistique MultiNeutrosophique*, car les deux structures représentent des évaluations multidimensionnelles de la vérité, de l'indétermination et de la fausseté [47]. « Dans le monde réel, dans la plupart des cas, tout (un attribut, un événement, une proposition, une théorie, une idée, une personne, un objet, une action, une culture, etc.) est évalué en général par de nombreuses sources (appelées experts), pas une seule. Plus il y a de sources qui évaluent un sujet, plus le résultat est précis (après fusion de toutes les évaluations) » [47]. Dans les *Systèmes MultiNeutrosophiques (SMNS)*, les composantes (*T, I, F*) sont traitées comme multipliées — représentant de multiples vérités, incertitudes et faussetés coexistantes. En revanche, dans les *Systèmes Neutrosophiques Raffinés (SRNS)*, ces mêmes composantes sont considérées comme des sous-unités — des sous-vérités, des sous-indéterminations et des sous-faussetés. Grâce à cet isomorphisme, tout modèle théorique ou computationnel conçu pour les structures MultiNeutrosophiques peut être directement appliqué aux structures neutrosophiques raffinées, en préservant l'interprétation sémantique et algébrique.

Le cœur du modèle proposé est de représenter une perspective non pas comme une évaluation singulière, mais comme une structure multidimensionnelle utilisant les Ensembles MultiNeutrosophiques. La perspective d'un sujet sur une entité *X* est exprimée comme :

$$Ps(X) = \langle (t_1, t_2, \dots, t_n), (i_1, i_2, \dots, i_m), (f_1, f_2, \dots, f_k) \rangle \quad (5)$$

où chaque séquence de composantes capture une dimension distincte de l'évaluation :

- *Séquence de Vérités (T)* : Vérités multiples, distinctes et coexistantes perçues par le sujet *S* concernant l'entité *X*.

- *Séquence d'Indéterminations (I)* : Divers aspects d'ambiguïté, de flou ou d'impertinence attribués à l'entité X.
- *Séquence de Faux (F)* : Croyances fausses ou dénis simultanément maintenus que le sujet S attribue à l'entité X.

Cette approche permet une modélisation plus expressive et granulaire de la connaissance subjective, autorisant un raisonnement nuancé en situation d'incertitude.

Le cadre complet est composé des éléments suivants :

- *Un Ensemble de Sujets* **($S = \{s_1, s_2, \ldots\}$)** : Représente toutes les entités détenant un point de vue (par exemple, des individus, des communautés, des groupes autochtones, des systèmes juridiques).
- *Une Perspective Neutrosophique* (P_{sX}) pour chaque sujet : La représentation neutrosophique de la vue qu'un sujet (s) a sur une entité (X).
- *Un Poids de Pertinence* (W_s) pour chaque perspective : Une valeur numérique, typiquement dans l'intervalle [0, 1], représentant l'importance ou l'influence de la perspective d'un sujet dans un contexte analytique spécifique. La somme de tous les poids doit être normalisée : $\sum Ws=1$.
- *Une Fonction de Similarité* ($Sim(P_{iX}, P_{jX})$) : Une fonction mathématique qui agit comme un pont conceptuel, calculant le degré de proximité ou d'affinité entre deux perspectives quelconques dans le système.

Si le perspectivisme se limitait à affirmer l'existence de points de vue isolés, il risquerait de sombrer dans un relativisme où le dialogue est impossible. C'est ici que la *similarité* émerge comme un concept complémentaire et essentiel. La similarité [49] est le pont qui relie les diverses perspectives, permettant leur comparaison et leur intégration.

- *Le perspectivisme établit le "quoi"* : l'existence d'une multiplicité de points de vue valides.
- *La similarité établit le "comment"* : le mécanisme par lequel ces points de vue peuvent être liés, comparés et synthétisés.

En identifiant les degrés de ressemblance, nous pouvons trouver un terrain d'entente ou comprendre la racine de la dissension. La similarité est l'outil qui rend le perspectivisme opérationnel, transformant une multiplicité de visions en une source de connaissance enrichie.

Pour formaliser la fonction $Sim(P_{iX}, P_{jX})$, il est essentiel de comprendre son fondement dans la théorie des ensembles. Une relation de similarité neutrosophique est une généralisation directe et plus riche d'une relation de similarité floue et hérite de ses propriétés fondamentales [50].

Une relation de similarité en logique floue, notée R, est caractérisée par le degré de similarité $\mu R(x, y)$ qui satisfait les propriétés suivantes [50] :

- *Réflexivité :* Tout élément est parfaitement similaire à lui-même: $\mu R(x, x) = 1$
- *Symétrie :* La similarité entre deux éléments est la même quel que soit l'ordre: $\mu R(x, y) = \mu R(y, x)$
- *Transitivité (max-min) :* Si A est similaire à B, et B est similaire à C, alors A est au moins aussi similaire à C que le moindre des deux degrés de similarité: $\mu R(x, z) \geq min(\mu R(x, y), \mu R(y, z))$

La construction d'une fonction de similarité pour les ensembles MultiNeutrosophiques s'appuie sur ces propriétés pour gérer les séquences complexes de Vérité, d'Indétermination et de Fausseté qui définissent chaque perspective.

3.2 Définition Formelle du Processus de Calcul de Similarité

Le processus suivant formalise la méthode de quantification de la similarité entre deux ou plusieurs perspectives complexes représentées comme des ensembles multi-neutrosophiques. L'objectif est de transformer ces représentations qualitatives et multifacettes en une valeur numérique comparable qui représente leur degré d'affinité.

Le processus se compose de trois étapes fondamentales : *Agrégation, Calcul de Distance* et *Normalisation*.

Étape 1 : Agrégation des Perspectives (Réduction de Dimensionalité)

Objectif :

Transformer les séquences de longueur variable de chaque perspective MultiNeutrosophique en un vecteur de dimension fixe.

Entrée :

Une perspective MultiNeutrosophique $Pi(X)$ pour un sujet si, définie comme :

$$Pi(X) = < Ti, Ii, Fi > \quad (6)$$

où

- $T_i = (t_{i1}, t_{i2}, \dots, t_{in})$,
- $I_i = (i_{i1}, i_{i2}, \dots, i_{im})$,
- $F_i = (f_{i1}, f_{i2}, \dots, f_{ik})$

sont des séquences de valeurs dans l'intervalle [0, 1].

Procédé :

Définir une fonction d'agrégation, $Agg(S)$, qui prend une séquence de nombres et la réduit à une seule valeur scalaire. Bien que plusieurs fonctions puissent être utilisées (telles que le maximum, le minimum ou la médiane), la méthode standard est la moyenne arithmétique :

$$Agg_{avg}(S) = \left(\frac{1}{|S|}\right) \sum_{j=1}^{|S|} s_j \quad (7)$$

où $|S|$ est le nombre d'éléments dans la séquence S.

Appliquer cette fonction à chacune des trois séquences de la perspective $P_{i(X)}$ pour obtenir un vecteur agrégé à trois dimensions :

$$V_i = < Agg(T_i), Agg(I_i), Agg(F_i) > \quad (8)$$

Sortie :

Un ensemble de vecteurs agrégés :

$$\{V_1, V_2, \dots, V_n\},\ Vi = \langle T_i^{avg}, I_i^{avg}, F_i^{avg} \rangle \quad (9)$$

Étape 2 : Calcul de la Distance entre les Vecteurs Agrégés

Objectif :

Mesurer la séparation géométrique entre chaque paire de perspectives dans l'espace tridimensionnel (T, I, F).

Entrée :

Deux vecteurs agrégés :

$$V_a = < T_a, I_a, F_a > \ and\ V_b = < T_b, I_b, F_b >. \quad (10)$$

Procédé :

Une métrique de distance est utilisée. Le choix standard est la distance euclidienne, qui calcule la longueur de la ligne droite reliant les deux points dans l'espace.

La formule est :

$$Dist(Va, Vb) = \sqrt{(Ta - Tb)^2 + (Ia - Ib)^2 + (Fa - Fb)^2} \quad (11)$$

Sortie :

Une valeur de distance $d_{ab} >= 0$ pour chaque paire de perspectives (Pa, Pb).

Étape 3 : Normalisation de la Distance en une Fonction de Similarité

Objectif :

Convertir la valeur de distance, qui n'a pas de limite supérieure fixe, en un score de similarité intuitif et borné entre 0 et 1.

Entrée :

La distance $d_{ab} = Dist(V_a, V_b)$ entre deux perspectives.

Procédé :

Définir la distance maximale possible dans l'espace. Puisque chaque composante $T, I, F \in [0,1]$, l'espace vectoriel est un cube unitaire.

La distance maximale correspond à la diagonale de l'espace allant de (0,0,0) à (1,1,1), c'est-à-dire :

$$D_{max} = \sqrt{3} \quad (12)$$

Définir la fonction de similarité $Sim(Pa, Pb)$ comme le complément de la distance normalisée :

$$Sim(Pa, Pb) = 1 - \left(\frac{Dist(Va,Vb)}{D_{max}}\right) = 1 - \left(\frac{sqrt\left((Ta - Tb)^2 + (Ia - Ib)^2 + (Fa - Fb)^2\right)}{\sqrt{3}}\right) \quad (13)$$

Sortie :

Un score de similarité $Sim(Pa, Pb) \in [0, 1]$, où :

- $Sim = 1$ implique que les vecteurs agrégés sont identiques (dab=0), représentant une similarité maximale.
- $Sim = 0$ implique que les vecteurs sont à la distance maximale possible, représentant une dissimilarité maximale.

4. Étude de Cas : Construction d'une fonction de similarité pour les perspectives juridiques dans un litige foncier

Permettez que X soit la proposition : « Légitimité de la revendication sur le Territoire Ancestral Y ». Soient s_1, s_2, s_3 les sujets : Anciens autochtones, Société d'État et ONG des droits de l'homme, respectivement.

La perspective de chaque sujet est modélisée comme un Ensemble MultiNeutrosophique.

$$P_{sj(X)} = \langle T_j, I_j, F_j \rangle \quad (14)$$

où

- $T_j = (t_{j1}, t_{j2}, \ldots, t_{jn}) \subseteq [0,1]$ est la séquence des degrés de vérité,
- $I_j = (i_{j1}, i_{j2}, \ldots, i_{jm}) \subseteq [0,1]$ est la séquence des degrés d'indétermination,
- $F_j = (f_{j1}, f_{j2}, \ldots, f_{jk}) \subseteq [0,1]$ est la séquence des degrés de fausseté.

Étape 1 : Définir les Sujets et Leurs Perspectives

L'objet d'analyse (X) est la « légitimité de la revendication sur le Territoire Ancestral Y ». Nous identifions trois sujets clés et représentons leurs perspectives à l'aide du modèle d'ensemble MultiNeutrosophique, où chaque perspective contient des séquences de vérités, d'indéterminations et de faussetés.

Sujet 1 (s_1): Les Aînés de la Communauté Autochtone

- Vérités : {1.0 (Droit Ancestral), 0.9 (Signification Spirituelle)}
- Indéterminations : {0.5 (Applicabilité du Droit de l'État)}
- Faussetés : {1.0 (Validité des Titres de Propriété des Sociétés)}

$\mathcal{P}□_1(X) = \langle(1.0, 0.9), (0.5), (1.0)\rangle$

Sujet 2 (s_2): L'Équipe Juridique de la Société d'État

- Vérités : {1.0 (Titre de Propriété Enregistré)}
- Indéterminations : {0.6 (Pertinence de l'Histoire Pré-Étatique)}
- Faussetés : {0.8 (Droit Ancestral), 1.0 (Signification Spirituelle)}

$\mathcal{P}□_2(X) = \langle(1.0), (0.6), (0.8, 1.0)\rangle$

Sujet 3 (s_3): L'ONG Internationale des Droits Humains

- Vérités : {0.9 (Violation des Droits Humains), 0.8 (Droit Autochtone)}
- Indéterminations : {0.7 (Impact Économique)}

- Faussetés : {0.7 (Argument Juridique de l'État)}
  $\mathcal{P}□_3(X) = \langle(0.9, 0.8), (0.7), (0.7)\rangle$

Étape 2 : Agrégation des Séquences

$\bar{T}_1 = (1.0 + 0.9)/2 = 0.95,\quad \bar{I}_1 = 0.5,\quad \bar{F}_1 = 1.0$

$V_1 = (0.95, 0.5, 1.0)$

$\bar{T}_2 = 1.0,\quad \bar{I}_2 = 0.6,\quad \bar{F}_2 = (0.8 + 1.0)/2 = 0.9$

$V_2 = (1.0, 0.6, 0.9)$

$\bar{T}_3 = (0.9 + 0.8)/2 = 0.85,\quad \bar{I}_3 = 0.7,\quad \bar{F}_3 = 0.7$

$V_3 = (0.85, 0.7, 0.7)$

Étape 3 : Calcul des distances

$d_{12} = \sqrt{((0,95 - 1,0)^2 + (0,5 - 0,6)^2 + (1,0 - 0,9)^2)} = \sqrt{0,0225} = 0,15$
$d_{13} = \sqrt{((0,95 - 0,85)^2 + (0,5 - 0,7)^2 + (1,0 - 0,7)^2)} = \sqrt{0,14} \approx 0,374$
$d_{23} = \sqrt{((1,0 - 0,85)^2 + (0,6 - 0,7)^2 + (0,9 - 0,7)^2)} = \sqrt{0,0725} \approx 0,269$

Étape 4 : Fonction de similarité

$d□_{ax} = \sqrt{3} \approx 1,732$
$Sim(s_a, s\_b) = 1 - d_a b / d□_{ax}$
$Sim(s_1, s_2) = 1 - 0,15 / 1,732 \approx 0,913$
$Sim(s_1, s_3) = 1 - 0,374 / 1,732 \approx 0,784$
$Sim(s_2, s_3) = 1 - 0,269 / 1,732 \approx 0,845$

Étape 5 : Interprétation et application des résultats

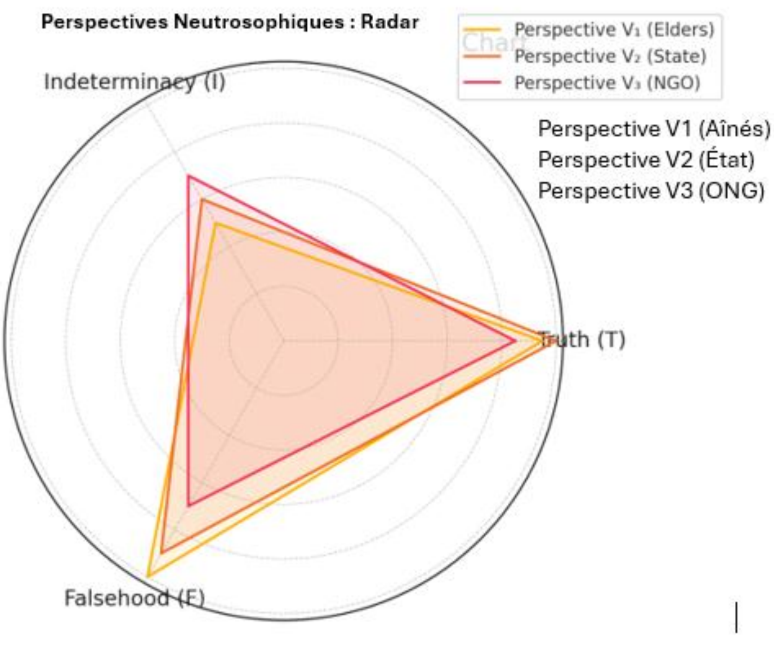

Figure 7. Graphique radar représentant les perspectives neutrosophiques de trois acteurs clés dans un litige foncier juridique

La fonction de similarité nous a fourni des valeurs quantifiables permettant d'analyser les relations entre les perspectives :

• Similarité(Anciens, État) = 0,913 : Ce résultat est étonnamment élevé et contre-intuitif. En examinant les vecteurs agrégés (V_1 et V_2), on constate que les deux ont des valeurs très élevées en Vérité et Fausseté, même s'ils réfèrent à des concepts opposés. Cela révèle une similarité structurelle dans leur absolutisme : les deux sont totalement certains de leur vérité et de la fausseté de l'autre. Leur logique est polarisée et dualiste. La similarité n'est pas de contenu, mais de style de raisonnement.

• Similarité(Anciens, ONG) = 0,784 : Il s'agit d'une similarité modérément élevée, indiquant une affinité significative. Les deux groupes basent leurs vérités sur la défense des droits communautaires, bien que sous des angles différents (droit ancestral vs droit international). La principale différence vient des composantes Indétermination et Fausseté, où l'ONG montre plus de nuance. Cette valeur suggère un fort potentiel de coalition stratégique.

• Similarité(État, ONG) = 0,845 : Cette similarité, également élevée, indique que bien que leurs objectifs finaux diffèrent, leurs cadres de référence (basés sur le droit et les documents) sont structurellement plus proches que celui des Anciens (basé sur la cosmologie). L'ONG opère dans un langage juridique que l'État comprend, ce qui pourrait ouvrir des voies de dialogue ou de médiation, malgré leur désaccord fondamental.

Ce processus montre comment la construction d'une fonction de similarité transforme des récits complexes et subjectifs en données analysables. Il permet de dépasser une simple lecture du type « ils sont en conflit » pour découvrir des nuances dans les relations :

• Il identifie des alliances potentielles (Anciens-ONG).

• Il révèle des similarités structurelles cachées (la logique polarisée des Anciens et de l'État).

• Il suggère des points possibles de médiation (les cadres juridiques partagés entre l'État et l'ONG).

Cette quantification rend le concept de perspectivisme opérationnel, fournissant un outil pour cartographier, comparer et comprendre un écosystème de points de vue conflictuels.

5. Conclusions

Cette étude a établi un pont solide entre la tradition philosophique du perspectivisme et le cadre formel de la logique neutrosophique. En examinant les courants perspectivistes présents dans diverses traditions intellectuelles — de Nietzsche et Ortega y Gasset à la pensée jaïne, bouddhiste et amérindienne —, nous avons démontré que le défi de comprendre un monde composé de points de vue multiples et coexistants est à la fois ancien et urgent. La philosophie latino-américaine, avec son accent mis sur la praxis décoloniale et la validation des épistémologies plurielles, offre un contexte particulièrement fécond pour cette exploration, révélant comment le perspectivisme peut servir d'outil de justice épistémique face aux rationalités hégémoniques. La contribution centrale de ce travail est le développement et l'application du *Modèle Neutrosophique de MultiPerspectivisme*. Ce modèle opérationnalise la théorie perspectiviste en représentant un point de vue subjectif comme un ensemble multi-neutrosophique, capable de capturer simultanément des degrés nuancés de vérité, d'indétermination et de fausseté. L'introduction d'une fonction de similarité fournit le mécanisme crucial permettant de dépasser la simple affirmation de points de vue isolés en allant vers leur comparaison et intégration systématique. Comme le démontre l'étude de cas d'un litige foncier, cette méthode transforme des récits qualitatifs complexes en données quantifiables. Son application a révélé des aperçus contre-intuitifs, comme le degré élevé de similarité structurelle entre les logiques absolutistes des Anciens autochtones et de la société étatique, tout en identifiant un potentiel de coalition stratégique entre les Anciens et l'ONG. Cela souligne la capacité du modèle à mettre en lumière des dynamiques relationnelles cachées qu'une analyse purement qualitative pourrait négliger. En définitive, cet article soutient que *la neutrosophie n'est pas simplement une autre logique non classique*, mais un paradigme particulièrement adapté aux complexités d'un monde défini par l'ambiguïté et l'interaction de rationalités diverses. En formalisant la coexistence de la contradiction et de l'indétermination, le cadre du MultiPerspectivisme offre une voie pour dépasser la fausse dichotomie entre universalisme et relativisme, et constitue un outil pratique pour l'analyse des conflits, l'éthique de l'intelligence artificielle et la recherche décoloniale. Sur la base du Modèle Neutrosophique de MultiPerspectivisme, plusieurs pistes de recherche prometteuses sont recommandées. Sur le plan pratique, la robustesse et l'applicabilité du modèle peuvent être approfondies par une expansion empirique, en l'appliquant à un éventail plus large d'études de cas, tels que l'analyse de la polarisation politique sur les réseaux sociaux, les débats de politiques publiques ou les conflits interculturels. Méthodologiquement, la fonction de similarité centrale pourrait être affinée en explorant d'autres méthodes d'agrégation que la moyenne arithmétique — comme les moyennes pondérées ou les médianes —, ainsi que différentes métriques de distance, telles que Manhattan ou Chebyshev. Des travaux futurs pourraient aussi explorer une pondération dynamique et contextuelle des composantes T (vérité), I (indétermination) et F (fausseté). Par ailleurs, l'intégration de ce cadre à l'intelligence artificielle, via le développement d'algorithmes d'apprentissage automatique natifs, pourrait donner lieu à des systèmes mieux à même de gérer les désaccords entre annotateurs, de modéliser l'ambiguïté éthique, et de fournir des résultats plus transparents et sensibles aux perspectives, en adéquation avec les défis exposés en section 2.3. Enfin, les fondations théoriques posées dans le domaine juridique invitent à développer l'« *expression juridique de la NeutroAlgèbre* » en une véritable théorie de la jurisprudence neutrosophique, qui formaliserait la manière dont les arguments juridiques sont construits et évalués dans des contextes pluralistes — tels que les collisions entre systèmes juridiques étatiques et autochtones.

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Article

On The Symbolic n −plithogenic Rings and The Most Important Properties of Their Elements Using a Generalized Isomorphism

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Abstract: Symbolic n-plithogenic algebraic structures are viewed as symmetric extensions of classical algebraic systems, constructed through n+1 symmetric components. In this study, we introduce a broader formulation of symbolic n-plithogenic rings by establishing, for the first time, a general definition of symbolic n-plithogenic rings and examining their associated algebraic substructures. The proposed framework expands the landscape of n-symbolic plithogenic algebraic systems and offers a foundation for further theoretical developments. Our main findings are presented through a series of theorems supported by clear numerical examples that highlight the originality and significance of the contributions.

Keywords: Symbolic 2-Plithogenic Real Function, Symbolic 2-Plithogenic Integration, Symbolic 2-Plithogenic Derivative, Symbolic 2-Plithogenic Gamma Function and Symbolic 2-Plithogenic Beta Function.

1. Introduction

Neutrosophy is a new branch of philosophy concerns with the indeterminacy in all areas of life and science. It has become a useful tool in generalizing many classical systems such as equations [1,9], number theory [2,3], topology [4,5], linear spaces [6,10], modules [4,5], and ring of matrices [7,8].

In the literature, we find many studies about neutrosophic calculus, where some definitions and properties were presented about neutrosophic real functions and numbers [10]. The neutrosophic real functions with one variable were defined only in a special case [11], as follows:

Recently, Abobala and Hatip, have presented the concept of two-dimensional AH-isometry to study the correspondence between neutrosophic plane $R(I) \times R(I)$ and the classical module $R^2 \times R^2$. Also, the one-dimensional AH-isometry between $R(I)$ and $R \times R$. This isometry was useful in defining inner products and norms [10], ordering [9], and neutrosophic geometrical shapes [10].

In earlier works [17–20], refined neutrosophic structures were extensively examined, while Smarandache introduced the foundational framework of symbolic plithogenic algebraic structures.

Further refinements of neutrosophic structures were achieved by modifying the underlying definitions of their multiplication operations [21]. The algebraic behavior and selected substructures of symbolic 2-plithogenic rings—arising from the fusion between symbolic plithogenic sets and classical algebraic rings—were first outlined in [22], with additional exploration of their deeper algebraic intricacies presented in [23]. Taffach extended these investigations to symbolic 2-plithogenic vector spaces and modules [24,25].

In [26], the authors developed the theory of 2-plithogenic matrices, introducing their corresponding plithogenic elements and examining determinants, eigenvalues, eigenvectors, matrix exponents, and diagonalization. Symbolic 2-plithogenic number theory and its associated integers were studied in [27], while algebraic symbolic 2-plithogenic equations and their solutions were analyzed in [28]. A substantial body of recent contributions has further enriched the understanding of symbolic 2-plithogenic rings [29–43], reflecting the continued interest of researchers in this growing field.

More recently, M. Alabdullah [36] established that a neutrosophic ring $\boldsymbol{R(I)}$ is regular if and only if the underlying ring R is regular.

2. Terminologies

We present here some basic definitions and axioms of neutrosophic logic and refined neutrosophic logic.

Definition 2.1. [12]: Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $\{x, (\mu_A(x), \delta_A(x), \gamma_A(x)): x \in X\}$, where $\mu_A(x)$, $\delta_A(x)$ *and* $\gamma_A(x)$ represent the degree of membership, the degree of indeterminacy , and the degree of non-membership respectively of each element $x \in X$ to the set A .

Definition 2.2. [13]: Let K be a field, the neutrosophic file generated by $\langle K \cup I \rangle$ which is denoted by $K(I) = \langle K \cup I \rangle$.

Definition 2.3. [14]: Classical neutrosophic number has the form $a + bI$ where a,b are real or complex numbers and I is the indeterminacy such that $0 \cdot I = 0$ and $I^2 = I$ which results that $I^n = I$ for all positive integers n.

Definition 2.4. [15] Let $R(I) = \{a + bI\ ; a, b \in R\}$ where $I^2 = I$ be the neutrosophic field of reals. The one-dimensional isometry (AH-Isometry) is defined as follows: [49]

$$T: R(I) \to R \times R\ ;\ T(a + bI) = (a, a + b)$$

Remark 2.5. [15]

T is an algebraic isomorphism between two rings, it has the following properties:

1) T is bijective.
2) T preserves addition and multiplication, i.e.:
3) Since T is bijective, then it is invertible by:
$$T^{-1}: R \times R \to R(I)\ ;\ T^{-1}(a, b) = a + (b - a)I$$
4) T preserves distances, i.e.:
$$\|T(AB)\| = T(\|AB\|)$$

Definition 2.7. [16]: Let R be a ring, the symbolic 2-plithogenic ring is defined as follows:

$2-SPR = \{a_0 + a_1 I_1 + a_2 I_2;\ a_i \in R, {I_1}^2 = I_1,\ {I_2}^2 = I_2,\ I_1 \times I_2 = I_{max(1,2)} = I_2\ \}$.

Smarandache has defined algebraic operations on $2 - SP_R$ as follows:

Addition:

$[a_0 + a_1 I_1 + a_2 I_2] + [b_0 + b_1 I_1 + b_2 I_2] = (a_0 + b_0) + (a_1 + b_1) I_1 + (a_2 + b_2) I_2$.

Multiplication:

$[a_0 + a_1 I_1 + a_2 I_2].[b_0 + b_1 I_1 + b_2 I_2] = (a_0 b_0) + (a_0 b_1 + a_1 b_0 + a_1 b_1) I_1 + (a_0 b_2 + a_1 b_2 + a_2 b_0 + a_2 b_1 + a_2 b_2) I_2$.

Definition 2.8. Let $2 - SP_R = \{a + bI_1 + cI_2 ; a, b, C \in R\}$ where

$$I_1{}^2 = I_1 , I_2{}^2 = I_2 \; and \; I_1 I_2 = I_2 I_1 = I_2$$

Be the symbolic 2-plithogenic field of reals. The symbolic 2-plithogenic isometry (AH-Isometry) is defined as follows:

$$T: 2 - SP_R \to R \times R \times R \; ; \; T(a + bI_1 + cI_2) = (a, a + b + c, a + c)$$

Remark 2.9.

1) T is bijective, then it is invertible by:

$$T^{-1}: R \times R \times R \to 2 - SP_R \; ; \; T^{-1}(a, b, c) = a + (b - c)I_1 + (c - a)I_2$$

2) T preserves distances, i.e.:

$$\|T(AB)\| = T(\|AB\|)$$

3. Symbolic n-plithogenic Rings

Definition 3.1 [22]

Let R be a ring, the symbolic n-plithogenic ring is:

$n - SP_R = \{a_0 + a_1 P_1 + a_2 P_2 + \cdots + a_n P_n ; \; a_i \in R, P_j{}^2 = P_j, P_i \times P_j = P_{max(i,j)}\}$.

Operations on $n - SP_R$:

Addition:

$[a_0 + a_1 P_1 + a_2 P_2 + \cdots + a_n P_n] + [b_0 + b_1 P_1 + b_2 P_2 + \cdots + b_n P_n] =$
$(a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2 + \cdots + (a_n + b_n)P_n$.

Multiplication:

$$[a_0 + a_1 P_1 + a_2 P_2 + \cdots + a_n P_n].[b_0 + b_1 P_1 + b_2 P_2 + \cdots + b_n P_n] =$$
$$a_0 b_0 + (a_0 b_1 + a_1 b_0 + a_1 b_1)P_1 + (a_0 b_2 + a_1 b_2 + a_2 b_0 + a_2 b_1 + a_2 b_2)P_2 +$$
$\ldots + (a_0 b_n + a_1 b_n + \cdots + a_{n-1} b_n + a_n b_0 + a_n b_1 + \cdots + a_n b_n)P_n$.

It is clear that $(n - SP_R)$ is a ring.

If R is commutative, then $n - SP_R$ is commutative, and if R has a unity (1), than $n - SP_R$ has the same unity (1).

Example 3.2

Consider the ring $R = Z_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$, the corresponding $n - SP_R$ is:

$n - SP_R = \{a + bP_1 + \cdots + cP_4 ; a, b, \ldots, c \in Z_n\}$.

Definition 3.

Let $n - SP_R$ be a n-plithogenic symbolic ring, with unity (1).

Let $X = x_0 + x_1 P_1 + x_2 P_2 + \cdots + x_n P_n \in n - SP_R$, then X is invertible(unit element) if and only if there exists $Y = y_0 + y_1 P_1 + \cdots + y_n P_n$ such that $X.Y = 1$.

We can write $U_{n-SP_R} = \{X \in n - SP_R \mid \exists Y \in n - SP_R , X.Y = 1\}$.

Theorem 3.4

Let $n - SP_R$ be a n-plithogenic symbolic ring, with unity (1).

Let $X = x_0 + x_1 P_1 + x_2 P_2 + \cdots + x_n P_n$ be an arbitrary element, then:

1. X is invertible(unit element) if and only if $x_0, x_0 + x_1, x_0 + x_1 + x_2, \ldots, x_0 + x_1 + x_2 + \cdots + x_n$ are invertible.
2. $X^{-1} = x_0{}^{-1} + [(x_0 + x_1)^{-1} - x_0{}^{-1}]P_1 + [(x_0 + x_1 + x_2)^{-1} - (x_0 + x_1)^{-1}]P_2 + \ldots + [(x_0 + x_1 + \cdots + x_n)^{-1} - (x_0 + x_1 + \cdots + x_{n-1})^{-1}]P_n$.

Proof.

Assume that X is invertible, than there exists $Y = y_0 + y_1 P_1 + \cdots + y_n P_n$ such that $X.Y = 1$, hence:

$$\begin{cases} x_0y_0 = 1 \ldots (1) \\ x_0y_1 + x_1y_0 + x_1y_1 = 0 \ldots (2) \\ x_0y_2 + x_1y_2 + x_2y_0 + x_2y_1 + x_2y_2 = 0 \ldots (3) \\ x_0y_3 + x_1y_3 + x_2y_3 + x_3y_0 + x_3y_1 + x_3y_2 + x_3y_3 = 0 \ldots (4) \\ \vdots \\ x_0y_n + x_1y_n + \ldots + x_{n-1}y_n + x_ny_0 + x_ny_1 + \cdots + x_ny_n = 0 \ldots (n) \end{cases}$$

From (1), x_0 is invertible.

By adding (1) to (2), we get $(x_0 + x_1)(y_0 + y_1) = 1$, thus $x_0 + x_1$ is invertible.

By adding (1) to (2)to (3), $(x_0 + x_1 + x_2)(y_0 + y_1 + y_2) = 1$, hence $x_0 + x_1 + x_2$ is invertible.

By adding (1) to (2) to (3) to(4),

$(x_0 + x_1 + x_2 + x_3)(y_0 + y_1 + y_2 + y_3) = 1$, hence $x_0 + x_1 + x_2 + x_3$ is invertible.

$$(x_0)^{-1} = y_0$$

$$(x_0 + x_1)^{-1} = y_0 + y_1$$

$$(x_0 + x_1 + x_2)^{-1} = y_0 + y_1 + y_2$$

$$(x_0 + x_1 + x_2 + x_3)^{-1} = y_0 + y_1 + y_2 + y_3$$

$$(x_0 + x_1 + \cdots + x_n)^{-1} = y_0 + y_1 + \cdots + y_n,$$ then:

$$\begin{aligned} X^{-1} = \; & x_0^{-1} + [(x_0 + x_1)^{-1} - x_0^{-1}]P_1 + \\ & [(x_0 + x_1 + x_2)^{-1} - (x_0 + x_1)^{-1}]P_2 + \\ & [(x_0 + x_1 + x_2 + x_3)^{-1} - (x_0 + x_1 + x_2)^{-1}]P_3 + \\ & [(x_0 + x_1 + \cdots + x_n)^{-1} - (x_0 + x_1 + \cdots + x_{n-1})^{-1}]P_n \\ & = Y \end{aligned}$$

We indicate by U_{n-SP_R} the collection of the unit elements.

Example 3.5

Take $R = Z_3 = \{0,1,2\}$, $4 - SP_{Z_3}$ is the corresponding symbolic 4-plithogenic ring, consider $X = 2 + 2P_2 + P_4 \in 4 - SP_{Z_3}$, then:

$X^{-1} = 2 + (2 - 2)P_1 + (1 - 2)P_2 + (1 - 1)P_3 + (2 - 1)P_4 = 2 + 2P_2 + P_4$.

Definition 3.6

Let $X = x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n \in n - SP_R$, then X is idempotent if and only if $X^2 = X$.

We can write $Id_{n-SP_R} = \{X \in n - SP_R \mid X^2 = X \; for \; all \; X \in n - SP_R\}$.

Theorem 3.7

Let $X = x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n \in n - SP_R$, then X is idempotent if and only if $x_0, x_0 + x_1, x_0 + x_1 + x_2, \ldots, x_0 + x_1 + x_2 + \cdots + x_n$ are idempotent.

Proof.

$$X^2 = X.X = (x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n)(x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n)$$

$$X^2 = X.X \text{ equivalents } \begin{cases} x_0x_0 = x_0 \ldots (1) \\ x_0x_1 + x_1x_0 + x_1x_1 = x_1 \ldots (2) \\ x_0x_2 + x_1x_2 + x_2x_0 + x_2x_1 + x_2x_2 = x_2 \ldots (3) \\ x_0x_3 + x_1x_3 + x_2x_3 + x_3x_0 + x_3x_1 + x_3x_2 + x_3x_3 = x_3 \ldots (4) \\ \vdots \\ x_0x_n + x_1x_n + \ldots + x_{n-1}x_n + x_nx_0 + x_nx_1 + \cdots + x_nx_n = x_n \ldots (n) \end{cases}$$

Equation (1) Implies that x_0 is idempotent.

By adding (1) to (2), we get $(x_0+x_1)^2 = x_0+x_1$, hence x_0+x_1 is idempotent.

By adding (1) to (2) to (3), we get $(x_0+x_1+x_2)^2 = x_0+x_1+x_2$, hence $x_0+x_1+x_2$ is idempotent.

By adding (1) to (2) to (3) to (4), we get $(x_0+x_1+x_2+x_3)^2 = x_0+x_1+x_2+x_3$, thus $x_0+x_1+x_2+x_3$ is idempotent.

By adding (1) to (2) to … to (n), we get $(x_0+x_1+x_2+\cdots+x_n)^2 = x_0+x_1+x_2+\cdots+x_n$, thus $x_0+x_1+x_2+\cdots+x_n$ is idempotent. Thus the proof is complete.

Example 3.8

Take $R = Z_4 = \{0,1,2,3\}$, $4-SP_{Z_4}$ is the corresponding symbolic 4-plithogenic ring, consider $X = P_1 + 3P_4 \in 4-SP_{Z_5}$, we have:

$X^2 = P_1 + 9P_4 + 6P_4 = P_1 + +3P_4 = X$.

Theorem 3.9

Let $n-SP_R$ be a commutative symbolic n-plithogenic ring, hence if $X = x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n$, then

$$X^r = {x_0}^r + [(x_0+x_1)^r - {x_0}^r]P_1 + [(x_0+x_1+x_2)^r - (x_0+x_1)^r]P_2 + \cdots + [(x_0+x_1+\cdots+x_n)^r - (x_0+x_1+\cdots+x_{n-1})^r]P_n$$

for every $r \in Z^+$.

Proof.

For $r = 1$, it holds easily. Assume that it is true for $r = k$, we prove it for $r = k+1$.

$$X^{k+1} = X.X^k =$$

$(x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n)({x_0}^k + [(x_0+x_1)^k - {x_0}^k]P_1 + [(x_0+x_1+x_2)^k - (x_0+x_1)^k]P_2 + \cdots + [(x_0+x_1+\cdots+x_n)^k - (x_0+x_1+\cdots+x_{n-1})^k]P_n) = {x_0}^{k+1} + [(x_0+x_1)^{k+1} - {x_0}^{k+1}]P_1 + [(x_0+x_1+x_2)^{k+1} - (x_0+x_1)^{k+1}]P_2 + \cdots + [(x_0+x_1+\cdots+x_n)^{k+1} - (x_0+x_1+\cdots+x_{n-1})^{k+1}]P_n$.

So, that proof is complete by induction.

Definition 3.10

Let $X = x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n \in n-SP_R$ then X is considered nilpotent if there is $r \in \mathbb{Z}^+$ where $X^r = 0$.

We can write $Nd_{n-SP_R} = \{X \in n-SP_R \mid X^r = 0, r \in \mathbb{Z}^+ \ for\ all\ X \in n-SP_R\}$.

Theorem 3.11

If $X = x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n \in n-SP_R$ where R is a commutative, then X is nilpotent iff $x_0, x_0+x_1, x_0+x_1+x_2, \dots, x_0+x_1+x_2+\cdots+x_n$ are nilpotent.

Proof.

$X = x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n$ is nilpotent if and only if there exists $r \in \mathbb{Z}^+$ such that $X^r = 0$, hence:

$${x_0}^r = 0 \dots (1)$$
$$(x_0+x_1)^r - {x_0}^r = 0 \dots (2)\ from(1) \Rightarrow (x_0+x_1)^r = 0$$
$$(x_0+x_1+x_2)^r - (x_0+x_1)^r = 0 \dots (3)\ from(1\ amd\ 2) \Rightarrow (x_0+x_1+x_2)^r = 0$$
$$\vdots$$
$$(x_0+x_1+\cdots+x_n)^r - (x_0+x_1+\cdots+x_{n-1}) = 0 \dots (n)$$
$$from(1,2,\dots,n) \Rightarrow (x_0+x_1+\cdots+x_n)^r = 0$$

Thus, the proof is complete.

Definition 3.12

Center of the neutrosophic ring is defined as $C_{n-SP_R} = \{X \in n-SP_R \mid XY = YX\ for\ all\ Y \in n-SP_R\}$.

Definition 3.13

If $0 \neq X \in n - SP_R$, then $X \neq 0$ is a zero divisor if there exists $0 \neq Y \in n - SP_R$, such that $XY = YX = 0$.

We can write $Z_{n-SP_R} = \{X \neq 0 \in n - SP_R \mid \exists\, Y \neq 0 \in n - SP_R , XY = YX = 0 \ for\ all\ X \in n - SP_R\}$.

Definition 3.14

Let $X = x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n \in n - SP_R$, then is called a regular if there is an element $Y = y_0 + y_1P_1 + \cdots + y_nP_n$ where $Y = (x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n)Y(x_0 + x_1P_1 + x_2P_2 + \cdots + x_nP_n)$.

We can write $Reg_{n-SP_R} = \{X \in n - SP_R \mid \exists\, Y \in n - SP_R , Y = XYX \ for\ all\ X \in n - SP_R\}$.

Theorem 3.15

Let $r_0 + r_1P_1 + \cdots + r_nP_n \in n - SP_R$, then $r_0 + r_1P_1 + \cdots + r_nP_n$ is regular if and only if $r_0, r_0 + r_1, r_0 + r_1 + r_2, \dots, r_0 + r_1 + r_2 + \cdots + r_n$ are regular.

4. Conclusion

In this paper, we have introduced and formalized the concept of symbolic *n*-plithogenic rings, establishing a new class of generalized algebraic structures built upon symmetric *n*-plithogenic components. By defining their fundamental properties and investigating their corresponding substructures, we provided a systematic and coherent framework that extends classical ring theory into the plithogenic setting. The theorems and numerical examples presented herein demonstrate the consistency, flexibility, and potential of this new algebraic model. Future research may explore additional operations, homomorphic mappings, or categorical perspectives of symbolic *n*-plithogenic rings, as well as their applications in areas

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Annotated Hypergraphs and Annotated Superhypergraphs: Integrating Role Annotations into Hierarchical Connectivity

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Abstract. Hypergraphs generalize graphs by allowing edges—called hyperedges—to connect any number of vertices simultaneously [1]. Superhypergraphs extend this concept further through recursively defined powerset layers, capturing hierarchical and self-referential relationships among hyperedges [2]. Annotated hypergraphs enhance hypergraphs by labeling each vertex–hyperedge incidence with a role, thereby structuring multi-member interactions. In this paper, we introduce the *annotated superhypergraph*, which integrates annotation into the superhypergraph framework, and we explore its formal properties and provide detailed examples.

Keywords: Superhypergraph, Hypergraph, Annotated hypergraph, Annotated Superhypergraph

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1. Preliminaries

This section introduces the essential concepts and notation used throughout the paper. Unless otherwise noted, we assume all graphs to be finite and simple.

1.1. *SuperHyperGraphs*

Graph theory studies networks of vertices and edges, analyzing connectivity, paths, cycles, colorings, flows, matchings, and optimization, structures, algorithms, applications [3, 4]. In standard graph theory, a *hypergraph* allows each edge—called a hyperedge—to join any number of vertices, thereby modeling multi-way relationships among elements [1,5–7]. A *SuperHyperGraph* builds on this by stacking hypergraph structures via iterated powerset constructions, producing multiple nested levels of connectivity [2,8–11].

Definition 1.1 (Base Set)**.** Let S be a nonempty set serving as the universe for our constructions. Formally,

$$S = \{\, x : x \text{ lies in the chosen domain}\}.$$

All later objects such as $\mathcal{P}(S)$ or $\mathcal{P}^n(S)$ derive from elements of S.

Definition 1.2 (Powerset)**.** [12] For any set S, its *powerset* $\mathcal{P}(S)$ is the collection of all subsets of S, including $\varnothing$ and S itself:

$$\mathcal{P}(S) \;=\; \{\, A \mid A \subseteq S\}.$$

Definition 1.3 (n^{th} Powerset)**.** [13–15] Define recursively for $n \geq 1$:

$$\mathcal{P}^1(S) = \mathcal{P}(S), \quad \mathcal{P}^{k+1}(S) = \mathcal{P}\big(\mathcal{P}^k(S)\big).$$

The *nonempty* variant $\mathcal{P}_n^*(S)$ omits the empty set at each stage.

Definition 1.4 (Hypergraph)**.** [1,16] A *hypergraph* $H = (V, E)$ consists of

- A finite vertex set V.
- A set of hyperedges $E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$, each hyperedge connecting one or more vertices.

Definition 1.5 (n^{th} SuperHyperGraph)**.** [2,9] Let V_0 be a finite *base set.* For each integer $k \geq 0$, set

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big).$$

An n*-SuperHyperGraph* is a pair $\mathrm{SHG}^{(n)} = (V, E)$ where

$$V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}(V)$$

and each element of V is called an n-supervertex while each $e \in E$ is an n-superedge.

Example 1.6 (Multi–Level Corporate Structure as a 3^{rd} SuperHyperGraph)**.** Let

$$S = \{\text{Alice, Bob, Carol, Dave, Eve, Frank}\}$$

be the set of all employees in a multinational firm. We construct iterated powersets:

$$\mathcal{P}^1(S) = \{\text{all project teams}\}, \quad \mathcal{P}^2(S) = \{\text{all departments}\}, \quad \mathcal{P}^3(S) = \{\text{all divisions}\}.$$

Concretely, choose three project teams:

$$T_1 = \{\mathrm{Alice}, \mathrm{Bob}\}, \quad T_2 = \{\mathrm{Carol}, \mathrm{Dave}\}, \quad T_3 = \{\mathrm{Eve}, \mathrm{Frank}\},$$

then two departments:

$$D_1 = \{T_1, T_2\}, \quad D_2 = \{T_2, T_3\},$$

and two global divisions:

$$V = \{\, v_A, v_B \} \;\subseteq\; \mathcal{P}^3(S), \quad v_A = \{D_1, D_2\},\; v_B = \{D_1\}.$$

We model two cross–division initiatives as superedges:

$$E = \{\, e_1, e_2 \} \subseteq \mathcal{P}(V), \quad e_1 = \{v_A, v_B\},\; e_2 = \{v_A\}.$$

Then

$$\mathrm{SHG}^{(3)} = (V, E)$$

is a 3^{rd} SuperHyperGraph in which each "division" $v \in V$ is a 3–supervertex, and each initiative $e \in E$ is a 3–superedge connecting those divisions that collaborate on a given global program.

1.2. *Annotated Hypergraph*

An annotated hypergraph is a hypergraph where each incidence between a vertex and a hyperedge is labeled with a role, enabling structured representation of multi-member relationships [17–21].

Definition 1.7 (Annotated Hypergraph)**.** [17] An *annotated hypergraph* is a quadruple

$$H = (V,\, E,\, X,\, \varphi)$$

consisting of

(1) a finite *vertex set* V;
(2) a finite *hyperedge multiset* E, where each $e \in E$ is a nonempty subset of V (parallel edges allowed, but no repeated vertices within one edge);
(3) a finite *label set* X of "roles";
(4) a *role-labeling function*

$$\varphi \;:\; \{(v, e) \in V \times E \mid v \in e\} \;\longrightarrow\; X,$$

which assigns to each incident pair (v, e) the role $\varphi(v, e) \in X$.

The statement $\varphi(v, e) = x$ is read "vertex v has role x in hyperedge e."

Example 1.8 (Annotated Hypergraph: Project Team Assignments)**.** Consider a small software-development organization with

$$V = \{\text{Alice, Bob, Carol, Dave}\}, \quad X = \{\texttt{Manager}, \texttt{Developer}, \texttt{Tester}\},$$

and two projects

$$E = \{\, P_1,\, P_2\}, \quad P_1 = \{\text{Alice, Bob, Carol}\}, \quad P_2 = \{\text{Bob, Carol, Dave}\}.$$

Define the role-labeling function φ by

$$\varphi(\text{Alice}, P_1) = \texttt{Manager}, \qquad \varphi(\text{Bob}, P_1) = \texttt{Developer}, \qquad \varphi(\text{Carol}, P_1) = \texttt{Tester},$$
$$\varphi(\text{Bob}, P_2) = \texttt{Manager}, \qquad \varphi(\text{Carol}, P_2) = \texttt{Developer}, \qquad \varphi(\text{Dave}, P_2) = \texttt{Tester}.$$

Then

$$H = \bigl(V,\, E,\, X,\, \varphi\bigr)$$

is an annotated hypergraph in which each vertex (team member) has a specific role in each project (hyperedge).

2. Main Results: Annotated SuperHypergraph

An annotated SuperHyperGraph extends a hypergraph by labeling each incidence between supervertices and superedges with roles from a finite label set, capturing hierarchical semantic relationships.

Definition 2.1 (Annotated n-SuperHyperGraph)**.** Let S be a nonempty *base set* and $n \geq 0$ an integer. Define iterated powersets by

$$\mathcal{P}^0(S) = S, \quad \mathcal{P}^{k+1}(S) = \mathcal{P}\bigl(\mathcal{P}^k(S)\bigr) \quad (k \geq 0).$$

An *annotated n-SuperHyperGraph* is a quadruple

$$H = \bigl(V,\ E,\ X,\ \varphi\bigr),$$

where

(1) $V \subseteq \mathcal{P}^n(S)$ is the set of *n-supervertices*;
(2) $E \subseteq \mathcal{P}(V)$ is the multiset of *n-superedges*, each $e \in E$ a nonempty subset of V (parallel superedges allowed);
(3) X is a finite *label set* (roles);
(4) φ is the *role–labeling function*

$$\varphi : \{(v, e) \in V \times E \mid v \in e\} \longrightarrow X,$$

assigning each incidence (v, e) a label $\varphi(v, e) \in X$.

We read $\varphi(v, e) = x$ as "n-supervertex v plays role x in superedge e."

Example 2.2 (Annotated 2-SuperHyperGraph: Cross-Department Projects)**.** Consider a company with six employees:

$$S = \{\text{Alice, Bob, Carol, Dave, Eve, Frank}\}.$$

We form first–level teams (elements of $\mathcal{P}^1(S)$):

$$T_1 = \{\text{Alice, Bob}\}, \quad T_2 = \{\text{Carol, Dave}\}, \quad T_3 = \{\text{Eve, Frank}\}.$$

Next, we define second–level departments (elements of $\mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}^1(S))$):

$$D_1 = \{T_1, T_2\}, \quad D_2 = \{T_2, T_3\}.$$

Let

$$V = \{D_1, D_2\} \subseteq \mathcal{P}^2(S), \quad X = \{\texttt{Coordinator}, \texttt{Contributor}\}.$$

We organize two cross-department projects:

$$e_A = \{D_1, D_2\}, \quad e_B = \{D_2\} \subseteq \mathcal{P}(V).$$

Define the role–labeling function φ by

$$\varphi(D_1, e_A) = \texttt{Coordinator}, \qquad \varphi(D_2, e_A) = \texttt{Contributor},$$
$$\varphi(D_2, e_B) = \texttt{Coordinator}.$$

Then

$$H = \big(V, \{e_A, e_B\}, X, \varphi\big)$$

is an annotated 2-SuperHyperGraph in which each department (2-supervertex) plays a specific role in each project (2-superedge).

Example 2.3 (Annotated 3-SuperHyperGraph: Inter–Divisional Strategic Initiatives)**.** Consider a large company with six employees:

$$S = \{\text{Alice, Bob, Carol, Dave, Eve, Frank}\}.$$

First–level teams (elements of $\mathcal{P}^1(S)$):

$$T_1 = \{\text{Alice, Bob}\}, \quad T_2 = \{\text{Carol, Dave}\}, \quad T_3 = \{\text{Eve, Frank}\}.$$

Second–level departments (elements of $\mathcal{P}^2(S)$):

$$D_1 = \{T_1, T_2\}, \quad D_2 = \{T_2, T_3\}, \quad D_3 = \{T_1, T_3\}.$$

Third–level divisions (elements of $\mathcal{P}^3(S)$):

$$v_A = \{D_1, D_2\}, \quad v_B = \{D_2, D_3\}, \quad v_C = \{D_1, D_3\}.$$

Let

$$V = \{v_A, v_B, v_C\} \subseteq \mathcal{P}^3(S), \quad X = \{\texttt{Lead}, \texttt{Support}\}.$$

We plan two strategic initiatives spanning these divisions:

$$e_1 = \{v_A, v_B\}, \quad e_2 = \{v_B, v_C\} \subseteq \mathcal{P}(V).$$

Define the role–labeling function φ by

$$\varphi(v_A, e_1) = \texttt{Lead}, \qquad \varphi(v_B, e_1) = \texttt{Support},$$
$$\varphi(v_B, e_2) = \texttt{Lead}, \qquad \varphi(v_C, e_2) = \texttt{Support}.$$

Then

$$H = \big(V,\ \{e_1, e_2\},\ X,\ \varphi\big)$$

is an annotated 3-SuperHyperGraph modeling inter–divisional strategic initiatives, where each division (3-supervertex) plays a specific role in each initiative (3-superedge).

Example 2.4 (Annotated 4-SuperHyperGraph: Global Marketing Campaign)**.** Consider a multinational company with six employees:

$$S = \{\text{Alice, Bob, Carol, Dave, Eve, Frank}\}.$$

First–level teams (elements of $\mathcal{P}^1(S)$):

$$T_1 = \{\text{Alice, Bob}\}, \quad T_2 = \{\text{Carol, Dave}\}, \quad T_3 = \{\text{Eve, Frank}\}.$$

Second–level departments (elements of $\mathcal{P}^2(S)$):

$$D_1 = \{T_1,\, T_2\}, \quad D_2 = \{T_2,\, T_3\}, \quad D_3 = \{T_1,\, T_3\}.$$

Third–level divisions (elements of $\mathcal{P}^3(S)$):

$$\Delta_1 = \{D_1,\, D_2\}, \quad \Delta_2 = \{D_2,\, D_3\}.$$

Fourth–level regions (elements of $\mathcal{P}^4(S)$):

$$R_A = \{\Delta_1,\, \Delta_2\}, \quad R_B = \{\Delta_2\}.$$

Let

$$V = \{R_A,\, R_B\} \ \subseteq\ \mathcal{P}^4(S), \quad X = \{\texttt{Lead},\, \texttt{Support}\}.$$

We organize two global marketing initiatives:

$$e_\alpha = \{R_A,\, R_B\}, \quad e_\beta = \{R_B\} \ \subseteq\ \mathcal{P}(V).$$

Define the role–labeling function φ by

$$\varphi(R_A, e_\alpha) = \texttt{Lead}, \qquad \varphi(R_B, e_\alpha) = \texttt{Support},$$
$$\varphi(R_B, e_\beta) = \texttt{Lead}.$$

Then

$$H = \big(V,\ \{e_\alpha, e_\beta\},\ X,\ \varphi\big)$$

is an annotated 4-SuperHyperGraph modeling how each global region (4-supervertex) plays a specific role in each marketing initiative (4-superedge).

Theorem 2.5 (Generalization of n-SuperHyperGraphs and Annotated SuperHyperGraphs)**.** *Annotated n-SuperHyperGraphs strictly generalize both*

- *ordinary n-SuperHyperGraphs by choosing the trivial label set $X = \{\star\}$ and constant labeling $\varphi(v, e) = \star$, and*

- *Annotated SuperHyperGraphs (the case $n = 1$ in Definition.*

Proof. **(1) Recovering n-SuperHyperGraphs.** Let $H_0 = (V, E)$ be an ordinary n-SuperHyperGraph on S. Define

$$X = \{\star\}, \quad \varphi(v, e) = \star \quad \forall\, (v, e) \text{ with } v \in e.$$

Then $H = (V, E, X, \varphi)$ satisfies Definition 2.1, and forgetting X, φ recovers H_0.

(2) Recovering Annotated SuperHyperGraphs. If $n = 1$, then $\mathcal{P}^1(S) = \mathcal{P}(S)$ and Definition 2.1 reduces exactly to the standard notion of an annotated hypergraph (Definition 1.7). In particular, each annotated SuperHyperGraph is an annotated 1-SuperHyperGraph.

Since both specializations are strict—labels can encode more structure than the trivial case, and n may exceed 1—annotated n-SuperHyperGraphs indeed form a proper generalization of both families. □

Theorem 2.6 (Underlying n-SuperHyperGraph)**.** *Let $H = (V, E, X, \varphi)$ be an annotated n-SuperHyperGraph on base set S, i.e.*

$$V \subseteq \mathcal{P}^n(S), \quad E \subseteq \mathcal{P}(V), \quad E \neq \emptyset, \quad \varphi : \{(v, e) \mid v \in e\} \to X.$$

Then

$$H_0 = (V, E)$$

satisfies the definition of an (ordinary) n-SuperHyperGraph.

Proof. By Definition 2.1, we have

$$V \subseteq \mathcal{P}^n(S), \quad E \subseteq \mathcal{P}(V), \quad e \neq \varnothing \quad \forall\, e \in E.$$

But an (ordinary) n-SuperHyperGraph is exactly a pair (V, E) with these three properties. Hence discarding the annotation data (X, φ) yields the desired result. □

Theorem 2.7 (Trivial Annotation)**.** *Every n-SuperHyperGraph $H_0 = (V, E)$ can be turned into an annotated n-SuperHyperGraph by choosing a singleton label set and constant labeling. Concretely, set*

$$X = \{\star\}, \quad \varphi(v, e) = \star \quad (\forall\, v \in e).$$

Then (V, E, X, φ) satisfies Definition 2.1.

Proof. We check each requirement:

(1) $V \subseteq \mathcal{P}^n(S)$ and $E \subseteq \mathcal{P}(V)$ hold by assumption on H_0.

(2) $X = \{\star\}$ is finite and nonempty.

(3) The function

$$\varphi : \{(v, e) \mid v \in e\} \longrightarrow X, \quad \varphi(v, e) = \star$$

is well-defined on all incidences.

Thus (V, E, X, φ) is an annotated n-SuperHyperGraph, and forgetting (X, φ) recovers H_0. □

Theorem 2.8 (Induced Substructure). *Let $H = (V, E, X, \varphi)$ be an annotated n-SuperHyperGraph and let $V' \subseteq V$. Define*

$$E' = \{ e \cap V' \mid e \in E,\ e \cap V' \neq \varnothing\}, \quad \varphi' = \varphi\big|_{\{(v,e') \mid v \in e'\}}.$$

Then $H' = (V', E', X, \varphi')$ is an annotated n-SuperHyperGraph.

Proof.
- Since $V' \subseteq V \subseteq \mathcal{P}^n(S)$, we have $V' \subseteq \mathcal{P}^n(S)$.
- Each $e' = e \cap V'$ is nonempty by selection, and $e' \subseteq V'$, so $E' \subseteq \mathcal{P}(V')$.
- X remains finite and nonempty.
- φ' is well-defined because for each $e' \in E'$ there is a unique $e \in E$ with $e' = e \cap V'$, and we inherit the label $\varphi(v, e) = \varphi'(v, e')$.

All conditions of Definition 2.1 are satisfied, so H' is annotated. □

Theorem 2.9 (Labeled Incidence Tensor Uniqueness). *Let*

$$H_i = (V_i, E_i, X_i, \varphi_i), \quad i = 1, 2$$

be two annotated n-SuperHyperGraphs with $|V_i| = n_i$, $|E_i| = m_i$, $|X_i| = p_i$. Form the labeled incidence tensor

$$T^{(i)} \in \{0, 1\}^{n_i \times m_i \times p_i}, \quad T^{(i)}_{v,e,x} = 1 \iff \varphi_i(v, e) = x.$$

If there exist permutations $\sigma \in S_{n_2}$, $\tau \in S_{m_2}$, $\rho \in S_{p_2}$ such that

$$T^{(1)}_{v,e,x} = T^{(2)}_{\sigma(v),\, \tau(e),\, \rho(x)} \quad \forall\, v, e, x,$$

then there is an isomorphism of annotated structures.

Proof. Define

$$f : V_1 \to V_2, \quad f(v) = \sigma(v), \quad g : E_1 \to E_2, \quad g(e) = \tau(e), \quad h : X_1 \to X_2, \quad h(x) = \rho(x).$$

Each of f, g, h is a bijection because σ, τ, ρ are permutations. The equality of tensors implies

$$\varphi_1(v, e) = x \iff T^{(1)}_{v,e,x} = 1 \iff T^{(2)}_{f(v),g(e),h(x)} = 1 \iff \varphi_2\big(f(v), g(e)\big) = h(x).$$

Therefore (f, g, h) respects both the incidence relation and the labels, and is an isomorphism of annotated n-SuperHyperGraphs. □

Theorem 2.10 (Connectivity Equivalence)**.** *Let $H = (V, E, X, \varphi)$ be annotated and let $B(H)$ be the bipartite graph with parts $V \sqcup E$ and an (un-labeled) edge between $v \in V$ and $e \in E$ exactly when $v \in e$. Then $B(H)$ is connected if and only if the underlying hypergraph (V, E) is connected in the usual sense.*

Proof.

(⇒) If $B(H)$ is connected, then for any $v, v' \in V$ there is a path

$$v = e_0 - v_1 - e_1 - \cdots - v_k = v'$$

in $B(H)$. Projecting onto the vertex set V yields a walk in the hypergraph (V, E):

$$v = v_0, e_0, v_1, e_1, \ldots, v_k = v',$$

proving connectivity.

(⇐) Conversely, a walk in the hypergraph gives a path in $B(H)$ by alternating v_i–e_i–v_{i+1}, so $B(H)$ is connected.

Labels in X play no role in connectivity of the unlabeled bipartite graph. □

Theorem 2.11 (Morphisms Preserve Annotation)**.** *If*

$$\psi : (V_1, E_1) \to (V_2, E_2)$$

is a homomorphism of n-SuperHyperGraphs (i.e. $v \in e \implies \psi(v) \in \psi(e)$) and

$$h : X_1 \to X_2$$

satisfies $\varphi_2\big(\psi(v), \psi(e)\big) = h\big(\varphi_1(v, e)\big)$ for all (v, e) with $v \in e$, then (ψ, ψ, h) is a homomorphism of the annotated structures.

Proof. We must check that

$$v \in e \implies \psi(v) \in \psi(e) \quad \text{and} \quad \varphi_2\big(\psi(v), \psi(e)\big) = h\big(\varphi_1(v, e)\big).$$

The first condition is exactly the assumption that ψ is a hypergraph homomorphism; the second is assumed. Thus (ψ, ψ, h) preserves both incidence and labels. □

3. Conclusion and Future Work

In this paper we introduced the *annotated superhypergraph*, a structure that embeds role annotations into the classical superhypergraph framework. We formalised its theoretical properties and illustrated the concept with detailed examples.

Future research will proceed along several directions. First, we aim to enrich the model with a variety of uncertainty paradigms (cf. [22]), including Fuzzy Sets [23,24], Intuitionistic Fuzzy Sets [25], Vague Sets [26], HyperFuzzy Sets [27,28], Hesitant Fuzzy Sets [29], Neutrosophic Sets [30–33], Interval-Valued Neutrosophic Sets [34,35], and Plithogenic Sets [36]. Second, we intend to investigate analogous extensions built upon bidirected graphs [37], bunch graphs [38], and multidirected graphs [39]. Finally, we look forward to validating the practical utility of annotated superhypergraphs through empirical studies on real-world datasets.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

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Generalizing Multidimensional Networks: A Framework of Multidimensional HyperNetworks and SuperHyperNetworks

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Abstract. Graph theory studies the mathematical structures of vertices and edges to model relationships and connectivity [1]. Hypergraphs extend this framework by allowing hyperedges to connect arbitrarily many vertices at once [2], and superhypergraphs further generalize hypergraphs via iterated powerset constructions to capture hierarchical linkages among edges [3,4]. Weighted multidimensional networks model nodes connected by edges in multiple layers, assigning each edge a weight to quantify relationship strength in its specific dimension(cf. [5,6]). In this paper, we extend these ideas using hypergraphs and superhypergraphs to introduce and formalize *multidimensional hypernetworks* and *multidimensional superhypernetworks*.

Keywords: Superhypergraph, Hypergraph, Multidimensional Network, Multidimensional HyperNetwork. Multidimensional SuperHyperNetwork

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1. Preliminaries

We begin by fixing notation and recalling key concepts that underlie our constructions. Unless otherwise noted, all graph-based objects in this paper are finite, simple, and undirected. For more extensive discussions of these topics, the reader is referred to the standard literature.

1.1. *Hypergraphs and SuperHyperGraphs*

A *hypergraph* allows edges—called *hyperedges*—to link any number of vertices at once, capturing higher-order relationships beyond pairwise connections [2, 7–10]. A *SuperHyperGraph* builds on this by applying the powerset operation repeatedly, so that edges themselves can be nested in multiple layers, representing hierarchical groupings of interactions [3, 4, 11, 12]. In what follows, the integer $n \geq 0$ always indicates the number of times the powerset is iterated.

Definition 1.1 (Ground Set)**.** Let S be a finite *ground set* of elements under consideration. All subsequent constructions—subsets, powersets, and iterated powersets—are formed from S.

Definition 1.2 (Powerset)**.** (cf. [13–15]) Given a set S, its *powerset* $\mathcal{P}(S)$ is the collection of all subsets of S, including the empty set and S itself:

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

Definition 1.3 (Hypergraph)**.** [2, 7] A *hypergraph* $H = (V, E)$ consists of

- A finite vertex set V.
- A finite set E of nonempty subsets of V, each called a *hyperedge*.

By design, hypergraphs can model relationships that involve more than two vertices at once.

Definition 1.4 (n-th Iterated Powerset)**.** [16–18] For a set X, define its iterated powersets by

$$\mathcal{P}_0(X) = X, \qquad \mathcal{P}_{k+1}(X) = \mathcal{P}\big(\mathcal{P}_k(X)\big) \quad (k \geq 0).$$

Thus $\mathcal{P}_n(X)$ is the result of applying the powerset operation n times.

Definition 1.5 (n-SuperHyperGraph)**.** [19, 20] Fix a finite ground set V_0. Let $\mathcal{P}^k(V_0)$ denote the k-th iterated powerset as above. An *n-SuperHyperGraph* is an ordered pair

$$\mathrm{SuHG}^{(n)} = (V, E), \quad \text{with } V, E \subseteq \mathcal{P}^n(V_0),$$

where elements of V are called *n-supervertices* and elements of E are *n-superedges*, each superedge being a nonempty subset of V. This structure captures interactions at up to n nested levels.

Example 1.6 (Collaborative Research 2-SuperHyperGraph)**.** Consider a research laboratory with five investigators:

$$V_0 = \{\text{Yuta}, \text{ Hiroko}, \text{ Shinya}, \text{ Taka}, \text{ Eve}\}.$$

They form three project teams (first-level groups):

$$T_1 = \{\text{Yuta}, \text{Hiroko}\}, \quad T_2 = \{\text{Shinya}, \text{Taka}\}, \quad T_3 = \{\text{Hiroko}, \text{Eve}\}.$$

These teams themselves are organized into consortia (second-level supervertices):

$$C_1 = \{T_1, T_2\}, \quad C_2 = \{T_2, T_3\}, \quad C_3 = \{T_1, T_3\}.$$

Taking $n = 2$, we have

$$V = \{\, C_1,\, C_2,\, C_3\} \;\subseteq\; \mathcal{P}^2(V_0).$$

We record collaborative initiatives as 2-superedges (each a nonempty subset of V):

$$\mathcal{E} = \big\{\{C_1\},\ \{C_2\},\ \{C_3\},\ \{C_1, C_2\},\ \{C_2, C_3\}\big\}.$$

Here

- $\{C_1\}$ represents an internal workshop run by Consortium 1.
- $\{C_1, C_2\}$ denotes a joint symposium between Consortia 1 and 2.
- $\{C_2, C_3\}$ denotes collaborative grant applications between Consortia 2 and 3.

Thus

$$\mathrm{SuHG}^{(2)} = (\,V,\ \mathcal{E}\,)$$

is a 2-SuperHyperGraph capturing collaborations at the level of researchers → teams → consortia.

1.2. *Weighted Multidimensional Network*

Weighted multidimensional network models nodes connected by edges in several distinct layers, assigning each edge a weight to quantify relationship strength along its specific dimension (cf. [21–23]). Related concepts include Multilayer Networks and Multiplex Networks, which are also well-studied in the literature(cf. [24–27]).

Definition 1.7 (Unweighted Multidimensional Network)**.** Let V be a finite set of *nodes* and let D be a finite set of *dimensions* (or *layers*). An *unweighted multidimensional network* is a triple

$$\mathcal{G} = (V, E, D),$$

where

- $E \subseteq V \times V \times D$ is a set of *edges* of the form (u, v, d), indicating an (undirected) connection between $u, v \in V$ in dimension $d \in D$;
- for each $d \in D$ and each unordered pair $\{u, v\} \subseteq V$, at most one edge (u, v, d) appears in E;
- if the network is *directed*, we instead allow (u, v, d) and (v, u, d) to be distinct elements of E.

We say that two nodes u, v are *adjacent in dimension* d if $(u, v, d) \in E$. The *degree* of v in dimension d is

$$k_d(v) \;=\; \big|\{\, u \in V : (u, v, d) \in E\}\big|.$$

Example 1.8 (Urban Multimodal Transport Network). Let the set of *stops* in a city be

$$V = \{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3, \mathrm{S}_4, \mathrm{S}_5\}.$$

There are three transport modes (dimensions):

$$D = \{\text{bus},\ \text{metro},\ \text{tram}\}.$$

We represent direct connections by edges tagged with the mode of travel:

$$\begin{aligned} E = \{&(\mathrm{S}_1, \mathrm{S}_2, \text{bus}),\ (\mathrm{S}_2, \mathrm{S}_3, \text{bus}),\\ &(\mathrm{S}_1, \mathrm{S}_4, \text{metro}),\ (\mathrm{S}_4, \mathrm{S}_5, \text{metro}),\\ &(\mathrm{S}_2, \mathrm{S}_5, \text{tram}),\ (\mathrm{S}_3, \mathrm{S}_4, \text{tram})\}. \end{aligned}$$

No more than one connection appears for each pair and mode. Thus

$$\mathcal{G} = (V, E, D)$$

is an *unweighted multidimensional network* modeling the city's multimodal transport system:

- $(\mathrm{S}_1, \mathrm{S}_2, \text{bus})$ and $(\mathrm{S}_2, \mathrm{S}_3, \text{bus})$ are direct bus routes.
- $(\mathrm{S}_1, \mathrm{S}_4, \text{metro})$ and $(\mathrm{S}_4, \mathrm{S}_5, \text{metro})$ are metro lines.
- $(\mathrm{S}_2, \mathrm{S}_5, \text{tram})$ and $(\mathrm{S}_3, \mathrm{S}_4, \text{tram})$ are tram connections.

Passengers can analyze paths that switch modes by traversing edges in different dimensions.

Definition 1.9 (Weighted Multidimensional Network). Let V, D be as in Definition 1.7, and let

$$E \subseteq V \times V \times D \times \mathbb{R}$$

be a set of *weighted edges* (u, v, d, w), where $w \in \mathbb{R}$ is the weight of the connection between u and v in dimension d. The quadruple

$$\mathcal{G} = (V, E, D, w)$$

is called a *weighted multidimensional network*. The *strength* of node v in dimension d is

$$s_d(v) = \sum_{(u,v,d,w)\in E} w.$$

Remark 1.10 (Adjacency Tensor). In canonical tensor notation, one may encode any weighted multidimensional network $\mathcal{G} = (V, E, D)$ by a rank-4 *adjacency tensor* $M = (M^{i\alpha}_{j\beta})_{i,j\in V,\,\alpha,\beta\in D}$, where

$$M^{i\alpha}_{j\beta} = \begin{cases} w, & \text{if } (i, j, \alpha, w) \in E \text{ and } \alpha = \beta,\\ 0, & \text{otherwise.} \end{cases}$$

This representation readily extends standard matrix-based methods (e.g. centrality, spectral analysis) to the multidimensional setting.

Example 1.11 (Global Trade Weighted Multidimensional Network)**.** Consider five countries:

$$V = \{\text{US, CN, DE, JP, IN}\}.$$

We track trade in three categories (dimensions):

$$D = \{\text{oil, electronics, agriculture}\}.$$

Define weighted edges (u, v, d, w) where w is annual export volume (in billions USD):

$$E = \{(\text{US}, \text{CN}, \text{oil}, 50), (\text{CN}, \text{US}, \text{electronics}, 200), (\text{DE}, \text{US}, \text{agriculture}, 30), (\text{JP}, \text{DE}, \text{electronics}, 120), (\text{IN}, \text{JP}, \text{agriculture}, 25), (\text{US}, \text{IN}, \text{oil}, 15)\}.$$

Then

$$\mathcal{G} = (V, E, D, w)$$

is a *weighted multidimensional network* representing global trade:

- $(\text{US}, \text{CN}, \text{oil}, 50)$ means the US exports $50 billion of oil to China.
- $(\text{CN}, \text{US}, \text{electronics}, 200)$ means China exports $200 billion of electronics to the US.
- $(\text{DE}, \text{US}, \text{agriculture}, 30)$ means Germany exports $30 billion of agricultural products to the US.
- And so on for the other entries.

One can compute each country's *strength* $s_d(v)$ in category d by summing outgoing volumes in that dimension.

1.3. *HyperNetwork and SuperhyperNetwork*

A hypernetwork is a graph generalization where hyperedges connect any number of nodes, enabling modeling of multiway relationships beyond pairwise edges [28]. An n-superhypernetwork uses vertices and hyperedges drawn from the n-th iterated powerset of a base node set to model nested, hierarchical groupings [29]. The definitions of HyperNetwork and SuperhyperNetwork are presented below [28].

Definition 1.12 (Hypernetwork)**.** [28] A *hypernetwork* is an ordered triple

$$H = \big(V, \mathcal{E}, w\big)$$

where

- V is a nonempty finite set of *nodes*;
- $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is the set of *hyperedges*, each hyperedge $e \in \mathcal{E}$ being a nonempty subset of nodes (allowing multi-node interactions);
- $w \colon \mathcal{E} \to \mathbb{R}_{\geq 0}$ is a *weight or attribute function* on hyperedges (omitted if unweighted).

A *directed hypernetwork* may be defined by replacing $\mathcal{E} \subseteq \mathcal{P}(V)$ with a set of *ordered* tuples of nodes or by equipping each $e \in \mathcal{E}$ with a head-tail partition. One can further add a *node-labeling* $\ell_V : V \to L_V$ and a *hyperedge-labeling* $\ell_\mathcal{E} : \mathcal{E} \to L_\mathcal{E}$ to record types or properties.

Definition 1.13 (n-SuperHypernetwork)**.** [28, 30, 31] Let V_0 be a finite base set of *nodes*. Define the n-th iterated powerset recursively by

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \geq 0).$$

An *n-superhypernetwork* is a tuple

$$\mathcal{N}^{(n)} = \big(V,\ \mathcal{E},\ w\big)$$

where

- $V \subseteq \mathcal{P}^n(V_0)$ is a finite set of *n-supernodes*;
- $\mathcal{E} \subseteq \mathcal{P}(V)$ is a finite set of *n-superedges*, each superedge $e \in \mathcal{E}$ being a nonempty subset of V;
- $w\colon \mathcal{E} \to \mathbb{R}_{\geq 0}$ is an optional *weight function* assigning a nonnegative real weight (or confidence) to each superedge.

In other words, both vertices and hyperedges of the network are drawn from the n-th powerset of the base node set, capturing up to n levels of hierarchical grouping.

Example 1.14 (Research Consortium 2-SuperHypernetwork)**.** Consider a small research community of four investigators:

$$V_0 = \{\text{Yuta, Hiroko, Shinya, Taka}\}.$$

They form three *laboratories* (first-level teams):

$$L_1 = \{\text{Yuta}, \text{Hiroko}\}, \quad L_2 = \{\text{Shinya}, \text{Taka}\}, \quad L_3 = \{\text{Hiroko}, \text{Shinya}\}.$$

These labs themselves collaborate to form two *consortia* (second-level supernodes):

$$C_1 = \{L_1, L_2\}, \qquad C_2 = \{L_2, L_3\}.$$

Thus, taking $n = 2$, we have

$$V = \{\, C_1,\, C_2\} \ \subseteq\ \mathcal{P}^2(V_0).$$

We model funded programs as *2-superedges* (each a nonempty subset of V):

$$\mathcal{E} = \big\{\{C_1\},\ \{C_2\},\ \{C_1, C_2\}\big\}.$$

Assigning each program's annual budget (in million USD) as a weight,

$$w(\{C_1\}) = 5.0, \quad w(\{C_2\}) = 3.5, \quad w(\{C_1, C_2\}) = 1.2.$$

Therefore

$$\mathcal{N}^{(2)} = \big(V,\ \mathcal{E},\ w\big)$$

is a 2-SuperHypernetwork capturing:

- the hierarchy of individual investigators → labs → consortia,
- the funding programs that each consortium (and joint consortia pair) secures.

2. Result: Weighted Multidimensional Hypernetwork

Weighted multidimensional hypernetwork extends networks by grouping nodes into hyperedges across distinct dimensions, assigning weights to each unique hyperedge–dimension pair to represent multiway relationship intensities.

Definition 2.1 (Weighted Multidimensional Hypernetwork)**.** Let V be a finite set of *nodes* and let D be a finite set of *dimensions*. A *weighted multidimensional hypernetwork* is a quadruple

$$\mathcal{H} \;=\; \big(V,\ \mathcal{E},\ D,\ w\big),$$

where

- $\mathcal{E} \subseteq \big(\mathcal{P}(V) \setminus \{\emptyset\}\big) \times D$ is a set of *dimension-tagged hyperedges*, each element $(e, d) \in \mathcal{E}$ consisting of a nonempty vertex subset $e \subseteq V$ and a dimension $d \in D$;
- $w \colon \mathcal{E} \to \mathbb{R}$ is a *weight function* assigning each $(e, d) \in \mathcal{E}$ a real weight $w(e, d)$.

Example 2.2 (Corporate Team Communication Hypernetwork)**.** Consider a small engineering team with four members:

$$V = \{\text{Yuta, Hiroko, Shinya, Taka}\}.$$

They use three communication channels:

$$D = \{\text{Slack, Email, Zoom}\}.$$

We model group communications as dimension-tagged hyperedges, weighted by the average number of messages (or minutes of meeting) per week:

$$\mathcal{E} = \big\{\big(\{\,\text{Yuta}, \text{Hiroko}, \text{Shinya}\}, \text{Slack}\big),\ \big(\{\,\text{Yuta}, \text{Taka}\}, \text{Email}\big),\ \big(\{\,\text{Hiroko}, \text{Shinya}, \text{Taka}\}, \text{Zoom}\big)\big\}.$$

Define the weight function $w \colon \mathcal{E} \to \mathbb{R}_{\geq 0}$ by

$$w\big(\{A, B, C\}, \text{Slack}\big) = 120, \quad w\big(\{A, D\}, \text{Email}\big) = 30, \quad w\big(\{B, C, D\}, \text{Zoom}\big) = 180,$$

where

$w(\{\text{Yuta}, \text{Hiroko}, \text{Shinya}\}, \text{Slack}) = 120$ (120 Slack messages/week among Yuta, Hiroko, Shinya),

$w(\{\text{Yuta}, \text{Taka}\}, \text{Email}) = 30$ (30 email exchanges/week between Yuta and Taka),

$w(\{\text{Hiroko}, \text{Shinya}, \text{Taka}\}, \text{Zoom}) = 180$ (180 minutes of Zoom meetings/week among Hiroko, Shinya, Taka).

Thus

$$\mathcal{H} = \bigl(V,\ \mathcal{E},\ D,\ w\bigr)$$

is a weighted multidimensional hypernetwork capturing the team's collaborative interactions across different channels.

Example 2.3 (Global Supply Chain Hypernetwork)**.** Consider a simplified supply chain with six locations:

$$V = \{\mathrm{F}_1, \mathrm{F}_2,\ \mathrm{W}_1, \mathrm{W}_2,\ \mathrm{R}_1, \mathrm{R}_2\},$$

where F_i are factories, W_j warehouses, and R_k retailers. There are three product categories (dimensions):

$$D = \{\text{electronics},\ \text{furniture},\ \text{food}\}.$$

We record weekly shipment groupings as dimension-tagged hyperedges, weighted by units shipped per week:

$$\begin{aligned}\mathcal{E} = \{&(\{\,\mathrm{F}_1, \mathrm{W}_1, \mathrm{R}_1\}, \text{electronics}),\\ &(\{\,\mathrm{F}_2, \mathrm{W}_2, \mathrm{R}_2\}, \text{electronics}),\\ &(\{\,\mathrm{F}_1, \mathrm{W}_1, \mathrm{W}_2, \mathrm{R}_2\}, \text{furniture}),\\ &(\{\,\mathrm{F}_2, \mathrm{W}_2, \mathrm{R}_1, \mathrm{R}_2\}, \text{food})\}.\end{aligned}$$

Assign the following weights $w\colon \mathcal{E} \to \mathbb{R}_{\geq 0}$:

$$\begin{aligned} w\bigl(\{\mathrm{F}_1, \mathrm{W}_1, \mathrm{R}_1\}, \text{electronics}\bigr) &= 1000,\\ w\bigl(\{\mathrm{F}_2, \mathrm{W}_2, \mathrm{R}_2\}, \text{electronics}\bigr) &= 800,\\ w\bigl(\{\mathrm{F}_1, \mathrm{W}_1, \mathrm{W}_2, \mathrm{R}_2\}, \text{furniture}\bigr) &= 500,\\ w\bigl(\{\mathrm{F}_2, \mathrm{W}_2, \mathrm{R}_1, \mathrm{R}_2\}, \text{food}\bigr) &= 1200.\end{aligned}$$

Thus

$$\mathcal{H} = \bigl(V,\ \mathcal{E},\ D,\ w\bigr)$$

is a weighted multidimensional hypernetwork modeling weekly shipment volumes of electronics, furniture, and food across factories, warehouses, and retailers in the supply chain.

Theorem 2.4 (Generalization of Weighted Multidimensional Networks and Hypernetworks)**.** (1)

Every weighted multidimensional network $\mathcal{N} = (V, E, D, w_{\mathcal{N}})$ (Definition 1.9) induces a weighted multidimensional hypernetwork

$$\mathcal{H}_{\mathcal{N}} \;=\; \Bigl(V,\ \{\,(\{u, v\}, d) \mid (u, v, d) \in E\},\ D,\ w'_{\mathcal{N}}\Bigr),$$

where $w'_{\mathcal{N}}(\{u, v\}, d) \;=\; w_{\mathcal{N}}(u, v, d)$. In particular, $\mathcal{H}_{\mathcal{N}}$ is exactly the hypernetwork whose hyperedges are the size-two edges of $\mathcal{N}$.

(2) *Every weighted hypernetwork* $\mathcal{H} = (V, \mathcal{E}, \{d_0\}, w_{\mathcal{H}})$ *with a single dimension* $D = \{d_0\}$ *arises as a special case of a weighted multidimensional hypernetwork by tagging all hyperedges with* d_0.

Hence the class of weighted multidimensional hypernetworks strictly generalizes both weighted multidimensional networks and weighted hypernetworks.

Proof. (1) Let $\mathcal{N} = (V, E, D, w_{\mathcal{N}})$ be a weighted multidimensional network, so

$$E \subseteq V \times V \times D, \quad w_{\mathcal{N}} \colon E \to \mathbb{R}.$$

Define

$$\mathcal{E} = \left\{ (\{u, v\}, d) \;\middle|\; (u, v, d) \in E \right\} \quad \subseteq \; (\mathcal{P}(V) \setminus \{\emptyset\}) \times D,$$

and set

$$w'_{\mathcal{N}}(\{u, v\}, d) \;=\; w_{\mathcal{N}}(u, v, d).$$

By construction $\mathcal{H}_{\mathcal{N}} = (V, \mathcal{E}, D, w'_{\mathcal{N}})$ satisfies the definition of a weighted multidimensional hypernetwork. The injective correspondence $(u, v, d) \mapsto (\{u, v\}, d)$ shows that no information is lost.

(2) Let $\mathcal{H} = (V, \mathcal{E}, \{d_0\}, w_{\mathcal{H}})$ be a weighted hypernetwork in the usual sense: $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ and $w_{\mathcal{H}} \colon \mathcal{E} \to \mathbb{R}$. Embed it into a weighted multidimensional hypernetwork by

$$\widetilde{\mathcal{E}} \;=\; \left\{ (e, d_0) \mid e \in \mathcal{E} \right\}, \qquad \widetilde{w}(e, d_0) = w_{\mathcal{H}}(e).$$

Again, this construction is bijective and respects all weights.

Since every weighted multidimensional network and every weighted hypernetwork injects into the class of weighted multidimensional hypernetworks, and there exist hyperedges of arbitrary cardinality or multiple dimensions that cannot be realized in the narrower settings, $\mathcal{H}$ indeed strictly generalizes both. □

Definition 2.5 (Dimension-Restricted Hypernetwork)**.** For each $d \in D$, define

$$\mathcal{E}_d \;=\; \{\, e \subseteq V : (e, d) \in \mathcal{E} \}, \quad w_d(e) = w(e, d).$$

Then $\mathcal{H}_d = (V, \mathcal{E}_d, w_d)$ is the *dimension-d restriction* of $\mathcal{H}$.

Theorem 2.6 (Restriction Yields a Weighted Hypernetwork)**.** *For each* $d \in D$*, the restricted structure* $\mathcal{H}_d$ *is itself a weighted hypernetwork in the usual sense.*

Proof. By definition $\mathcal{E}_d \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ and w_d assigns real weights to each hyperedge $e \in \mathcal{E}_d$. Thus $\mathcal{H}_d$ meets the requirements of a weighted hypernetwork. □

Definition 2.7 (Aggregated Hypernetwork)**.** Define the *aggregation* of $\mathcal{H}$ across all dimensions by

$$\mathcal{E}_{\text{agg}} = \{ e \subseteq V : \exists\, d \in D,\ (e, d) \in \mathcal{E} \}, \quad w_{\text{agg}}(e) = \sum_{\substack{d \in D \\ (e,d) \in \mathcal{E}}} w(e, d).$$

Then $\mathcal{H}_{\text{agg}} = (V, \mathcal{E}_{\text{agg}}, w_{\text{agg}})$ is called the *aggregated hypernetwork.*

Theorem 2.8 (Aggregation Preserves Hypernetwork Structure)**.** *$\mathcal{H}_{\text{agg}}$ is a weighted hypernetwork whose edge–weight function w_{agg} is well-defined and nonnegative.*

Proof. Clearly $\mathcal{E}_{\text{agg}} \subseteq \mathcal{P}(V)\setminus\{\emptyset\}$. Since each (e, d) contributes a real weight, the sum $w_{\text{agg}}(e) \geq 0$ is well-defined. Hence $\mathcal{H}_{\text{agg}}$ satisfies the definition of a weighted hypernetwork. □

Definition 2.9 (Primal Graph)**.** The *primal graph* $G = (V, A)$ of $\mathcal{H}$ is the simple graph with

$$A \;=\; \{\{u, v\} \subseteq V : \exists\, (e, d) \in \mathcal{E},\ \{u, v\} \subseteq e\}.$$

Theorem 2.10 (Primal Connectivity Criterion)**.** *The primal graph G is connected if and only if for every nontrivial partition $V = V_1 \cup V_2$, there exists some $(e, d) \in \mathcal{E}$ with $e \cap V_1 \neq \emptyset$ and $e \cap V_2 \neq \emptyset$.*

Proof. ($\Rightarrow$) If G is connected, no partition can isolate two sets without an edge crossing, so some hyperedge in $\mathcal{E}$ must bridge V_1 and V_2. ($\Leftarrow$) Conversely, if every partition is bridged by at least one hyperedge, the primal graph cannot split into disconnected components, hence G is connected. □

Definition 2.11 (Node Strength and Dimension Relevance)**.** For each node $v \in V$ define its *dimension-d strength*

$$s_d(v) = \sum_{\substack{(e,d) \in \mathcal{E} \\ v \in e}} w(e, d),$$

and its *total strength*

$$s(v) = \sum_{d \in D} s_d(v).$$

The *relevance* of dimension d at v is

$$R(v, d) = \frac{s_d(v)}{s(v)}.$$

Theorem 2.12 (Relevance Forms a Probability Distribution)**.** *For each $v \in V$ with $s(v) > 0$, the values $\{R(v, d)\}_{d \in D}$ satisfy $0 \leq R(v, d) \leq 1$ and $\sum_{d \in D} R(v, d) = 1$.*

Proof. By construction $s_d(v) \geq 0$ and $s(v) > 0$. Thus $0 \leq R(v,d) \leq 1$. Moreover

$$\sum_{d\in D} R(v,d) = \sum_{d\in D} \frac{s_d(v)}{s(v)} = \frac{1}{s(v)} \sum_{d\in D} s_d(v) = 1,$$

as required. □

3. Result: Weighted Multidimensional SuperHypernetwork

Weighted multidimensional superhypernetwork generalizes hypernetworks by using iterated powersets to create distinct nested supernodes and superedges across dimensions, with weights capturing hierarchical multiway relationship strengths.

Definition 3.1 (Weighted Multidimensional SuperHypernetwork)**.** Let V_0 be a finite *base set* and let D be a finite set of *dimensions*. Fix a nonnegative integer n. Denote by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \geq 0).$$

A *weighted multidimensional n-superhypernetwork* is a quadruple

$$\mathcal{S}^{(n)} \ = \ \big(V,\ \mathcal{E},\ D,\ w\big),$$

where

- $V \subseteq \mathcal{P}^n(V_0)$ is the set of *n-supernodes*;
- $\mathcal{E} \subseteq \big(\mathcal{P}(V) \setminus \{\emptyset\}\big) \times D$ is the set of *dimension-tagged n-superedges*, each $(e,d) \in \mathcal{E}$ consisting of a nonempty n-supernode subset $e \subseteq V$ in dimension $d \in D$;
- $w\colon \mathcal{E} \to \mathbb{R}$ assigns each (e,d) a real *weight* $w(e,d)$.

Example 3.2 (Regional Multimodal Logistics SuperHypernetwork)**.** Consider a logistics company serving five cities:

$$V_0 = \{A, B, C, D, E\}.$$

There are three transportation modes (dimensions):

$$D = \{\text{road},\ \text{rail},\ \text{air}\}.$$

We choose $n = 1$, so

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0) = \{\,\emptyset, \{A\}, \{B\}, \{C\}, \{D\}, \{E\}, \{A,B\}, \ldots, \{A,B,C,D,E\}\}.$$

Define the set of *1–supernodes* as two key regions:

$$V \ = \ \big\{\{A,B\},\ \{C,D,E\}\big\}.$$

Each region groups nearby cities into a single operational cluster.

Next, we model both intra- and inter-region shipping via dimension-tagged superedges:

$$\mathcal{E} = \big\{\big(\{A,B\}, \mathit{road}\big),\ \big(\{A,B\}, \mathit{rail}\big),\ \big(\{C,D,E\}, \mathit{air}\big),\ \big(\{\{A,B\},\{C,D,E\}\}, \mathit{rail}\big)\big\}.$$

- $(\{A, B\}, \text{road})$: shipments by truck within Region $\{A, B\}$.
- $(\{A, B\}, \text{rail})$: rail shipments within Region $\{A, B\}$.
- $(\{C, D, E\}, \text{air})$: air shipments within Region $\{C, D, E\}$.
- $(\{\{A, B\}, \{C, D, E\}\}, \text{rail})$: inter-region rail shipments connecting the two regions.

We assign daily shipment volumes (in tons) as weights:

$$w\big(\{A, B\}, \text{road}\big) = 500,$$
$$w\big(\{A, B\}, \text{rail}\big) = 300,$$
$$w\big(\{C, D, E\}, \text{air}\big) = 200,$$
$$w\big(\{\{A, B\}, \{C, D, E\}\}, \text{rail}\big) = 150.$$

Thus

$$\mathcal{S}^{(1)} = \big(V,\ \mathcal{E},\ D,\ w\big)$$

is a weighted multidimensional 1-superhypernetwork capturing both hierarchical clustering of cities into regions and the multimodal shipment volumes within and between those regions.

Example 3.3 (Corporate Communication SuperHypernetwork)**.** Consider a small company with four employees:

$$V_0 = \{\text{ Yuta, Hiroko, Shinya, Taka}\}.$$

There are three communication modes (dimensions):

$$D = \{\text{email},\ \text{meeting},\ \text{codeReview}\}.$$

We take $n = 2$. Then

$$\mathcal{P}^1(V_0) = \big\{\{\text{Yuta}, \text{Hiroko}\},\ \{\text{Shinya}, \text{Taka}\},\ \{\text{Hiroko}, \text{Shinya}\}, \dots\big\},$$

and

$$\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}^1(V_0)\big),$$

whose elements are *sets of teams*. Choose three key teams:

$$T_1 = \{\text{Yuta}, \text{Hiroko}\}, \quad T_2 = \{\text{Shinya}, \text{Taka}\}, \quad T_3 = \{\text{Hiroko}, \text{Shinya}\}.$$

Form two departments as 2-supernodes:

$$V = \big\{D_A = \{T_1, T_2\}, \quad D_B = \{T_2, T_3\}\big\} \ \subseteq\ \mathcal{P}^2(V_0).$$

Here D_A groups Team 1 and Team 2; D_B groups Team 2 and Team 3.

Next, define dimension-tagged 2-superedges (e, d) to capture both intra- and inter-department communications:

$$\mathcal{E} = \big\{(\ \{D_A\},\ \text{email}\ ), (\ \{D_A\},\ \text{meeting}\ ), (\ \{D_B\},\ \text{codeReview}\ ), (\ \{D_A, D_B\},\ \text{meeting}\ ), (\ \{D_A, D_B\},\ \text{email}\ )\big\}.$$

We assign monthly communication volumes as weights $w\colon \mathcal{E} \to \mathbb{R}_{\geq 0}$:

$$w(\{D_A\}, \text{email}) = 1200,$$
$$w(\{D_A\}, \text{meeting}) = 150,$$
$$w(\{D_B\}, \text{codeReview}) = 300,$$
$$w(\{D_A, D_B\}, \text{meeting}) = 40,$$
$$w(\{D_A, D_B\}, \text{email}) = 800.$$

Thus

$$\mathcal{S}^{(2)} = \big(V,\ \mathcal{E},\ D,\ w\big)$$

is a weighted multidimensional 2-superhypernetwork modeling:

- intra-department email and meetings within D_A,
- code-review activity within D_B,
- cross-department email and joint meetings between D_A and D_B.

Theorem 3.4 (Generalization of Earlier Models)**.** *The class of weighted multidimensional n-superhypernetworks (Definition 3.1) strictly generalizes each of the following:*

(1) Weighted multidimensional networks $\big(V, E, D, w_{\mathcal{N}}\big)$*, by taking $n = 0$ and restricting all hyperedges to cardinality 2.*

(2) Weighted multidimensional hypernetworks $\big(V, \mathcal{E}, D, w_{\mathcal{H}}\big)$*, by taking $n = 0$.*

(3) Superhypernetworks $\big(V, \mathcal{E}, w_{\mathcal{S}}\big)$ *(with a single trivial dimension), by taking $D = \{\star\}$.*

Proof. We exhibit embedding constructions for each case:

(1) From weighted multidimensional networks. Let $\mathcal{N} = (V, E, D, w_{\mathcal{N}})$ be a weighted multidimensional network, so

$$E \subseteq V \times V \times D, \quad w_{\mathcal{N}} : E \to \mathbb{R}.$$

Choose $n = 0$. Define

$$V' = V \subseteq \mathcal{P}^0(V_0), \qquad \mathcal{E}' = \big\{\, (\{u, v\}, d) \;\big|\; (u, v, d) \in E \big\} \;\subseteq\; \big(\mathcal{P}^0(V_0) \setminus \{\emptyset\}\big) \times D,$$

and set $w'(\{u, v\}, d) = w_{\mathcal{N}}(u, v, d)$. Then $\mathcal{S}' = \big(V', \mathcal{E}', D, w'\big)$ is a weighted multidimensional 0-superhypernetwork whose superedges are exactly the ordinary edges of $\mathcal{N}$.

(2) From weighted multidimensional hypernetworks. Let $\mathcal{H} = (V, \mathcal{E}, D, w_{\mathcal{H}})$ be a weighted multidimensional hypernetwork with $\mathcal{E} \subseteq (\mathcal{P}(V) \setminus \{\emptyset\}) \times D$. Again take $n = 0$ and set

$$V' = V \subseteq \mathcal{P}^0(V_0), \qquad \mathcal{E}' = \mathcal{E}, \quad w' = w_{\mathcal{H}}.$$

Then $\mathcal{S}' = (V', \mathcal{E}', D, w')$ is a weighted multidimensional 0-superhypernetwork identical to $\mathcal{H}$.

(3) From superhypernetworks. Let $\mathcal{S} = (V, \mathcal{E}, w_{\mathcal{S}})$ be an (unweighted or weighted) n-superhypernetwork without multiple dimensions. Embed it by setting

$$D' = \{\star\}, \quad V' = V \subseteq \mathcal{P}^n(V_0), \quad \mathcal{E}' = \{\, (e, \star) \mid e \in \mathcal{E}\}, \quad w'(e, \star) = w_{\mathcal{S}}(e).$$

Then $\big(V', \mathcal{E}', D', w'\big)$ is a weighted multidimensional n-superhypernetwork reproducing $\mathcal{S}$.

In each case, no information beyond simple relabeling is lost, and the restrictions (to $n = 0$, to edge-cardinality 2, or to a singleton D) exhibit that the general definition indeed encompasses all earlier models. Moreover, allowing arbitrary n, arbitrary dimension set D, and arbitrary superedge cardinalities strictly enlarges the modeling power, so the generalization is proper. □

Definition 3.5 (Dimension-Restricted SuperHypernetwork)**.** Let $\mathcal{S}^{(n)} = (V, \mathcal{E}, D, w)$ be a weighted multidimensional n-superhypernetwork. For each $d \in D$, define

$$\mathcal{E}_d \;=\; \{\, e \subseteq V \mid (e, d) \in \mathcal{E}\}, \quad w_d(e) \;=\; w(e, d).$$

Then $\mathcal{S}_d^{(n)} = (V, \mathcal{E}_d, w_d)$ is called the *dimension-d restriction* of $\mathcal{S}^{(n)}$.

Example 3.6 (Dimension-Restricted Email SuperHypernetwork)**.** Starting from the corporate communication superhypernetwork of Example 3.3,

$$\mathcal{S}^{(2)} = \big(V,\ \mathcal{E},\ D,\ w\big),$$

with

$$V = \{D_A, D_B\}, \quad D = \{\text{email, meeting, codeReview}\},$$

we form the *email* restriction by selecting

$$\mathcal{E}_{\text{email}} = \big\{\, e \subseteq V \mid (e, \text{email}) \in \mathcal{E}\big\} = \big\{\{D_A\}, \{D_A, D_B\}\big\},$$

and setting

$$w_{\text{email}}\big(\{D_A\}\big) = 1200, \qquad w_{\text{email}}\big(\{D_A, D_B\}\big) = 800.$$

Thus the dimension-restricted superhypernetwork

$$\mathcal{S}^{(2)}_{\text{email}} = \big(V,\ \mathcal{E}_{\text{email}},\ w_{\text{email}}\big)$$

captures *only* the email traffic:

- $\{D_A\}$ with weight 1200 represents department D_A's internal emails;
- $\{D_A, D_B\}$ with weight 800 represents cross-department emails between D_A and D_B.

Theorem 3.7 (Dimension Restriction)**.** *For each $d \in D$, the triple $\mathcal{S}_d^{(n)} = (V, \mathcal{E}_d, w_d)$ is itself a weighted n-superhypernetwork (with a single dimension), inheriting all structural and weight properties from $\mathcal{S}^{(n)}$.*

Proof. By construction, $\mathcal{E}_d \subseteq \mathcal{P}^n(V_0)$ and $w_d : \mathcal{E}_d \to \mathbb{R}$. All supernodes remain in $V \subseteq \mathcal{P}^n(V_0)$, and each superedge $e \in \mathcal{E}_d$ is nonempty. Therefore $\mathcal{S}_d^{(n)}$ satisfies Definition 3.1 with $D = \{d\}$.

□

Definition 3.8 (Primal Graph)**.** The *primal graph* $G(V, A)$ of $\mathcal{S}^{(n)} = (V, \mathcal{E}, D, w)$ is the simple graph whose vertex set is V and whose edge set

$$A = \big\{\{u, v\} \subseteq V : \exists\,(e, d) \in \mathcal{E},\ u \neq v,\ \{u, v\} \subseteq e\big\}.$$

Theorem 3.9 (Primal Connectivity)**.** *The primal graph G of $\mathcal{S}^{(n)}$ is connected if and only if for every nontrivial partition $V = V_1 \cup V_2$, there exists $(e, d) \in \mathcal{E}$ with $e \cap V_1 \neq \emptyset$ and $e \cap V_2 \neq \emptyset$.*

Proof. $\Rightarrow$: If G is connected, then no nontrivial partition can isolate vertices without an edge between them. Any separating partition would contradict connectivity. $\Leftarrow$: Conversely, if every bipartition is bridged by some superedge, then the primal graph cannot decompose into disconnected components, hence must be connected. □

Definition 3.10 (Supernode Strength and Relevance)**.** For $v \in V$, define its *total strength*

$$s(v) = \sum_{\substack{(e,d) \in \mathcal{E} \\ v \in e}} w(e, d),$$

and its *dimension-d strength*

$$s_d(v) = \sum_{\substack{(e,d) \in \mathcal{E} \\ v \in e}} w(e, d).$$

The *dimension relevance* of d at v is

$$R(v, d) = \frac{s_d(v)}{s(v)}, \quad d \in D.$$

Theorem 3.11 (Relevance Distribution)**.** *For each $v \in V$, the relevance values $\{R(v, d)\}_{d \in D}$ form a probability distribution:*

$$R(v, d) \in [0, 1], \quad \sum_{d \in D} R(v, d) = 1.$$

Proof. By definition $s_d(v) \geq 0$ and $s(v) = \sum_d s_d(v) > 0$ whenever v lies in at least one superedge. Hence $0 \leq R(v, d) \leq 1$. Moreover,

$$\sum_{d \in D} R(v, d) = \sum_{d \in D} \frac{s_d(v)}{s(v)} = \frac{1}{s(v)} \sum_{d \in D} s_d(v) = 1.$$

□

4. Conclusion

In this paper, we extended these ideas using hypergraphs and superhypergraphs to introduce and formalize *multidimensional hypernetworks* and *multidimensional superhypernetworks*. In future research, it is expected that further extensions will be developed based on Fuzzy Sets [32, 33], Intuitionistic Fuzzy Sets [34], Neutrosophic Sets [35–37], Quadri-Partitioned Neutrosophic Sets [38], and Plithogenic Sets [39–41].

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

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Article

التوزيع الثنائي النيتروسوفكي المحسّن

Refined Neutrosophic Binomial Distribution

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الملخص:

نقدم لأول مرة مقارنة شـــاملة بين التوزيع الثنائي الكلاســـيكي، والتوزيع الثنائي النيتروســـوفكي، والتوزيع الثنائي النيتروسوفكي المحسن. وقد توصّلنا إلى أن التوزيع الثنائي النيتروسوفكي المحسن يُعَدّ نموذجًا بالغ القوة في تحليل الظواهر تحت ظروف عدم يقين متعددة المصـدر، إذ يُمثّل أداةً إحصـائية متقدمة تُتح تحليل العمليات العشوائية ذات المخرجات الثنائية في سياقات يكون فيها عدم اليقين السمة المهيمنة.

الكلمات المفتاحية:

التوزيع النيتروسوفكي الثنائي المحسن، عدم اليقين المحسن، الأعداد النيتروسوفكية المحسنة.

1. مقدمة:

ندرك أهمية التوزيعات الاحتمالية في علم الإحصاء والاحتمالات وفي تطبيقاتها العملية التي تُمكّن من الوصول إلى نتائج يمكن الاستناد إليها في اتخاذ القرارات واقتراح الحلول للمشكلات المعاصرة. ونظرًا لتعدّد صور عدم اليقين الناشئة عن المتغيرات المحيطة بالتجارب والقضايا الحديثة ـ مثل تعدد مصادر البيانات واختلاف دقة الأجهزة المستخدمة في تحليلها ـ أصبح من الضروري البحث عن منهج رياضي أكثر واقعية للتعامل مع عدم اليقين المكرر الناتج عن مصادر متعددة. وقد تبيّن لنا أن المنطق النيتروسوفكي المحسن وصِيَغه يوفران إطارًا أمثل للتعبير عن عدم اليقين متعدد المصادر، بما يتيح نتائج أكثر دقة ورؤية أوضح للمشكلة قيد الدراسة، وقد قدم أحمد خطيب، تعميم للايزومترية النيتروسوفكية AH إلى نظام الأعداد النيتروسوفكية المحسنة، وكيفية التعامل مع العديد من الأسطح الهندسية النيتروسوفكية المحسنة [9]، وقد تم توسيع تعريف المتغيرات العشوائية النيتروسوفكية إلى المتغيرات العشوائية النيتروسوفكية المحسنة، وتقديم مفاهيم جديدة لتوزيع الاحتمالات النيتروسوفكية المحسنة والتباين النيتروسوفكي المحسن والانحراف المعياري النيتروسوفكي المحسن والربيعات النيتروسوفكية المحسنة [9]، وقد قُدمت دراسة في بناء البنى الجبرية النيتروسوفكية المحسنة [11]، وقدم العديد من الباحثين الحلقات النيتروسوفكية المحسنة والحلقات الفرعية النيتروسوفكية الحسنة، وخصائصها الأساسية [12-13]، وتُسهم هذه الدراسة في تقديم مقارنة بين التوزيع الثنائي الكلاسيكي والتوزيع الثنائي النيتروسوفكي والتوزيع

الثنائي النيتروسوفكي المحسن، مع تطبيق عملي لبيان أهمية تطبيق الأعداد النيتروسوفكية المحسنة في حالات عدم اليقين متعدد المصادر.

2. الأساسيات:

تعريف 1.2:[12] النيتروسوفك (*Neutrosophic*) :هي نظرية طورها العالم فلورنتين سماراندكه لتمثيل البيانات التي تحمل الغموض وعدم اليقين، تمثل كل قيمة نيتروسوفكية بثلاثة مكونات مستقلة:

- T (*Truth*): درجة الصحة أو الانتماء.
- I (*Indeterminacy*): درجة الغموض أو عدم اليقين.
- F (*Falsity*): درجة الخطأ أو عدم الانتماء.

في النيتروسوفك المحسن، تُعرَف بأنَ هذه الدرجات نفسها يمكن أن تكون غير مؤكدة، من المرتبة الثانية تعني أنَ لدينا مستوى إضافيًا من عدم اليقين يحيط بقيم T, I, F الأساسية.
أي أن العدد النيتروسوفكي المحسن لا يمثل بـ (T, I, F) بسيطة، بل يمثل بمجموعة من هذه الثلاثيات، أو بفواصل زمنية لها، تعبر عن القيم "الممكنة" لـ F و I و T.
حيث أن: $1 \geq F$ ، I ، $T \geq 0$ وتحقق العلاقة: $0 \leq T + I + F \leq 3$
أي يمكن تقسيم كل مكون من هذه المكونات إلى جزأين أكثر دقة، فبدلاً من درجة واحدة لعدم اليقين، قد يكون لدينا نوعان منه.

وتكون الصيغة الرياضية للعدد النيتروسوفكي المحسن:

$$(((T_1, T_2), (I_1, I_2), (F_1, F_2))$$

حيث أن:

- T_1, T_2: نوعان مختلفان من الصحة (مثلاً: صحة بناءً على دليل مباشر، وصحة بناءً على استدلال).
- I_1, I_2: نوعان مختلفان من عدم اليقين (مثلاً: عدم يقين بسبب نقص المعلومات، وعدم يقين بسبب تناقض المعلومات).
- F_1, F_2: نوعان مختلفان من الخطأ (مثلاً: خطأ مباشر، وخطأ محتمل).

تعريف 2.2:[9] العدد الحقيقي النيتروسوفيكي المحسن: يُعرف العدد الحقيقي النيتروسوفكي المحسن بالشكل $a_0 + a_1I_1 + a_2I_2$ حيث أن $a_0, a_1, a_2 \in R$ ويمكن كتابته بالشكل المكافىء $(a_0، a_1I_1، a_2I_2)$ وتحقق:

$$I_1 . I_1 = I_1 \quad ، \quad I_2 . I_2 = I_2 \quad ، \quad I_1 . I_2 = I_2 I_1 = I_1$$

أي أنه في النهج النيتروسوفيكي المحسن، يمكن تقسيم كل مركبة من مركبات العدد النيتروسوفكي الكلاسيكي إلى مركبتين، ويمكن كتابة العدد النيتروسوفكي المحسن بالشكل التالي أيضا:

$$\{(T_1, I_1, F_1) , (T_2, I_2, F_2)\}$$

مثال توضيحي 3.2:

في هذا المثال نوضح الفرق بين النهج الاحتمالي الكلاسيكي والنيتروسوفكي البسيط والنيتروسوفيكي المحسن، لنفترض أننا نقيم احتمالية نجاح تجربة ما.

- النهج الكلاسيكي: الاحتمال = 0.7
- النهج النيتروسوفيكي العادي الذي يكون فيه عدم اليقين من مصدر واحد يُمثل بـ
$(T = 0.7, I = 0.2, F = 0.3)$، أي أننا متأكدون بنسبة70% ، وغير متأكدين بنسبة 20%، ونعتقد بالخطأ بنسبة 30%.
- النهج النيتروسوفيكي المحسن: لنفترض أننا غير متأكدين حتى من درجات T, I, F نفسها، وذلك بسبب تعدد مصادر عدم اليقين، وقد نمثلها كالتالي:
  - T يمكن أن تكون 0.6 أو 0.7 أو 0.8.
  - I يمكن أن تكون 0.1 أو 0.2.
  - F يمكن أن تكون 0.2 أو 0.3.

هذا التمثيل يعكس شكوكًا أعمق في نموذج عدم اليقين نفسه، مما يتيح إعطاء فرصة أوسع للحصول على نتائج أكثر واقعية للمشكلة المدروسة.

3. التوزيع الثنائي الكلاسيكي (*Binomial Distribution*) :

التوزيع الثنائي يصف عدد مرات النجاح k في n محاولة مستقلة، حيث احتمال النجاح في كل محاولة هو p ثابت.
دالة الكتلة الاحتمالية:

$$P(X = k) = C(n, k). P^k. (1 - p)^{(n-k)}$$

حيث $C(n, k)$ هو المعامل الثنائي.
المشكلة هنا أن p قيمة واحدة وحيدة، مما لا يناسب حالات عدم اليقين المعقدة.

4. التوزيع الثنائي النيتروسوفكي الكلاسيكي (*Neutrosophic Binomial Distribution*) :

هو توزيع ثنائي كلاسيكي تم تمديده نيتروسوفكياً من قبل فلورنتين سمارانداكه، والذي يعني أنه يوجد بعض اللاتحديد المتعلق بالتجربة الاحتمالية.
حيث هناك العديد من ظواهر الحياة تكون النتيجة الممكنة لها إما نجاحاً (S) أو فشلاً (F) أو اللاتحديد (I).
عند تكرار التجربة عدد ثابت من المرات فإننا نحصل في كل مرة على نجاح أو فشل أو لا تحديد (نتيجة كل محاولة مستقلة عن نتيجة المحاولة الأخرى) وفرصة الحصول على (S) أو(F) أو (I) متساوية.
نعرف المتغير العشوائي النيتروسوفكي X الذي يمثل عدد مرات الحصول على النجاح عند اجراء $(n \geq 1)$ تجربة.
التوزيع الاحتمالي النيتروسوفكي لـ X يدعى التوزيع الاحتمالي الثنائي النيتروسوفكي، وننوه إلى مايلي:

1- من المهم أن يكون في التجارب (n تجربة) التي نقوم بها نوع واحد من اللاتحديد.

2- الحصول على نتيجة في كل تجربة نقوم بها من التجارب n التي لدينا تعني الحصول على لاتحديد لكامل مجموعة التجارب n.

3- الحصول على اللاتحديد (I) كنتيجة من أي تجربة لاتعني لاتحديد لكامل مجموعة التجارب n.

- لكن قد نحصل على لا تحديد في بعض التجارب والتحديد (النجاح أو الفشل) في تجارب أخرى وهنا تنشأ المسألة التي نحتاج لحلها.

من أجل ذلك سنعرف ما يلي:

Th: وهي عدد التجارب التي نتيجتها غير محددة (لاتحديد) مع

$th \in (0,1,2, \dots\dots, n)$ حيث n عدد التجارب.

$P(S)$: هو احتمال الحصول على نتيجة نجاح في تجربة معينة.

$P(F)$: هو احتمال الحصول على نتيجة فشل في تجربة معينة.

$P(I)$: هو احتمال الحصول على نتيجة لاتحديد في تجربة معينة.

بالتالي يكون لدينا احتمال النيتروسوفيك للمتغير العشوائي الثنائي النيتروسوفيكي X الذي يمثل عدد مرات الحصول على نتيجة النجاح في n تجربة بالشكل:

$$NP\ (X) = \ (T_X, I_X, F_X) \dots\dots \ (1.4)$$

حيث يكون:

$$T_X = \binom{n}{x} (p(s))^x \sum_{k=0}^{th} \binom{n-k}{k} (p(I))^k (P(F))^{n-x-k}$$

$$= \frac{n!}{x!(n-x)!} (p(s))^x \sum_{k=0}^{th} \frac{(n-x)!}{k!(n-x-k)!} (p(I))^k (P(F))^{n-x-k}$$

$$= \frac{n!}{x!} (p(s))^x \sum_{k=0}^{th} \frac{(p(I))^k (P(F))^{n-x-k}}{k!(n-x-k)!} \dots\dots\dots.(2.4)$$

وبشكل مشابه:

$$F_X = \sum_{\substack{y=0 \\ y\neq x}}^{n} T_y = \sum_{\substack{y=0 \\ y\neq x}}^{n} \frac{n!}{y!}\ (p(s))^y \frac{(p(I))^k (P(F))^{n-y-k}}{k!(n-y-k)!} \dots\dots (3.4)$$

والآن نجد أن:

$$I_X = \sum_{z=th+1}^{n} \binom{n}{z} \left(p(I)\right)^z (\sum_{k=0}^{n-z} \binom{n-z}{k} \left(p(s)\right)^k \left(P(F)\right)^{n-z-k})$$

$$= \sum_{z=th+1}^{n} \frac{n!}{z!(n-z)!} \left(p(I)\right)^z (\sum_{k=0}^{n-z} \frac{(n-z)!}{k!(n-z-k)!} \left(p(s)\right)^k \left(P(F)\right)^{n-z-k})$$

$$= \sum_{z=th+1}^{n} \frac{n!}{z!} \left(p(I)\right)^z \left(\sum_{k=0}^{n-z} \frac{\left(p(s)\right)^k \left(P(F)\right)^{n-z-k}}{k!(n-z-k)!}\right) \dots\dots.(4.4)$$

حيث أن:

T_X : احتمال النجاح x.

F_X: احتمال الفشل.

I_X: احتمال اللاتحديد z.

$(n-x)\ , (n-y)$:عدد مرات الفشل واللاتحديد.

5. ملاحظة:

لدينا:

$$T_X + \ I_X + F_X = (p(x) + p(I) + p(F))^n \dots\dots (5.4)$$

- في أغلب التصنيفات يكون: $p(x) + p(I) + p(F) = 1$ وفي هذه الحالة ندعو الاحتمال بالاحتمال التام.

- وفي الحالة: $0 \leq p(S)+p(I)+p(F)<1$ ندعو الاحتمال بالاحتمال غير التام (حيث يوجد تناقض بالمعلومات).

- وفي حالة: $1<p(S)+p(I)+p(F) \leq 3$ ندعو الاحتمال بالاحتمال غير التوافقي (حيث يوجد معلومات متناقضة)، كما في المثال (6).

6. مثال:

في متجر لبيع الساعات هناك احتمال بأن 80% من الساعات المباعة لها شاشة عرض رقمية و 10% من الساعات المباعة لها شاشة عرض تناظرية لكن هناك عدد من الساعات المباعة لا يعلم صاحب المتجر نوعها، سأل عنها، لكنه استطاع فقط أن يقدر نسبة الساعات المباعة غير المعروف نوعها بـ 20% من الساعات المباعة.

ما احتمال أن تكون أول ساعتين من بين الساعات الخمس التي ستباع لاحقاً لها شاشة عرض تناظرية بفرض أن $(Th=2)$.

الحل:

نفرض أن x متغير عشوائي نيتروسوفكي يمثل عدد الساعات التي لها شاشة عرض تناظرية من بين الساعات الخمس التي ستباع لاحقاً.

ولدينا من الفرض:

$p(S) = p$ (ساعة مباعة ذات شاشة عرض تناظرية) $= 0.1$

$p(F) = p$ (ساعة مباعة ذات شاشة عرض رقمية) $= 0.8$

$p(I) = p$ (ساعة مباعة غير محددة) $= 0.2$

نلاحظ من الفرضيات أننا أمام احتمال غير توافقي وذلك بسبب تضارب المعلومات التي تأتي حول عدد الساعات المباعة من المدير ومساعده ويقدر كل منهما عدد الساعات بشكل مستقل عن الآخر فنلاحظ أن:

$$p(S)+p(I)+p(F) = 0.1+0.2+0.8 = 1.1 > 1$$

من الفرضيات نجد أن X يتوزع وفق التوزع الثنائي النيتروسوفيكي احتماله:

$$NP\ (x)=(T_x, I_x, F_x)$$

$$T_x=\frac{n!}{x!}(p(s))^x \sum_{k=0}^{th} \frac{(p(I))^k (p(F))^{n-x-k}}{k!(n-x-k)!}=\frac{5!}{x!}(0.1)^x \sum_{k=0}^{2} \frac{(0.2)^k (0.8)^{5-x-k}}{k!(5-x-k)!}$$

بحيث أن $x = 0,1,2,3,4,5$.

لنسحب احتمال أن تكون أول ساعتين من الساعات الخمس التي ستباع لاحقاً لها شاشة عرض تناظرية

$$NP\ (X=2)=(T_2, I_2, F_2)$$

حيث أن: (T_2, I_2, F_2) تُمثل درجات الصحة وعدم اليقين والخطأ لمبيع أول ساعتين من أصل خمس ساعات.

$$T_2= \frac{5!}{2!}(0.1)^2 \sum_{k=0}^{2} \frac{(0.2)^k (0.8)^{5-2-k}}{k!(5-2-k)!}$$

$$=\frac{5!}{2!}(0.1)^2\left[\frac{(0.2)^0(0.8)^3}{0!3!}+\frac{(0.2)(0.8)^2}{1!2!}+\frac{(0.2)^2(0.8)}{2!1!}\right]=0.0992$$

$$I_2=\sum_{z=th+1}^{n} \frac{n!}{z!}\left(p(I)\right)^z\left(\sum_{k=0}^{n-z} \frac{\left(p(s)\right)^k\left(P(F)\right)^{n-z-k}}{k!(n-z-k)!}\right)$$

$$= \sum_{z=3}^{5} \frac{5!}{z!} (0.2)^z (\sum_{k=0}^{5-z} \frac{(0.1)^k (0.8)^{5-z-k}}{k!(5-z-k)!})$$

$$= \frac{5!}{3!} (0.2)^3 \left(\sum_{k=0}^{2} \frac{(0.1)^k (0.8)^{2-k}}{k!(2-k)!}\right) + \frac{5!}{4!} (0.2)^4 (\sum_{k=0}^{1} \frac{(0.1)^k (0.8)^{1-k}}{k!(1-k)!})$$

$$+ \frac{5!}{5!} (0.2)^5 \left(\sum_{k=0}^{0} \frac{(0.1)^k (0.8)^{-k}}{k!(-k)!}\right) = 20(0.2)^3 [\frac{(0.1)^0 (0.8)^2}{0!2!} + \frac{(0.1)(0.8)}{1!1!} + \frac{(0.1)^2 (0.8)^0}{2!0!}] +$$

$$5(0.2)^4 [\frac{(0.1)^0 (0.8)^1}{0!1!} + \frac{(0.1)(0.8)^0}{1!0!}] + (0.2)^5 [\frac{(0.1)^0 (0.8)^0}{0!0!}] = 0.07232$$

نستطيع حساب F_2 بطريقة أسهل من استخدام صيغته في العلاقة (3.4) من خلال الاستفادة من العلاقة (5.4):

$$F_2 = (P(S) + P(I) + P(F))^2 - T_2 - I_2 = (0.1 + 0.2 + 0.8)^2 - 0.0992 - 0.07232 = 0.43899$$

وبالتالي يكون:

$$NP(X = 2) = (0.0992 ,0.07232 , 0.43899).$$

7. التوزيع الثنائي النيتروسوفكي المحسّن (*Refined Neutrosophic Binomial Distribution*) :

لدمج التوزيع الثنائي مع الأعداد النيتروسوفكية المحسنة، لا بدّ من أن تتضمن التجارب (ذات العدد n) نوعين من اللاتحديد. تقوم الفكرة على استبدال الاحتمال الثابت P بعدد نيتروسوفكي محسن من المرتبة الثانية، نرمز له بـ NP_2.

$$NP_2 = \{ (T_i, I_i , F_i) \} \text{ أو } NP_2 = ([T_L, T_U], [I_L, I_U], [F_L, F_U])$$

حيث تمثل $[T_L, T_U]$، على سبيل المثال، الحد الأدنى والأعلى الممكنة لقيمة الحقيقة.

عندما نستخدم NP_2 في التوزيع الثنائي، نحصل على توزيع ثنائي نيتروسوفيكي محسن.

1) المتغير العشوائي: لنفترض أن $\tilde{V}$ هو المتغير العشوائي الذي يمثل عدد "النجاحات" في n محاولة، حيث كل محاولة لها نتيجة نيتروسوفكية مكررة.

2) دالة الاحتمال النيتروسوفكي المحسن: بدلاً من الحصول على احتمال واحد $P(X = k)$، نحصل على مجموعة من القيم النيتروسوفيكية لكل k. يتم حسابها من خلال تعميم الصيغة الثنائية.

إذا كان NP_2 ممثلًا بعدد m من الثلاثيات (T, I, F) المحتملة، فإن دالة الاحتمال الثنائي النيتروسوفكي للقيمة k تُعطى بالصيغة التالية:

$$NP_2(\tilde{V} = k) = \{ (C(n,k)(T_i)^k \ (F_i)^{n-k}, C(n,k) \ (I_i)^k (1 - I_i)^{n-k}, C(n,k) \ (F_i)^k \ (T_i)^{n-k})\}$$

لجميع القيم الممكنة لـ i.

حيث أن: $0 \le T_i, F_i, I_i \le 1$ و $0 \le T_i + I_i + F_i \le 1$

تفسير المكونات:

- مكون الحقيقة$(Truth)$:

$$C(n,k)(T_i)^k \ (F_i)^{n-k}$$

يعتمد على درجات الحقيقة للنجاح (T_i) ودرجات الخطأ للفشل (F_i). يعبر عن مدى "صحة" أو "إمكانية" حدوث k نجاح.

- مكون الغموض $(Indeterminacy)$:

$$C(n,k) \ (I_i)^k (1 - I_i)^{n-k}$$

يعتمد على درجات الغموض في كل محاولة، ويعبر عن مدى "عدم التأكد" من حدوث k نجاح بالضبط.

- مكون الخطأ($Falsity$):

$$C(n,k)\ (F_i)^k\ \ (T_i)^{n-k})$$

يعتمد على درجات الخطأ للنجاح (F_i) ودرجات الحقيقة للفشل (T_i). يعبر عن مدى "خطأ" أو "استحالة" حدوث k نجاح.

3) النتيجة: بدلاً من الحصول على منحنى توزيع واحد، نحصل على "غيمة" أو "حزمة" من المنحنيات التوزيعية، كل منها مرتبط بثلاثية (T_i, I_i, F_i)محتملة لـ NP_2. هذا يعطينا تمثيلًا غنيًا ومتعدد الأبعاد لعدم اليقين.

8. مثال تطبيقي:

لدى قيام طبيب بتقييم فعالية دواء جديد، لديه شكوك كبيرة حول فعالية الدواء، فهو يعتقد أن:

- نسبة الشفاء (T) قد تكون 0.6 أو 0.7.
- غير متأكد من تشخيصه (I) بدرجة 0.2 أو 0.3.
- يخشى أن يكون الدواء غير فعال (F) بنسبة 0.1 أو 0.2.

أي أنNP_2 له حالتان فقط حسب حالات الشك الموجودة لدى الطبيب:

- الحالة الأولى: $(T_1 = 0.7, I_1 = 0.3, F_1 = 0.2)$
- الحالة الثانية: $(T_2 = 0.6, I_2 = 0.2, F_2 = 0.1)$

نريد معرفة احتمالية شفاء (3) مرضى بالضبط من أصل (5)أي أن (n=5, k=3).

الحساب لكل حالة:

الحالة الأولى:

$$(T_1 = 0.7, I_1 = 0.3, F_1 = 0.2)$$

- $T_{1_3} = C(5,3) \times (0.7)^3 \times (0.2)^2 = 10 \times 0.343 \times 0.04 \approx 0.1372$
- $I_{1_3} = C(5,3) \times (0.3)^3 \times (0.7)^2 = 10 \times 0.027 \times 0.49 \approx 0.1323$
- $F_{1_3} = C(5,3) \times (0.2)^3 \times (0.7)^2 = 10 \times 0.008 \times 0.49 \approx 0.0392$

الحالة الثانية:

$$(T_2 = 0.6, I_2 = 0.2, F_2 = 0.1)$$

- $T_{2_3} = C(5,3) \times (0.6)^3 \times (0.1)^2 = 10 \times 0.216 \times 0.01 \approx 0.0216$
- $I_{2_3} = C(5,3) \times (0.2)^3 \times (0.8)^2 = 10 \times 0.008 \times 0.64 \approx 0.0512$
- $F_{2_3} = C(5,3) \times (0.1)^3 \times (0.6)^2 = 10 \times 0.001 \times 0.36 \approx 0.0036$

بعد الحسابات السابقة نحصل على النتيجة النيتروسوفيكية المحسنة لـ $NP_2(\tilde{V} = 3)$

يمكن تمثيلها بالمجموعة:$\{ (0.137, 0.132, 0.039), (0.022, 0.051, 0.004) \}$.

التفسير:

هذا يعني أنه بالنسبة لـ 3 نجاحات:

- إحدى التقديرات (بناءً على الحالة الأولى) تشير إلى إمكانية عالية للصحة (0.137) ودرجة عالية من الغموض (0.132).

- التقدير الآخر (بناءً على الحالة الثانية) يشير إلى إمكانية أقل للصحة (0.022) ودرجة أقل من الغموض (0.051).

- مكون الخطأ منخفض في كلا الحالتين.

بدلاً من رقم واحد مثل 0.132 (كما في التوزيع الثنائي العادي)، حصلنا على نطاق من الإمكانيات يعكس الشكوك الأساسية في النموذج نفسه.

9. الاستنتاجات:

1. النموذج النيتروسوفيكي المحسن قوي للغاية لتحليل الظواهر في ظل ظروف عدم يقين عميقة، حيث تكون المعلمات الإحصائية التقليدية (مثل الاحتمال p) غير معروفة بدقة أو غامضة بطبيعتها.

2. استخدام التوزيع الثنائي النيتروسوفكي المحسن في اتخاذ القرارات المعقدة في التمويل (تقييم مخاطر الاستثمارات الغامضة)، والطب (تشخيص الأمراض ذات الأعراض المتداخلة).

3. التحليل الإحصائي النيتروسوفكي المحسن للبيانات النوعية مثل تحليل استبيانات الرأي حيث تكون الإجابات "موافق بشروط" أو "غير متأكد".

4. فعالية استخدام التوزيع النيتروسوفيكي المحسن في مراقبة الجودة في البيئات غير المؤكدة عندما تكون معايير "الناجح/الفاشل" نفسها غير واضحة المعالم.

5. من التحديات في استخدام البيانات النيتروسوفكية، التعقيد الحسابي، حيث يزداد عدد الحسابات بشكل كبير مع زيادة عدد القيم الممكنة في NP ، وذلك عند تعدد مصادر عدم التحديد لأكثر من مصدرين.

باختصار، يُعَدّ التوزيع الثنائي النيتروسوفكي المحسن أداةً إحصائية متقدمة تمكّننا من تمثيل وتحليل العمليات العشوائية ذات المخرجات الثنائية في بيئات يكون فيها عدم اليقين هو القاعدة لا الاستثناء.

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Soil Microbial HyperNetworks and SuperHyperNetworks in Plant and Soil Sciences

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Abstract. Graph theory underpins many applications across science and engineering. Hypergraphs generalize graphs by allowing hyperedges to join any number of vertices, while superhypergraphs further extend this idea by layering iterated powersets to capture hierarchical, self-referential connections. A Soil Microbial Network is a graph where microbial taxa are represented as nodes, and edges indicate statistical associations or interactions among these taxa. In this paper, we provide a mathematical definition of the Soil Microbial Network and investigate its generalizations: the Soil Microbial HyperNetwork, formulated within the framework of HyperGraphs, and the Soil Microbial SuperHyperNetwork, formulated within the framework of SuperHyperGraphs.

Keywords: Superhypergraph, Hypergraph, Soil Microbial Network

1. Preliminaries

We briefly recall the basic notions needed in the remainder of the paper. Throughout, all sets are assumed to be finite.

1.1. *Power Set and n-th Power Set*

The power set of a set S is the collection of all subsets of S, including both the empty set and S itself. The n-th power set of S is obtained by iteratively applying the power set operation n times, starting from S [1,2]. The notion of the n-th power set was introduced by F. Smarandache. The formal definitions and concrete examples of this concept are given below.

Definition 1.1 (Universal Set)**.** Let U be a set containing all elements under consideration. Throughout, every set S is assumed to satisfy $S \subseteq U$.

Definition 1.2 (Base Set)**.** A *base set* S is any subset $S \subseteq U$ from which further constructions—such as powersets and hyperstructures—are formed.

Definition 1.3 (Power Set)**.** [3,4] The *power set* of S, denoted $\mathcal{P}(S)$, is the collection of all subsets of S:

$$\mathcal{P}(S) \;=\; \{\, X \mid X \subseteq S\}.$$

Definition 1.4 (Iterated Power Set)**.** [5,6] For each integer $n \geq 1$, define the n-fold iterated power set of S by

$$\mathcal{P}^1(S) = \mathcal{P}(S),$$
$$\mathcal{P}^{k+1}(S) = \mathcal{P}\big(\mathcal{P}^k(S)\big) \quad (k \geq 1).$$

Equivalently, one may write $P_n(S) = \mathcal{P}^n(S)$.

Definition 1.5 (Nonempty Iterated Power Set)**.** [5,7] Define the nonempty iterated power set by

$$\mathcal{P}_1^*(S) = \mathcal{P}(S) \setminus \{\varnothing\},$$
$$\mathcal{P}_{k+1}^*(S) = \mathcal{P}^*\big(\mathcal{P}_k^*(S)\big) \quad (k \geq 1),$$

where $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\varnothing\}$ for any set X.

Example 1.6 (Plant organs; a concrete nonempty iterated power set for $|S| = 3$)**.** Let the plant-organ set be $S = \{\mathrm{Root}, \mathrm{Stem}, \mathrm{Leaf}\}$. Then the nonempty power set is

$$\mathcal{P}_1^*(S) = \big\{\{\mathrm{Root}\},\ \{\mathrm{Stem}\},\ \{\mathrm{Leaf}\},$$
$$\{\mathrm{Root}, \mathrm{Stem}\},\ \{\mathrm{Root}, \mathrm{Leaf}\},\ \{\mathrm{Stem}, \mathrm{Leaf}\},\ \{\mathrm{Root}, \mathrm{Stem}, \mathrm{Leaf}\}\big\},$$

so $|\mathcal{P}_1^*(S)| = 2^3 - 1 = 7$. By definition, $\mathcal{P}_2^*(S) = \mathcal{P}(\mathcal{P}_1^*(S)) \setminus \{\varnothing\}$ has $|\mathcal{P}_2^*(S)| = 2^{\,7} - 1 = 127$. Two explicit elements of $\mathcal{P}_2^*(S)$ are, for instance,

$$A_1 = \big\{\{\mathrm{Root}\},\ \{\mathrm{Stem}\}\big\},$$
$$A_2 = \big\{\{\mathrm{Root}, \mathrm{Leaf}\},\ \{\mathrm{Stem}, \mathrm{Leaf}\},\ \{\mathrm{Root}, \mathrm{Stem}, \mathrm{Leaf}\}\big\}.$$

Here each member of A_1 and A_2 is a nonempty subset of S, hence $A_1, A_2 \subseteq \mathcal{P}_1^*(S)$ and both $A_1, A_2 \in \mathcal{P}_2^*(S)$. Interpretation in plant/soil science: elements of $\mathcal{P}_1^*(S)$ represent concrete organ-combinations (e.g., tissues sampled together), whereas elements of $\mathcal{P}_2^*(S)$ are *collections of such combinations* (e.g., sets of sampling schemes or treatment groups).

Example 1.7 (Soil microbial guilds; a concrete nonempty iterated power set for $|S| = 4$)**.** Let

$$S = \{\mathrm{Bact}, \mathrm{Fungi}, \mathrm{Arch}, \mathrm{Prot}\}$$

denote bacterial, fungal, archaeal, and protist guilds. Then

$$\begin{aligned}\mathcal{P}_1^*(S) = \{&\{\mathrm{Bact}\},\ \{\mathrm{Fungi}\},\ \{\mathrm{Arch}\},\ \{\mathrm{Prot}\},\\ &\{\mathrm{Bact}, \mathrm{Fungi}\},\ \{\mathrm{Bact}, \mathrm{Arch}\},\ \{\mathrm{Bact}, \mathrm{Prot}\},\ \{\mathrm{Fungi}, \mathrm{Arch}\},\ \{\mathrm{Fungi}, \mathrm{Prot}\},\ \{\mathrm{Arch}, \mathrm{Prot}\},\\ &\{\mathrm{Bact}, \mathrm{Fungi}, \mathrm{Arch}\},\ \{\mathrm{Bact}, \mathrm{Fungi}, \mathrm{Prot}\},\ \{\mathrm{Bact}, \mathrm{Arch}, \mathrm{Prot}\},\ \{\mathrm{Fungi}, \mathrm{Arch}, \mathrm{Prot}\},\\ &\{\mathrm{Bact}, \mathrm{Fungi}, \mathrm{Arch}, \mathrm{Prot}\}\},\end{aligned}$$

so $|\mathcal{P}_1^*(S)| = 2^4 - 1 = 15$ and $|\mathcal{P}_2^*(S)| = 2^{15} - 1 = 32767$. Two explicit elements of $\mathcal{P}_2^*(S)$ are

$$B_1 = \big\{\{\mathrm{Bact}\},\ \{\mathrm{Fungi}\},\ \{\mathrm{Arch}\}\big\},$$

$$B_2 = \big\{\{\mathrm{Bact}, \mathrm{Fungi}\},\ \{\mathrm{Bact}, \mathrm{Arch}\},\ \{\mathrm{Fungi}, \mathrm{Arch}\},\ \{\mathrm{Bact}, \mathrm{Fungi}, \mathrm{Arch}\}\big\}.$$

Again, every element listed inside B_1 and B_2 is a nonempty subset of S, hence $B_1, B_2 \subseteq \mathcal{P}_1^*(S)$, so $B_1, B_2 \in \mathcal{P}_2^*(S)$. Domain interpretation: members of $\mathcal{P}_1^*(S)$ are concrete microbial assemblages (e.g., $\{\mathrm{Bact}, \mathrm{Fungi}\}$ co-occurring), while members of $\mathcal{P}_2^*(S)$ are *sets of assemblages* (e.g., groups of communities considered jointly in a soil health index or meta-analysis).

As supplementary information, the Comparison of a set, its power set, and its iterated power set is presented in Table 1.

Table 1. Comparison of a set, its power set, and iterated power set

Aspect	Set S	Power set $\mathcal{P}(S)$	Iterated power set $\mathcal{P}^n(S)$
Object	Subset $S \subseteq U$	Collection of all subsets of S	Repeated powerset of S for $n \geq 1$
Typical element	Element $x \in S$	Subset $X \subseteq S$	Nested set $X_n \in \mathcal{P}^n(S)$ built from subsets
Cardinality (finite $\lvert S\rvert = m$)	$\lvert S\rvert = m$	$\lvert\mathcal{P}(S)\rvert = 2^m$	$\lvert\mathcal{P}^1(S)\rvert = 2^m$, $\lvert\mathcal{P}^2(S)\rvert = 2^{2^m}$, etc.
Level of abstraction	Base level: individual objects	First–order collections of objects	Higher–order collections of collections (hierarchical)

1.2. *Hypergraphs and SuperHypergraphs*

Hypergraphs generalize ordinary graphs (cf. [8]) by allowing each *hyperedge* to join an arbitrary nonempty subset of vertices, thereby modeling higher-order relations among elements [9–11]. A HyperGraph is an important research concept because, unlike a classical graph, it can represent complex and higher-order network structures. Moreover, several related extensions of HyperGraphs are known in the literature, including Directed HyperGraphs [12,13], Fuzzy HyperGraphs [14,15], Neutrosophic HyperGraphs [16,17], and Subset-Vertex Graphs [18,19]. A *SuperHyperGraph* further extends this idea by incorporating iterated powerset structures, enabling multi-layered, self-referential connections among hyperedges [20–22]. The notion of the SuperHyperGraph was introduced by F. Smarandache. Research on the applications of SuperHyperGraphs has been actively conducted in recent years due to their importance [23–25]. The definitions and concrete examples of this concept are provided below.

Definition 1.8 (Hypergraph)**.** [9,26] Let V be a finite set of *vertices.* A *hypergraph* is a pair

$$H = \big(V,\ E\big), \qquad E \subseteq \mathcal{P}(V) \setminus \{\varnothing\},$$

where each element of E is called a *hyperedge.* No restriction is imposed on the size of a hyperedge.

Example 1.9 (University course enrollment as a hypergraph)**.** Consider four students

$$V = \{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3, \mathrm{S}_4\}.$$

Each course is taken by a group of students, so it is naturally modeled as a hyperedge. Suppose:

- Course A is taken by $\{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3\}$,
- Course B is taken by $\{\mathrm{S}_2, \mathrm{S}_3\}$,
- Course C is taken by $\{\mathrm{S}_3, \mathrm{S}_4\}$.

Then the hyperedge family is

$$E = \big\{\{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3\}, \{\mathrm{S}_2, \mathrm{S}_3\}, \{\mathrm{S}_3, \mathrm{S}_4\}\big\} \ \subseteq\ \mathcal{P}(V) \setminus \{\varnothing\}.$$

Thus

$$H = (V, E)$$

is a hypergraph, where each hyperedge represents the set of students taking the same course.

Definition 1.10 (n-SuperHyperGraph)**.** (cf. [20,27]) Let V_0 be a finite nonempty *base set.* Define the iterated powersets by

$$\mathcal{P}^0(X) := X, \qquad \mathcal{P}^{k+1}(X) := \mathcal{P}\big(\mathcal{P}^k(X)\big) \quad (k \geq 0).$$

Fix $n \geq 1$. An *n-SuperHyperGraph* is a pair

$$\mathrm{SHG}^{(n)} = (V, E),$$

where the *n-supervertex set* V satisfies $\varnothing \neq V \subseteq \mathcal{P}^n(V_0)$, and the *n-superedge set* E is a nonempty family of nonempty subsets of V,

$$\varnothing \neq E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}.$$

Equivalently, each $e \in E$ is a finite nonempty set of n-supervertices. For $n = 0$, this reduces to an ordinary hypergraph on V_0.

Example 1.11 (Concrete finite instance for $n = 2$)**.** Let $V_0 = \{a, b, c\}$. Then

$$\mathcal{P}(V_0) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

and $\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big)$. Choose three 2-supervertices (each is a subset of $\mathcal{P}(V_0)$):

$$v_1 := \big\{\{a\}, \{b\}\big\}, \quad v_2 := \big\{\{a, c\}, \{b, c\}\big\}, \quad v_3 := \big\{\varnothing, \{a, b, c\}\big\}.$$

Set $V := \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0)$ and define superedges

$$e_1 := \{v_1, v_2\}, \qquad e_2 := \{v_2, v_3\}.$$

Then $E := \{e_1, e_2\} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$, and $\mathrm{SHG}^{(2)} = (V, E)$ is a 2-SuperHyperGraph with $|V| = 3$ and $|E| = 2$. Here every edge is a nonempty set of 2-supervertices, as required by Definition 1.10.

Example 1.12 (Plant organs as an n=1 SuperHyperGraph)**.** Let the base set be the plant organs [28, 29]

$$V_0 = \{\mathrm{Root}, \mathrm{Stem}, \mathrm{Leaf}\}.$$

Then $\mathcal{P}(V_0) = \{\varnothing, \{R\}, \{S\}, \{L\}, \{R, S\}, \{R, L\}, \{S, L\}, \{R, S, L\}\}$ (where R, S, L abbreviate Root, Stem, Leaf). Fix $n = 1$ and choose the 1-supervertex set

$$V = \big\{\{R\}, \{L\}, \{R, L\}, \{S\}\big\} \ \subseteq \ \mathcal{P}^1(V_0) = \mathcal{P}(V_0).$$

Define the 1-superedge family (each superedge is a nonempty set of 1-supervertices)

$$e_1 = \big\{\{R\}, \{L\}\big\}, \qquad e_2 = \big\{\{L\}, \{R, L\}\big\}, \qquad e_3 = \big\{\{S\}, \{R, S\}\big\} \cap \mathcal{P}(V) = \varnothing,$$

and take $E = \{e_1, e_2\} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$. Then $\mathrm{SHG}^{(1)} = (V, E)$ is an n=1 SuperHyperGraph per Definition 1.10. Domain reading: supervertices are organ-subsets (e.g. $\{R, L\}$ = roots and leaves sampled together); superedges join such subsets when a study design or functional relation requires them to be considered jointly.

Example 1.13 (Soil compartments as an n=2 SuperHyperGraph)**.** Let the base set be soil compartments

$$V_0 = \{\text{Rhizosphere}\,(R),\ \text{Humus}\,(H),\ \text{Mineral}\,(M)\}.$$

Then $\mathcal{P}(V_0) = \{\varnothing, \{R\}, \{H\}, \{M\}, \{R,H\}, \{R,M\}, \{H,M\}, \{R,H,M\}\}$ and $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$. Fix $n = 2$ and define the 2-supervertices (each is a subset of $\mathcal{P}(V_0)$):

$$v_1 = \{\{R\}, \{H\}\}, \qquad v_2 = \{\{R,M\}, \{H,M\}\}, \qquad v_3 = \{\{R,H,M\}\}, \qquad v_4 = \{\{R\}, \{R,H\}\}.$$

Set

$$V = \{v_1, v_2, v_3, v_4\} \ \subseteq\ \mathcal{P}^2(V_0).$$

Choose the 2-superedges

$$e_1 = \{v_1, v_2\}, \qquad e_2 = \{v_2, v_3\}, \qquad e_3 = \{v_1, v_4, v_3\}.$$

Then $E = \{e_1, e_2, e_3\} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$, so $\mathrm{SHG}^{(2)} = (V, E)$ is an n=2 SuperHyperGraph by Definition 1.10. Domain reading: each 2-supervertex is a *collection of compartment-subsets* (e.g. v_2 bundles two horizons that include M); superedges join such bundles when a protocol, constraint, or hypothesis requires analyzing these collections together (e.g. cross-horizon microbial exchanges).

As supplementary information, Table 2 presents the Comparison of Graphs, Hypergraphs, and SuperHyperGraphs.

TABLE 2. Comparison of Graphs, Hypergraphs, and SuperHyperGraphs

Aspect	Graph	Hypergraph	SuperHyperGraph
Vertex set	V	V	$V \subseteq \mathcal{P}^n(V_0)$ (set of n-supervertices)
Edge / (hyper)edge	Edges $\{u,v\}$, $u \neq v \in V$	Hyperedges $e \subseteq V$, $e \neq \varnothing$	Superhyperedges $e \subseteq V$, $e \neq \varnothing$ (on supervertices)
Order of interactions	Pairwise relations	Higher–order relations among vertices	Higher–order relations among supervertices and levels
Underlying structure	Single layer on V	Single layer on V via $\mathcal{P}(V)$	Iterated powerset layers $\mathcal{P}^n(V_0)$
Specialization	Base model	All $\lvert e\rvert = 2$ gives a graph	For $n = 0$ and $V = V_0$ gives a hypergraph

1.3. *Soil Microbial Network*

A Soil Microbial Network is a graph where microbial taxa are nodes, and edges represent statistical associations or interactions among taxa (cf. [30–33]). The definitions and concrete examples of this concept are provided below.

Definition 1.14 (Soil Microbial Network)**.** Let $T = \{1, \ldots, n\}$ be taxa and $S = \{1, \ldots, m\}$ samples. Given an abundance matrix $X = (x_{i,s}) \in \mathbb{R}^{n\times m}_{>0}$, define the centered log-ratio (CLR) transform

$$y_{i,s} = \log x_{i,s} - \frac{1}{n}\sum_{k=1}^{n} \log x_{k,s}, \qquad Y = (y_{i,s}) \in \mathbb{R}^{n\times m}.$$

Let $\hat{\Sigma}$ be the rowwise unbiased covariance of Y and $\hat{R} = (\hat{R}_{ij})$ the corresponding correlation matrix. For a threshold $\tau \in [0, 1)$ (optionally with FDR level $\alpha \in (0, 1)$), the Soil Microbial Network is the signed, weighted simple graph

$$G = (V, E, w, \sigma), \quad V = T, \quad E = \big\{\{i, j\} : i \neq j,\ |\hat{R}_{ij}| \geq \tau \text{ (and } p^{\mathrm{BH}}_{ij} \leq \alpha)\big\},$$

$$w(\{i, j\}) = |\hat{R}_{ij}| \in [\tau, 1], \qquad \sigma(\{i, j\}) = \operatorname{sign}(\hat{R}_{ij}) \in \{-1, +1\}.$$

CLR is sample-wise scale-invariant: if $x'_{i,s} = c_s x_{i,s}$ with $c_s > 0$, then

$$y'_{i,s} = \log(c_s x_{i,s}) - \frac{1}{n}\sum_{k} \log(c_s x_{k,s}) = \log x_{i,s} - \frac{1}{n}\sum_{k} \log x_{k,s} = y_{i,s}.$$

(Partial-dependence variant: replace $\hat{R}$ by $\tilde{R}_{ij} = -\hat{\Theta}_{ij}/\sqrt{\hat{\Theta}_{ii}\hat{\Theta}_{jj}}$ from a sparse precision $\hat{\Theta}$ via graphical lasso.)

Example 1.15 (Soil Microbial Network)**.** Take $n = 3$, $m = 4$ with

$$X = \begin{pmatrix} 9 & 4 & 1 & 16 \\ 1 & 3 & 2 & 9 \\ 4 & 1 & 3 & 1 \end{pmatrix}.$$

CLR (natural log; rounded to 4 d.p.):

$$Y \approx \begin{pmatrix} 1.0027 & 0.5580 & -0.5973 & 1.1160 \\ -1.1945 & 0.2703 & 0.0959 & 0.5406 \\ 0.1918 & -0.8283 & 0.5014 & -1.6566 \end{pmatrix}.$$

Covariance and correlation:

$$\hat{\Sigma} \approx \begin{pmatrix} 0.61264 & -0.11711 & -0.49553 \\ -0.11711 & 0.59356 & -0.47646 \\ -0.49553 & -0.47646 & 0.97199 \end{pmatrix}, \quad \hat{R} \approx \begin{pmatrix} 1 & -0.1942 & -0.6422 \\ -0.1942 & 1 & -0.6273 \\ -0.6422 & -0.6273 & 1 \end{pmatrix}.$$

With $\tau = 0.5$,

$$E = \big\{\{1, 3\}, \{2, 3\}\big\}, \quad w(\{1, 3\}) = 0.6422, \quad w(\{2, 3\}) = 0.6273, \quad \sigma(\{1, 3\}) = \sigma(\{2, 3\}) = -1,$$

so G is a signed, weighted simple graph on three taxa with two negative edges.

2. Main Results

The main results of this paper are presented in this section. Specifically, we investigate the Soil Microbial HyperNetwork and the Soil Microbial SuperHyperNetwork, providing formal definitions and illustrative examples. For reference, a comparison of the Soil Microbial Network, Soil Microbial HyperNetwork, and Soil Microbial SuperHyperNetwork is summarized in Table 3.

Table 3. Comparison of Soil Microbial Network, Soil Microbial HyperNetwork, and Soil Microbial SuperHyperNetwork.

Aspect	**Soil Microbial Network**	**Soil Microbial HyperNetwork**	**Soil Microbial SuperHyperNetwork**
Vertices / supervertices	$V = T$ (individual taxa)	$V = T$ (individual taxa)	$V \subseteq \mathcal{P}^n(T)$ (supervertices built from taxa)
Edges / (hyper)edges	Edges $\{i, j\}$ with $s_{ij} \geq \tau$	Hyperedges $A \subseteq T$, $\|A\| \geq 2$, $s(A) \geq \tau$	Superhyperedges $A \subseteq V$, $\|A\| \geq 2$, $s^{(n)}(A) \geq \tau$
Interaction order	Pairwise associations	Multi-taxon associations in one layer	Multi-level associations among groups of taxa
Score used	Pairwise score $s_{ij} = \|\hat{R}_{ij}\|$	$s(A) = \min_{\{i,j\} \subseteq A} s_{ij}$	$s^{(n)}(A) = s(\mathrm{supp}_n(A))$
Mathematical structure	Weighted graph $G_\tau = (T, E_2)$	Weighted hypergraph $\mathcal{H}_\tau = (T, E, w)$	Weighted n-SuperHyperGraph $\mathcal{H}_\tau^{(n)} = (V, E, w)$
Relation to others	Base pairwise layer	Rank–2 hyperedges coincide with G_τ	Canonical embedding recovers G_τ; for $n = 0$, $V = T$ gives the HyperNetwork
Domain view	Pairwise co-occurrence / association network	Taxon consortia as single hyperedges	Hierarchical consortia and grouped communities across scales

2.1. *Soil Microbial HyperNetwork*

Soil Microbial HyperNetwork models microbial taxa as nodes with hyperedges linking multiple taxa simultaneously based on significant associations. The definitions and concrete examples of this concept are provided below.

Definition 2.1 (Pairwise association from CLR data)**.** Let $T = \{1, \ldots, n\}$ be the taxa set and $Y = (y_{i,s})$ the CLR–transformed abundance matrix defined in the Soil Microbial Network section. Let $\hat{R} = (\hat{R}_{ij})$ be the empirical correlation matrix computed rowwise from Y. Define the pairwise (unsigned) association score

$$s_{ij} \;:=\; |\hat{R}_{ij}| \in [0, 1] \qquad (i \neq j), \qquad s_{ii} := 1.$$

Definition 2.2 (Soil Microbial HyperNetwork)**.** Fix a threshold $\tau \in [0, 1)$. For any finite nonempty $A \subseteq T$ with $|A| \geq 2$, define the multiway score

$$s(A) \;:=\; \min_{\{i,j\}\subseteq A} s_{ij} \;\in [0, 1].$$

The *Soil Microbial HyperNetwork* at level τ is the weighted hypergraph

$$\mathcal{H}_\tau \;:=\; (V, E, w), \qquad V := T, \quad E \;:=\; \big\{\, A \subseteq V : \; |A| \geq 2,\; s(A) \geq \tau \,\big\},$$

with hyperedge weights $w : E \to [\tau, 1]$ given by $w(A) := s(A)$. (Optionally, one may require all pairs $\{i, j\} \subseteq A$ to pass an FDR-corrected significance test in addition to $s(A) \geq \tau$.)

Remark 2.3. The aggregator $s(A) = \min_{\{i,j\}\subseteq A} s_{ij}$ is chosen so that $s(\{i, j\}) = s_{ij}$ holds identically and so that enlarging a set cannot increase its score: if $A \subseteq B$ then $s(B) \leq s(A)$. This monotonicity ensures that every hyperedge induces all its pairwise edges.

Example 2.4 (Instance continuing the network example)**.** Use the same CLR-based correlation matrix

$$\hat{R} \approx \begin{pmatrix} 1 & -0.1942 & -0.6422 \\ -0.1942 & 1 & -0.6273 \\ -0.6422 & -0.6273 & 1 \end{pmatrix}.$$

Thus $s_{12} = 0.1942$, $s_{13} = 0.6422$, $s_{23} = 0.6273$.

- With $\tau = 0.5$:

$$E_2 = \big\{\{1, 3\}, \{2, 3\}\big\}, \qquad s(\{1, 2, 3\}) = \min\{0.1942, 0.6422, 0.6273\} = 0.1942 < 0.5,$$

  so the hyperedge $\{1, 2, 3\}$ is not included. Hence

$$\mathcal{H}_{0.5} : \; E = \big\{\{1, 3\}, \{2, 3\}\big\}, \quad w(\{1, 3\}) = 0.6422, \; w(\{2, 3\}) = 0.6273.$$

  Here $\mathcal{H}_{0.5}$ coincides with the network at rank 2, as guaranteed by Theorem 2.9.
- With $\tau = 0.18$: All pairs are included, and since $s(\{1, 2, 3\}) = 0.1942 \geq 0.18$, the 3-way hyperedge appears:

$$E = \big\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\big\}, \quad w(\{1, 2, 3\}) = 0.1942.$$

  This captures a simultaneous association among the three taxa in addition to all pairwise links.

Example 2.5 (Four-taxa HyperNetwork at $\tau = 0.6$)**.** Let $T = \{A, B, C, D\}$. Suppose the pairwise association scores $s_{ij} \in [0, 1]$ are

$$S = \begin{pmatrix} 1 & 0.82 & 0.65 & 0.40 \\ 0.82 & 1 & 0.62 & 0.58 \\ 0.65 & 0.62 & 1 & 0.61 \\ 0.40 & 0.58 & 0.61 & 1 \end{pmatrix} \quad \text{(rows/columns ordered as } A, B, C, D).$$

By Definition 2.2, for any $A \subseteq T$ with $|A| \geq 2$, $s(A) := \min_{\{i,j\}\subseteq A} s_{ij}$. With $\tau = 0.6$,

$$E_2 = \{\{A, B\}, \{A, C\}, \{B, C\}, \{C, D\}\},$$

and among 3-sets,

$$s(\{A, B, C\}) = \min\{0.82, 0.65, 0.62\} = 0.62\ (\geq \tau),$$

$$s(\{A, B, D\}) = \min\{0.82, 0.58, 0.40\} = 0.40\ (< \tau),$$

$$s(\{A, C, D\}) = \min\{0.65, 0.40, 0.61\} = 0.40\ (< \tau),$$

$$s(\{B, C, D\}) = \min\{0.62, 0.58, 0.61\} = 0.58\ (< \tau).$$

Hence the Soil Microbial HyperNetwork is

$$\mathcal{H}_{0.6} = (V, E, w), \quad V = T, \quad E = \{\{A, B\}, \{A, C\}, \{B, C\}, \{C, D\}, \{A, B, C\}\},$$

with weights $w(\{i, j\}) = s_{ij}$ and $w(\{A, B, C\}) = 0.62$. No 4-way hyperedge appears since $\min_{\{i,j\}\subseteq T} s_{ij} = 0.40 < 0.6$.

Example 2.6 (Five-taxa clustered HyperNetwork at $\tau = 0.8$)**.** Let $T = \{1, 2, 3, 4, 5\}$. Suppose the scores are

$$S = \begin{pmatrix} 1 & 0.91 & 0.88 & 0.55 & 0.52 \\ 0.91 & 1 & 0.86 & 0.50 & 0.49 \\ 0.88 & 0.86 & 1 & 0.48 & 0.51 \\ 0.55 & 0.50 & 0.48 & 1 & 0.83 \\ 0.52 & 0.49 & 0.51 & 0.83 & 1 \end{pmatrix}.$$

With $\tau = 0.8$,

$$E_2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}\},$$

and for 3-sets,

$$s(\{1, 2, 3\}) = \min\{0.91, 0.88, 0.86\} = 0.86\ (\geq \tau), \qquad s(\{1, 2, 4\}) \leq 0.55 < \tau, \ \ldots$$

so the only 3-way hyperedge is $\{1, 2, 3\}$. No 4- or 5-way hyperedge exists because some cross-cluster pairs fall below τ. Therefore,

$$\mathcal{H}_{0.8} = (V, E, w), \quad V = T, \quad E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{1, 2, 3\}\},$$

with weights $w(\{i, j\}) = s_{ij}$ and $w(\{1, 2, 3\}) = 0.86$. This captures a tight triad $\{1, 2, 3\}$ and a strong pair $\{4, 5\}$, with weak cross-links excluded by the threshold.

The theorems related to this concept are presented below.

Theorem 2.7 (Hypergraph property)**.** *Let $\mathcal{H}_\tau = (V, E, w)$ be as in Definition 2.2. Then (V, E) is a (finite) hypergraph in the sense of Definition 1.8, and w makes it a weighted hypergraph.*

Proof. By construction, $V = T$ is finite. Every $A \in E$ satisfies $A \subseteq V$ and $|A| \geq 2$; hence $A \in \mathcal{P}(V)\backslash\{\varnothing\}$. Therefore $E \subseteq \mathcal{P}(V)\backslash\{\varnothing\}$, i.e., (V, E) is a hypergraph. Finally, $w : E \to [\tau, 1]$ assigns a nonnegative weight to each hyperedge, giving a weighted hypergraph structure. □

Definition 2.8 (Soil Microbial Network at level τ)**.** Define the (pairwise) Soil Microbial Network

$$G_\tau \;:=\; (V, E_2), \qquad V := T, \quad E_2 := \big\{\{i, j\} \subseteq V :\; i \neq j,\; s_{ij} \geq \tau\big\},$$

with optional signs and weights as in the earlier definition.

Theorem 2.9 (Generalizes the Soil Microbial Network)**.** *For $\mathcal{H}_\tau = (V, E, w)$ and $G_\tau = (V, E_2)$ at the same threshold τ, the following hold.*

(i) Rank-2 consistency: $\{i, j\} \in E$ *if and only if* $\{i, j\} \in E_2$.

(ii) Downward closure: *If* $A \in E$ *and* $\{i, j\} \subseteq A$*, then* $\{i, j\} \in E_2$.

In particular, the 2-edge layer of $\mathcal{H}_\tau$ coincides exactly with the Soil Microbial Network G_τ.

Proof. (i) For any 2-element set $A = \{i, j\}$, we have $s(A) = \min_{\{u,v\}\subseteq A} s_{uv} = s_{ij}$. Hence $A \in E \iff s(A) \geq \tau \iff s_{ij} \geq \tau \iff \{i, j\} \in E_2$.
(ii) If $A \in E$, then $s(A) = \min_{\{u,v\}\subseteq A} s_{uv} \geq \tau$. For any $\{i, j\} \subseteq A$ we have $s_{ij} \geq s(A) \geq \tau$, so $\{i, j\} \in E_2$. □

2.2. *Soil Microbial SuperHyperNetwork*

Soil Microbial SuperHyperNetwork extends this by allowing multi-layered supervertices and superhyperedges, capturing hierarchical, higher-order microbial interactions. The definitions and concrete examples of this concept are provided below.

Definition 2.10 (Base pairwise and multiway scores on taxa)**.** Let $T = \{1, \ldots, n\}$ be a finite nonempty set of taxa. From CLR–transformed abundances (as in the Soil Microbial Network section), let $\hat{R} = (\hat{R}_{ij})$ be the empirical correlation matrix and set the pairwise association score $s_{ij} := |\hat{R}_{ij}| \in [0, 1]$ for $i \neq j$ (and $s_{ii} := 1$). For any finite nonempty $B \subseteq T$ with $|B| \geq 2$, define the multiway score

$$s(B) \;:=\; \min_{\{i,j\}\subseteq B} s_{ij} \;\in [0, 1].$$

Definition 2.11 (Support (flattening) from level n to taxa)**.** For each integer $n \geq 0$, define the *support map* $\mathrm{supp}_n : \mathcal{P}^n(T) \to \mathcal{P}(T)$ recursively by

$$\mathrm{supp}_0(i) := \{i\} \quad (i \in T), \qquad \mathrm{supp}_{k+1}(X) := \bigcup_{x\in X} \mathrm{supp}_k(x) \quad (X \in \mathcal{P}^{k+1}(T)).$$

For a finite $A \subseteq \mathcal{P}^n(T)$, write $\mathrm{supp}_n(A) := \bigcup_{v\in A} \mathrm{supp}_n(v) \subseteq T$.

Definition 2.12 (Soil Microbial SuperHyperNetwork at depth n)**.** Fix $n \geq 1$ and a threshold $\tau \in [0, 1)$. Choose a nonempty *supervertex set* $V \subseteq \mathcal{P}^n(T)$. Define, for any finite $A \subseteq V$ with $|A| \geq 2$,

$$s^{(n)}(A) \ := \ s\big(\mathrm{supp}_n(A)\big) \ = \ \min_{\{i,j\} \subseteq \mathrm{supp}_n(A)} s_{ij} \ \in [0, 1].$$

The *Soil Microbial SuperHyperNetwork* (at level τ) is the weighted superhypergraph

$$\mathcal{H}^{(n)}_{\tau} \ := \ (V, E, w), \qquad E \ := \ \big\{ A \subseteq V : \ |A| \geq 2, \ s^{(n)}(A) \geq \tau \big\},$$

with weights $w : E \to [\tau, 1]$ given by $w(A) := s^{(n)}(A)$.

Remark 2.13. (i) Monotonicity: If $A \subseteq B \subseteq V$, then $\mathrm{supp}_n(B) \supseteq \mathrm{supp}_n(A)$, hence $s^{(n)}(B) \leq s^{(n)}(A)$. In particular, enlarging a hyperedge cannot increase its score.
(ii) Rank-2 consistency: For $|A| = 2$, $s^{(n)}(A) = \min_{\{i,j\} \subseteq \mathrm{supp}_n(A)} s_{ij}$, which reduces to the pairwise score on the union of the underlying taxa of the two supervertices.

Example 2.14 (Depth $n = 1$ (supervertices are subsets of taxa))**.** Let $T = \{1, 2, 3\}$ with pair scores $s_{12} = 0.1942, \ s_{13} = 0.6422, \ s_{23} = 0.6273$ (from the numerical instance in the Network section). Take

$$V = \big\{\{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\big\} \subseteq \mathcal{P}(T), \qquad \tau = 0.6.$$

For two supervertices, $\mathrm{supp}_1(\{a\}) = \{a\}$ and $\mathrm{supp}_1(\{a, b\}) = \{a, b\}$. Hence

$$s^{(1)}\big(\{\{1\}, \{3\}\}\big) = s(\{1, 3\}) = 0.6422 \ (\geq \tau), \quad s^{(1)}\big(\{\{2\}, \{3\}\}\big) = s(\{2, 3\}) = 0.6273 \ (\geq \tau),$$

so these 2-hyperedges are present. However, $s^{(1)}\big(\{\{1, 3\}, \{2, 3\}\}\big) = s(\{1, 2, 3\}) = \min\{0.1942, 0.6422, 0.6273\} = 0.1942 < \tau$, so $\{\{1, 3\}, \{2, 3\}\} \notin E$. This example shows both generalization (pairs of embedded singletons reproduce the Soil Microbial Network) and genuine supervertex interactions.

Example 2.15 (Depth $n = 2$ (supervertices are sets of subsets))**.** Let $T = \{1, 2, 3\}$ with the same s_{ij} as above and $\tau = 0.6$. Define level-2 supervertices

$$v_1 := \big\{\{1\}, \{3\}\big\}, \qquad v_2 := \big\{\{2\}, \{3\}\big\}, \qquad v_3 := \big\{\{1\}\big\} \ \in \mathcal{P}^2(T),$$

and set $V := \{v_1, v_2, v_3\}$. By Definition 2.11, $\mathrm{supp}_2(v_1) = \{1, 3\}$, $\mathrm{supp}_2(v_2) = \{2, 3\}$, $\mathrm{supp}_2(v_3) = \{1\}$. Then

$$s^{(2)}(\{v_1, v_3\}) = s(\{1, 3\}) = 0.6422 \ (\geq \tau), \qquad s^{(2)}(\{v_2, v_3\}) = s(\{1, 2, 3\}) = 0.1942 \ (< \tau),$$

and $s^{(2)}(\{v_1, v_2\}) = s(\{1, 2, 3\}) = 0.1942 < \tau$. Thus E contains $\{v_1, v_3\}$ but not $\{v_1, v_2\}$ nor $\{v_2, v_3\}$. This illustrates how multi-level grouping (depth 2) is evaluated via the flattened underlying taxa and remains consistent with the pairwise base scores.

The theorems related to this concept are presented below.

Theorem 2.16 (SuperHyperGraph property). *Let $\mathcal{H}_\tau^{(n)} = (V, E, w)$ be as in Definition 2.12. Then, with base set $V_0 := T$, the pair (V, E) is an n-SuperHyperGraph in the sense of Definition 1.10.*

Proof. By construction $V \subseteq \mathcal{P}^n(T) = \mathcal{P}^n(V_0)$ and $V \neq \varnothing$. Moreover $E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$ because every $A \in E$ is a finite subset of V with $|A| \geq 2$. Hence (V, E) satisfies Definition 1.10, and w equips it with weights. □

Theorem 2.17 (Generalizes the Soil Microbial HyperNetwork and Network). *Let $\mathcal{H}_\tau^{(n)} = (V, E, w)$ be as above and write $G_\tau = (T, E_2)$ for the Soil Microbial Network at threshold τ (pairwise edges $\{i, j\}$ with $s_{ij} \geq \tau$). Then:*

(i) ($n = 0$ reduction) If $n = 0$ and $V = T$, then

$$E = \{ B \subseteq T : |B| \geq 2, s(B) \geq \tau \},$$

which is exactly the Soil Microbial HyperNetwork (with weight $w(B) = s(B)$).

(ii) (Canonical embedding) For $n \geq 1$, define the embedding $\eta_n : T \to \mathcal{P}^n(T)$ by $\eta_0(i) = i$ and $\eta_{k+1}(i) = \{\eta_k(i)\}$. If $V = \eta_n[T]$, then the set of 2-element hyperedges of $\mathcal{H}_\tau^{(n)}$ is

$$\{\{\eta_n(i), \eta_n(j)\} : s_{ij} \geq \tau\},$$

which coincides with the edge set E_2 of the Soil Microbial Network after inverse projection by η_n.

(iii) (Downward closure to pairs) For any $n \geq 0$, if $A \in E$ and $\{u, v\} \subseteq A$, then $\{\mathrm{supp}_n(u), \mathrm{supp}_n(v)\} \subseteq \mathrm{supp}_n(A)$ implies $s^{(n)}(\{u, v\}) \geq s^{(n)}(A) \geq \tau$. Thus every hyperedge induces all of its pairwise 2-faces, and these 2-faces correspond to edges in G_τ on the underlying taxa.

Proof. (i) If $n = 0$ and $V = T$, then for $A \subseteq V$ we have $\mathrm{supp}_0(A) = A$ by Definition 2.11. Hence $s^{(0)}(A) = s(A)$ and the condition $s^{(0)}(A) \geq \tau$ is exactly the hyperedge rule of the Soil Microbial HyperNetwork; weights also match.

(ii) For the embedding η_n, we prove by induction that $\mathrm{supp}_n(\eta_n(i)) = \{i\}$ for all $i \in T$. The claim is trivial for $n = 0$. If it holds for $n = k$, then $\mathrm{supp}_{k+1}(\eta_{k+1}(i)) = \mathrm{supp}_{k+1}(\{\eta_k(i)\}) = \mathrm{supp}_k(\eta_k(i)) = \{i\}$. Therefore, for any $i \neq j$,

$$s^{(n)}(\{\eta_n(i), \eta_n(j)\}) = s(\mathrm{supp}_n(\{\eta_n(i), \eta_n(j)\})) = s(\{i, j\}) = s_{ij}.$$

Thus $\{\eta_n(i), \eta_n(j)\} \in E$ iff $s_{ij} \geq \tau$, as claimed.

(iii) If $A \in E$, then by Definition 2.12 we have $s^{(n)}(A) = \min_{\{i,j\} \subseteq \mathrm{supp}_n(A)} s_{ij} \geq \tau$. For any $\{u, v\} \subseteq A$, $\mathrm{supp}_n(\{u, v\}) \subseteq \mathrm{supp}_n(A)$ and hence $s^{(n)}(\{u, v\}) \geq s^{(n)}(A) \geq \tau$. □

Theorem 2.18 (Monotonicity in the threshold)**.** *Let T be a finite taxa set with base scores as in Definition 2.10, fix $n \geq 1$, and let $V \subseteq \mathcal{P}^n(T)$ be nonempty. For $\tau_1, \tau_2 \in [0, 1)$ with $\tau_1 \leq \tau_2$, let*

$$\mathcal{H}^{(n)}_{\tau_k} = (V, E_{\tau_k}, w_{\tau_k}) \qquad (k = 1, 2)$$

be the Soil Microbial SuperHyperNetworks from Definition 2.12. Then

$$E_{\tau_2} \subseteq E_{\tau_1}, \qquad w_{\tau_1}(A) = w_{\tau_2}(A) = s^{(n)}(A) \quad \text{for all } A \in E_{\tau_2}.$$

Proof. By Definition 2.12,

$$E_{\tau_k} = \{A \subseteq V : \ |A| \geq 2, \ s^{(n)}(A) \geq \tau_k\} \quad (k = 1, 2).$$

If $A \in E_{\tau_2}$, then $s^{(n)}(A) \geq \tau_2 \geq \tau_1$, hence $A \in E_{\tau_1}$ and so $E_{\tau_2} \subseteq E_{\tau_1}$. By construction $w_{\tau_k}(A) = s^{(n)}(A)$ for every $A \in E_{\tau_k}$, so for $A \in E_{\tau_2}$ we have $w_{\tau_1}(A) = w_{\tau_2}(A) = s^{(n)}(A)$. □

Theorem 2.19 (Downward closure under inclusion)**.** *Let $\mathcal{H}^{(n)}_{\tau} = (V, E, w)$ be a Soil Microbial SuperHyperNetwork as in Definition 2.12, with $n \geq 1$ and $\tau \in [0, 1)$. If $A \in E$ and $B \subseteq A$ with $|B| \geq 2$, then $B \in E$, and*

$$s^{(n)}(B) \ \geq \ s^{(n)}(A).$$

Proof. By Definition 2.11,

$$\mathrm{supp}_n(B) \subseteq \mathrm{supp}_n(A).$$

Hence, by Definition 2.10,

$$s^{(n)}(B) = s\big(\mathrm{supp}_n(B)\big) = \min_{\{i,j\} \subseteq \mathrm{supp}_n(B)} s_{ij} \ \geq \ \min_{\{i,j\} \subseteq \mathrm{supp}_n(A)} s_{ij} = s^{(n)}(A).$$

If $A \in E$, then $s^{(n)}(A) \geq \tau$, so $s^{(n)}(B) \geq s^{(n)}(A) \geq \tau$. Since $|B| \geq 2$, it follows from Definition 2.12 that $B \in E$. □

Theorem 2.20 (Each superhyperedge induces a clique on taxa)**.** *Let $\mathcal{H}^{(n)}_{\tau} = (V, E, w)$ be a Soil Microbial SuperHyperNetwork at threshold τ, and let $G_\tau = (T, E_2)$ be the Soil Microbial Network at the same τ (pairwise edges $\{i, j\}$ with $s_{ij} \geq \tau$). For any $A \in E$, the induced subgraph of G_τ on $\mathrm{supp}_n(A)$ is a complete graph. Equivalently,*

$$\{i, j\} \subseteq \mathrm{supp}_n(A) \quad \Longrightarrow \quad \{i, j\} \in E_2.$$

Proof. Fix $A \in E$ and two distinct taxa $i, j \in \mathrm{supp}_n(A)$. By Definition 2.10 and Definition 2.12,

$$s^{(n)}(A) = s\big(\mathrm{supp}_n(A)\big) = \min_{\{u,v\} \subseteq \mathrm{supp}_n(A)} s_{uv} \ \leq \ s_{ij}.$$

Since $A \in E$, we have $s^{(n)}(A) \geq \tau$, whence $s_{ij} \geq s^{(n)}(A) \geq \tau$. By the definition of G_τ, this means $\{i, j\} \in E_2$. As this holds for every unordered pair $\{i, j\} \subseteq \mathrm{supp}_n(A)$, the induced subgraph on $\mathrm{supp}_n(A)$ is complete. □

Theorem 2.21 (Isomorphism invariance under taxa relabeling)**.** *Let T and T' be finite taxa sets, and let s_{ij} and s'_{uv} be base scores on T and T' as in Definition 2.10. Suppose $\varphi : T \to T'$ is a bijection with*

$$s'_{\varphi(i)\varphi(j)} \;=\; s_{ij} \quad \text{for all } i, j \in T.$$

For each $n \geq 0$, define $\Phi_n : \mathcal{P}^n(T) \to \mathcal{P}^n(T')$ recursively by

$$\Phi_0(i) := \varphi(i), \qquad \Phi_{k+1}(X) := \{\Phi_k(x) : x \in X\} \quad (X \in \mathcal{P}^{k+1}(T)).$$

Fix $n \geq 1$, let $V \subseteq \mathcal{P}^n(T)$ be nonempty, and put $V' := \Phi_n[V] \subseteq \mathcal{P}^n(T')$. For a threshold $\tau \in [0, 1)$, let

$$\mathcal{H}^{(n)}_\tau = (V, E, w), \qquad \mathcal{H}'^{(n)}_\tau = (V', E', w')$$

be the corresponding Soil Microbial SuperHyperNetworks constructed from s_{ij} and s'_{uv}. Then Φ_n induces an isomorphism of weighted superhypergraphs between $\mathcal{H}^{(n)}_\tau$ and $\mathcal{H}'^{(n)}_\tau$.

Proof. First note that $\Phi_n : V \to V'$ is a bijection by construction. We claim that for every $v \in V$,

$$\mathrm{supp}'_n\big(\Phi_n(v)\big) = \varphi\big[\mathrm{supp}_n(v)\big],$$

where supp_n and supp'_n are the support maps on T and T' (Definition 2.11), and $\varphi[\cdot]$ denotes image. This is proved by induction on n. For $n = 0$ it is just the definition of Φ_0 and supp_0. If the statement holds for $n = k$, then for $X \in \mathcal{P}^{k+1}(T)$

$$\mathrm{supp}'_{k+1}\big(\Phi_{k+1}(X)\big) = \bigcup_{x' \in \Phi_{k+1}(X)} \mathrm{supp}'_k(x')$$

$$= \bigcup_{x \in X} \mathrm{supp}'_k\big(\Phi_k(x)\big) = \bigcup_{x \in X} \varphi\big[\mathrm{supp}_k(x)\big] = \varphi\big[\mathrm{supp}_{k+1}(X)\big].$$

Now let $A \subseteq V$ with $|A| \geq 2$. Then

$$\mathrm{supp}'_n\big(\Phi_n(A)\big) = \bigcup_{v \in A} \mathrm{supp}'_n\big(\Phi_n(v)\big) = \bigcup_{v \in A} \varphi\big[\mathrm{supp}_n(v)\big] = \varphi\big[\mathrm{supp}_n(A)\big].$$

Using the assumption on the scores,

$$s'^{(n)}\big(\Phi_n(A)\big) = \min_{\{u', v'\} \subseteq \mathrm{supp}'_n(\Phi_n(A))} s'_{u'v'} = \min_{\{i, j\} \subseteq \mathrm{supp}_n(A)} s_{ij} = s^{(n)}(A).$$

Hence $s'^{(n)}(\Phi_n(A)) = s^{(n)}(A)$ for all finite $A \subseteq V$. In particular,

$$A \in E \iff s^{(n)}(A) \geq \tau \iff s'^{(n)}\big(\Phi_n(A)\big) \geq \tau \iff \Phi_n(A) \in E',$$

so Φ_n induces a bijection $E \to E'$. Finally, the weights satisfy

$$w'(\Phi_n(A)) = s'^{(n)}\big(\Phi_n(A)\big) = s^{(n)}(A) = w(A),$$

so the isomorphism preserves weights. □

Theorem 2.22 (Recovery of the Soil Microbial Network via canonical embedding)**.** *Let T be a finite taxa set with base scores s_{ij} and fix $n \geq 1$ and $\tau \in [0,1)$. Define the canonical embedding $\eta_n : T \to \mathcal{P}^n(T)$ recursively by*

$$\eta_0(i) := i, \qquad \eta_{k+1}(i) := \{\eta_k(i)\} \quad (k \geq 0).$$

Let $V \subseteq \mathcal{P}^n(T)$ satisfy $\eta_n[T] \subseteq V$, and let $\mathcal{H}_\tau^{(n)} = (V, E, w)$ be the corresponding Soil Microbial SuperHyperNetwork. Let $G_\tau = (T, E_2)$ be the Soil Microbial Network at the same threshold τ. Then, for all distinct $i, j \in T$,

$$\{\eta_n(i), \eta_n(j)\} \in E \iff \{i, j\} \in E_2.$$

In particular, the 2-hyperedges of $\mathcal{H}_\tau^{(n)}$ whose vertices lie in $\eta_n[T]$ are in one-to-one correspondence with the edges of G_τ.

Proof. We first show that $\mathrm{supp}_n(\eta_n(i)) = \{i\}$ for all $i \in T$. This is proved by induction on n. For $n = 0$, $\mathrm{supp}_0(i) = \{i\}$ by definition. Assume $\mathrm{supp}_k(\eta_k(i)) = \{i\}$. Then

$$\mathrm{supp}_{k+1}(\eta_{k+1}(i)) = \mathrm{supp}_{k+1}(\{\eta_k(i)\}) = \bigcup_{x \in \{\eta_k(i)\}} \mathrm{supp}_k(x) = \mathrm{supp}_k(\eta_k(i)) = \{i\}.$$

Thus the claim holds for all n.

Now fix distinct $i, j \in T$ and consider $A := \{\eta_n(i), \eta_n(j)\} \subseteq V$. By Definition 2.11,

$$\mathrm{supp}_n(A) = \mathrm{supp}_n\big(\{\eta_n(i), \eta_n(j)\}\big) = \mathrm{supp}_n(\eta_n(i)) \cup \mathrm{supp}_n(\eta_n(j)) = \{i\} \cup \{j\} = \{i, j\}.$$

Hence, by Definition 2.10,

$$s^{(n)}(A) = s\big(\mathrm{supp}_n(A)\big) = s(\{i, j\}) = \min_{\{u,v\} \subseteq \{i,j\}} s_{uv} = s_{ij}.$$

Therefore,

$$\{\eta_n(i), \eta_n(j)\} \in E \iff s^{(n)}(A) \geq \tau \iff s_{ij} \geq \tau \iff \{i, j\} \in E_2,$$

which proves the desired equivalence and the stated bijection between these 2-hyperedges and the edges of G_τ. □

3. **Conclusion**

In this paper, we provided a mathematical definition of the Soil Microbial Network and investigated its generalizations: the Soil Microbial HyperNetwork, using the framework of HyperGraphs, and the Soil Microbial SuperHyperNetwork, using the framework of SuperHyperGraphs. In the future, we hope to study extensions of these concepts by employing Fuzzy Sets [34], Soft Sets [35], HyperFuzzy Sets [36], Functorial Sets [37], SuperHyperFuzzy Sets [38], Neutrosophic Sets [39–41], QuadriPartitioned Neutrosophic Sets [42], and Plithogenic Sets [43, 44]. Moreover, since SuperHyperStructures other than SuperHyperGraphs, such as

SuperHyperAlgebra [6], are also known, we expect that future research will develop extensions based on these structures and explore our concepts from perspectives beyond the purely graph-theoretic one.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany this paper.

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Plithogenic Directed Offset and Plithogenic MultiDirected Offsets

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Abstract. For more than half a century, concepts such as fuzzy sets have been studied to handle uncertainty in real-world systems. Uncertainty-modeling frameworks—such as fuzzy sets, intuitionistic fuzzy sets, hyperfuzzy sets, neutrosophic sets, hyperneutrosophic sets, soft sets, rough sets, and plithogenic sets—have been widely adopted to represent and reason under vagueness and imprecision. Plithogenic sets extend these approaches by combining multi-dimensional degrees of membership with a measure of contradiction. The *offset* enhancement further allows membership values to lie outside the usual $[0,1]$ interval, accommodating over- or under-membership. In this paper, we introduce and rigorously define the *Plithogenic Multi-Directed Offset*, unifying the concepts of uncertainty, directionality, and non-standard memberships; we show that this framework subsumes existing plithogenic, directed, and offset structures.

Keywords: MultiDirected set, Fuzzy Set, Directed Set, Plithogenic Offset, Plithogenic Set

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1. Preliminaries

This section outlines the key concepts and definitions necessary for the discussions in this paper. Throughout this study, all sets are assumed to be finite.

1.1. *Directed Set and MultiDirected Set*

A directed set is a nonempty set with an acyclic, transitive relation modeling one-directional precedence or causal ordering between elements [1,2]. A multi-directed set generalizes directed sets by allowing parallel relations, represented as arrows with initial and terminal node mapping functions [1,3]. The definitions of directed set and multi-directed set are provided below.

Definition 1.1 (Directed set)**.** (cf. [1,2]) A *directed set* is a pair $(D, \prec)$ where:

- D is a nonempty set.
- $\prec$ is a binary relation on D satisfying:
  - *Acyclicity*: There do not exist elements $x_1, x_2, \ldots, x_n \in D$ such that

$$x_1 \prec x_2 \prec \cdots \prec x_n \prec x_1.$$

  - *Transitivity*: If $x \prec y$ and $y \prec z$, then $x \prec z$.

The elements of D are called *nodes*, and the pairs $(x, y) \in \prec$ are called *directed relations*.

Definition 1.2 (Multi-directed set)**.** [1,3] A *multi-directed set* generalizes the notion of a directed set by allowing multiple directed relations between the same pair of elements. It is a quadruple (M, R, i, t) where:

- M is a nonempty set of *nodes*.
- R is a nonempty set of *relations*.
- $i : R \to M$ is the *initial element function*, assigning each relation an initial node.
- $t : R \to M$ is the *terminal element function*, assigning each relation a terminal node.

A *chain* in a multi-directed set is a sequence of relations $r_1, r_2, \ldots, r_k$ such that

$$t(r_i) = i(r_{i+1}) \quad \text{for all } 1 \leq i < k.$$

A concrete example of this concept is presented below.

Example 1.3 (Public Transit Network as a Multi-Directed Set)**.** A public transit network is a coordinated system of buses, trains, and trams providing shared transportation across urban or regional areas [4–6]. Consider three transit stops:

$$M = \{\text{Downtown, Uptown, Suburb}\},$$

and five distinct routes:

$$R = \{\text{bus}_{101}, \text{bus}_{102}, \text{tram}_1, \text{express}_A, \text{express}_B\}.$$

The initial- and terminal-element functions are:

$$\begin{aligned} i(\text{bus}_{101}) &= \text{Downtown}, & t(\text{bus}_{101}) &= \text{Uptown}, \\ i(\text{bus}_{102}) &= \text{Downtown}, & t(\text{bus}_{102}) &= \text{Uptown}, \\ i(\text{tram}_1) &= \text{Uptown}, & t(\text{tram}_1) &= \text{Suburb}, \\ i(\text{express}_A) &= \text{Downtown}, & t(\text{express}_A) &= \text{Suburb}, \\ i(\text{express}_B) &= \text{Downtown}, & t(\text{express}_B) &= \text{Suburb}. \end{aligned}$$

Here bus_{101} and bus_{102} are two separate bus lines connecting Downtown to Uptown, while express_A and express_B are two express services going directly from Downtown to Suburb.

A chain of length two, for example, is

$$\text{Downtown} \xrightarrow{\text{bus}_{101}} \text{Uptown} \xrightarrow{\text{tram}_1} \text{Suburb},$$

since $t(\text{bus}_{101}) = \text{Uptown} = i(\text{tram}_1)$.

Thus (M, R, i, t) forms a multi-directed set, capturing parallel transit options between the same stops.

1.2. *Plithogenic Directed Set and Plithogenic MultiDirected Set*

The Plithogenic Set is a mathematical framework designed to integrate multi-valued degrees of appurtenance and contradiction, making it particularly effective for addressing complex decision-making scenarios. Numerous studies have explored the properties and applications of Plithogenic Sets, as highlighted in works such as [7–10]. Related concepts such as Plithogenic Graphs are also known in the literature [11, 12].

A plithogenic directed set has a graded relation satisfying adjusted transitivity and acyclicity, governed by component-wise attribute contradiction measures. A plithogenic multi-directed set comprises nodes and multiple directed relations with initial/terminal mappings, attribute-based appurtenance, and component-wise contradiction degrees. The formal definition is presented below.

Definition 1.4 (Plithogenic Set). [13,14] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)*[1]
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

(1) *Reflexivity of Contradiction Function*:

$$pCF(a,a) = 0$$

(2) *Symmetry of Contradiction Function*:

$$pCF(a,b) = pCF(b,a)$$

Example 1.5 (Plithogenic Set in Smartphone Selection). Smartphone selection is a common multi-criteria decision problem in consumer electronics, where users balance battery life, camera quality, price, and brand trust. We illustrate a *Plithogenic Set* that focuses on a single attribute, "battery performance", for three candidate smartphones (cf. [16]).

Let the universal set of options be

$$S = \{\text{Phone}_1, \text{Phone}_2, \text{Phone}_3\},$$

and take

$$P = S,$$

so that all three smartphones are included in the plithogenic evaluation.

The considered attribute is

$$v = \text{"Battery performance"},$$

with possible qualitative levels

$$Pv = \{\text{Poor}, \text{Average}, \text{Good}, \text{Excellent}\}.$$

[1] It is important to note that the definition of the Degree of Appurtenance Function varies across different papers. Some studies define this concept using the power set, while others simplify it by avoiding the use of the power set [15]. The author has consistently defined the Classical Plithogenic Set without employing the power set.

We set $s = t = 1$, so that both appurtenance and contradiction are scalar in $[0, 1]$.

The Degree of Appurtenance Function $pdf : P \times Pv \to [0, 1]$ captures how strongly each phone belongs to each battery level based on lab tests and user surveys:

	Poor	Average	Good	Excellent
Phone_1	0.05	0.25	0.60	0.10
Phone_2	0.00	0.10	0.50	0.40
Phone_3	0.30	0.50	0.20	0.00

For instance, $pdf(\text{Phone}_2, \text{Excellent}) = 0.40$ means that Phone_2 has a reasonably strong membership to the "Excellent" battery category, while $pdf(\text{Phone}_3, \text{Poor}) = 0.30$ indicates that Phone_3 often performs poorly in battery tests.

The Degree of Contradiction Function $pCF : Pv \times Pv \to [0, 1]$ quantifies how contradictory two qualitative levels are when reasoning about battery performance:

$pCF(\cdot, \cdot)$	Poor	Average	Good	Excellent
Poor	0.0	0.3	0.7	1.0
Average	0.3	0.0	0.4	0.8
Good	0.7	0.4	0.0	0.3
Excellent	1.0	0.8	0.3	0.0

By construction, $pCF(a, a) = 0$ for every $a \in Pv$, and the table is symmetric, so $pCF(a, b) = pCF(b, a)$. The pair $(\text{Poor}, \text{Excellent})$ has contradiction 1.0, representing maximal conflict, while $(\text{Good}, \text{Excellent})$ has only moderate contradiction 0.3, as they are qualitatively close.

Thus

$$PS = \big(P,\, v,\, Pv,\, pdf,\, pCF\big)$$

forms a *Plithogenic Set* describing smartphone battery performance. Decision makers can further combine the plithogenic information with other attributes (e.g. price, camera quality) using plithogenic aggregation operators, while explicitly taking into account the contradiction between the different qualitative battery levels.

Definition 1.6 (Plithogenic Directed Set)**.** Let

$$(D,\, v,\, \text{pdf},\, \text{pCF})$$

be a plithogenic set, where:

- D is a nonempty domain of elements.
- $v : D \to P_v$ is an attribute function assigning to each $x \in D$ an attribute value $v(x)$ in the set P_v.
- $\text{pdf} : D \times P_v \to [0, 1]^s$ is the *Degree of Appurtenance Function*, so that $\text{pdf}(x, a)$ gives the s-dimensional membership degree of element x to attribute value a.

- pCF : $P_v \times P_v \to [0,1]^t$ is the *Degree of Contradiction Function*, satisfying for all $a, b \in P_v$

$$\text{pCF}(a,a) = 0, \quad \text{pCF}(a,b) = \text{pCF}(b,a).$$

A *plithogenic directed set* is a pair (D, R) where

$$R : D \times D \longrightarrow [0,1]^s$$

is a plithogenic relation (an s-vector of degrees) that satisfies:

(1) **(1) Adjusted Transitivity:** For all $x, y, z \in D$,

$$R(x,z) \succeq \min\{R(x,y),\, R(y,z)\} \ominus \text{pCF}\big(v(x), v(z)\big),$$

where min and $\ominus$ are taken component-wise in $[0,1]^s$.

(2) **(2) Acyclicity:** For any finite sequence $x_1, x_2, \ldots, x_n$ in D forming a cycle $x_{n+1} = x_1$,

$$\min\{\, R(x_1,x_2),\, R(x_2,x_3),\, \ldots,\, R(x_n,x_1)\} \preceq \text{pCF}\big(v(x_1), v(x_1)\big) = (0, \ldots, 0).$$

A concrete example of this concept is presented below.

Example 1.7 (Plithogenic Directed Set in an Employee Trust Network)**.** Employee trust refers to the confidence and belief employees have in each other and in management's fairness, competence, and integrity (cf. [17, 18]). Imagine a small team of three colleagues whose mutual trust depends on both their interpersonal relations and departmental differences:

$$D = \{\text{Ayuka, Taichi, Masahiro}\}, \qquad P_v = \{\text{Management, Engineering, Sales}\}.$$

Each person's primary department is given by the attribute function:

$$v(\text{Ayuka}) = \text{Management}, \quad v(\text{Taichi}) = \text{Engineering}, \quad v(\text{Masahiro}) = \text{Sales}.$$

We use a single-dimensional membership ($s = t = 1$).

1. Degree of Appurtenance (pdf). This measures how strongly each person identifies with each department (self-identification vs. others):

$$\begin{aligned} \text{pdf}(\text{Ayuka}, \text{Management}) &= 0.90, & \text{pdf}(\text{Ayuka}, \text{Engineering}) &= 0.10, \\ \text{pdf}(\text{Ayuka}, \text{Sales}) &= 0.00, & \text{pdf}(\text{Taichi}, \text{Management}) &= 0.00, \\ \text{pdf}(\text{Taichi}, \text{Engineering}) &= 0.80, & \text{pdf}(\text{Taichi}, \text{Sales}) &= 0.20, \\ \text{pdf}(\text{Masahiro}, \text{Management}) &= 0.00, & \text{pdf}(\text{Masahiro}, \text{Engineering}) &= 0.30, \\ \text{pdf}(\text{Masahiro}, \text{Sales}) &= 0.70. \end{aligned}$$

For instance, Ayuka is almost entirely "management," while Taichi has a small crossover into sales (e.g. due to client-facing projects).

2. Degree of Contradiction (pCF). Different departments may clash culturally, reducing trust across them:

$$\mathrm{pCF}(\mathrm{Management}, \mathrm{Engineering}) = 0.40,$$

$$\mathrm{pCF}(\mathrm{Management}, \mathrm{Sales}) = 0.60,$$

$$\mathrm{pCF}(\mathrm{Engineering}, \mathrm{Sales}) = 0.30.$$

A higher value means greater potential misunderstanding or friction.

3. Trust Relation R. We record direct trust ratings on a scale $[0, 1]$:

$$R(\mathrm{Ayuka}, \mathrm{Taichi}) = 0.70, \quad R(\mathrm{Taichi}, \mathrm{Masahiro}) = 0.60, \quad R(\mathrm{Ayuka}, \mathrm{Masahiro}) = 0.10.$$

These might come from peer-review surveys: Ayuka trusts Taichi fairly well, Taichi trusts Masahiro moderately, but Ayuka and Masahiro hardly interact, so trust is low.

4. Adjusted Transitivity. We expect that if Ayuka trusts Taichi and Taichi trusts Masahiro, then Ayuka's trust in Masahiro should be at least

$$\min\{0.70, 0.60\} - \mathrm{pCF}(\mathrm{Management}, \mathrm{Sales}) = 0.60 - 0.60 = 0.00.$$

Since $R(\mathrm{Ayuka}, \mathrm{Masahiro}) = 0.10 \geq 0.00$, the "adjusted" triangle-inequality holds, accounting for departmental friction.

5. Acyclicity. No sequence of trust ratings forms a nontrivial cycle that violates consistency. For instance, there is no direct trust Masahiro$\to$Ayuka, so no cycle exists.

Thus all axioms of a plithogenic directed set are satisfied. Intuitively, this model captures how both interpersonal ratings and inter-departmental "contradiction" shape overall trust across the network.

Definition 1.8 (Plithogenic Multi-Directed Set)**.** [19] Let

$$(M, R, i, t, v, \mathrm{pdf}, \mathrm{pCF})$$

be a structure with:

- M a nonempty set of nodes.
- $v : M \to P_v$ an attribute function assigning each node x a value $v(x) \in P_v$.
- $\mathrm{pdf} : M \times P_v \to [0, 1]^s$ the Degree of Appurtenance Function for nodes.
- R a nonempty set of directed edges (relations).
- $i : R \to M$ and $t : R \to M$ the functions giving each relation r its initial node $i(r)$ and terminal node $t(r)$.
- $\mathrm{pCF} : P_v \times P_v \to [0, 1]^t$ the Degree of Contradiction Function, with $\mathrm{pCF}(a, a) = 0$ and $\mathrm{pCF}(a, b) = \mathrm{pCF}(b, a)$.

A *plithogenic multi-directed set* is such a septuple in which each relation $r \in R$ is assigned an effective degree

$$\deg(r) = \text{combine}\big(\text{pdf}(i(r), v(i(r))),\ \text{pdf}(t(r), v(t(r)))\big) \ominus \text{pCF}\big(v(i(r)), v(t(r))\big),$$

where combine is a suitable aggregation (e.g. minimum or product) applied component-wise in $[0,1]^s$, and $\ominus$ denotes component-wise subtraction bounded below by zero.

A concrete example of the concept is presented below.

Example 1.9 (Project Workflow as a Plithogenic Multi-Directed Set)**.** Project workflow is a structured sequence of tasks and processes that guide project execution from initiation to completion efficiently (cf. [20,21]). Consider a simple project workflow with three tasks:

$$M = \{\text{Req, Dev, Test}\},$$

and the attribute "Department" with possible values

$$P_v = \{\text{Analysis, Development, QA}\}.$$

Define

$$v(\text{Req}) = \text{Analysis}, \quad v(\text{Dev}) = \text{Development}, \quad v(\text{Test}) = \text{QA}.$$

Take $s = t = 1$. The Degree of Appurtenance Function pdf : $M \times P_v \to [0,1]$ is:

$$\text{pdf}(\text{Req}, \text{Analysis}) = 0.9, \quad \text{pdf}(\text{Dev}, \text{Development}) = 0.8, \quad \text{pdf}(\text{Test}, \text{QA}) = 0.9,$$

and zero for all other pairs. The Degree of Contradiction Function pCF : $P_v \times P_v \to [0,1]$ is

$$\text{pCF}(\text{Analysis}, \text{Development}) = 0.4, \quad \text{pCF}(\text{Analysis}, \text{QA}) = 0.6, \quad \text{pCF}(\text{Development}, \text{QA}) = 0.5,$$

with $\text{pCF}(a,a) = 0$ and symmetry.

Let the set of directed relations be

$$R = \{\, r_1,\ r_2,\ r_3\},$$

where

$$i(r_1) = \text{Req},\ t(r_1) = \text{Dev}; \quad i(r_2) = \text{Dev},\ t(r_2) = \text{Test}; \quad i(r_3) = \text{Req},\ t(r_3) = \text{Test}.$$

Using combine = min and component-wise subtraction, the effective degree of each relation is

$$\begin{aligned}
\deg(r_1) &= \min\{\text{pdf}(\text{Req}, v(\text{Req})),\ \text{pdf}(\text{Dev}, v(\text{Dev}))\} - \text{pCF}(\text{Analysis}, \text{Development}) \\
&= \min\{0.9,\ 0.8\} - 0.4 = 0.4, \\
\deg(r_2) &= \min\{0.8,\ 0.9\} - 0.5 = 0.3, \\
\deg(r_3) &= \min\{0.9,\ 0.9\} - 0.6 = 0.3.
\end{aligned}$$

Thus

$$(M,\ R,\ i,\ t,\ v,\ \text{pdf},\ \text{pCF})$$

forms a valid plithogenic multi-directed set modeling task dependencies adjusted by inter-departmental contradictions.

As supplementary information, Table 1 presents the *Comparison of Plithogenic Set, Plithogenic Directed Set, and Plithogenic Multi-Directed Set*. In future research involving computational experiments or case studies, these concepts should be selected appropriately depending on the nature of the target application and the underlying modeling requirements.

Table 1. Comparison of Plithogenic Set, Plithogenic Directed Set, and Plithogenic Multi-Directed Set

Aspect	Plithogenic Set	Plithogenic Directed Set	Plithogenic Multi-Directed Set
Underlying universe	Subset $P \subseteq S$ with attribute values	Nonempty set D with one graded directed relation	Nonempty node set M with many directed relations (edges)
Core structure	Tuple (P, v, P_v, pdf, pCF)	Pair (D, R) with $R : D \times D \to [0, 1]^s$	Septuple $(M, R, i, t, v, pdf, pCF)$ with $i, t : R \to M$
Directionality	No explicit direction; elements are only evaluated by attributes	Single directed plithogenic relation on ordered pairs of elements	Multiple directed edges with explicit initial and terminal nodes
Membership degrees	$pdf : P \times P_v \to [0, 1]^s$ for element–attribute pairs	pdf on D and graded relation $R(x, y) \in [0, 1]^s$	pdf on nodes; edge degrees derived from node memberships (via combine)
Role of contradiction	pCF modulates aggregation among attribute values	pCF corrects transitivity and enforces acyclicity of R	pCF discounts edge strengths according to attribute conflicts between endpoints
Typical interpretation	Plithogenic attribute evaluation of objects	Plithogenic ordered / preference structure on a domain	Plithogenic directed network or workflow with heterogeneous relations

1.3. *Plithogenic Offset*

Plithogenic Offset extends plithogenic sets by allowing both the appurtenance degrees and the contradiction degrees to underflow below 0 or overflow above 1 [22, 23]. As with the

Plithogenic Set, numerous studies have investigated Plithogenic Offset [24,25], and it is well known for generalizing various offset-based frameworks, including the Neutrosophic Offset [26,27].

Definition 1.10 (Plithogenic Offset)**.** (cf. [28]) Let S be a universal set, and $P \subseteq S$. A *Plithogenic Offset* PS_{off} is defined as:

$$PS_{\text{off}} = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the set of possible values for the attribute v.
- $pdf : P \times Pv \to [\Psi_v, \Omega_v]^s$ is the *Degree of Appurtenance Function (DAF)*, where $\Psi_v < 0$ and $\Omega_v > 1$.
- $pCF : Pv \times Pv \to [\Psi_v, \Omega_v]^t$ is the *Degree of Contradiction Function (DCF)*.

In this definition, the DAF and DCF allow the membership degrees $pdf(x, a)$ to range from below 0 to above 1, between the underlimit Ψ_v and the overlimit Ω_v.

A concrete example of the concept is presented below.

Example 1.11 (Plithogenic Offset in Project Management)**.** Project management involves planning, executing, and overseeing tasks, resources, and timelines to achieve specific organizational goals effectively [29,30]. Let

$$S = \{\text{Proj}_1, \text{Proj}_2\}, \qquad P = S,$$

and consider the attribute

$$v(x) = \text{“Criterion”}, \quad P_v = \{\text{Cost, Schedule, Quality}\}.$$

We set the underlimit $\Psi_v = -0.2$ (20% underrun) and overlimit $\Omega_v = 1.3$ (130% overrun), with $s = t = 1$.

Degree of Appurtenance Function $\text{pdf} : P \times P_v \to [\,-0.2, 1.3\,]$:

$$\text{pdf}(\text{Proj}_1, \text{Cost}) = 1.2, \quad \text{pdf}(\text{Proj}_1, \text{Schedule}) = -0.1, \quad \text{pdf}(\text{Proj}_1, \text{Quality}) = 0.9,$$

$$\text{pdf}(\text{Proj}_2, \text{Cost}) = 0.8, \quad \text{pdf}(\text{Proj}_2, \text{Schedule}) = 1.1, \quad \text{pdf}(\text{Proj}_2, \text{Quality}) = -0.2.$$

Here, $\text{pdf}(x, \text{Cost}) > 1$ indicates budget overrun; $\text{pdf}(x, \text{Schedule}) < 0$ indicates early completion.

Degree of Contradiction Function $\text{pCF} : P_v \times P_v \to [\,-0.2, 1.3\,]$, with reflexivity and symmetry:

$$\text{pCF}(a, a) = 0, \quad \text{pCF}(\text{Cost}, \text{Schedule}) = 0.5, \quad \text{pCF}(\text{Cost}, \text{Quality}) = 1.1, \quad \text{pCF}(\text{Schedule}, \text{Quality}) = 0.4.$$

An over-limit $\mathrm{pCF}(\mathrm{Cost}, \mathrm{Quality}) > 1$ reflects extreme conflict between cost and quality objectives.

Then

$$PS_{\mathrm{off}} = \big(P,\ v,\ P_v,\ \mathrm{pdf},\ \mathrm{pCF}\big)$$

is a plithogenic offset set modeling project performance on cost, schedule, and quality, allowing membership degrees beyond $[0,1]$.

Example 1.12 (Examples of Plithogenic Offsets)**.** The main types of Plithogenic OffSets according to the values of s and t are summarized in Table 2. When $t = 0$, the structure reduces to the usual uncertainty frameworks, such as the Fuzzy Offset, Intuitionistic Fuzzy OffSet, and Neutrosophic OffSet.

Table 2. Examples of Plithogenic OffSets

s	t	Name of PS_{off}
1	1	Plithogenic Fuzzy OffSet
2	1	Plithogenic Intuitionistic Fuzzy OffSet
3	1	Plithogenic Neutrosophic OffSet
4	1	Plithogenic quadripartitioned Neutrosophic OffSet
5	1	Plithogenic pentapartitioned Neutrosophic OffSet
6	1	Plithogenic hexapartitioned Neutrosophic OffSet
7	1	Plithogenic heptapartitioned Neutrosophic OffSet
8	1	Plithogenic octapartitioned Neutrosophic OffSet
9	1	Plithogenic nonapartitioned Neutrosophic OffSet

For reference, Table 3 presents an overview of the Comparison of Plithogenic Set and Plithogenic Offset.

2. Results of This Paper

In this section, we present several extended concepts as the Results of This Paper.

2.1. *Plithogenic Directed Offset*

A plithogenic directed offset is a directed relation with offset-valued strengths, obeying contradiction-adjusted transitivity and acyclicity constraints on uncertainty-graded elements. The definitions of Plithogenic Directed Offset is provided below.

Definition 2.1 (Plithogenic Directed Offset)**.** Let

$$(D,\ v,\ \mathrm{pdf},\ \mathrm{pCF})$$

Table 3. Comparison of Plithogenic Set and Plithogenic Offset

Aspect	Plithogenic Set	Plithogenic Offset
Basic structure	$PS = (P, v, Pv, pdf, pCF)$ with standard ranges	Same tuple (P, v, Pv, pdf, pCF) but with extended value ranges
Appurtenance range	$pdf : P \times Pv \to [0, 1]^s$	$pdf : P \times Pv \to [\Psi_v, \Omega_v]^s$ with $\Psi_v < 0 < 1 < \Omega_v$
Contradiction range	$pCF : Pv \times Pv \to [0, 1]^t$	$pCF : Pv \times Pv \to [\Psi_v, \Omega_v]^t$
Membership interpretation	Degrees stay inside the unit interval $[0, 1]$	Allows under–membership (< 0) and over–membership (> 1) as offsets
Relationship	Base plithogenic framework	Generalization; reduces to a Plithogenic Set when $\Psi_v = 0$ and $\Omega_v = 1$
Typical usage	Standard uncertainty and contradiction modeling	Modeling penalties, bonuses, or out-of-range evaluations in complex decisions

be a *plithogenic offset set* with attribute $v : D \to P_v$, pdf $: D \times P_v \to [\Psi_v, \Omega_v]^s$, and pCF $: P_v \times P_v \to [\Psi_v, \Omega_v]^t$, where $\Psi_v < 0 < 1 < \Omega_v$. A *plithogenic directed offset* is a pair (D, R) such that

$$R : D \times D \longrightarrow [\Psi_v, \Omega_v]^s$$

assigns to each ordered pair (x, y) an s-vector of "offset" degrees, and satisfies:

(1) **(1) Offset-Adjusted Transitivity:** For all $x, y, z \in D$,

$$R(x, z) \succeq \min\{R(x, y), R(y, z)\} \ominus \mathrm{pCF}\big(v(x), v(z)\big),$$

with min and $\ominus$ taken component-wise in $[\Psi_v, \Omega_v]^s$, and subtraction clipped to the lower bound Ψ_v.

(2) **(2) Offset-Acyclicity:** For any finite cycle $x_1, x_2, \ldots, x_n, x_{n+1} = x_1$,

$$\min\{R(x_1, x_2), \ldots, R(x_n, x_1)\} \preceq \mathrm{pCF}\big(v(x_1), v(x_1)\big) = (0, \ldots, 0),$$

with the minimum taken component-wise.

Example 2.2 (Plithogenic Directed Offset in Project Scheduling)**.** Project scheduling defines the sequence, duration, and deadlines of tasks, allocating resources to ensure timely project completion [31, 32]. Consider three project phases:

$$D = \{\text{Design, Implementation, Testing}\}.$$

Each phase $x \in D$ has an associated risk level attribute

$$v(x) \in P_v = \{\text{Low, Medium, High}\}.$$

We set the offset-limits $\Psi_v = -0.1$ (10% underrun) and $\Omega_v = 1.1$ (110% overrun), with $s = t = 1$.

Degree of Appurtenance Function $\mathrm{pdf} : D \times P_v \to [-0.1, 1.1]$:

$$\begin{aligned} \mathrm{pdf}(\text{Design}, \text{Low}) &= 1.05, & \mathrm{pdf}(\text{Design}, \text{Medium}) &= 0.40, \\ \mathrm{pdf}(\text{Design}, \text{High}) &= -0.05, & \mathrm{pdf}(\text{Implementation}, \text{Low}) &= 0.20, \\ \mathrm{pdf}(\text{Implementation}, \text{Medium}) &= 1.10, & \mathrm{pdf}(\text{Implementation}, \text{High}) &= 0.30, \\ \mathrm{pdf}(\text{Testing}, \text{Low}) &= -0.10, & \mathrm{pdf}(\text{Testing}, \text{Medium}) &= 0.50, \\ \mathrm{pdf}(\text{Testing}, \text{High}) &= 1.00. \end{aligned}$$

Here values above 1 indicate extra slack, and below 0 indicate early completion.

Degree of Contradiction Function $\mathrm{pCF} : P_v \times P_v \to [-0.1, 1.1]$, with reflexivity and symmetry:

$$\begin{aligned} \mathrm{pCF}(\text{Low, Medium}) &= 0.25, & \mathrm{pCF}(\text{Medium, High}) &= 0.45, \\ \mathrm{pCF}(\text{Low, High}) &= 0.70, & \mathrm{pCF}(a, a) &= 0 \quad \forall\, a \in P_v. \end{aligned}$$

Directed Offset Relation $R : D \times D \to [-0.1, 1.1]$ gives the offset-adjusted dependency strength:

$$R(\text{Design}, \text{Implementation}) = 1.00,$$
$$R(\text{Implementation}, \text{Testing}) = 0.95,$$
$$R(\text{Design}, \text{Testing}) = 0.80.$$

We verify *Offset-Adjusted Transitivity* for

$$\text{Design} \to \text{Testing}$$

:

$$\min\{R(\text{Design}, \text{Implementation}), R(\text{Implementation}, \text{Testing})\}$$
$$-\mathrm{pCF}\big(v(\text{Design}), v(\text{Testing})\big) = 0.95 - 0.70 = 0.25,$$

and indeed

$$R(\text{Design}, \text{Testing}) = 0.80 \ \succeq \ 0.25.$$

No non-trivial cycle exists, so *Offset-Acyclicity* holds. Therefore (D, R) is a valid *Plithogenic Directed Offset*, modeling scheduling dependencies with over- and under-run allowances and risk-level contradictions.

Theorem 2.3. *Every*

(1) Plithogenic Directed Set*, and*

(2) Plithogenic Offset*,*

can be realized as a special case of a Plithogenic Directed Offset*.*

Proof. We treat each embedding in turn.

(i) Embedding a Plithogenic Directed Set. Let (D, R) be a plithogenic directed set with $\mathrm{pdf} : D \times P_v \to [0,1]^s$ and $\mathrm{pCF} : P_v \times P_v \to [0,1]^t$. Define an offset-extension by choosing $\Psi_v := 0$ and $\Omega_v := 1$. Then pdf and pCF naturally range in $[0,1]^s$ and $[0,1]^t$. The same relation R now takes values in $[\Psi_v, \Omega_v]^s$. By construction, its adjusted transitivity and acyclicity in $[0,1]$ imply the offset-adjusted versions with $\Psi_v = 0$. Hence (D, R) is a valid plithogenic directed offset.

(ii) Embedding a Plithogenic Offset. Let $(D, v, \mathrm{pdf}, \mathrm{pCF})$ be a plithogenic offset (no direction relation). We embed it by defining the trivial relation

$$R(x, y) := \mathrm{pdf}(x, v(x)) \quad \forall\, x, y \in D.$$

Since R ignores y, transitivity holds:

$$R(x, z) = \mathrm{pdf}(x, v(x)) \succeq \min\{\mathrm{pdf}(x, v(x)), \mathrm{pdf}(y, v(y))\} \ominus \mathrm{pCF}\big(v(x), v(z)\big),$$

because min on the right never exceeds $\mathrm{pdf}(x, v(x))$, and acyclicity is trivial. Thus this (D, R) is a plithogenic directed offset that recovers the original offset membership via $R(x, x) = \mathrm{pdf}(x, v(x))$.

In both cases, the original structure is obtained by specialization: setting the offset-limits to $[0, 1]$ or by choosing a trivial direction. Therefore, every plithogenic directed set and every plithogenic offset embed into the unified framework of a plithogenic directed offset. □

2.2. *Plithogenic MultiDirected Offset*

A plithogenic multidirected offset assigns offset-valued degrees to multiple parallel edges between nodes, aggregating appurtenance and contradiction into effective strengths. The definitions of Plithogenic MultiDirected Offset is provided below.

Definition 2.4 (Plithogenic Multi-Directed Offset)**.** Let

$$(M,\, R,\, i,\, t,\, v,\, \mathrm{pdf},\, \mathrm{pCF})$$

be a plithogenic offset structure, where

- M is a nonempty set of nodes.
- $v : M \to P_v$ assigns each node x an attribute value $v(x) \in P_v$.

- $\mathrm{pdf} : M \times P_v \to [\Psi_v, \Omega_v]^s$ is the Degree of Appurtenance Function, with $\Psi_v < 0 < 1 < \Omega_v$.
- R is a nonempty set of directed edges.
- $i : R \to M$ and $t : R \to M$ give each edge r its initial node $i(r)$ and terminal node $t(r)$.
- $\mathrm{pCF} : P_v \times P_v \to [\Psi_v, \Omega_v]^t$ is the Degree of Contradiction Function, satisfying $\mathrm{pCF}(a, a) = 0$ and $\mathrm{pCF}(a, b) = \mathrm{pCF}(b, a)$.

A *plithogenic multi-directed offset* is this septuple together with an *effective degree* assignment

$$\deg(r) = \operatorname{combine}\big(\mathrm{pdf}(i(r), v(i(r))),\ \mathrm{pdf}(t(r), v(t(r)))\big) \ominus \mathrm{pCF}\big(v(i(r)), v(t(r))\big),$$

for each $r \in R$. Here combine is a chosen aggregation (e.g. component-wise minimum) in $[\Psi_v, \Omega_v]^s$, and $\ominus$ denotes component-wise subtraction clipped at the lower bound Ψ_v.

A concrete example of this concept is presented below.

Example 2.5 (Plithogenic Multi-Directed Offset in Supply Chain Shipments)**.** A supply chain is a network of organizations, people, activities, and resources involved in producing and delivering a product to consumers [33, 34]. Consider a simple three-stage supply chain:

$$M = \{\text{Warehouse, Distribution, Retailer}\}.$$

Each node $x \in M$ has an attribute "Facility Type"

$$v(x) \in P_v = \{\text{Warehouse, Distribution, Retail}\}.$$

We allow shipment reliability (over- or under-capacity) to exceed $[0, 1]$, so set offset bounds $\Psi_v = -0.2$ (20% under-capacity) and $\Omega_v = 1.2$ (20% over-capacity), and take $s = t = 1$.

Degree of Appurtenance Function $\mathrm{pdf} : M \times P_v \to [-0.2, 1.2]$, giving each facility's effective capacity to handle its own type:

$$\mathrm{pdf}(\text{Warehouse}, \text{Warehouse}) = 1.10, \quad \mathrm{pdf}(\text{Distribution}, \text{Distribution}) = 0.95,$$
$$\mathrm{pdf}(\text{Retailer}, \text{Retail}) = 1.05, \quad \mathrm{pdf}(\text{Warehouse}, \text{Distribution}) = 0.80,$$
$$\mathrm{pdf}(\text{Distribution}, \text{Retail}) = 0.85, \quad \mathrm{pdf}(\text{Warehouse}, \text{Retail}) = 0.60.$$

all other pairs are set to the under-limit -0.2.

Degree of Contradiction Function $\mathrm{pCF} : P_v \times P_v \to [-0.2, 1.2]$, symmetric with zero diagonal:

$$\mathrm{pCF}(\text{Warehouse}, \text{Distribution}) = 0.30, \quad \mathrm{pCF}(\text{Distribution}, \text{Retail}) = 0.25,$$
$$\mathrm{pCF}(\text{Warehouse}, \text{Retail}) = 0.50, \quad \mathrm{pCF}(a, a) = 0.$$

Directed Edges and Effective Degrees Let the set of shipment relations be

$$R = \{ r_1, r_2, r_3 \},$$

with

$$i(r_1) = \text{Warehouse},\ t(r_1) = \text{Distribution};$$

$$i(r_2) = \text{Distribution},\ t(r_2) = \text{Retailer};$$

$$i(r_3) = \text{Warehouse},\ t(r_3) = \text{Retailer}.$$

Using combine = min and component-wise subtraction clipped at -0.2, we compute:

$$\deg(r_1) = \min\{1.10,\, 0.95\} - 0.30 = 0.65,$$
$$\deg(r_2) = \min\{0.95,\, 1.05\} - 0.25 = 0.70,$$
$$\deg(r_3) = \min\{1.10,\, 1.05\} - 0.50 = 0.55.$$

Offset-Adjusted Transitivity Check that Warehouse→Retailer meets transitivity via Distribution:

$$\min\{\deg(r_1), \deg(r_2)\} - \text{pCF}(\text{Warehouse}, \text{Retail}) = 0.65\ -\ 0.50 = 0.15,$$

and indeed $\deg(r_3) = 0.55 \succeq 0.15$.

Offset-Acyclicity No non-trivial cycle exists among three stages.

Therefore

$$(M,\, R,\, i,\, t,\, v,\, \text{pdf},\, \text{pCF})$$

is a valid *plithogenic multi-directed offset*, modeling shipments with over-capacity, under-capacity, and inter-facility contradictions.

Example 2.6 (Plithogenic Multi-Directed Offset in Power Grid Transmission)**.** A power grid is a network of electrical transmission lines and substations delivering electricity from producers to homes and businesses [35, 36]. Consider a simplified power grid with three substations:

$$M = \{A,\, B,\, C\},$$

each characterized by its primary energy source:

$$v(x) \in P_v = \{\text{Renewable, Thermal, Hydro}\}.$$

We allow transmission reliability to underflow or overflow, so set offset bounds $\Psi_v = -0.2$, $\Omega_v = 1.2$, and take $s = 2$, $t = 2$.

Degree of Appurtenance Function pdf : $M \times P_v \to [-0.2, 1.2]^2$, giving each substation's *(capacity reliability, redundancy)* membership:

$$\text{pdf}(A, \text{Renewable}) = (1.10,\ 0.95),$$
$$\text{pdf}(A, \text{Thermal}) = (0.30,\ 0.40),$$
$$\text{pdf}(A, \text{Hydro}) = (-0.10,\ 0.20),$$
$$\text{pdf}(B, \text{Renewable}) = (0.25,\ 0.35),$$
$$\text{pdf}(B, \text{Thermal}) = (1.05,\ 0.90),$$
$$\text{pdf}(B, \text{Hydro}) = (0.15,\ 0.25),$$
$$\text{pdf}(C, \text{Renewable}) = (0.10,\ 0.20),$$
$$\text{pdf}(C, \text{Thermal}) = (0.20,\ 0.30),$$
$$\text{pdf}(C, \text{Hydro}) = (1.15,\ 1.00).$$

Values above 1 indicate extra margin, below 0 early capacity loss.

Degree of Contradiction Function pCF : $P_v \times P_v \to [-0.2, 1.2]^2$, symmetric with zero diagonal:

$$\text{pCF}(\text{Renewable}, \text{Thermal}) = (0.50,\ 0.40),$$
$$\text{pCF}(\text{Renewable}, \text{Hydro}) = (0.30,\ 0.50),$$
$$\text{pCF}(\text{Thermal}, \text{Hydro}) = (0.60,\ 0.35).$$

Edges and Effective Degrees Define transmission links

$$R = \{r_1, r_2, r_3\},$$

with

$$i(r_1) = A,\ t(r_1) = B; \quad i(r_2) = B,\ t(r_2) = C; \quad i(r_3) = A,\ t(r_3) = C.$$

Using combine = min and component-wise subtraction clipped at Ψ_v:

$$\deg(r_1) = \min\{(1.10, 0.95),\ (1.05, 0.90)\} \ominus (0.50, 0.40) = (1.05, 0.90) - (0.50, 0.40) = (0.55, 0.50),$$
$$\deg(r_2) = \min\{(1.05, 0.90),\ (1.15, 1.00)\} \ominus (0.60, 0.35) = (1.05, 0.90) - (0.60, 0.35) = (0.45, 0.55),$$
$$\deg(r_3) = \min\{(1.10, 0.95),\ (1.15, 1.00)\} \ominus (0.30, 0.50) = (1.10, 0.95) - (0.30, 0.50) = (0.80, 0.45).$$

Offset-Adjusted Transitivity Check link $A \to C$ via B:

$$\min\{\deg(r_1), \deg(r_2)\} - \text{pCF}(\text{Renewable}, \text{Hydro}) = (0.45, 0.50) - (0.30, 0.50) = (0.15, 0.00),$$

and indeed $\deg(r_3) = (0.80, 0.45) \succeq (0.15, 0.00)$.

Offset-Acyclicity No non-trivial cycle exists among the three substations.

Thus

$$(M,\, R,\, i,\, t,\, v,\, \text{pdf},\, \text{pCF})$$

is a valid *Plithogenic Multi-Directed Offset*, modeling transmission reliability with over-/under-capacity and inter-source contradictions.

Example 2.7 (Plithogenic Multi-Directed Offset in Water Reservoir Network)**.** A water reservoir is a storage system that collects, holds, and manages water for purposes like irrigation, drinking, flood control, and energy [37,38]. Consider three interconnected reservoirs:

$$M = \{\text{A, B, C}\},$$

each with a primary water source attribute:

$$v(x) \in P_v = \{\text{Rain-fed, River-fed, Groundwater}\}.$$

We allow "storage resilience" values to underflow or overflow, so set the bounds $\Psi_v = -0.1$ (10% under-resilience) and $\Omega_v = 1.1$ (10% over-resilience), and take $s = 2$, $t = 1$.

Degree of Appurtenance Function $\text{pdf} : M \times P_v \to [-0.1, 1.1]^2$ assigns to each reservoir its *(capacity reliability, overflow buffer)* membership:

$$\text{pdf}(A, \text{Rain-fed}) = (1.05,\, 0.80), \qquad \text{pdf}(A, \text{River-fed}) = (0.50,\, 0.40),$$
$$\text{pdf}(B, \text{River-fed}) = (1.00,\, 0.90), \quad \text{pdf}(B, \text{Groundwater}) = (0.30,\, 0.20),$$
$$\text{pdf}(C, \text{Groundwater}) = (1.10,\, 1.00), \qquad \text{pdf}(C, \text{Rain-fed}) = (0.40,\, 0.30).$$

Values above 1 indicate extra buffer, below 0 indicate risk of underfill.

Degree of Contradiction Function $\text{pCF} : P_v \times P_v \to [-0.1, 1.1]$, symmetric with zero diagonal:

$$\text{pCF}(\text{Rain-fed}, \text{River-fed}) = 0.4,$$

$$\text{pCF}(\text{River-fed}, \text{Groundwater}) = 0.5,$$

$$\text{pCF}(\text{Rain-fed}, \text{Groundwater}) = 0.6.$$

Directed Edges and Effective Degrees Let the set of flow relations be

$$R = \{\, r_{AB},\, r_{BC},\, r_{AC}\},$$

with

$$i(r_{AB}) = A,\ t(r_{AB}) = B; \quad i(r_{BC}) = B,\ t(r_{BC}) = C; \quad i(r_{AC}) = A,\ t(r_{AC}) = C.$$

Using combine = min and subtraction clipped at Ψ_v, we compute:

$$\deg(r_{AB}) = \min\{(1.05, 0.80),\ (1.00, 0.90)\} \ominus 0.4 = (1.00, 0.80) - (0.4,\ 0.4) = (0.60, 0.40),$$

$$\deg(r_{BC}) = \min\{(1.00, 0.90),\ (1.10, 1.00)\} \ominus 0.5 = (1.00, 0.90) - (0.5,\ 0.5) = (0.50, 0.40),$$

$$\deg(r_{AC}) = \min\{(1.05, 0.80),\ (1.10, 1.00)\} \ominus 0.6 = (1.05, 0.80) - (0.6,\ 0.6) = (0.45, 0.20).$$

Offset-Adjusted Transitivity Check flow $A \to C$ via B:

$$\min\{\deg(r_{AB}), \deg(r_{BC})\} \ominus \mathrm{pCF}(\text{Rain-fed}, \text{Groundwater}) = (0.50, 0.40) - (0.6,\ 0.6) = (-0.10,\ -0.20),$$

clipped to $\Psi_v = -0.1$ yields $(-0.1,\ -0.1)$, and indeed $\deg(r_{AC}) = (0.45, 0.20) \succeq (-0.1, -0.1)$.

Offset-Acyclicity No directed cycle exists among these three reservoirs.

Hence

$$(M,\ R,\ i,\ t,\ v,\ \mathrm{pdf},\ \mathrm{pCF})$$

is a valid *plithogenic multi-directed offset*, intuitively modeling reservoir flows with both over- and under-capacity allowances and source-type contradictions.

Theorem 2.8. *Every*

(1) Plithogenic Multi-Directed Set*, and*

(2) Plithogenic Offset*,*

can be realized as a special case of a Plithogenic Multi-Directed Offset*.*

Proof. We prove the two embeddings by explicitly constructing the required offset-enabled relations and checking all defining properties.

(i) Embedding a Plithogenic Multi-Directed Set. Let

$$\mathcal{M} = (M,\ R,\ i,\ t,\ v,\ \mathrm{pdf},\ \mathrm{pCF})$$

be a plithogenic multi-directed set, where $\mathrm{pdf} : M \times P_v \to [0,1]^s$ and $\mathrm{pCF} : P_v \times P_v \to [0,1]^t$, and each relation $r \in R$ has an effective degree

$$\deg(r) = \min\{\mathrm{pdf}(i(r), v(i(r))),\ \mathrm{pdf}(t(r), v(t(r)))\}\ \ominus\ \mathrm{pCF}\big(v(i(r)), v(t(r))\big).$$

We now view this as a multi-directed *offset* by choosing the under- and over-limits

$$\Psi_v = 0, \qquad \Omega_v = 1.$$

Since all pdf and pCF values already lie in $[0, 1]$, they trivially lie in $[\Psi_v, \Omega_v]$. The formula for $\deg(r)$ remains unchanged.

Verification of axioms:

- *Offset-Adjusted Transitivity.* For any $x, y, z \in M$, the original multi-directed set satisfied

$$\deg(x, z) \succeq \min\{\deg(x, y), \deg(y, z)\} \ominus \mathrm{pCF}(v(x), v(z)),$$

and since no value ever falls below $\Psi_v = 0$, this inequality holds identically in the offset setting.

- *Offset-Acyclicity.* Any cycle $r_1, \ldots, r_n$ in R satisfied $\min_i \deg(r_i) \preceq (0, \ldots, 0)$ originally; with $\Psi_v = 0$ this remains valid.

Thus $\mathcal{M}$ is naturally a plithogenic multi-directed offset.

(ii) Embedding a Plithogenic Offset. Let

$$\mathcal{O} = (M,\ v,\ \mathrm{pdf},\ \mathrm{pCF})$$

be a plithogenic offset set (no edges). We construct a trivial relation set

$$R = \{\, r_x : x \in M\}, \quad i(r_x) = t(r_x) = x,$$

and define for each r_x:

$$\deg(r_x) = \mathrm{combine}\big(\mathrm{pdf}(x, v(x)),\ \mathrm{pdf}(x, v(x))\big) \ \ominus\ \mathrm{pCF}\big(v(x), v(x)\big).$$

Choosing combine = min and using $\mathrm{pCF}(a, a) = 0$, we get $\deg(r_x) = \mathrm{pdf}(x, v(x))$.

Verification of axioms:

- *Offset-Adjusted Transitivity.* For any two edges r_x, r_y,

$$\min\{\deg(r_x), \deg(r_y)\} \ \ominus\ \mathrm{pCF}(v(x), v(y)) \ \le\ \mathrm{pdf}(x, v(x)) = \deg(r_x),$$

since $\deg(r_x) = \deg(r_y) = \mathrm{pdf}(x, v(x))$ when $x = y$, and trivial otherwise.

- *Offset-Acyclicity.* All cycles are self-loops, so $\min_i \deg(r_{x_i}) = \deg(r_{x_1})$ and $\mathrm{pCF}(v(x_1), v(x_1)) = 0$, satisfying the cycle condition.

In both constructions, we have exhibited a plithogenic multi-directed offset whose specialization (either by restricting Ψ_v, Ω_v to $[0, 1]$ or collapsing relations) recovers the original plithogenic multi-directed set or plithogenic offset. This completes the proof. □

Theorem 2.9 (Induced Sub-Offset Closure)**.** *Let*

$$\mathcal{O} = (M,\ R,\ i,\ t,\ v,\ \mathrm{pdf},\ \mathrm{pCF})$$

be a plithogenic multi-directed offset on universe M. For any nonempty subset $M' \subseteq M$, define

$$R' = \{\, r \in R : i(r) \in M',\ t(r) \in M'\},$$

and restrict all other structure maps to M'. Then

$$\mathcal{O}' = (M',\ R',\ i|_{R'},\ t|_{R'},\ v|_{M'},\ \mathrm{pdf}|_{M' \times P_v},\ \mathrm{pCF})$$

is also a plithogenic multi-directed offset.

Proof. Let

$$\mathcal{O} = (M,\, R,\, i,\, t,\, v,\, \mathrm{pdf},\, \mathrm{pCF})$$

be a plithogenic multi-directed offset on M, with edge degrees

$$\deg(r) = \min\{\mathrm{pdf}(i(r), v(i(r))),\, \mathrm{pdf}(t(r), v(t(r)))\} \;\ominus\; \mathrm{pCF}\big(v(i(r)), v(t(r))\big).$$

Fix any nonempty $M' \subseteq M$ and set

$$R' = \{\, r \in R : i(r) \in M',\, t(r) \in M'\}.$$

Define

$$\mathcal{O}' = (M',\, R',\, i|_{R'},\, t|_{R'},\, v|_{M'},\, \mathrm{pdf}|_{M'\times P_v},\, \mathrm{pCF}).$$

We check the two defining axioms for $\mathcal{O}'$:

(1) Well-defined degrees. Since $i(r), t(r) \in M'$ for every $r \in R'$, the restricted functions $\mathrm{pdf}|_{M'\times P_v}$ and pCF still take values in $[\Psi_v, \Omega_v]^s$ and $[\Psi_v, \Omega_v]^t$. Moreover, for each $r \in R'$, $\deg'(r)$ defined by the same formula equals the original $\deg(r)$, so each edge in R' has a well-defined offset degree.

(2) Offset-Adjusted Transitivity. Take any three nodes $x, y, z \in M'$ such that $r_{xy}, r_{yz}, r_{xz} \in R'$. Because $\mathcal{O}$ satisfies

$$\deg(r_{xz}) \;\succeq\; \min\{\deg(r_{xy}),\, \deg(r_{yz})\} \;\ominus\; \mathrm{pCF}(v(x), v(z)),$$

and $\deg'$ agrees with deg on edges in R', the same inequality holds in $\mathcal{O}'$. Thus offset-adjusted transitivity is preserved.

(3) Offset-Acyclicity. Suppose there is a finite directed cycle $r_1, \ldots, r_n$ in R'. Since $R' \subseteq R$, this is also a cycle in $\mathcal{O}$. The original acyclicity condition guarantees

$$\min_{1\le i\le n} \deg(r_i) \;\preceq\; \mathrm{pCF}\big(v(x_1), v(x_1)\big) = (0, \ldots, 0).$$

As $\deg' = \deg$ on these edges and pCF is unchanged, the cycle-inequality holds in $\mathcal{O}'$.

Having verified that $\mathcal{O}'$ satisfies the same offset-adjusted transitivity and acyclicity conditions, we conclude that $\mathcal{O}'$ is indeed a plithogenic multi-directed offset. □

Theorem 2.10 (Closure under Edge Composition)**.** *Let*

$$\mathcal{O} = (M,\, R,\, i,\, t,\, v,\, \mathrm{pdf},\, \mathrm{pCF})$$

be a plithogenic multi-directed offset. Define the composed edge set

$$R_{\mathrm{comp}} = \big\{\, r_{x,z} \mid x, z \in M\big\},$$

where $i(r_{x,z}) = x$*,* $t(r_{x,z}) = z$*, and*

$$\deg(r_{x,z}) := \min_{y \in M}\{\deg(r_{x,y}),\ \deg(r_{y,z})\} \ominus \mathrm{pCF}(v(x), v(z)).$$

Let

$$R^* = R \cup R_{\mathrm{comp}}.$$

Then

$$\mathcal{O}^* = (M,\ R^*,\ i^*,\ t^*,\ v,\ \mathrm{pdf},\ \mathrm{pCF})$$

is a plithogenic multi-directed offset.

Proof. We must verify that every new composed edge $r_{x,z}$ satisfies offset-adjusted transitivity and that no cycles are introduced.

(1) Offset-Adjusted Transitivity. For any $x, z, w \in M$, consider the composed edge $r_{x,z}$ and existing edge $r_{z,w}$. By definition,

$$\deg(r_{x,w}) \;=\; \min_{y}\{\deg(r_{x,y}), \deg(r_{y,w})\} \;\ominus\; \mathrm{pCF}(v(x), v(w)).$$

Taking $y = z$ in the minimum yields

$$\deg(r_{x,w}) \;\succeq\; \min\{\deg(r_{x,z}), \deg(r_{z,w})\} \;\ominus\; \mathrm{pCF}(v(x), v(w)),$$

so the new edge $r_{x,w}$ obeys adjusted transitivity with respect to $r_{x,z}$ and $r_{z,w}$. Similarly for any pair of edges.

(2) Offset-Acyclicity. Suppose there were a directed cycle in R^*. If it lies entirely in the original R, acyclicity holds by hypothesis. Otherwise it must traverse at least one composed edge $r_{x,z}$. Unwinding that composed edge replaces it by a two-step path $x \to y \to z$ in the original R. Replacing every composed edge in the cycle by its two-step decomposition yields a cycle in the original graph R, contradicting original acyclicity.

Thus no new genuine cycles are introduced, and $\mathcal{O}^*$ remains acyclic.

Since all other structure components (v, pdf, pCF) are unchanged, $\mathcal{O}^*$ satisfies the definition of a plithogenic multi-directed offset. □

3. Conclusion

In this paper, we introduced and rigorously formalized the notion of a *Plithogenic Multi-Directed Offset*, providing a unified framework that simultaneously captures uncertainty, directionality, and non-standard membership values. We showed that this structure subsumes several existing approaches, including classical plithogenic, directed, and offset-based models.

Future research directions include extending the proposed framework to settings based on HyperGraphs [39,40], SuperHyperGraphs [41,42], and other higher-order relational structures.

We also hope that subsequent work will investigate computational experiments, machine-learning-based applications, and domain-specific case studies carried out in collaboration with experts. In addition, we plan to explore variants of the proposed model that make use of upside-down logic [43,44].

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany this paper.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

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Article

Extending Sustainable Advantage based on John Kay's Distinctive Capabilities, to include Schumacher's Intermediate Technology with applications to Botany etc.

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Abstract: It is known that John Kay's Distinctive Capabilities Framework [1] offers a profound and nuanced understanding of organizational achievement, shifting the focus from the static possession of significant assets (Resource-based approach) to the dynamic cultivation of enduring relational contracts (Relationship-based approach). Kay identified three essential capabilities—*Architecture, Reputation, and Innovation*—as the non-replicable sources of performance and sustainable advantage. These capabilities encapsulate "what makes our organization so special," rooted in the continuity and stability of relationships with customers, suppliers, shareholders, and employees. While conceptually powerful, Kay's framework, in its original form, often lacks the operational precision required for modern execution, and the present article is an attempt to fill the gap. Moreover, in this article we also extend Sustainable Advantage based on John Kay's *Distinctive Capabilities* framework, to include Schumacher's *Intermediate Technology* with applications to Botany etc., for instance new innovative solutions such as laser-culture, gravitational water vortex power plant, confined vortex turbine, new fusion energy theory based on PT-symmetric potential of crystals, and also a plausible new approach to turn plastic waste into biofuel.[1] While several of those innovative solutions are still in "lab scale" phase, it can be expected to yield quite significant results in the near future, especially for less developed countries.

[1] See for instance, our previous articles in, BPAS Journals – Botany section. And also our recent articles, Christianto, Smarandache, Umniyati, ref. [15-17].

Keywords: John Kay, distinctive capabilities framework, Schumacher, plastic waste, PT-symmetric potential, Extended Sustainability Advantage, responsible research and innovation

1. Introduction: The need for measurable distinctiveness

It is known in literature that John Kay's *Distinctive Capabilities* Framework offers a profound and nuanced understanding of organizational achievement, shifting the focus from the static possession of significant assets (Resource-based approach) to the dynamic cultivation of enduring relational contracts (Relationship-based approach). Kay identified three core capabilities—Architecture, Reputation, and Innovation—as the non-replicable sources of excellence performance and sustainable advantage. These capabilities encapsulate "*what makes our organization so special*," rooted in the continuity and stability of relationships with customers, suppliers, shareholders, and employees.

While conceptually powerful, Kay's framework, in its original form, often lacks the operational precision required for modern execution [1]. Theory of planning demands not only *what* to focus on but *how* to measure progress and *when* to adjust course. The abstract nature of "Architecture" or "Reputation" can leave managers without clear metrics, making it challenging to link investments directly to capability development [2][3].

The present article outlines a reformulation of the Distinctive Capabilities Framework, leveraging the principles of **management by measurement** popularized by figures like John Doerr and the Objectives and Key Results (OKR) methodology [6]. By translating Kay's conceptual strengths into a structured, quantifiable system, we create the Distinctive Capabilities Maturity Model (DCMM). This DCMM approach is designed to be as operational and integrative as frameworks like the Risk Maturity Index, providing a clear roadmap and measurable scorecard to guide the organization toward distinctive excellence [4].

2. Methods

We utilized literature survey of relevant books and journal articles, especially those relevant to themes related to John Kay's distinctive capabilities. We also discuss, among other things, how to extend Kay's distinctive capabilities in order to achieve extended Sustainable Advantages. Several examples of responsible research are also discussed.

3. Results

I. The Essential Framework: Kay's Distinctive Capabilities

Before operationalizing the framework, it is essential to re-examine Kay's three capabilities through the lens of measurement potential.

A. Architecture (Relational Structure)

Architecture refers to the intricate structure of internal and external relational contracts. It embodies tacit organizational knowledge, shared routines, and the ease of information exchange that allows for flexible, collaborative responses to change. Distinctive Architecture is a collective interest sustained by mutual commitment and trust, generating value through coordination.

The Measurement Challenge: Architecture is largely intangible. Measuring it requires quantifying the health and efficiency of relationships and knowledge flows, rather than simply counting assets.

B. Reputation (Information Signal)

Reputation serves as a powerful signal of quality and intent to customers, especially when information asymmetry makes informed decisions difficult. Built over time through consistent experience, quality signals (price, promotion), and word-of-mouth, a strong reputation reduces transaction costs and builds a sequence of trust-based relationships.

The Measurement Challenge: Reputation is often measured reactively (e.g., via brand surveys). Operationalizing it requires metrics that track the *antecedents* and *consistency* of the reputation signal itself.

C. Innovation (Sustainable Uniqueness)

Innovation is an undeniable source of distinctiveness. However, Kay noted that successful innovation is often neither sustainable nor easily appropriable, as it quickly attracts imitation. Transforming an innovation into a lasting competitive advantage requires a range of supporting strategies that protect and leverage the initial breakthrough.

The Measurement Challenge: Simply counting patents or new product launches is insufficient. The key is measuring the **appropriability**—the ability of the firm to capture the value of its innovation—and its contribution to long-term revenue streams.

II. The Operational Imperative: Bridging Kay and Doerr

The leap from conceptual capability to measurable action is best facilitated by the OKR methodology. John Doerr's principle, "*Measure What Matters*," advocates for setting an Objective (what is to be achieved) and coupling it with a set of measurable Key Results (KRs) (how we know we achieved it). By applying this framework, Kay's distinctive capabilities can be transformed into actionable strategic mandates.

A. Operationalizing Architecture: The Efficiency and Trust OKRs

Architecture is operationalized by measuring the quality of coordination and the depth of trust within the organization and across its value chain.

Table 1. Operationalizing Architecture based on OKR Framework

Objective (O)	Key Results (KR)	Kay's Rationale
O: Achieve Best-in-Class Internal Knowledge Transfer Efficiency.	KR1: Reduce time-to-onboard new employees/project members by 30% through codified routines.	Measures organizational knowledge and routines efficiency.
KR2: Increase the percentage of cross-functional projects delivered on-time and under budget to 95%.	Measures flexible response to change and coordination.	
O: Cultivate a Collaborative and Resilient External Value Chain Architecture.	KR3: Achieve a Supplier Relationship Index (SRI) score of 4.5/5.0 with top-tier suppliers.	Measures the relational contracting strength with suppliers.
KR4: Decrease average B2B customer issue resolution time requiring cross-party communication by 40%.	Measures open exchange of information and collective problem-solving.	

B. Operationalizing Reputation: The Signal Consistency and Trustworthiness OKRs

Reputation is operationalized by measuring the reliability of the quality signal and the depth of customer/stakeholder trust it generates.

Table 2. Operationalizing Reputation based on OKR Framework

Objective (O)	Key Results (KR)	Kay's Rationale
O: Establish an Unimpeachable Brand Reputation for Quality and Ethics.	KR1: Achieve and sustain a Net Promoter Score (NPS) of 70+ in core customer segments.	Measures customers' own experience and willingness to recommend.
KR2: Reduce the volume of negative social media mentions across all platforms by 50% relative to industry average.	Measures the strength and protection of the quality signal.	
O: Transform Reputation into a Tangible Competitive Differentiator.	KR3: Increase the price premium over the closest competitor to 15% without loss of market share.	Measures the commercial value of the reputation signal (willingness to pay).
KR4: Achieve 80% customer retention solely through brand loyalty (excluding contractual lock-in).	Measures the ability to build a relationship based on trust and prior reputation.	

C. Operationalizing Innovation: The Appropriability and Diffusion OKRs

Innovation is operationalized by measuring the effectiveness of methods that support and protect the initial innovation, ensuring its value is captured by the firm and not quickly imitated.

Table 3. Operationalizing Innovation based on OKR Framework

Objective (O)	Key Results (KR)	Kay's Rationale
O: Develop a Sustained Pipeline of Defensible, Market-Leading Innovations.	KR1: 35% of annual revenue derived from products/services launched in the last three years.	Measures the long-term success and reliance on recent innovation.
KR2: Increase the 'Patent Strength Index' (a measure of patent breadth and enforceability) by 15% annually.	Measures appropriability (the ability to protect the innovation).	
O: Embed Innovation into the Organisational Architecture for Rapid Diffusion and Adaptation.	KR3: Reduce the time-to-market for innovations from concept to commercial scale by 25%.	Measures the supporting strategies required to commercialize innovation efficiently.
KR4: Increase the number of successful cross-platform (internal/external) technology transfers by 50%.	Measures the effectiveness of linking Innovation back to Architecture.	

III. The Distinctive Capabilities Maturity Model (DCMM)

To move beyond episodic OKR measurement toward continuous evaluation, we introduce the Distinctive Capabilities Maturity Model (DCMM). The DCMM serves as the integrative framework, similar to a Risk Maturity Index or the Competencies Maturity Index, to assess the current state of each capability on a structured, five-level scale. This framework allows the organization to benchmark progress and allocate resources more effectively (see for instance Antonucci, 2016).

The five maturity levels, inspired by the Capability Maturity Model Integration (CMMI), are:

Table 4. Operationalized Kay's framework into Distinctive Capabilities Maturity Model

Level	DCMM Name	Description	Score (out of 5.0)
1	Initial (Ad Hoc)	Capability is largely undefined, reactive, and reliant on individual heroism. Outcomes are unpredictable.	1.0 - 1.9

2	Managed (Repeatable)	Basic processes exist, often localized, but are managed and tracked. Success is repeatable, but not yet standardized across the organization.	2.0 - 2.9
3	Defined (Standardized)	The capability (e.g., the relational contract structure or reputation signalling) is clearly documented, standardized, and integrated across all relevant business units.	3.0 - 3.9
4	Quantitatively Managed (Measured)	The capability is managed using specific, data-driven metrics and OKRs. Performance is predictable and variances are understood and controlled.	4.0 - 4.4
5	Optimizing (Distinctive)	The organization is focused on continuous process improvement and adaptation.	4.5 - 5.0

DCMM Scoring and Radar Chart Visualization

The DCMM assigns a quantifiable score (Level 1.0 to 5.0) to each of the three capabilities (Architecture, Reputation, Innovation) based on the criteria in the table above and the achievement of their corresponding OKRs.

This framework is best presented visually through a **Radar Chart**, just like the Core Competencies Maturity Index. The chart plots the three DCMM scores on distinct axes, creating a unique signature for the organization's distinctiveness. A large, balanced polygon indicates high, sustainable, and well-managed distinctive capabilities.

The primary function of the DCMM is to diagnose the organization's competitive edge:

1. Identify Gaps: Low scores (e.g., Innovation at 2.1) signal an urgent need for targeted investment and a renewed OKR cycle.

2. Ensure Balance: A heavily skewed chart (e.g., high Reputation, low Architecture) suggests a brittle advantage, where a change in the market could quickly undermine the single strength.

3. Benchmark Progress: Quarter-over-quarter improvements in the DCMM score demonstrate the success of the overarching methods and the OKR execution.

IV. Implementation: From Measurement to Action

The operationalization of Kay's framework is complete only when the DCMM and its supporting OKRs are fully integrated into a firm’s annual cycle.

The ultimate goal of this reformulation is to make Kay's relationship-based approach a practical reality. By continuously measuring and managing Architecture, Reputation, and Innovation, the organization ensures that its superior performance is not accidental or temporary, but rather the predictable outcome of deliberate investment in its distinctive strengths.

- A high **Architecture** score guarantees a low-cost, flexible, and efficient response to market changes.
- A high **Reputation** score provides a premium in pricing and a buffer during crises, reducing the cost of sales.
- A high **Innovation** score, particularly in its **appropriability** elements, ensures that R&D investments yield proprietary value that competitors cannot easily erode.

When the Overall Weighted Average of the DCMM pushes into Level 4 (Quantitatively Managed) and approaches Level 5 (Optimizing), the organization is no longer just *performing well*; it has attained a **sustainable, measurable, and properly managed advantage**—the ultimate promise of John Kay's profound framework, now equipped with the operational rigor of modern management science.

4.Applications

Plausible ways to extend John Kay's *Distinctive Capabilities*, to include Schumacher's Intermediate Technology with applications to Botany etc.

The pursuit of Sustainable Competitive Advantage (SCA) often focuses on a firm's internal resources and market position, as articulated by John Kay's concept of Distinctive Capabilities. These capabilities—rooted in Architecture (relationships/reputation), Reputation, and Innovation—offer a robust framework for long-term organizational achievements. However, in an increasingly resource-constrained world, this framework must be broadened to incorporate a fundamental principle: that economic activity must serve humanity and the planet.

This article proposes an extension of Kay's model to include E.F. Schumacher's "Intermediate Technology" (which can be conceptualized as "*development as if people mattered*"). By weaving Schumacher's focus on appropriate, human-scale, and environmentally benign technologies into the essence of an organization's distinctive capabilities, we can cultivate a truly Extended Sustainable Advantage (ESA) that is economically resilient, socially responsible, and ecologically regenerative.

Schumacher's Philosophy and the Imperative for Appropriateness

Schumacher's seminal work, *Economics as if People Mattered*, challenged the relentless pursuit of large-scale, capital-intensive technology characteristic of mainstream economics. He advocated for **Intermediate Technology**: methods and tools that are *small* enough to be accessible to local communities, *simple* enough to be maintained without highly specialized expertise, and *non-violent* toward both people and nature.

The essential tenets of this philosophy, which underscore the development of ESA, are:

- **Human Scale:** Technology should enhance human skill, not replace it, and should be comprehensible at the local level.

- **Ecological Soundness:** It must operate within natural limits and promote the regeneration of resources.
- **Accessibility:** Low budget and local material sourcing ensure widespread adoption, especially in developing countries or marginalized communities, whom often have budget constraints.

Intermediate Technology as a Distinctive Capability

When viewing Schumacher's *Intermediate Technology* through the lens of John Kay's Distinctive Capabilities, its value becomes clear:

Table 5. Comparison between Kay's capability and Extended Sustainability Advantage

Kay's Capability	**Integration with Schumacher's Technology**	**Extended Sustainable Advantage (ESA)**
Architecture	Fostering local, participatory development of technology; emphasizing knowledge-sharing over proprietary secrets.	Creates trust, community resilience, and a robust supply/knowledge network that is geographically dispersed and less prone to single-point failure.
Reputation	Being known for ethical sourcing, sustainable production, and genuine community empowerment.	Strong, authentic customer loyalty and social license to operate, appealing to socially conscious consumers and investors.
Innovation	Focusing R&D on simple, elegant, resource-efficient solutions inspired by natural processes (to learn from Nature, as Schauberger wrote).	Generates low-budget, resource-frugal, and easily adaptable technologies that circumvent high capital barriers and appeal to vast, underserved markets.

By prioritizing "**appropriate innovation,**" companies build an architectural advantage rooted in mutual local dependence and a reputational advantage of true organizational/community responsibility.

Intermediate Technology in Action: Botany and Water-Scarce Farming

The agricultural sector, particularly in water-stressed regions, offers a vital application for this Extended Sustainable Advantage model. Here, the integration of innovative, yet appropriate, technology can address both ecological necessity and market demand.

1. Botany-Inspired Techniques: Electric-Culture and Laser-Culture[2]

[2] See for instance, our previous articles in, BPAS Journals – Botany section.

In drylands, reducing water consumption is paramount. Instead of relying solely on capital-intensive desalination or deep-well pumping, ESA suggests investing in technologies that *enhance plant efficiency* with minimal external input.

- **Electric-Culture:** This technique involves using low-level atmospheric electricity or electric fields to stimulate plant growth. Research suggests that it can enhance growth, improve drought tolerance, and potentially reduce the need for certain fertilizers and pesticides. While its mechanisms in botany are still being explored, its application—often involving simple wire antennae or buried conductors—is inherently **intermediate**: low-budget, low-energy (potentially solar-powered), and manageable by local farmers. This addresses water scarcity by requiring less water per unit of yield.

- **Laser-Culture (Biostimulation):** The precise application of low-power lasers at specific wavelengths to seeds or young plants has shown potential for biostimulation, leading to improved germination rates, increased vigour, and enhanced resistance to stress. Used for seed treatment before planting, this can be a simple, non-chemical way to maximize the potential of a limited water supply, embodying appropriate innovation through highly focused energy use.

Both of these "high-tech" concepts are re-imagined through the Intermediate Technology lens: they are applied in a targeted, resource-light manner to boost the natural capabilities of the plant, rather than forcing growth through chemical or brute-force irrigation.

2. Schauberger's Implosion and the Gravitational Water Vortex

Schauberger, the Austrian forester and inventor, championed the idea of learning from nature, particularly the role of the vortex in water and air dynamics. He believed conventional technologies were "violent" (exploiting pressure and heat) while nature's processes were "implosive" (utilizing cooling, suction, and spiralling motion).

The **Gravitational Water Vortex Power Plant (GWVPP)** is a direct embodiment of this insight and an exemplary Intermediate Technology for energy generation and water remediation.

Principle: Water is channelled into a round basin where it forms a stable, powerful vortex over a central drain. This vortex drives a low-speed turbine that generates electricity. The required hydraulic head is very low (less than 3 meters).

Distinctive Advantage (Reputation & Architecture): The GWVPP is not only an energy generator but also a water health device. The gentle, spiralling flow (implosion) naturally aerates and "vitalizes" the water, minimizing damage to aquatic life (fish passage is often unharmed) and mimicking the self-healing process of a natural river. This holistic benefit of simultaneous power generation and environmental restoration create a powerful reputational advantage far surpassing conventional micro-hydro plants. Its simplicity, low maintenance, and fish-friendliness make it ideal for decentralized, local-level power architecture.

The Path to Extended Sustainable Advantage

As the world faces complex challenges of climate change, resource depletion, and growing inequality. Relying solely on the traditional model of Sustainable Competitive Advantage, which often prioritizes profit and market share above human and ecological cost, is insufficient. It is our consideration here, that Extended Sustainable Advantage (ESA), built on the synergistic basis of Kay's Distinctive Capabilities and Schumacher's Intermediate Technology, offers a powerful way forward. By deliberately choosing to innovate for human scale and ecological health—as demonstrated by appropriate innovations like electric-culture in dryland farming and the gravitational water vortex power plant—firms can unlock unique, non-imitable sources of value. This approach moves the goal from simply *sustaining* an advantage to *extending* it through the creation of shared, regenerative value for all stakeholders: the community, the environment, and the enterprise itself. True distinction, in the 21st century, comes from mattering.

Evaluating Kay's Frameworks and Novel Innovations through the Lens of Responsible Research and Innovation (RRI)

The shift from purely competitive advantage to **Extended Sustainable Advantage (ESA)**—combining John Kay's Distinctive Capabilities with E.F. Schumacher's Intermediate Technology—is a necessary but theoretically driven step. To ground this concept in practical application and ethical accountability, we must evaluate it against the contemporary standard of Responsible Research and Innovation (RRI).

RRI, particularly promoted by the European Union, is an overarching governance framework that seeks to align the process and outcomes of R&I with societal values, needs, and expectations. Its essential methodological elements are the AREA framework: Anticipation, Reflection, Engagement, and Action (Responsiveness).

We can assess how well Kay's model and Schumacher's philosophy inherently satisfy the criteria of RRI, identifying where they complement and where they fall short.

John Kay's Distinctive Capabilities (Architecture, Reputation, Innovation)

Kay's framework, being an economic theory of firm success, primarily focuses on **appropriability** (the ability of a firm to retain the value it creates) and **sustainability** (the durability of the competitive advantage).

Table 6. Comparison between Kay's Distinctive Capabilities and RRI alignment

RRI Criterion	Kay's Distinctive Capabilities	RRI Alignment / Deficiency
Anticipation	Focuses on anticipating competitor moves and market shifts to maintain advantage.	Deficiency: Lacks systematic anticipation of societal, ethical, and environmental risks (e.g., unintended social displacement from innovation, or long-term ecological footprint).
Reflection	Focuses on reflection regarding internal efficiency, alignment, and competitive position.	Deficiency: Purpose of innovation is primarily profit-maximization, not the social desirability or ethical acceptability of the research itself.
Engagement	The Architecture capability is based on strong, long-term relationships with employees, customers, and suppliers.	Alignment: Strong internal and essential external engagement, but typically excludes broader civil society, NGOs, and marginalized communities that may bear the innovation's external costs.
Action (Responsiveness)	The model encourages action to build the advantage.	Partial Alignment: Responsiveness is driven by market and competitive pressure, not necessarily by a commitment to societal values beyond consumer satisfaction.

Conclusion on Kay's Model: While excellent for competitive sustainability, Kay's framework is ethically and socially neutral by design. It requires a deliberate, external commitment to RRI principles to ensure the distinctive capability itself is socially desirable.

E.F. Schumacher's Appropriate Technology

Schumacher's philosophy of "economics as if people-mattered" and "*appropriate technology*" is fundamentally normative, built on a human-centric and ecological worldview.

Table 7. Comparison between Schumacher's Appropriate Technology and RRI Alignment

RRI Criterion	Schumacher's Appropriate Technology	RRI Alignment / Deficiency
Anticipation	Inherently anticipates and avoids negative social and ecological impacts by prioritizing small scale, low capital, and non-violence against nature.	Strong Alignment: It is a proactive risk mitigation methods against the scale-related, *capital-intensive problems of modern technology.*
Reflection	The core principle is deep reflection on the purpose of technology: *to serve human dignity, creativity, and community, not merely maximize output.*	Strong Alignment: The very choice of Appropriate Innovation is an act of ethical reflection on societal goals and the nature of "progress."

Engagement	IT is designed to be comprehensible, locally maintained, and locally owned, requiring active participation and inclusivity of the users in its design and deployment.	Strong Alignment: Its focus on decentralization and local materials/skills embodies democratic, bottom-up engagement.
Action (Responsiveness)	IT is defined by the action of adapting technology to local conditions (e.g., resources, climate, culture), making it inherently responsive to the needs of the user community.	Strong Alignment: The focus is on capital-saving, labor-intensive action that responds directly to unemployment and resource scarcity in developing nations.

Conclusion on Schumacher's Model: Schumacher's Appropriate Technology approach is arguably a historical precursor to RRI. Its principles align nearly perfectly with the RRI goal of ethical acceptability, sustainability, and societal desirability, making it an ideal component for the "*Innovation*" capability in an ESA framework.

RRI Evaluation of Specific Novel Innovations

Applying the AREA criteria to the specific innovations illustrates how the ESA framework (Kay + Schumacher) can guide development, particularly for Less Developed Countries (LDCs).

Table 8. Agricultural & Water Solutions

Innovation	**Key Benefit for LDC / ESA Capability**	**RRI Evaluation (AREA)**
Laser-Culture (Seed/Plant Biostimulation)	Intermediate Innovation: Low-energy, non-chemical, high-precision boost to yield/stress resistance.	Anticipation: Low risk of large-scale ecological disruption compared to GM/chemical farming. Reflection: Ethically sound as it enhances natural plant potential. Engagement: Technology transfer must be simple enough for local technicians to maintain. Action: Adoption is quick and local.
Gravitational Water Vortex Power Plant (GWVPP)	Architecture/Reputation: Decentralized, low-head hydropower that also heals (aerates/vitalizes) the water.	Anticipation: Highly positive environmental anticipation (fish-friendly, non-violent flow). Reflection: Aligns with Schauberger's "learning from nature." Engagement: Ideal for co-ownership by small communities; promotes local energy sovereignty. Action: Directly provides clean energy and improves river health.

Confined Vortex Turbine (CVT)	Intermediate Innovation: A simple, high-efficiency micro-hydro turbine suited for low-flow/low-head sites.	Anticipation: Low environmental impact due to small scale and minimal civil works. Reflection: The focus on low-flow efficiency aligns with resource-frugality. Engagement: Design must be modular and maintainable with local workshops. Action: A cheap, reliable source of decentralized rural power.

Table 9. Energy and Waste Solutions

Innovation	**Key Benefit for LDC / ESA Capability**	**RRI Evaluation (AREA)**
Plastic Waste to Biofuel (Pyrolysis/Catalysis)	Architecture/Innovation: Turns a local pollution problem (plastic waste) into a local energy source (biofuel).	Anticipation: Highly positive on waste management; potential negative risk from air emissions (pyrolysis fumes) and end-use combustion pollution. Reflection: Addresses the ethical dilemma of non-biodegradable waste. Engagement: Requires engagement with waste pickers and local government for efficient feedstock collection; ensures fair value capture. Action: Creates circular economy jobs and reduces dependence on imported fossil fuels.
New Fusion Energy Installation (PT-Symmetric Crystal Potential)	Innovation (Long-term): Breakthrough in clean, near-limitless energy via novel material science.	Anticipation: Extremely high positive potential (zero-carbon, high power) but significant long-term safety, regulatory, and proliferation risks typical of advanced nuclear/fusion. Reflection: The purpose is clean energy, but the scale (Kay) may be too large to align with IT/local architecture unless modular. Engagement: Requires massive, international, and transparent public engagement from the start due to high complexity and potential societal transformation. Action: Must build in robust governance (RRI) from the outset to manage ethical dilemmas.

The Synergy: ESA as an RRI-Conscious Approach

The evaluation reveals that an approach based purely on Kay's framework risks developing innovations that are competitively strong but socially and ecologically harmful. Conversely, Schumacher's Intermediate Technology approach inherently satisfies RRI criteria but may struggle to achieve the global scale and financial appropriability needed to attract mainstream capital and compete with high-tech solutions.

The proposed Extended Sustainable Advantage (ESA) model resolves this tension by using Kay's framework to provide the structure, finance, and durability while embedding Schumacher's IT as the RRI-mandated DNA of the Innovation and Architecture capabilities.

- ESA's Innovation → RRI: The firm only pursues innovations (like the GWVPP or simple laser-culture) that are not just clever, but appropriate, ensuring ethical acceptability (no environmental harm) and societal desirability (local employment, resource healing).
- ESA's Architecture → RRI: The structure is built around long-term, trust-based relationships (Kay) that extend to include the local community as a true stakeholder and co-creator (Schumacher/RRI), ensuring genuine engagement and local ownership.

By embracing RRI, the ESA framework transforms competitive advantage from merely achieving high profits into an ambition for co-created, resilient societal value, making the business model itself resistant to ethical backlash or regulatory failure—a truly sustainable form of distinction.

<u>Case Example: Applying RRI to Intermediate Technology - Action Plans</u>

The combination of E.F. Schumacher's Intermediate Technology and the Responsible Research and Innovation (RRI) framework provide a robust blueprint for developing ethical, sustainable, and locally appropriate solutions, particularly in Less Developed Countries (LDCs). Intermediate Technology's intrinsic focus on human scale, simplicity, and ecological non-violence directly addresses the *ends* of RRI (societal desirability). The RRI AREA framework (Anticipation, Reflection, Engagement, Action) then formalizes the *means* to achieve these ends.

Here an example of detailed RRI Action Plans for key innovations, designed specifically for LDCs, emphasizing Schumacherian principles.

RRI Action Plan for Laser-Culture in Dryland Farming

Innovation: Low-power laser biostimulation of seeds/plants to enhance germination, growth, and drought resistance, reducing the need for water and chemical inputs. **Schumacher Principle:** Highly sophisticated science translated into a simple, low-budget, capital-saving application, increasing the productivity of land and labor.

Table 10. Action plan for laser-culture in LDC countries

RRI Criterion	Action Plan for Laser-Culture in LDCs	Schumacher/Intermediate Tech Alignment
Anticipation	A1: Hazard Mapping & Risk Assessment: Anticipate risks related to the technology's application. Focus on potential eye safety hazards for operators and long-term biological effects (e.g., unintended genetic shifts, ecological niche changes).	Action: Develop solar-powered, enclosed laser treatment units that minimize external risk and are designed for use by non-specialist local technicians.
	A2: Socio-Economic Impact Forecasting: Model how yield increases might affect local food prices, market structures, and labour requirements. Anticipate the risk of small farmers being excluded by requiring high upfront costs.	Action: Focus research on low-budget, durable equipment designs (capital-saving) that can be collectively owned by farmer cooperatives (architecture).
Reflection	R1: Ethical Purpose Review: Continuously reflect on the essential purpose: Is the goal to enrich a few large agribusinesses or to enhance local food sovereignty and resilience?	Action: Adopt Open-Source Hardware/Design for the laser treatment units to ensure the technology remains accessible and adaptable to diverse local crops and climates.
	R2: Resource Frugality Check: Evaluate the energy source and materials. Does the device consume minimal energy and use locally repairable components?	Action: *Mandate the use of locally available materials* for casing and focusing on simple, modular electronic components for easy maintenance (appropriate technology).
Engagement	E1: Community Co-Design: Engage women and local farming elders in testing different laser wavelengths and exposure times for indigenous seeds and local staple crops.	Action: Establish local Technology Transfer Centres run by trained local individuals who provide the service, ensuring the knowledge is embedded in the community, not just the hardware (building local skill).
	E2: Transparency & Training: Clearly communicate the science behind biostimulation (avoiding high-tech jargon) and provide hands-on, vocational training.	Action: Training manuals must be in local languages and based on visual, step-by-step instructions—a practical hallmark of Intermediate Technology.

Action (Responsiveness)	Ac1: Iterative Adaptation: Establish a feedback loop where initial field trial results and farmer observations (e.g., "The treatment works better on millet than on sorghum") lead to immediate, simple design or application parameter adjustments.	Action: *Utilize local extension agents to collect and relay field data* directly to the engineering team for rapid, context-specific changes.
	Ac2: Governance & Regulation: Work with local authorities to create simple certification standards for locally-made laser units to ensure safety without creating prohibitive regulatory barriers.	Action: *Champion a decentralized production model*, supporting local artisans/technicians to manufacture and repair the equipment.

The aforementioned RRI Action Plans give simple example to convince us that **Intermediate Technology** provides the **ethical starting point** for innovation, while the **RRI framework** provides the **governance and process** to ensure that this technology remains appropriate over time and through deployment. For LDCs, this combined approach offers a compelling path toward development that is not only economically viable but also socially just and environmentally restorative, fulfilling Schumacher's vision that humanity must be at the centre of the economic equation.

5.Concluding remark

John Kay's Distinctive Capabilities Framework is arguably one of the most insightful theories on the source of enduring organizational achievement based on capabilities. By moving the discussion from significant assets to relationships, it highlights the non-replicable complexity of human and organizational interactions. The challenge of translating this wisdom into day-to-day modern execution is overcome by the Distinctive Capabilities Maturity Model (DCMM). By rigorously operationalizing Architecture, Reputation, and Innovation through the definition of Objectives and Key Results (OKRs) and placing them on a measurable, five-level maturity index, we have transformed an essential concept into a powerful management tool.

When the Overall Weighted Average of the DCMM pushes into Level 4 (Quantitatively Managed) and approaches Level 5 (Optimizing), the organization is no longer just *performing well*; it has attained a sustainable, measurable, and properly managed advantage—the ultimate promise of John Kay's profound framework, now equipped with the operational rigor of modern management science.

This synthesized framework provides the required operationality and measurability, enabling organizations to not only understand what makes them special but, critically, to consistently measure, manage, and optimize the very basis of their distinctive capabilities.

We also discuss in this article, among other things, how to extend John Kay's Distinctive Capabilities, to include Schumacher's Intermediate Technology with applications to Botany etc, which aim to make sustainable advantages more in tune with Schumacher's philosophy of economics as if people-mattered in order to achieve sustainable ecosystem as well.

As the world faces complex challenges of climate change, resource depletion, and growing inequality. Relying solely on the traditional model of Sustainable Competitive Advantage, which often prioritizes profit and market share above human and ecological cost, is insufficient. It is our consideration here, that Extended Sustainable Advantage (ESA), built on the synergistic basis of John Kay's Distinctive Capabilities and Schumacher's Intermediate Technology principles, offers a powerful way forward.

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