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Article

Invertible Neutrosophic Functions on Neutrosophic Set Theory of Three Types

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Abstract:

The aim of this paper is to present the reasons or arguments for believing that the transformation from classical set theory into neutrosophic set theory may be possible for the neutrosophic set of three types. For this reason, I present the invertible neutrosophic function, Bijective Neutrosophic function, Neutrosophic Image, investigate some examples, and prove theorems.

Keywords: Neutrosophic sets $H_i^t[I], i = 1,2,3$;Inverse Neutrosophic Function on $H_i^t[I]$; Bijective Neutrosophic function on $H_i^t[I]$; Neutrosophic Image; Properties of Neutrosophic Function on $H_i^t[I]$.

1. Introduction

This paper is about invertible neutrosophic functions on neutrosophic sets of three types with their properties. I will try by the sequence of humble contributions to continue our work in [3,5,6,7,9]. The aim of our work is to present the reasons or arguments for believing that the transformation from classical set theory into neutrosophic set theory may be possible for the neutrosophic set of three types. Of course, if this goal is achieved, we will be able to build mathematical systems that follow the new neutrosophic system depend on neutrosophic set of three types, such as: neutrosophic graph of three neutrosophic set of three types, neutrosophic topology of three neutrosophic set of three types, and so on. The indeterminacy concept arises from our world itself is indeterminate and lack of knowledge that requires a judgment, it is a type of thinking that takes a middle-term value between two extreme values, for instance, The Mu'tazilite Mental School of Islamic Thought, from a theological perspective. In 1921 Jan Lukasiewicz was proposed a third truth value indeterminacy, to treat "future contingent" sentences in [11,12]. The extension of the indeterminacy concept by Smarandache linked with the neutrosophic set as an extension of fuzzy set / intuitionistic fuzzy. It is depending on the concept of neutrality which means everything between the opposites $\langle A\rangle$ and $\langle antiA\rangle$ there is a $\langle neutA\rangle$ see for instance [16-19]. Finally, this work coincides with others that focus on creating a neutrosophic group theory of a neutrosophic set of three types in [2,4,10], and a neutrosophic linear algebra theory in[1,8]. With concerning the classical set theory, I refer to [13,14,20].

2. Invertible Neutrosophic Functions on Neutrosophic Sets of Three Types

In this section, we investigate the invertible neutrosophic function on neutrosophic set of three types with their properties.

Definition 1.2. Let $f_n^i: X_i^t[I] \mapsto Y_i^t[I]$ be the neutrosophic function of three types, where, f_n^i generated by a classical function $f_c: X \mapsto Y$, $X_i^t[I]$, and $Y_i^t[I]$ are genarted by a classical set. X and Y, we say that f_n^i is a neutrosophic invertible, if ${f_n^i}^{-1}: Y_i^t[I] \mapsto X_i^t[I]$is a neutrosophic function, and it's called the invertible neutrosophic function of f_n^i.

Observation. $\langle x, y\rangle \in f_n^i \Leftrightarrow \langle y, x\rangle \in {f_n^i}^{-1}$ and $f_n^i(I) = I \Leftrightarrow {f_n^i}^{-1}(I) = I$.

Theorem 1.2. If $f_n^i: X_i^t[I] \mapsto Y_i^t[I]$ is a neutrosophic function of three types, then ${f_n^i}^{-1}: Y_i^t[I] \mapsto X_i^t[I]$

May be a neutrosophic function or non neutrosophic function, and vice versa.

Proof. By counter example. Let $X_1^t[I] = \begin{Bmatrix} 1 + 1I, 1 + 2I, \\ 2 + 1I, 2 + 2I, \end{Bmatrix}$ be a neutrosophic set of type-1, and

$Y_1^t[I] = \{-1 - 1I\}$. Define $f_n^1(1 + 1I) = f_n^1(1 + 2I) = f_n^1(2 + 1I) = f_n^1(2 + 2I) = -1 - 1I$

It is clear that $f_n^1: X_1^t[I] \mapsto Y_1^t[I]$ is a neutrosophic function, and it's known that the constant neutrosophic function, but ${f_n^1}^{-1}: Y_1^t[I] \mapsto X_1^t[I]$ is not neutrosophic function. Also, if $f_n^1: Y_1^t[I] \mapsto X_1^t[I]$, where $f_n^1(-1 - 1I) = 1 + 1I, f_n^1(-1 - 1I) = 1 + 2I$

$f_n^1(-1 - 1I) = 2 + 1I, and\, f_n^1(-1 - 1I) = 2 + 2I$, we see that f_n^1 is not neutrosophic function, while

${f_n^1}^{-1}: X_1^t[I] \mapsto Y_1^t[I]$ is a neutrosophic function. The following theorem give us the necessary and sufficient condition for existence the inverse neutrosophic function for any neutrosophic function.

Theorem 2.2. Let $f_n^i: X_i^t[I] \mapsto Y_i^t[I]$ be the neutrosophic function of three types, then f_n^i is an invertible neutrosophic function **iff** $f_n^i: X_i^t[I] \mapsto Y_i^t[I]$ is a neutrosophic bijective function.

Proof. Let $f_n^i: X_i^t[I] \mapsto Y_i^t[I]$ be a neutrosophic invertible function, that is ${f_n^i}^{-1}: Y_i^t[I] \mapsto X_i^t[I]$, we want to show that $f_n^i: X_i^t[I] \mapsto Y_i^t[I]$ is a bijective neutrosophic function.

Suppose that $x, z \in X_i^t[I]$ such that $f_n^i(x) = f_n^i(z)$. Consider $f_n^i(x) = f_n^i(z) = y$, we have

$(\langle x, y\rangle \in f_n^i) \wedge (\langle z, y\rangle \in f_n^i) \Leftrightarrow \left(\langle y, x\rangle \in {f_n^i}^{-1}\right) \wedge \left(\langle y, z\rangle \in {f_n^i}^{-1}\right)$, since ${f_n^i}^{-1}$ is a neutrosophic function,

$$\Rightarrow y = z. \text{ Therefore } f_n^i \text{ a neutrosophic injective function.}$$

Second, assume that $y \in Y_i^t[I]$, and ${f_n^i}^{-1}: Y_i^t[I] \mapsto X_i^t[I]$ is a neutrosophic function, so

$\Rightarrow \exists x \in X_i^t[I]$ such that $\langle y, x\rangle \in {f_n^i}^{-1}$

$\Rightarrow \exists x \in X_i^t[I]$ such that $\langle x, y\rangle \in f_n^i$

$\Rightarrow \exists x \in X_i^t[I]$ such that $f_n^i(x) = y$

$\Rightarrow f_n^i$ is a neutrosophic surjective function.

$\Rightarrow f_n^i$ is a neutrosophic bijective (one-to-one onto) function. Conversely, Suppose that

$f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ is a neutrosophic bijective function, we need to show that $f_n^{i^{-1}}: Y_i^t[I] \longmapsto X_i^t[I]$ is neutrosophic inverse function, Suppose that $y \in Y_i^t[I]$, but $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ (bijective)
$\Longrightarrow \exists x \in X_i^t[I]$ such that $f_n^i(x) = y$
$\Longrightarrow \exists x \in X_i^t[I]$ such that $\langle x, y \rangle \in f_n^i$
$\Longrightarrow \exists x \in X_i^t[I]$ such that $\langle y, x \rangle \in f_n^{i^{-1}}$
$\Longrightarrow NeuDom\left(f_n^{i^{-1}}\right) = Y_i^t[I]$. Now, suppose that $\left(\langle y, x \rangle \in f_n^{i^{-1}}\right) \wedge \left(\langle y, z \rangle \in f_n^{i^{-1}}\right)$,
$\Longrightarrow (\langle x, y \rangle \in f_n^i) \wedge (\langle z, y \rangle \in f_n^i)$
$\Longrightarrow (f_n^i(x) = y) \wedge (f_n^i(z) = y)$
$\Longrightarrow f_n^i(x) = f_n^i(z)$
$\Longrightarrow x = \mathrm{z}$, since $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ is a neutrosophic bijective function,
$\Longrightarrow f_n^{i^{-1}}: Y_i^t[I] \longmapsto X_i^t[I]$ is a neutrosophic function.

Theorem 3.2. Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ be a neutrosophic function of three types. If f_n^i is an invertible function, then $f_n^{i^{-1}}: Y_i^t[I] \longmapsto X_i^t[I]$ is a neutrosophic bijective function.
Proof. Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ be a neutrosophic is invertible function, then $f_n^{i^{-1}}: Y_i^t[I] \longmapsto X_i^t[I]$ is a neutrosophic function by definition 1.2. Suppose that $y, z \in Y_i^t[I]$ such that $f_n^{i^{-1}}(y) = f_n^{i^{-1}}(z) = x$
$\Longrightarrow \left(\langle y, x \rangle \in f_n^{i^{-1}}\right) \wedge \left(\langle z, x \rangle \in f_n^{i^{-1}}\right)$
$\Longrightarrow (\langle x, y \rangle \in f_n^i) \wedge (\langle x, z \rangle \in f_n^i)$
$\Longrightarrow (f_n^i(x) = y) \wedge (f_n^i(x) = z)$
$\Longrightarrow y = z \Longrightarrow f_n^{i^{-1}}$ is a neutrosophic injective (one-to-one) function. Also, suppose that $x \in X_i^t[I]$.
Since, $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ is a neutrosophic function.
$\Longrightarrow \exists y \in Y_i^t[I]$ such that $f_n^i(x) = y$
$\Longrightarrow \exists y \in Y_i^t[I]$ such that $\langle x, y \rangle \in f_n^i$
$\Longrightarrow \exists y \in Y_i^t[I]$ such that $\langle y, x \rangle \in f_n^{i^{-1}}$
$\Longrightarrow \exists y \in Y_i^t[I]$ such that $f_n^{i^{-1}}(y) = x$
$\Longrightarrow f_n^{i^{-1}}$ is a neutrosophic surjective (onto) function
$\Longrightarrow f_n^{i^{-1}}$ is a neutrosophic bijective (one-to-one +onto) function.

Theorem 4.2. Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ be an invertible neutrosophic function, then:

1. $f_n^{i^{-1}} \circ f_n^i = I_{ndX}^i$, and
2. $f_n^i \circ f_n^{i^{-1}} = I_{ndY}^i$

Proof. (1) Suppose that $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ is an invertible neutrosophic function, then $f_n^{i^{-1}}: Y_i^t[I] \longmapsto X_i^t[I]$ is a neutrosophic function, then $f_n^{i^{-1}} \circ f_n^i: X_i^t[I] \longmapsto X_i^t[I]$ is a neutrosophic function. Assume that $x \in X_i^t[I]$ such that $f_n^i(x) = y$, hence $\left(f_n^{i^{-1}} \circ f_n^i\right)(x) = f_n^{i^{-1}}\left(f_n^i(x)\right) = f_n^{i^{-1}}(\mathrm{y}) = x$ In addition, $I_{ndX}^i: X_i^t[I] \longmapsto X_i^t[I]$ is a neutrosophic identity function, therefore $I_{ndX}^i(x) = x$, for all $x \in X_i^t[I]$, we have, $f_n^{i^{-1}} \circ f_n^i(x) = I_{ndX}^i(x)$, for all $x \in X_i^t[I]$, that is $f_n^{i^{-1}} \circ f_n^i = I_{ndX}^i$ by definition 6.2 in (Al-Odhari A. M., Some Aspects of Neutrosophic Functions on Neutrosphic set theory of three types, 2025).

Theorem 5.2. Let $f_n^i: X_i^t[I] \mapsto Y_i^t[I]$ and $g_n^i: Y_i^t[I] \mapsto Z_i^t[I], i = 1,2,3$ be two one-to-one and onto neutrosophic functions. Then $g_n^i \circ f_n^i$ is one-to-one and onto neutrosophic function.

Proof. Let $f_n^i: X_i^t[I] \mapsto Y_i^t[I]$ and $g_n^i: Y_i^t[I] \mapsto Z_i^t[I], i = 1,2,3$ be two one-to-one and onto neutrosophic functions. Suppose that $(g_n^i \circ f_n^i)(x) = (g_n^i \circ f_n^i)(y), \forall x, y \in X_i^t[I]$. We conclude that

$\Rightarrow \left(g_n^i(f_n^i(x)\right) = \left(g_n^i f_n^i(y)\right) \Rightarrow g_n^i(x) = g_n^i(y) \Rightarrow x = y,$ Therefore $g_n^i \circ f_n^i$ is a one-to-one.

Now consider, $z \in Z_i^t[I] \Rightarrow \exists y \in Y_i^t[I] \ni g_n^i(y) = z$

$\Rightarrow \exists x \in X_i^t[I] \ni f_n^i(x) = y$

$\Rightarrow \exists x \in X_i^t[I] \ni g_n^i(y) = g_n^i\left(f_n^i(x)\right) = (g_n^i \circ f_n^i)(x).$

$\Rightarrow$ $g_n^i \circ f_n^i$ is an onto, Thus it's injective neutrosophic function.

Theorem 6.2. Let $f_n^i: X_i^t[I] \mapsto Y_i^t[I]$ and $g_n^i: Y_i^t[I] \mapsto Z_i^t[I], i = 1,2,3$ be two one-to-one and onto neutrosophic functions. Then:

1. $\left(f_n^{i^{-1}}\right)^{-1} = f_n^i$, and
2. $(g_n^i \circ f_n^i)^{-1} = f_n^{i^{-1}} \circ g_n^{i^{-1}}$.

Proof. (1). Assume that $(x, y) \in \left(f_n^{i^{-1}}\right)^{-1}$

$\Leftrightarrow \left(f_n^{i^{-1}}\right)^{-1}(x) = y \Leftrightarrow x = f_n^{i^{-1}}(y) \Leftrightarrow f_n^i(x) = y \Leftrightarrow (x, y) \in f_n^i$, thus $\left(f_n^{i^{-1}}\right)^{-1} = f_n^i$.

(2). Let $f_n^i: X_i^t[I] \mapsto Y_i^t[I]$ and $g_n^i: Y_i^t[I] \mapsto Z_i^t[I], i = 1,2,3$ be two one-to-one and onto, then $g_n^i \circ f_n^i$ is one-to-one and onto by theorem 4.3, hence the inverse neutrosophic function is one-to-one and onto. Now, from the right-hand side, we can be construct:

$\left(f_n^{i^{-1}} \circ g_n^{i^{-1}}\right) \circ (g_n^i \circ f_n^i) = f_n^{i^{-1}} \circ \left(g_n^{i^{-1}} \circ \left((g_n^i \circ f_n^i)\right)\right)$

$= f_n^{i^{-1}} \circ \left(\left(g_n^{i^{-1}} \circ g_n^i\right) \circ f_n^i\right)$ " by the composition associative property".

$= f_n^{i^{-1}} \circ \left(I_{ndY}^i \circ f_n^i\right)$ " by theorem 4.2".

$= f_n^{i^{-1}} \circ f_n^i$

$= I_{ndX}^i$

By similar argument, we have:

$(g_n^i \circ f_n^i) \circ \left(f_n^{i^{-1}} \circ g_n^{i^{-1}}\right) = g_n^i \circ \left(f_n^i \circ \left(f_n^{i^{-1}} \circ g_n^{i^{-1}}\right)\right)$

$= g_n^i \circ \left(\left(f_n^i \circ f_n^{i^{-1}}\right) \circ g_n^{i^{-1}}\right)$

$= g_n^i \circ \left(I_{ndY}^i \circ g_n^{i^{-1}}\right)$

$= g_n^i \circ g_n^{i^{-1}}$

$= I_{ndZ}^i$

Finally, to check that $(g_n^i \circ f_n^i)^{-1}(z) = f_n^{i^{-1}} \circ g_n^{i^{-1}}(z)$.

By taking the right-hand side: ${f_n^i}^{-1} \circ {g_n^i}^{-1}(z) = {f_n^i}^{-1}\left({g_n^i}^{-1}(z)\right) = {f_n^i}^{-1}(y) = x = I_{ndX}^i$, and by taking

The left-hand side: $\left({f_n^i}^{-1} \circ {g_n^i}^{-1}\right)(z) = {f_n^i}^{-1}\left({g_n^i}^{-1}(z)\right) = {f_n^i}^{-1}(y) = x = I_{ndX}^i$. Moreover,

$(g_n^i \circ f_n^i)(x) = g_n^i\left(f_n^i(x)\right) = g_n^i(y) = z = I_{ndZ}^i$. Thus $(g_n^i \circ f_n^i)^{-1} = {f_n^i}^{-1} \circ {g_n^i}^{-1}$.

3. Properties of Invertible Neutrosophic Functions on Neutrosophic sets of Three Types

In this section, we investigate the properties of invertible neutrosophic functions on neutrosophic sets of three types, and we proved some theories with few examples explaining (or illustrating) the invertible of neutrosophic functions.

Definition 1.3 Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I](\ i = 1,2,3)$ be a the neutrosophic function of three types generated from a classical function $f_c: X \longmapsto Y$ and classical sets X and Y respectively.
Let $D_i^t[I]$ be a neutrosophic subset of $Y_i^t[I]$ generated from $D \subset Y$. Define the neutrosophic image of $D_i^t[I]$ under f_n^i , written ${f_n^i}^{-1}(D_i^t[I])$, as follows :${f_n^i}^{-1}(D_i^t[I]) = \{x \in X_i^t[I]: f_n^i(x) \in D_i^t[I]\}$.
Observation. If $D_i^t[I] = \{x\}$ consist of a singleton element x, the notation ${f_n^i}^{-1}(x)$ use instead of ${f_n^i}^{-1}(\{x\})$.

Example 1.3 Let $f_n^1: \mathbb{R}_1^t[I] \longmapsto \mathbb{R}_1^t[I]$ be a neutrosophic function of type-1 on neutrosophic set of real numbers defined by: $f_n^1(x) = f_n^1(x_1 + x_2 I) = f_c(x_1) + f_c(x_2) f_n^1(I) =$ where $f_c(x) = x^2 + 2$, then

$$
\begin{aligned}
{f_n^i}^{-1}(11 + 11I) &= \{x \in X_i^t[I]: f_n^i(x) = 11 + 11I \ \} \\
&= \{x \in X_i^t[I]: f_c(x_1) + f_c(x_2) f_n^1(I) = 11 + 11I \ \} \\
&= \{x \in X_i^t[I]: ({x_1}^2 + 2) + ({x_2}^2 + 2)I = 11 + 11I \ \} \\
&= \{x \in X_i^t[I]: (3 + 3I), (-3 - 3I) \ \}
\end{aligned}
$$

Theorem 1.3 Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ be a the neutrosophic function of three types generated from a classical function $f_c: X \longmapsto Y$ and classical sets X and Y respectively, and Let $C_i^t[I] \subset Y[I]$ and
$B_i^t[I] \subset Y_i^t[I]$, **if** $C_i^t[I] = B_i^t[I]$, then ${f_n^i}^{-1}(C_i^t[I]) = {f_n^i}^{-1}(B_i^t[I])$.
Proof. Suppose that $C_i^t[I] = B_i^t[I]$, and let $x \in {f_n^i}^{-1}(C_i^t[I])$,
$\Rightarrow f_n^i(x) \in C_i^t[I] \Rightarrow f_n^i(x) \in B_i^t[I] \Rightarrow x \in {f_n^i}^{-1}(B_i^t[I]) \Rightarrow {f_n^i}^{-1}(C_i^t[I]) \subset {f_n^i}^{-1}(B_i^t[I])$. By similar method,
${f_n^i}^{-1}(B_i^t[I]) \subset {f_n^i}^{-1}(C_i^t[I])$. Hence ${f_n^i}^{-1}(C_i^t[I]) = {f_n^i}^{-1}(B_i^t[I])$. The converse of the theorem is not true, by the following example.
Example 2.3 Let $f_n^1: \mathbb{R}_1^t[I] \longmapsto \mathbb{R}_1^t[I]$ be a neutrosophic function of type-1 from the neutrosophic set of real numbers to itself. Defined by $f_n^1(x) = |x_1| + |x_2| I$. Consider
$C_1^t[I] = \{x_1 + x_2 I: x_1, x_2 \in C = (1,2)\} = (1 + 1I, 2 + 2I)$, and
$B_1^t[I] = \{x_1 + x_2 I: x_1, x_2 \in B = (-1,2)\} = (-1 - 1I, 2 + 2I)$, here $C_1^t[I] \neq B_1^t[I]$, but
${f_n^1}^{-1}\big((1 + 1I, 2 + 2I)\big) = (1 + 1I, 2 + 2I)$ and ${f_n^1}^{-1}(= (-1 - 1I, 2 + 2I) = (1 + 1I, 2 + 2I)$, we have ${f_n^i}^{-1}(C_i^t[I]) = {f_n^i}^{-1}(B_i^t[I])$, but $C_1^t[I] \neq B_1^t[I]$.

Theorem 2.3 Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ be a the neutrosophic function of three types generated from a classical function $f_c: X \longmapsto Y$ and classical sets X and Y respectively, and Let $C_i^t[I] \subset Y_i^t[I]$ and $B_i^t[I] \subset Y_i^t[I]$, then:

1. ${f_n^i}^{-1}(C_i^t[I] \cup B_i^t[I]) = {f_n^i}^{-1}(C_i^t[I]) \cup {f_n^i}^{-1}(B_i^t[I])$,
2. ${f_n^i}^{-1}(C_i^t[I] \cap B_i^t[I]) = {f_n^i}^{-1}(C_i^t[I]) \cap {f_n^i}^{-1}(B_i^t[I])$,
3. ${f_n^i}^{-1}(C_i^t[I] - B_i^t[I]) = {f_n^i}^{-1}(C_i^t[I]) - {f_n^i}^{-1}(B_i^t[I])$, and
4. ${f_n^i}^{-1}\left(\overbrace{C_i^t[I]}^{c}\right) = \overbrace{\left({f_n^i}^{-1}(C_i^t[I])\right)}^{c}$

Proof. (1). Let $x \in {f_n^i}^{-1}(C_i^t[I] \cup B_i^t[I]) \Leftrightarrow f_n^i(x) \in (C_i^t[I] \cup B_i^t[I])$

$$\Leftrightarrow f_n^i(x) \in C_i^t[I] \vee f_n^i(x) \in B_i^t[I]$$
$$\Leftrightarrow x \in {f_n^i}^{-1}(C_i^t[I]) \vee x \in {f_n^i}^{-1}(B_i^t[I])$$
$$\Leftrightarrow x \in \left({f_n^i}^{-1}(C_i^t[I]) \cup {f_n^i}^{-1}(B_i^t[I])\right)$$
$$\Rightarrow {f_n^i}^{-1}(C_i^t[I] \cup B_i^t[I]) = {f_n^i}^{-1}(C_i^t[I]) \cup {f_n^i}^{-1}(B_i^t[I]),$$

(2). Let $x \in {f_n^i}^{-1}(C_i^t[I] \cap B_i^t[I]) \Leftrightarrow f_n^i(x) \in (C_i^t[I] \cap B_i^t[I])$

$$\Leftrightarrow f_n^i(x) \in C_i^t[I] \wedge f_n^i(x) \in B_i^t[I]$$
$$\Leftrightarrow x \in {f_n^i}^{-1}(C_i^t[I]) \wedge x \in {f_n^i}^{-1}(B_i^t[I])$$
$$\Leftrightarrow x \in \left({f_n^i}^{-1}(C_i^t[I]) \cap {f_n^i}^{-1}(B_i^t[I])\right)$$
$$\Rightarrow {f_n^i}^{-1}(C_i^t[I] \cap B_i^t[I]) = {f_n^i}^{-1}(C_i^t[I]) \cap {f_n^i}^{-1}(B_i^t[I]),$$

(3). Let $x \in {f_n^i}^{-1}(C_i^t[I] - B_i^t[I]) \Leftrightarrow f_n^i(x) \in (C_i^t[I] - B_i^t[I])$

$$\Leftrightarrow f_n^i(x) \in C_i^t[I] \wedge f_n^i(x) \notin B_i^t[I]$$
$$\Leftrightarrow x \in {f_n^i}^{-1}(C_i^t[I]) \wedge x \notin {f_n^i}^{-1}(B_i^t[I])$$
$$\Leftrightarrow x \in \left({f_n^i}^{-1}(C_i^t[I]) - {f_n^i}^{-1}(B_i^t[I])\right)$$
$$\Rightarrow {f_n^i}^{-1}(C_i^t[I] - B_i^t[I]) = {f_n^i}^{-1}(C_i^t[I]) - {f_n^i}^{-1}(B_i^t[I]),$$

(4). Suppose that $x \in {f_n^i}^{-1}\left(\overbrace{C_i^t[I]}^{c}\right) \Leftrightarrow f_n^i(x) \in \left(\overbrace{C_i^t[I]}^{c}\right)$

$$\Leftrightarrow f_n^i(x) \notin C_i^t[I]$$
$$\Leftrightarrow x \notin {f_n^i}^{-1}(C_i^t[I])$$
$$\Leftrightarrow x \in \overbrace{{f_n^i}^{-1}(C_i^t[I])}^{c}, \text{ hence } {f_n^i}^{-1}\left(\overbrace{C_i^t[I]}^{c}\right) = \overbrace{\left({f_n^i}^{-1}(C_i^t[I])\right)}^{c}.$$

Definition 2.3 Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ be a neutrosophic function of three types generated from a classical function $f_c: X \longmapsto Y$ and classical sets X and Y respectively, $\Im(X_i^t[I])$ and $\Im(Y_i^t[I])$ Are neutrosophic power sets of $X_i^t[I]$. A function f_n^i is called induce neutrosophic function, if $f_n^i: \Im(X_i^t[I]) \longmapsto \Im(Y_i^t[I])$ by $C_i^t[I] \longmapsto f_n^i(C_i^t[I])$, $C_i^t[I] \subset X_i^t[I]$, and
${f_n^i}^{-1}: \Im(Y_i^t[I]) \longmapsto X_i^t[I]$ by $B_i^t[I] \longmapsto {f_n^i}^{-1}(B_i^t[I])$, $B_i^t[I] \subset Y_i^t[I]$.

Theorem 3.3 Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ be a neutrosophic one-to-one function of three types generated from a classical function $f_c: X \longmapsto Y$ and classical sets X and Y respectively, then $f_n^i: \Im(X_i^t[I]) \longmapsto \Im(Y_i^t[I])$ is a neutrosophic one-to-one function.

Proof. Case1. Let $X_i^t[I] \neq \emptyset_i^t[I]$, then $\Im(X_i^t[I])$ contains at least two neutrosophic elements, say $C_i^t[I] \in \Im(X_i^t[I])$ and $B_i^t[I] \in \Im(X_i^t[I])$ such that $C_i^t[I] \neq B_i^t[I]$, hence there exists $x \in X_i^t[I]$ such that $x \in C_i^t[I]$ and $x \notin B_i^t[I]$, therefore $f_n^i(x) \in f_n^i(C_i^t[I])$ and $f_n^i(x) \notin f_n^i(B_i^t[I])$, we conclude that $X_i^t[I] \neq \emptyset_i^t[I] \Longrightarrow f_n^i(C_i^t[I]) \neq f_n^i(B_i^t[I])$, hence $f_n^i: \Im(X_i^t[I]) \longmapsto \Im(Y_i^t[I])$ is a neutrosophic one-to-one function.

Case2. Let $X_i^t[I] = \emptyset_i^t[I]$, that $\Im(X_i^t[I]) = \{\emptyset_1 + \emptyset_1 I\}$, it's obvious one-to-one function, since there is no two different elements have the same image.

Theorem 4.3 Let $f_n^i: X_i^t[I] \longmapsto Y_i^t[I]$ be a the neutrosophic function of three types generated by a classical function $f_c: X \longmapsto Y$ and classical sets X and Y respectively, then the neutrosophic induced function ${f_n^i}^{-1}: \Im(Y_i^t[I]) \longmapsto \Im(X_i^t[I])$ preserves the neutrosophic elementary operations.

1. ${f_n^i}^{-1}\left(\cup_{\alpha\in\mathbb{i}} \underbrace{B_i^t[I]}_{\alpha}\right) = \cup_{\alpha\in\mathbb{i}}\left({f_n^i}^{-1}\left(\underbrace{B_i^t[I]}_{\alpha}\right)\right)$, and

2. ${f_n^i}^{-1}\left(\cap_{\alpha\in\mathbb{i}} \underbrace{B_i^t[I]}_{\alpha}\right) = \cap_{\alpha\in\mathbb{i}}\left({f_n^i}^{-1}\left(\underbrace{B_i^t[I]}_{\alpha}\right)\right)$.

Proof (1). Suppose that $x \in {f_n^i}^{-1}\left(\cup_{\alpha\in\mathbb{i}} \underbrace{B_i^t[I]}_{\alpha}\right) \Leftrightarrow f_n^i(x) \in \cup_{\alpha\in\mathbb{i}} \underbrace{B_i^t[I]}_{\alpha}$

$\Leftrightarrow \exists\alpha \in \mathbb{i},\ f_n^i(x) \in \underbrace{B_i^t[I]}_{\alpha}$

$\Leftrightarrow \exists\alpha \in \mathbb{i},\ x \in {f_n^i}^{-1}\left(\underbrace{B_i^t[I]}_{\alpha}\right)$

$\Leftrightarrow x \in \cup_{\alpha\in\mathbb{i}} {f_n^i}^{-1}\left(\underbrace{B_i^t[I]}_{\alpha}\right)$, hence

$${f_n^i}^{-1}\left(\bigcup_{\alpha\in\mathbb{i}} \underbrace{B_i^t[I]}_{\alpha}\right) = \bigcup_{\alpha\in\mathbb{i}}\left({f_n^i}^{-1}\left(\underbrace{B_i^t[I]}_{\alpha}\right)\right).$$

(2). Suppose that $x \in {f_n^i}^{-1}\left(\cap_{\alpha\in\mathbb{i}} \underbrace{B_i^t[I]}_{\alpha}\right) \Leftrightarrow f_n^i(x) \in \cap_{\alpha\in\mathbb{i}} \underbrace{B_i^t[I]}_{\alpha}$

$\Leftrightarrow \forall\alpha \in \mathbb{i},\ f_n^i(x) \in \underbrace{B_i^t[I]}_{\alpha}$

$\Leftrightarrow \forall\alpha \in \mathbb{i},\ x \in {f_n^i}^{-1}\left(\underbrace{B_i^t[I]}_{\alpha}\right)$

$\Leftrightarrow x \in \cap_{\alpha\in\mathbb{i}} {f_n^i}^{-1}\left(\underbrace{B_i^t[I]}_{\alpha}\right)$, hence

$${f_n^i}^{-1}\left(\bigcap_{\alpha\in\mathbb{i}} \underbrace{B_i^t[I]}_{\alpha}\right) = \bigcap_{\alpha\in\mathbb{i}}\left({f_n^i}^{-1}\left(\underbrace{B_i^t[I]}_{\alpha}\right)\right).$$

Theorem 5.3 Let $f_n^i: X_i^t[I] \mapsto Y_i^t[I]$ be a the neutrosophic function of three types generated from a classical function $f_c: X \mapsto Y$ and classical sets X and Y respectively, let $C_i^t[I] \subset X_i^t[I]$ and $B_i^t[I] \subset Y_i^t[I]$, then:

1. $C_i^t[I] \subset f_n^{i^{-1}}\left(f_n^i(C_i^t[I])\right)$, and

2. $f_n^i\left(f_n^{i^{-1}}(B_i^t[I])\right) \subset B_i^t[I]$.

Proof (1). Consider $x \in C_i^t[I] \Rightarrow f_n^i(x) \in f_n^i(C_i^t[I]) \Rightarrow x \in f_n^{i^{-1}}\left(f_n^i(C_i^t[I])\right)$, hence,

$C_i^t[I] \subset f_n^{i^{-1}}\left(f_n^i(C_i^t[I])\right)$. Part 2 in the theorem by the same technique.

Example 3.3 Let $f_n^1: \mathbb{R}_1^t[I] \mapsto \mathbb{R}_1^t[I]$ be a neutrosophic function of type-1 from the neutrosophic set of real numbers to itself. Defined by $f_n^1(x) = {x_1}^2 + {x_2}^2 I$. Consider

$$C_1^t[I] = \{x_1 + x_2 I: x_1, x_2 \in C = \{1,2,3\} = \begin{Bmatrix} 1+1I, 1+2I, 1+3I, \\ 2+1I, 2+2I, 2+3I, \\ 3+1I, 3+2I, 3+3I, \end{Bmatrix}, \text{ and}$$

$$f_n^1(C_1^t[I]) = f_n^1 \begin{Bmatrix} 1+1I, 1+2I, 1+3I, \\ 2+1I, 2+2I, 2+3I, \\ 3+1I, 3+2I, 3+3I, \end{Bmatrix} = \begin{Bmatrix} 1+1I, 1+4I, 1+9I, \\ 4+1I, 4+4I, 4+9I, \\ 9+1I, 9+4I, 9+9I, \end{Bmatrix}, \text{ therefore,}$$

$$f_n^{1^{-1}}\left(f_n^1(C_1^t[I])\right) = f_n^{1^{-1}} \begin{Bmatrix} 1+1I, 1+4I, 1+9I, \\ 4+1I, 4+4I, 4+9I, \\ 9+1I, 9+4I, 9+9I, \end{Bmatrix} = \begin{Bmatrix} 1+1I, -1-1I, 1+2I, -1-2I, 1+3I, -1-3I, \\ 2+1I, -2-1I, 2+2I, -2-2I, 2+3I, -2-3I, \\ 3+1I, -3-1I, 3+2I, -3-2I, 3+3I, -3-3I \end{Bmatrix},$$

it is clear that $f_n^{1^{-1}}\left(f_n^1(C_1^t[I])\right) \neq C_1^t[I]$. Also, consider

$$f_n^{1^{-1}}(\{-4-4I\}) = \emptyset_1^t[I] = \{u_1 + u_2 I: u_1, u_1 \in \emptyset\}, \text{ then}$$

$$f_n^1\left(f_n^{1^{-1}}(\{-4-4I\})\right) = f_n^1\left(\emptyset_1^t[I]\right) = \emptyset_1^t[I] \neq B_1^t[I] = \{-4-4I\}.$$

4. Conclusions: In this paper, I studied the invertible neutrosophic set of three types with their properties along with some theories and examples to construct the neutrosophic set theory of the neutrosophic set of three types as an extension of previous work.

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