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Article

A Brief Comparative Study on HyperStructure, Super HyperStructure, and n- Super SuperHyperStructure

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Abstract:
This article compares Hyperfunction, Extra Hyperfunction, Super Hyperfunction, and Extra Super Hyperfunction depending on the nth-PowerSet. The nth-PowerSet reflects many real-world problems because any ontological object, like a system, organization, country, or entity, can be represented by systems and subsystems by the nth-PowerSet. This article will review the concepts and investigate some theorems and examples.

Keywords: nth-PowerSet, HyperFunction, Extra HyperFunction, SuperHyperFunction, and Extra SuperHyperFunction.

1. Introduction

The magic of mathematics is generated by the concept of an abstract set and its abstract concepts, such as abstract relations, functions, operations, etc. I will claim that - without hesitation- if we could imagine a human body without a skeleton, then one can imagine mathematics without the concept of a set. This is a sensory image, which provides us with the importance of a set in mathematics. This article represents a modest (or humble) contribution to the field of the hyper-structures of mathematics. The mathematical system of Hyper-Structure Theory appeared in 1934 when French mathematician Marty presented the Hyper-Groups structure. He extended the codomain of the binary operation on a non-empty set "H" into the power set of H. He defined the Hyper-Groups by taking the concept of left/right cosets on the set Hn, instead of the subset of "H" with associative property [4]. At that time, the concept of HyperSet was not known. After more than half a century, namely, in 1991, Barwise and Moss introduced the HyperSet [1]. So, there is no relationship between the HyperOperation and the HyperSet by the common word of hyper, and both have different structures. In 2016, Smarandache proposed the SuperHyperOperation and Super HyperAlgebra and their corresponding Neutrosophic SuperHyperOperation and Neutrosophic Super Hyper-Algebra [8-12]. This article will review HyperFunction and introduce Extra HyperFunction with some theorems. Moreover, we will present some facts about SuperHyperFunction and n-SuperHyperFunction.

2. Power of A Set and N[th]-Power of a Set

The concept of PowerSet was known in the classical books of Set Theory, for example [5,6]. In 2016, Smarandache proposed the nth-PowerSet to construct new mathematical structures such as SuperHyperOperation, Super HyperAlgebra, and Neutrosophic Super HyperAlgebra [8,-12]. This section will review this concept, which will be used in subsequent sections.

Definition 1.2. [5,6] A set is a collection of well-defined objects called elements. This concept is due to Georg Cantor (1845–1918).

Definition 2.2. [5,6] Let H be a universal set. A set $P(H) = \{S: S \subseteq H\}$is called the PowerSet of all subsets of a set H.

Theorem 1.2. [5,6] If H is a finite set of order n, then the order of $P(H)$ is equal to 2^n**.**

Definition 3.2. [5,6] Let H be any set and n be any positive integer**,** then the set $P_n(H)$ is the set of all n-elements of a subset of H with order $\boldsymbol{n}$.

Definition 4.2. [8,9] (n[th] -Power set**)** Let H be a universe of discourse set, and $n \in \mathbb{Z}^+$**.** Define the n^{th} -Power set of a set H as follows:

$$P^n(H) = P\big(P^{n-1}(H)\big),$$
$$= P\big(P^{n-1}(P^{n-2}(H)\big)$$
$$= P\big(P^{n-1}P^{n-2}(P^{n-3}(H)\big)$$
$$\vdots$$
$$= P\big(P^{n-1}P^{n-2}P^{n-3}\cdots P^1(P^0(H)\big),$$

where $P^0(H) = H$, and $P^1(H) = P(H)$ with the decreasing order relation of subsets, such as: $P^0(H) \subset P^1(H) \subset P^2(H) \cdots P^{n-1} \subset P^n$. If we excluded the empty set from $P(H)$,Then $P_n^*(H) = P^n(H)\backslash\emptyset$ defined in a similar way. The class $P^n(H)$ plays a crucial role in complex reality.

Theorem 2.2. [8,9] Let X be a discrete finite set of 2 or more elements, and $n \geq 1$ is an integer. Then:

$$P^0(H) \subset P^1(H) \subset P^2(H) \cdots P^{n-1} \subset P^n.$$

Remark. For any subset A, we identify {A} with A.

Example 1.2. Let $H = \{a\}$ be a singleton set. Then the 0-order power of a set H: $P^0(H) = \{a\}$, and The 1st-order power of a set $P^0(H)$: $P^1(H) = \{\emptyset, \{a\}\}$. Also, the 2nd-order power of a set $P^1(H)$: $P^2(H) = P\left(P^1(H)\right) = P\left(\{\emptyset, \{a\}\}\right)$

$P^2(H) = \begin{Bmatrix} \{\emptyset, \{a\}\} \\ \{\emptyset\}, \{\{a\}\} \\ \emptyset \end{Bmatrix}$, and the 3rd-order power of a set $P^2(H)$:

$$P^3(H) = P\left(P^2(H)\right) = P\left(\begin{Bmatrix} \{\emptyset, \{a\}\} \\ \{\emptyset\}, \{\{a\}\} \\ \emptyset \end{Bmatrix}\right)$$

$$P^3(H) = \begin{Bmatrix} \begin{Bmatrix} \{\emptyset, \{a\}\} \\ \{\emptyset\}, \{\{a\}\} \\ \emptyset \end{Bmatrix} \\ \{\{\emptyset\}, \{\{\emptyset\}\}, \{\{\{a\}\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}, \{\{\emptyset, \{a\}\}\}\}, \{\{\emptyset\}, \{\{\{a\}\}\}, \{\{\emptyset, \{a\}\}\}\}, \{\{\{\emptyset\}\}, \{\{\{a\}\}\}, \{\{\emptyset, \{a\}\}\}\} \\ \{\{\emptyset\}, \{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\{a\}\}\}\}, \{\{\emptyset\}, \{\{\emptyset, \{a\}\}\}\}, \{\{\{\emptyset\}\}, \{\{\{a\}\}\}\}, \{\{\{\emptyset\}\}, \{\{\emptyset, \{a\}\}\}\}, \{\{\{\{a\}\}\}, \{\{\emptyset, \{a\}\}\}\} \\ \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{a\}\}\}, \{\{\emptyset, \{a\}\}\} \\ \emptyset \end{Bmatrix}$$

To calculate $P^4(H)$, we have the order of $P^4(H) = 2^{16} = 65536$ elements by Theorem 1.2. In this case and beyond, we see the limitations of manual handling in classifying cases, and the role of the machine and algorithms comes to solve some of the required problems. If we exclude the empty set, we get $P_0^*(H) = \{a\}$, and the 1st-order power of a set $P_0^*(H): P_1^*(H) = \{\{a\}\}$, and the 2nd-order power of a set $P_1^*(H): P_2^*(H) = \{\{\{a\}\}\}$. We deduced that,

$$P_n^*(H) = \begin{Bmatrix} \underbrace{\{\{\cdots\{a\}\cdots\}\}}_{n+1} \\ \vdots \\ \{\{\{a\}\}\} \\ \{\{a\}\} \\ \{a\}. \end{Bmatrix}.$$

Example 2.2. Let $H = \{a, b, c, d\}$, then $P^0(H) = H = \{a, b, c, d\}$, and

$$P^1(H) = \begin{Bmatrix} \{a, b, c, d\} \\ \{a, b, c, \}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\} \\ \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\} \\ \{a\}, \{b\}, \{c\}, \{d\} \\ \emptyset \end{Bmatrix}, \text{ also,}$$

$$P_1^*(H) = \begin{Bmatrix} \{a, b, c, d\} \\ \{a, b, c, \}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\} \\ \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\} \\ \{a\}, \{b\}, \{c\}, \{d\} \end{Bmatrix}.$$

If $n = 2$, then by Theorem 1.2 tells us the order of $|P^2(H)| = 2^{16} = 65536$-elements, while the order of $|P_2^*(H)| = 32768$-elements.

3. HyperFunction and Inverse HyperFunction of one variable

The semantics of the word" HyperFunction" in mathematical language refers to the connection between the universal set under consideration, say H. In classical set theory, and its power set, written $\mathrm{P}(H)$, by unary or more operations. In 2016, Smarandache proposed the concept of a Super HyperOperation as an extension of a HyperFunction [8].

1.3. HyperFunction of One Variable

Definition 1.1.3. [11,12] Let H be a universal set, and $P(H)$ be the power set of H. A function $f^h: H \longmapsto P(H)$ is called a HyperFunction, if for all $h \in H$, then there exists an element $E_{\gamma \in J} \in P(H)$ such that $f^h(h) = E_{\gamma \in J}$, for some γ.

Observation. The codomain of HyperFunction includes the empty set. If we consider $P^*(H) = P(H) \backslash \emptyset$, then the codomain of HyperFunction $f^h: H \longmapsto P^*(H)$ does not include the empty set. If H is a finite with order n, i.e.,$O(H) = n$, then the order $\mathrm{O}(\mathrm{P}(H)) = 2^n$. The notation $\mathrm{P}(H)^H = \{f^h: H \longmapsto P(H)\}$ represents the set of all hyperfunctions from H into $\mathrm{P}(H)$.

Example 1.1.3. Let $H = \{1,2,3\}$ be a set and $P(H) = \{\emptyset, H, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ is the power set of H. Define the hyperfunction f^h by $f^h: H \longmapsto P(H)$ such that $f^h(1) = \{1,2\}$, $f^h(2) = \{1,3\}$, and $f^h(3) = \{2\}$. Then the $HypDom(f_1^h) = \{1,2,3\}$, and $Hypcd(f_1^h) = \{\{1,2\}, \{1,3\}, \{2\}\}$.

Example 2.1.3. Let $H = \{h\}$ be a set of singleton elements, and $P(H) = \{\emptyset, H\}$ be the power set of H. The function:

$$f^h(h) = \begin{cases} \emptyset, if\ h \notin H \\ \{h\}, if\ h \in H \end{cases}$$

Or $f^h(h) = \{h\}$, for all $h \in H$ is a hyperfunction.

Example 3.1.3. Let $H = \{h_1, h_2, \dots, h_n\}$ be a finite set, and $P(H)$ be the power set of H. Then the function:

$$f^h(h) = \begin{cases} \emptyset, if\ h \notin H \\ H - \{h\}, if\ h \in H \end{cases}$$

or $f^h(h) = H - \{h\}$, for all $h \in H$ is a hyperfunction. The following theorem tells us the main properties of subsets of H under operations, union, intersection, difference, and subset between any two subsets of H.

Theorem 1.1.3. Consider the hyperfunction $f^h: H \longmapsto P(H)$, A,B⊂H, then:

1. $f^h(A \cup B) = f^h(A) \cup f^h(B)$,
2. $f^h(A \cap B) \subset f^h(A) \cap f^h(B)$,
3. $f^h(A) - f^h(B) \subset f^h(A - B)$, and
4. If $A \subset B$, then $f^h(A) \subset f^h(B)$.

Proof. (1) Assume that $E_{\gamma\in J} \in f^h(A \cup B)$, for some $\gamma \Longrightarrow \exists \alpha \in (A \cup B)$ such that $f^h(\alpha) = E_{\gamma\in J}$ for some γ Since $\alpha \in (A \cup B) \Longrightarrow \alpha \in A \Longrightarrow f^h(\alpha) = E_{\gamma\in J} \in f^h(A)$ for some γ or $\alpha \in (A \cup B) \Longrightarrow \alpha \in B \Longrightarrow f^h(\alpha) = E_{\gamma\in J} \in f^h(B)$, for some γ, therefore, $E_{\gamma\in J} \in f^h(A) \cup f^h(B)$, we deduced that $f^h(A \cup B) \subset f^h(A) \cup f^h(B)$. Conversely, assume that $E_{\gamma\in J} \in f^h(A) \cup f^h(B) \Longrightarrow E_{\gamma\in J} \in f^h(A) \vee E_{\gamma\in J} \in f^h(B) \Longrightarrow \exists \alpha \in A$ such that $f^h(\alpha) = E_{\gamma\in J}$ or $\Longrightarrow \exists \alpha \in B$ such that $f^h(\alpha) = E_{\gamma\in J}$, since $f^h(\alpha) = E_{\gamma\in J}$ and $\alpha \in (A \cup B)$, hence $E_{\gamma\in J} \in f^h(A \cup B)$, that is $f^h(A) \cup f^h(B) \subset f^h(A \cup B)$. Therefore, $f^h(A \cup B) = f^h(A) \cup f^h(B)$.

(2). Suppose that $E_{\gamma\in J} \in f^h(A \cap B) \Longrightarrow \exists \alpha \in (A \cap B)$ such that $f^h(\alpha) = E_{\gamma\in J}$, for some γ Since $\alpha \in (A \cap B) \Longrightarrow \alpha \in A \Longrightarrow f^h(\alpha) = E_{\gamma\in J}$, for some $\gamma \in f^h(A)$ and $\alpha \in (A \cap B) \Longrightarrow \alpha \in B \Longrightarrow f^h(\alpha) = E_{\gamma\in J}$, for some $\gamma \in f^h(B)$, therefore, $E_{\gamma\in J} \in f^h(A) \cap f^h(B)$, we deduced that $f^h(A \cap B) \subset f^h(A) \cap f^h(B)$.

(3). Consider $E_{\gamma\in J} \in \left(f^h(A) - f^h(B)\right) \Longrightarrow E_{\gamma\in J}$, for some $\gamma \in f^h(A)$ and $E_{\gamma\in J}$, for all $\gamma \notin f^h(B)$. Since $E_{\gamma\in J}$, for some $\gamma \in f^h(A) \Longrightarrow \exists \alpha \in A$ such that $f^h(\alpha) = E_{\gamma\in J}$, for some γ. Also, $E_{\gamma\in J}$, for all $\gamma \notin f^h(B)$ $\Longrightarrow f^h(\alpha) \notin f^h(B) \Rightarrow \alpha \notin B$. We have $\exists \alpha \in A$ and $\alpha \notin B$ such that $f^h(\alpha) = E_{\gamma\in J}$, for some γ, therefore, $\alpha \in (A - B)$ such that $f^h(\alpha) = E_{\gamma\in J}$, for some γ. Hence $E_{\gamma\in J}$, for some $\gamma \in f^h(A - B)$. That is, $f^h(A) - f^h(B) \subset f^h(A - B)$.

(4). Suppose that $A \subset B$ and $E_{\gamma\in J} \in f^h(A)$, for some γ, then there exists an $\alpha \in A$ such that $f^h(\alpha) = E_{\gamma\in J}$ for some γ, therefore $\alpha \in B$ such that $f^h(\alpha) = E_{\gamma\in J}$ for some γ, hence $E_{\gamma\in J} \in f^h(B)$, and consequently, $f^h(A) \subset f^h(B)$. The next example illustrates that the previous theorem's equality in parts 2 and 3 does not hold.

Example 4.1.3. Consider the set $\mathrm{H} = \{\mathrm{a}, \mathrm{b}\}$ with its power set $\mathrm{P(H)} = \{\emptyset, \{\mathrm{a}\}, \{\mathrm{b}\}, \{\mathrm{a}, \mathrm{b}\}\}$, $A = \{a\}$ and $B = \{b\}$. Define the hyperfunction $f^h: H \longmapsto P(H)$ by:

$$f^h(h) = \begin{cases} \{a\}, if\ h \in H \\ \emptyset, if\ h \notin H \end{cases}.$$

In this case, we get,

$f^h(a) = \{a\}, f^h(b) = \{a\}, f^h(a) \cap f^h(b) = \{a\} \cap \{a\} = \{a\}$, and $A \cap B = \{a\} \cap \{b\} = \emptyset$, $f^h(A \cap B) = f^h(\emptyset) = \emptyset$, so $f^h(A \cap B) \subset f^h(A) \cap f^h(B)$. Also $A - B = \{a\} - \{b\} = \{a\}$. Moreover, $f^h(A - B) = f^h(\{a\}) = \{a\}$, while $f^h(a) - f^h(b) = \{a\} - \{a\} = \emptyset$.

Hence $f^h(A) - f^h(B) \subset f^h(A - B)$. The next theorem provides the properties of the family of subsets of H.

Theorem 2.1.3. Consider the hyperfunction $f^h: H \longmapsto P(H)$, for any family $\{A_{\rho\in J}\}$ of subsets of H, then:

1. $f^h(\bigcup_{\rho\in J} A_\rho) = \bigcup_{\rho\in J} f^h(A_\rho)$, and
2. $f^h(\bigcap_{\rho\in J} A_\rho) \subset \bigcap_{\rho\in J} f^h(A_\rho)$.

Proof. (1) Suppose that $E_{\rho\in J} \in \bigcup_{\rho\in J} f^h(A_\rho) \Leftrightarrow \exists \rho \in J, E_{\rho\in J} \in f^h(A_\rho)$

$$\Leftrightarrow \exists \rho \in J, h \in H \ni f^h(h) = E_{\rho\in J}$$

$$\Leftrightarrow \exists \rho \in J, h \in \bigcup_{\rho\in J} A_\rho \ni f^h(h) = E_{\rho\in J}$$

$$\Leftrightarrow f^h(h) = E_{\rho\in J} \in f^h(\bigcup_{\rho\in J} A_\rho).$$

(2). Suppose that $E_{\rho\in J} \in f^h(\bigcap_{\rho\in J} A_\rho) \Longrightarrow \exists h \in \bigcap_{\rho\in J} A_\rho \ni f^h(h) = E_{\rho\in J}$

$$\Longrightarrow \exists h, \forall \rho \in A_\rho \ni f^h(h) = E_{\rho\in J}$$

$$\Longrightarrow \exists h, \forall \rho, E_{\rho\in J} \in f^h(A_\rho)$$

$$\Longrightarrow, E_{\rho\in J} \in \bigcap_{\rho\in J} f^h(A_\rho).$$

Theorem 3.1.3. Let $f^h: H \longmapsto P(H)$ be a hyperfunction and $A, B \subset H$, then f^h is a one-to-one hyperfunction if and only if $f^h(A \cap B) = f^h(A) \cap f^h(B)$.

Proof. Let $f^h: H \longmapsto P(H)$ be a hyperfunction and $A, B \subset H$. Suppose that f^h is a one-to-one hyperfunction. To show that $f^h(A \cap B) = f^h(A) \cap f^h(B)$.

Let $E_{\rho\in J} \in f^h(A \cap B) \Leftrightarrow \exists h \in (A \cap B) \ni f^h(h) = E_{\rho\in J}$, for some ρ.

$\Leftrightarrow \exists h \in A \ni f^h(h) = E_{\rho\in J}$, for some $\rho \wedge \exists h \in B \ni f^h(h) = E_{\rho\in J}$, for some ρ.

$\Leftrightarrow E_{\rho\in J} \in f^h(A) \wedge E_{\rho\in J} \in f^h(B)$.

$\Leftrightarrow E_{\rho\in J} \in (f^h(A) \cap f^h(B))$.

Conversely, suppose that $f^h(A \cap B) = f^h(A) \cap f^h(B)$.

To show that the hyperfunction $f^h: H \longmapsto P(H)$ is a one-to-one. Let h_1, $h_2 \in H$ with $h_1 \neq h_2$ such that $f^h(h_1) = f^h(h_2) = E_{\rho\in J}$. Consider $A = h_1, B = h_2$, we get,

$f^h(A) \cap f^h(B) = f^h(h_1) \cap f^h(h_2) = E_{\rho\in J} \neq f^h(A \cap B) = f^h(\emptyset)$.

Definition 2.1.3. Let $f^h: H \longmapsto P(H)$ be a hyperfunction and $g^h: P(H) \to P^2(H)$ be a hyperfunction. Then the composition of the hyperfunction $g^h \circ f^h: H \longmapsto P^2(H)$ such that

$(g^h \circ f^h)(h) = g^h(f^h(h)), \forall\, h \in H$.

Example 4.1.3. Consider the set $\mathrm{H} = \{\mathrm{a}, \mathrm{b}\}$ with its power set $\mathrm{P(H)} = \{\emptyset, \{\mathrm{a}\}, \{\mathrm{b}\}, \{\mathrm{a}, \mathrm{b}\}\}$, and

$$\mathrm{P}^2(\mathrm{H}) = \left\{\begin{matrix} \{\{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a,b\}\}\} \\ \{\{\emptyset\}, \{\{a\}\}, \{\{b\}\}\}, \{\{\emptyset\}, \{\{a\}\}, \{\{a,b\}\}\}, \{\{\emptyset\}, \{\{b\}\}, \{\{a,b\}\}\}, \{\{\{a\}\}, \{\{b\}\}, \{\{a,b\}\}\} \\ \{\{\emptyset\}, \{\{a\}\}\}, \{\{\emptyset\}, \{\{b\}\}\}, \{\{\emptyset\}, \{\{a,b\}\}\}, \{\{a\}, \{b\}\}, \{\{a\}, \{a,b\}\}, \{\{b\}, \{a,b\}\} \\ \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a,b\}\} \\ \emptyset \end{matrix}\right\}$$

Define $f^h: H \longmapsto P(H) \ni f^h(a) = \{b\}$, $f^h(b) = \{a, b\}$, and $g^h: P(H) \to P^2(H) \ni g^h(\{\emptyset\}) = \emptyset$, $g^h(\{a\}) = \{\{b\}\}$, $g^h(\{b\}) = \{\{\mathrm{a}\}\}$ and $g^h(\{a, b\}) = \{\{\mathrm{b}\}\}$. Then the composition of the hyperfunction $g^h \circ f^h$ is given by:

$(g^h \circ f^h)(a) = g^h(f^h(a)) = g^h(\{\mathrm{b}\}) = \{\{\mathrm{a}\}\}$, and $(g^h \circ f^h)(b) = g^h(f^h(b)) = g^h(\{\mathrm{a}, \mathrm{b}\}) = \{\{\mathrm{b}\}\}$.

Definition 3.1.3. Let H be a universe of discourse set, $n \in \mathbb{Z}^+$, and $P^n(H)$ is n^{th}-Power set of a set H. Then there exists a sequence of hyperfunctions $f_i^h, i = 1,2, \dots, n$.

$f_1^h: H \to P(H), f_2^h: P(H) \longrightarrow P^2(H), f_3^h: P^2(H) \longrightarrow P^3(H), \dots, f_n^h: P^{n-1}(H) \longrightarrow P^n(H)$ such that

$$\begin{aligned}(f_n^h \circ f_{n-1}^h \circ \dots \circ f_2^h \circ f_1^h)(x) &= (f_n^h \circ f_{n-1}^h \circ \dots \circ f_2^h)(f_1^h(x)) \\ &= (f_n^h \circ f_{n-1}^h \circ \dots)\left(f_2^h\left(f_1^h(x)\right)\right) \\ &=\vdots \\ &= (f_n^h)\left(f_{n-1}^h\left(\dots f_2^h(f_1^h(x))\right)\right), \ \forall\, x \in H.\end{aligned}$$

2.3. Inverse HyperFunction of One Variable

Definition 1.2.3. Let H be a universal set, and $\mathrm{P}(H)$ be the power set of H. A function $f_h^{-1}: P(H) \longmapsto H$ is called the inverse hyperfunction, if for all γ, $E_{\gamma \in J} \in P(H)$, then there exists an element $h \in H$ such that $f_h^{-1}(E_{\gamma \in J}) = h$.

Observation. If $f^h: H \longmapsto P(H)$ is a hyperfunction, then $f_h^{-1}: P(H) \longmapsto H$ maybe not inverse hyperfunction by the following example.

Example 1.2.3. Let $H = \{1,2,3\}$ be a set, and $P(H) = \{\emptyset, H, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ is the power set of H. Consider $f^h: H \longmapsto P(H)$ such that $f^h(1) = f^h(2) = f^h(3) = \{2\}$. Then the $f_h^{-1}: P(H) \longmapsto H$ is given by: $f_h^{-1}(\{2\}) = f_h^{-1}(\{2\}) = f_h^{-1}(\{3\}) = \{1\}$. f_h^{-1} is not an inverse hyperfunction.

Example 2.2.3. Let $H = \{1,2,3\}$ be a set, and $P(H) = \{\emptyset, H, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ is the power set of H. Consider $f^h: H \longmapsto P(H)$ as in example 1.1.3. Then the $f_h^{-1}: P(H) \longmapsto H$ is given by: $f_h^{-1}(\{1,2\}) = 1$, $f_h^{-1}(\{1,3\}) = 2$, and $f_h^{-1}(\{2\}) = 3$. f_h^{-1} is an inverse hyperfunction.

Example 3.2.3. Let $H = \{1,2,3\}$ be a set, and $P(H) = \{\emptyset, H, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ is the power set of H. Here, some inverse HyperFunction are defined by:

1. $f_h^{-1}(A) = \begin{cases} H - A. if\ A \neq \emptyset \\ 1, if\ A = \emptyset \end{cases}$. Or
2. $f_h^{-1}(A) = \begin{cases} the\ smallest\ of\ A. if\ A \neq \emptyset \\ 2, if\ A = \emptyset \end{cases}$. Or

3. $f_h^{-1}(A) = \begin{cases} the\ largest\ of\ A. if\ A \neq \emptyset \\ 3, if\ A = \emptyset \end{cases}$

Theorem 1.2.3. If H is an infinite universal set or discourse, then there exists a denumerable set S such that $S \subset H$.

Proof. Let H be an infinite universal set or discourse, and $P(H)$ is a power set of a set H. Define the inverse hyperfunction $f_h^{-1}: P(H) \longmapsto H$ as follows:

$$f_h^{-1}(H) = h_1,$$
$$f_h^{-1}(H - \{h_1\}) = h_2,$$
$$f_h^{-1}(H - \{h_1, h_2\}) = h_3,$$
$$f_h^{-1}(H - \{h_1, h_2, h_3\}) = h_4,$$
$$\vdots$$
$$f_h^{-1}(H - \{h_1, h_2, h_3, \dots, h_n\}) = h_{n+1}$$

Since the set is infinite, hence the set $H - \{h_1, h_2, h_3, \dots, h_n\} \neq \emptyset$, for any $n \in \mathbb{N}$. Since the inverse hyperfunction f_h^{-1} is a choice function, we get $h_n \neq h_k$, for all $n \neq k$, we conclude that the set $S = \{h_1, h_2, \dots, h_n, h_{n+1}, \dots\}$ is a denumerable subset of H, and the elements h_n are distinct according to the hyperfunction f_h^{-1} is a choice function that chooses one element from H, say $f_h^{-1}(H) = h_1$, and so on from other sets. The next theorem gives us some properties of hyperfunctions.

4. Extra HyperFunction of One Variable

Definition 1.4. Let H be a universal set, and $P(H)$ be the power set of H. A function $f^{eh}: P(H) \longmapsto P(H)$ is called an extra hyperfunction, if for all $A_{\gamma \in J} \in P(H)$. Then there exists an element $B_{\delta \in I} \in P(H)$ such that $f^{eh}(A_{\gamma \in J}) = B_{\delta \in I}$. This is an extra hyperfunction including the empty set.

Definition 2.4. [8,9,11,12] Let H be a universal set, and $P^*(H)$ be the power set of H. A function $f^{eh}: P^*(H) \longmapsto P^*(H)$ is called an extra hyperfunction, if for all $A_{\gamma \in J} \in P^*(H)$. Then there exists an element $B_{\delta \in I} \in P^*(H)$ such that $f^{eh}(A_{\gamma \in J}) = B_{\delta \in I}$. This is an extra hyperfunction that does not include the empty set, where $P^*(H) = P(H) \backslash \emptyset$.

Example 1.4. Let $H = \{1,2,3\}$ be a set and $P(H) = \{\emptyset, H, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ is the power set of H. Define the extra hyperfunction f^{eh} by $f^{eh}: P(H) \longmapsto P(H)$ such that $f^{eh}(A_{\gamma \in J}) = A^c{}_{\gamma \in J}$, for all $A_{\gamma \in J} \in P(H)$. f^{eh} is an extra hyperfunction.

Example 2.4. Let $H = \{a, b\}$ be a set and $P(H) = \{\emptyset, H, \{a\}, \{b\}\}$ is the power set of H. Define the extra hyperfunction $f^{eh}: P(H) \longmapsto P(H)$ such that

$f^{eh}(\emptyset) = \emptyset$, $f^{eh}(\{a\}) = \{a\}, f^{eh}(\{b\}) = \{b\}$, and $f^{eh}(H) = H$ is an extra hyperfunction.

Theorem 1.4. Let H be a universal set, and $\mathrm{P}(H)$ be the power set of H. If $f: H \longrightarrow H$ is a one-to-one function, then the extra hyperfunction $f^{eh}: P(H) \longmapsto P(H)$ is a one-to-one.

Proof. Case-1. If $H = \emptyset$, then $P(H) = \{\emptyset\}$. In this case, $f^{eh}: P(H) \longmapsto P(H)$ is a one-to-one function, because no two different elements of $P(H)$ can have the same image as the set $P(H)$ consists of only one element.

Case-2. Suppose that $H \neq \emptyset$, then $P(H)$ has at least two elements. Let's say A and B, and $A \neq B$. Then there exists $x \in A$ and $x \notin B \Longrightarrow f(x) \in f(A)$ and $f(x) \notin f(B)$. Since f is a one-to-one, we get $f(A) \neq f(B)$, therefore, $f^{eh}(A) \neq f^{eh}(B)$. Hence f^{eh} is a one-to-one.

Theorem 2.4. Let H be a universal set, and $\mathrm{P}(H)$ be the power set of H. If $f: H \longrightarrow H$ is a function, then the extra hyperfunction $f^{eh}: P(H) \longmapsto P(H)$ preserving the elementary set operations as follows:

1. $f^{eh}\left(\bigcup_{\rho\in J} A_\rho\right) = \bigcup_{\rho\in J} f^{eh}(A_\rho)$,
2. $f^{eh}\left(\bigcap_{\rho\in J} A_\rho\right) = \bigcap_{\rho\in J} f^{eh}(A_\rho)$, and
3. $f^{eh}(A - B) = f^{eh}(A) - f^{eh}(B)$.

Proof (1). Suppose that $f^{eh}(E) \in f^{eh}\left(\bigcup_{\rho\in J} A_\rho\right) \Leftrightarrow E \in \bigcup_{\rho\in J} A_\rho$

$\Leftrightarrow E \in A_\rho$, for some $\rho \in J$.

$\Leftrightarrow f^{eh}(E) \in f^{eh}(A_\rho)$, for some $\rho \in J$.

$\Leftrightarrow f^{eh}(E) \in \bigcup_{\rho\in J} f^{eh}(A_\rho)$, for some $\rho \in J$. Hence,

$f^{eh}\left(\bigcup_{\rho\in J} A_\rho\right) = \bigcup_{\rho\in J} f^{eh}(A_\rho)$.

(2). Consider $f^{eh}(E) \in f^{eh}\left(\bigcap_{\rho\in J} A_\rho\right)$

$\Leftrightarrow E \in \bigcap_{\rho\in J} A_\rho$

$\Leftrightarrow E \in A_\rho$, for all $\rho \in J$.

$\Leftrightarrow f^{eh}(E) \in f^{eh}(A_\rho)$, for all $\rho \in J$.

$\Leftrightarrow f^{eh}(E) \in \bigcap_{\rho\in J} f^{eh}(A_\rho)$, for all $\rho \in J$. Therefore, $f^{eh}\left(\bigcap_{\rho\in J} A_\rho\right) = \bigcap_{\rho\in J} f^{eh}(A_\rho)$.

(3). Assume that $f^{eh}(E) \in f^{eh}(A - B) \Leftrightarrow E \in (A - B)$

$\Leftrightarrow E \in A \wedge E \notin B$

$\Leftrightarrow f^{eh}(E) \in f^{eh}(A) \wedge f^{eh}(E) \notin f^{eh}(B)$

$\Leftrightarrow f^{eh}(E) \in (f^{eh}(A) - f^{eh}(B))$. We deduced that,

$f^{eh}(A - B) = f^{eh}(A) - f^{eh}(B)$.

5. Super HyperFunction and Extra Super HyperFunction of One Variable

Definition 1.5. [3,8,9,11,12] Let H be a universal set, and $P^n(H)$ be the n[th]-PowerSet of H. A function $f^s: H \longmapsto P^n(H)$ is called a super hyperfunction, if for all $h \in H$. Then there exists an element $E_{\gamma\in J} \in P^n(H)$ such that $f^s(h) = E_{\gamma\in J}$, where $n \geq 2$. When $n = 1$, then $f^s(h) = f^h(h)$. The codomain of f^s is includes the empty set. But when $f^s: H \longmapsto P_n^*(H) = P^n(H)\backslash\emptyset$, then the codomain of f^s does not contain the empty set. The following theorem generalizes from hyperfunction to super hyperfunction, namely, Theorem 1.3.

Theorem 1.5. Consider the super hyperfunction $f^s: H \longmapsto P^n(H)$, A,B⊂H, then:

1. $f^s(A \cup B) = f^s(A) \cup f^s(B)$,
2. $f^s(A \cap B) \subset f^s(A) \cap f^s(B)$,
3. $f^s(A) - f^s(B) \subset f^s(A - B)$, and
4. If $A \subset B$, then $f^s(A) \subset f^s(B)$.

Proof. By the same argument as Theorem 1.3. The following theorem is a generalization of Theorem 2.3.

Theorem 2.5. Consider the super hyperfunction $f^s: H \longmapsto P^n(H)$, for any family $\{A_{\rho\in J}\}$ of subsets of H, then:

1. $f^s\left(\bigcup_{\rho\in J} A_\rho\right) = \bigcup_{\rho\in J} f^s(A_\rho)$, and
2. $f^s\left(\bigcap_{\rho\in J} A_\rho\right) \subset \bigcap_{\rho\in J} f^s(A_\rho)$.

Proof. By a similar method to Theorem 2.3.

Theorem 3.5. Let $f^s: H \longmapsto P^n(H)$ be a super hyperfunction and $A, B \subset H$, then f^s is a one-to-one SuperHyperFunction if and only if $f^s(A \cap B) = f^s(A) \cap f^s(B)$.

Proof. By a similar method to Theorem 3.3.

Definition 2.5.[3, 8-12,] Let H be a universal set, and $P^n(H)$ be the nth-PowerSet of H. A function $f^{es}: P^m(H) \longmapsto P^n(H)$ is called an Extra SuperHyperFunction, or n-SuperHyperFunction if for all $A_{\gamma\in J} \in P^n(H)$. Then there exists an element $B_{\delta\in I} \in P^n(H)$ such that $f^s\left(A_{\gamma\in J}\right) = B_{\delta\in I} \in P^n(H)$, where m, n ≥0 . The following remark gives us the relationship between f^h, f_h^{-1}, f^{es}, f^s, and f^{es}.

Remark.

- If $m = 0 \; and \; n = 0$, then f^{es} maybe a classical identity function, an ordinary function, or a permutation function.
- If $m = 0 \; and \; n = 1$, then $f^{es} = f^h$.
- If $m = 0 \; and \; n \geq 2$, then $f^{es} = f^s$.
- If $m = 1 \; and \; n = 0$, then $f^{es} = f_h^{-1}$.

- If $m = 1$ and $n = 1$, then $f^{es} = f^{eh}$.
- If $m = 1$ and $n \geq 2$, then we always get f^{es}.

Example 1.5. Let $H = \{a, b\}$ and $P^0(H) = H = \{a, b\}, P^1(H) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, then

$$P^2(H) = \left\{\begin{matrix} \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \\ \left\{\{\emptyset\}, \{\{a\}\}, \{\{b\}\}\right\}, \left\{\{\emptyset\}, \{\{a\}\}, \{\{a, b\}\}\right\}, \left\{\{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}\right\} \\ \{\{\emptyset\}, \{\{a\}\}\}, \left\{\{\emptyset\}, \{\{b\}\}\right\}, \left\{\{\emptyset\}, \{\{a, b\}\}, \left\{, \{\{a\}\}, \{\{b\}\}\right\}, \left\{\{\{a\}\}, \{\{a, b\}\}\right\}, \left\{\{\{b\}\}\right\}, \{\{a, b\}\}\right\} \\ \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\} \\ \emptyset \end{matrix}\right\}$$

Define a super hyperfunction as follows: $f^s: H \mapsto P^2(H)$ such that $f^s(a) = \{b\}$, and $f^s(b) = \{b\} = \left\{\{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}\right\}$. Also, we can define an extra super hyperfunction or 2-SuperHyperFunction as $f^{es}: P^1(H) \mapsto P^2(H)$ such that $f^{es}(\emptyset) = \emptyset$, $f^{es}(\{a\}) = \left\{\{b\}\right\}$, $f^{es}(\{b\}) = \left\{\{a\}\right\}$, and $f^{es}(\{a, b\}) = \left\{\{\{a\}\}, \{\{a, b\}\}\right\}$.

Theorem 3.5. Consider the extra super hyperfunction $f^{es}: P^n(H) \mapsto P^n(H)$, $A, B \subset P^n(H)$, then:

1. $f^{es}(A \cup B) = f^{es}(A) \cup f^{es}(B)$,
2. $f^{se}(A \cap B) \subset f^{es}(A) \cap f^{es}(B)$,
3. $f^{es}(A) - f^{es}(B) \subset f^{es}(A - B)$, and
4. If $A \subset B$, then $f^{es}(A) \subset f^{es}(B)$.

Theorem 3.5 is a generalization of Theorem 1.5 for a fixed $n \geq 2$.

Proof. By the same argument as Theorem 1.3.

Theorem 4.5. Consider the extra super hyperfunction $f^{es}: P^n(H) \mapsto P^n(H)$, for any family $\{A_{\rho \in J}\}$ of subsets of $P^n(H)$, then:

1. $f^{es}\left(\bigcup_{\rho \in J} A_\rho\right) = \bigcup_{\rho \in J} f^{es}(A_\rho)$, and
2. $f^{es}\left(\bigcap_{\rho \in J} A_\rho\right) \subset \bigcap_{\rho \in J} f^{es}(A_\rho)$.

Theorem 4.5 is an extension of Theorem 2.5, and the proof is similar to that of Theorem 2.3.

. Conclusions:

This paper presents a brief comparative study of hyperfunction, extra hyperfunction, and extra super hyperfunction, presenting some characteristics for developing the concepts mentioned in the reference list.

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References

1. Barwise, J., & Moss, L. (1991). Hypersets. *The Mathematical Intelligencer, 13*, 31-41. doi:10.1007/BF03028340
2. Extension of G-Algebras to SuperHyper G-Algebras. (2023). *Neutrosophic Sets and Systems, 55*.
3. Fujita, T. (2024, December). *A Theoretical Exploration of Hyperconcepts: Hyperfunctions, Hyperrandomness, Hyperdecision-Making, and Beyond (Including a Survey of Hyperstructures).* doi: 10.13140/RG.2.2.21658.96964
4. Marty, F. (1934). Sur une Généralisation de la Notion de Groupe. *Huitième Congrès des Mathématiciens Scand*, 45-49.
5. Pinter, C. C. (2014). *A Book of SET THEORY.* Mineola, New York: DOVER PUBLICATIONS, INC. Retrieved from www.doverpublications.com
6. Roitman, J. (2011). *Introduction to Modern Set Theory.* Retrieved from https://www.people.vcu.edu/~clarson/roitman-set-theory.pdf
7. Smaracenche, Florentin. (2020). Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra. *Neutrosophic Sets and Systems, 33*, 290–296.
8. Smarandache, F. (2016). SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra. In F. Smarandache, *Nidus Idearum* (II ed., pp. 107-108). Brussels: Pons Publishing.
9. Smarandache, F. (2022). Introduction to SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 1-8.
10. Smarandache, F. (2022). Introduction to the n-SuperHyperGraph - the most general form of graph today. *Neutrosophic Sets and Systems, 48*, 483-485.
11. Smarandache, F. (2022). The SuperHyperFunction and the Neutrosophic SuperHyperFunction (revisited again). *Neutrosophic Sets and Systems, 49*, 594-600. doi:10.5281/zenodo.6466524
12. Smarandache, F. (2023). SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions. *Neutrosophic Systems with Applications, 12*, 68-76. doi:https://orcid.org/0000-0002-5560-5926