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Toward a Unified Framework for Knot Theory, Hyperknot Theory, and Superhyperknot Theory via Superhyperstructures

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Abstract. Many real-world concepts exhibit hierarchical organization, and mathematics has explored numerous hierarchical structures. Mathematical frameworks can often be extended into hyperstructures and superhyperstructures by employing the powerset and the n-th iterated powerset constructions (cf. [21,22]). The concept of SuperHyperStructures was defined by F. Smarandache and has been studied in the context of graphs, algebraic structures, and functions [19,22,25]. These extensions are particularly well suited for modeling hierarchical relationships across diverse conceptual domains.

Knot theory studies smooth embeddings of circles in three-dimensional space, classifying knots by algebraic and geometric invariants and exploring their topological properties. Beyond its intrinsic theoretical interest, knot theory has found applications in chemistry, computer science, and other fields.

In this paper, we develop HyperKnot Theory and SuperHyperKnot Theory as extensions of classical knot theory. We provide rigorous definitions, examine fundamental properties, and present illustrative examples of these new frameworks. We anticipate that this work will foster further advances in both the mathematical theory and practical applications of knots.

Keywords: Hyperstructure, SuperHyperstructure, Knot Theory, HyperKnot Theory, SuperHyperKnot Theory

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1. **Preliminaries and Definitions**

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. Throughout this paper, we assume that all concepts and sets under consideration are finite.

1.1. *Classical Structures, Hyperstructures, and n-Superhyperstructures*

A *Classical Structure* is a fundamental algebraic framework defined over a base set. A *Hyperstructure* generalizes this notion by defining operations on the powerset of the base set [1,3,16]. An *n-Superhyperstructure* further extends the idea by employing the n-th iterated powerset [2,6,17,24]. Intuitively, each iteration applies the powerset operation to the result of the previous level [7,9,23].

The concept of SuperHyperStructures was introduced by F. Smarandache and has been studied in various mathematical contexts, including graphs, algebraic structures, and functions [19, 22, 25]. The parameter n is assumed to be a natural number. Related notions, such as superhyperalgebras [5, 20] and superhypergraphs [8, 10, 11, 19], have also been explored. Relevant definitions and illustrative examples follow.

Definition 1.1 (Set)**.** [12] A *set* is a well-defined collection of distinct objects, called its *elements*. We write $x \in A$ to indicate that x is an element of the set A.

Definition 1.2 (Subset)**.** [12] Given two sets A and B, we say that A is a *subset* of B, written $A \subseteq B$, if every element of A is also an element of B:

$$A \subseteq B \quad \iff \quad \forall x \, (x \in A \implies x \in B).$$

Definition 1.3 (Base Set)**.** A *base set* S is the underlying set from which more complex constructions—such as powersets and hyperstructures—are built:

$$S = \{ \, x \mid x \text{ belongs to the domain of interest} \}.$$

All elements of $\mathcal{P}(S)$ and $\mathcal{P}^n(S)$ are subsets whose members lie in S.

Definition 1.4 (Powerset)**.** The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all subsets of S, including the empty set and S itself:

$$\mathcal{P}(S) = \{ \, A \mid A \subseteq S \}.$$

Definition 1.5 (n-th Powerset)**.** [18, 24] The *n-th powerset* of a set H, denoted $\mathcal{P}^n(H)$, is defined recursively by

$$\mathcal{P}^1(H) = \mathcal{P}(H), \quad \mathcal{P}^{k+1}(H) = \mathcal{P}\big(\mathcal{P}^k(H)\big), \quad k \geq 1.$$

Analogously, the *n-th nonempty powerset* $\mathcal{P}^*_n(H)$ is given by

$$\mathcal{P}^*_1(H) = \mathcal{P}^*(H), \quad \mathcal{P}^*_{k+1}(H) = \mathcal{P}^*\big(\mathcal{P}^*_k(H)\big),$$

where $\mathcal{P}^*(H)$ denotes the powerset of H with the empty set removed.

Definition 1.6 (Classical Structure)**.** (cf. [18, 24]) A *Classical Structure* is a mathematical framework defined on a non-empty set H, equipped with one or more *Classical Operations* that satisfy specified *Classical Axioms*. Specifically:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ is a positive integer, and H^m denotes the m-fold Cartesian product of H. Common examples include addition and multiplication in algebraic structures such as groups, rings, and fields.

Definition 1.7 (Hyperoperation)**.** [26,27] A *hyperoperation* on a set S is a binary rule whose value is a subset of S rather than a single element. Formally, a hyperoperation $\circ$ is a map

$$\circ \;:\; S \times S \;\longrightarrow\; \mathcal{P}(S),$$

where $\mathcal{P}(S)$ denotes the powerset of S.

Definition 1.8 (Hyperstructure)**.** [18,24] A *hyperstructure* extends a classical algebraic structure by defining its operations on the powerset of a base set. Concretely, given a set S and a hyperoperation $\circ$, the pair

$$\mathcal{H} = \bigl(\mathcal{P}(S), \circ\bigr)$$

is called a hyperstructure, where $\circ$ acts on subsets of S.

Definition 1.9 (SuperHyperOperation)**.** [24] Let H be a nonempty set and define its iterated powersets by

$$\mathcal{P}^0(H) = H, \quad \mathcal{P}^{k+1}(H) = \mathcal{P}\bigl(\mathcal{P}^k(H)\bigr), \quad k \geq 0.$$

An (m,n)-*SuperHyperOperation* is an m-ary mapping

$$\circ^{(m,n)} \;:\; H^m \;\longrightarrow\; \mathcal{P}^n_*(H),$$

where $\mathcal{P}^n_*(H)$ denotes the n-th powerset of H, either excluding the empty set (classical type) or including it (neutrosophic type). Such operations generalize hyperoperations by producing outputs in higher–order powersets.

Definition 1.10 (n-Superhyperstructure)**.** [18, 24] An n-*Superhyperstructure* on a set S is given by the pair

$$\mathcal{SH}_n = \bigl(\mathcal{P}^n(S), \circ\bigr),$$

where $\mathcal{P}^n(S)$ is the n-th iterated powerset of S and $\circ$ is an operation defined on elements of $\mathcal{P}^n(S)$. This framework captures hierarchical algebraic behaviors across n levels of powerset iteration.

Example 1.11 (n-Superhyperstructure of integer "sum-difference" hyperoperations)**.** Let $S = \mathbb{Z}$. For each $k \geq 0$ define the k-th iterated powerset

$$\mathcal{P}^0(\mathbb{Z}) = \mathbb{Z}, \quad \mathcal{P}^{k+1}(\mathbb{Z}) = \mathcal{P}\big(\mathcal{P}^k(\mathbb{Z})\big).$$

We construct a family of binary *SuperHyperOperations* $\circ^{(k)}$ for $k = 0, 1, \ldots, n$ by induction:

$\circ^{(0)}$: Define

$$\circ^{(0)} : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathcal{P}^1(\mathbb{Z}), \qquad a \ \circ^{(0)} \ b \ = \ \{\, a+b, \ a-b\}.$$

Clearly $a \circ^{(0)} b \subseteq \mathbb{Z}$ is nonempty, so $\circ^{(0)}$ is a hyperoperation on $\mathbb{Z}$.

$\circ^{(k+1)}$: Suppose $\circ^{(k)} : \mathcal{P}^k(\mathbb{Z}) \times \mathcal{P}^k(\mathbb{Z}) \to \mathcal{P}^{k+1}(\mathbb{Z})$ is defined. Then set

$$\circ^{(k+1)} : \mathcal{P}^{k+1}(\mathbb{Z}) \times \mathcal{P}^{k+1}(\mathbb{Z}) \longrightarrow \mathcal{P}^{k+2}(\mathbb{Z}),$$

$$X \ \circ^{(k+1)} \ Y = \big\{\, A \ \circ^{(k)} \ B \ \big| \ A \in X, \ B \in Y \big\},$$

which is a well-defined map into $\mathcal{P}^{k+2}(\mathbb{Z})$ because each $A \circ^{(k)} B$ is itself a nonempty subset of $\mathcal{P}^k(\mathbb{Z})$.

Then

$$\mathcal{SH}_n = \big(\mathcal{P}^n(\mathbb{Z}), \circ^{(n)}\big)$$

is an n-Superhyperstructure:

(1) *Nonempty values:* By induction, for any $X, Y \in \mathcal{P}^n(\mathbb{Z})$, $X \circ^{(n)} Y$ is a nonempty collection of nonempty subsets of $\mathcal{P}^{n-1}(\mathbb{Z})$.
(2) *Closure:* $X \circ^{(n)} Y \subseteq \mathcal{P}^n(\mathbb{Z})$ by construction.
(3) *Hyperoperation property:* The result of $\circ^{(n)}$ is a set of elements of $\mathcal{P}^n(\mathbb{Z})$, not a single element.

Hence $\mathcal{SH}_n$ forms a concrete example of an n-Superhyperstructure on $\mathbb{Z}$.

1.2. *Knot Theory*

Knot theory studies embeddings of circles in three-dimensional spaces, classifying knots by algebraic and geometric invariants and understanding topological properties (cf. [4, 13–15]).

Definition 1.12 (Knot)**.** A *knot* is a smooth embedding

$$K : S^1 \hookrightarrow \mathbb{R}^3$$

considered up to ambient isotopy in $\mathbb{R}^3$. Equivalently, a knot is the image $K(S^1)$ of such an embedding, where two embeddings K_0, K_1 define the same knot if there exists a smooth one-parameter family of diffeomorphisms

$$\Phi_t : \mathbb{R}^3 \to \mathbb{R}^3, \quad t \in [0, 1],$$

with $\Phi_0 = \mathrm{id}$ and $\Phi_1 \circ K_0 = K_1$.

Example 1.13 (Unknot)**.** Let

$$K_{\text{unknot}}\colon S^1 \to \mathbb{R}^3, \quad K_{\text{unknot}}(\theta) = (\cos\theta, \sin\theta, 0), \quad \theta \in [0, 2\pi).$$

This is a smooth embedding of the circle into the xy–plane:

- *Smoothness*: Each coordinate is a smooth function of θ.
- *Injectivity*: If $\theta_0 \neq \theta_1 \bmod 2\pi$, then $(\cos\theta_0, \sin\theta_0) \neq (\cos\theta_1, \sin\theta_1)$.
- *No self-intersections*: The image is the unit circle, a simple closed curve.

Since any smooth simple closed curve in $\mathbb{R}^3$ that lies in a plane is ambient-isotopic to the standard unit circle, K_{unknot} represents the trivial (unknot) class.

Example 1.14 (Trefoil Knot)**.** Define the map

$$K_{\text{trefoil}}\colon S^1 \to \mathbb{R}^3, \quad K_{\text{trefoil}}(t) = \big(x(t),\, y(t),\, z(t)\big), \quad t \in [0, 2\pi),$$

with the parametric functions

$$\begin{cases} x(t) = \big(2 + \cos(3t)\big)\cos(2t), \\ y(t) = \big(2 + \cos(3t)\big)\sin(2t), \\ z(t) = \sin(3t). \end{cases}$$

Properties:

- *Smoothness*: $x(t), y(t), z(t)$ are all infinitely differentiable in t.
- *Injectivity*: One checks that for $t_0 \neq t_1 \bmod 2\pi$, the points $(x(t_0), y(t_0), z(t_0)) \neq (x(t_1), y(t_1), z(t_1))$, so there are no self-intersections.
- *Nontrivial knot type*: This embedding has three crossings in its minimal planar projection and is not ambient-isotopic to the unknot.

Thus K_{trefoil} is a smooth embedding whose image is the (right-handed) trefoil knot.

2. Result of this paper

As the main result of this paper, we investigate the definitions, properties, and examples of HyperKnots and SuperHyperKnots.

2.1. *HyperKnot*

We present the definition of a HyperKnot as follows.

Definition 2.1 (HyperKnot)**.** Let $C(\mathbb{R}^3)$ be the hyperspace of nonempty compact subsets of $\mathbb{R}^3$, equipped with the Hausdorff metric d_H. A *HyperKnot* is a continuous map

$$K_H\colon S^1 \to C(\mathbb{R}^3)$$

such that

(1) There exists a smooth classical knot embedding

$$K : S^1 \hookrightarrow \mathbb{R}^3 \quad \text{with} \quad \{K(t)\} \subseteq K_H(t) \quad (\forall\, t \in S^1),$$

(2) The map K_H is continuous in the Hausdorff metric: for all $t_0 \in S^1$, $\lim_{t \to t_0} d_H\big(K_H(t), K_H(t_0)\big) = 0$.

Two HyperKnots K_H^0, K_H^1 are *ambient hyperisotopic* if there exists a continuous family of homeomorphisms

$$\Phi_t : C(\mathbb{R}^3) \;\to\; C(\mathbb{R}^3), \quad t \in [0, 1],$$

with $\Phi_0 = \mathrm{id}$ and $\Phi_1 \circ K_H^0 = K_H^1$.

Example 2.2 (Arc-based HyperKnot)**.** Let

$$K : S^1 \to \mathbb{R}^3, \quad K(\theta) = (\cos\theta, \sin\theta, 0)$$

be the standard unit circle in the xy-plane. Fix a small $\varepsilon > 0$. Define

$$K_H(\theta) = \big\{\, K(\phi) \mid \phi \in [\theta - \varepsilon,\ \theta + \varepsilon] \big\}, \quad \theta \in S^1,$$

where intervals are taken mod 2π. Then:

(1) Each $K_H(\theta)$ is a nonempty compact arc in $\mathbb{R}^3$ containing the core point $K(\theta)$.

(2) As $\theta \to \theta_0$, the endpoints of the arc move continuously, so $d_H\big(K_H(\theta), K_H(\theta_0)\big) \leq \max\{\|K(\theta \pm \varepsilon) - K(\theta_0 \pm \varepsilon)\|\} \to 0$.

(3) A smooth core embedding is K itself, and obviously $\{K(\theta)\} \subseteq K_H(\theta)$.

Thus $K_H : S^1 \to C(\mathbb{R}^3)$ is a HyperKnot.

Example 2.3 (Tubular-neighborhood HyperKnot)**.** Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be any smooth knot embedding and choose a radius $r > 0$ smaller than the reach of K. Define

$$K_H(t) = \big\{\, x \in \mathbb{R}^3 \mid \|x - K(t)\| \leq r \big\}, \quad t \in S^1;$$

each $K_H(t)$ is the closed ball of radius r around the core point $K(t)$. Then:

(1) $K_H(t)$ is nonempty, compact, and contains $\{K(t)\}$.

(2) Continuity in the Hausdorff metric follows since

$$d_H\big(B(K(t), r), B(K(t_0), r)\big) = \|K(t) - K(t_0)\| \;\xrightarrow[t \to t_0]{}\; 0.$$

(3) The core inclusion condition is immediate from the construction.

Hence $K_H : S^1 \to C(\mathbb{R}^3)$ defines a HyperKnot which is a "thickening" of the classical knot.

Example 2.4 (Normal-Circle HyperKnot)**.** Let

$$K : S^1 \to \mathbb{R}^3, \quad K(t) = (\cos t, \sin t, 0)$$

be the standard unit circle, which has everywhere nonzero curvature. Let $T(t)$, $N(t)$, $B(t)$ be its Frenet frame, and fix a radius $r > 0$. Define for each $t \in S^1$

$$C_t \;=\; \{\,K(t)\} \;\cup\; \{\,K(t) + r\big(\cos\phi\, N(t) + \sin\phi\, B(t)\big) \;\big|\; \phi \in [0, 2\pi]\}.$$

Then $C_t \subset \mathbb{R}^3$ is a nonempty compact subset containing the core point $K(t)$. Define

$$K_H \colon S^1 \to C(\mathbb{R}^3), \qquad K_H(t) = C_t.$$

We check:

(1) **Core inclusion.** By construction $\{K(t)\} \subset C_t$.

(2) **Compactness.** Each circle $\{\,K(t) + r(\cos\phi\, N(t) + \sin\phi\, B(t))\}$ is compact, and the union with $\{K(t)\}$ remains compact.

(3) **Continuity in the Hausdorff metric.** As $t \to t_0$, the Frenet frame $\{T, N, B\}$ depends smoothly on t, so

$$d_H\big(C_t, C_{t_0}\big) \;\le\; \sup_{\phi}\big\|K(t){+}r(\cos\phi\, N(t){+}\sin\phi\, B(t)) \,-\, K(t_0){-}r(\cos\phi\, N(t_0){+}\sin\phi\, B(t_0))\big\| \;\xrightarrow[t\to t_0]{}\; 0.$$

(4) **Smooth core embedding.** The map $K \colon S^1 \to \mathbb{R}^3$ is a smooth knot embedding and $\{K(t)\} \subset K_H(t)$.

Therefore $K_H \colon S^1 \to C(\mathbb{R}^3)$ is a HyperKnot, giving a family of small normal-plane circles thickening the core.

Theorem 2.5 (HyperKnot generalizes classical knots)**.** *The assignment*

$$\iota \colon \{\textit{classical knots}\} \;\to\; \{\textit{HyperKnots}\}, \qquad \iota\big(K\big)(t) = \{K(t)\}$$

is injective and respects ambient isotopy. Hence every classical knot yields a HyperKnot, and distinct knot types give non-hyperisotopic HyperKnots.

Proof. Given a smooth embedding $K \colon S^1 \hookrightarrow \mathbb{R}^3$, define $K_H(t) = \{K(t)\}$. Clearly:

- $\{K(t)\} \in C(\mathbb{R}^3)$ for all t.
- Continuity in the Hausdorff metric follows since

$$d_H\big(\{K(t)\}, \{K(t_0)\}\big) \;=\; \|K(t) - K(t_0)\| \;\xrightarrow[t\to t_0]{}\; 0.$$

- If K_0 and K_1 are ambient-isotopic via $\Psi_t \colon \mathbb{R}^3 \to \mathbb{R}^3$, then $\Phi_t(A) = \Psi_t(A)$ for each compact $A \subset \mathbb{R}^3$ defines an ambient hyperisotopy between $\iota(K_0)$ and $\iota(K_1)$.

Injectivity follows because if $\iota(K_0)$ and $\iota(K_1)$ are hyperisotopic then their cores $\{K_0(t)\}$ and $\{K_1(t)\}$ are ambient-isotopic in $\mathbb{R}^3$, so K_0 and K_1 represent the same knot type. $\square$

Theorem 2.6 (HyperKnots form a hyperstructure)**.** *Let $\mathcal{HK}$ be the set of all HyperKnots. Define a binary* hyperoperation $\odot$ *on $\mathcal{HK}$ by*

$$(K_H^1 \odot K_H^2)(t) \;=\; K_H^1(t) \cup K_H^2(t) \quad (\forall\, t \in S^1).$$

Then $(\mathcal{HK}, \odot)$ is a hyperstructure: for any $K_H^1, K_H^2 \in \mathcal{HK}$, $\{K_H^1 \odot K_H^2\} \subseteq \mathcal{HK}$ and the union-map is well-defined.

Proof. We must check that $K_H^1 \odot K_H^2$ is again a HyperKnot:

(1) For each t, $K_H^1(t) \cup K_H^2(t)$ is a nonempty compact subset of $\mathbb{R}^3$.

(2) Continuity in the Hausdorff metric holds because union is continuous on $C(\mathbb{R}^3)$: if $d_H(K_H^i(t), K_H^i(t_0)) \to 0$ as $t \to t_0$ for $i = 1, 2$, then

$$d_H\bigl(K_H^1(t) \cup K_H^2(t),\, K_H^1(t_0) \cup K_H^2(t_0)\bigr) \;\le\; d_H\bigl(K_H^1(t), K_H^1(t_0)\bigr) + d_H\bigl(K_H^2(t), K_H^2(t_0)\bigr) \;\to\; 0.$$

(3) There exists a classical knot K whose image lies in $K_H^1(t) \cup K_H^2(t)$ (for instance, choose either core of K_H^1 or K_H^2), so the core-inclusion condition holds.

Thus $\mathcal{HK}$ is closed under $\odot$. By definition of a hyperstructure, $(\mathcal{HK}, \odot)$ is a hyperstructure. □

Theorem 2.7 (Classical knots embed faithfully into HyperKnots)**.** *The map*

$$\iota \colon \{\textit{classical knots up to ambient isotopy}\} \;\to$$

$$\{\textit{HyperKnots up to ambient hyperisotopy}\}, \quad \iota(K)(t) = \{K(t)\}$$

is well-defined and injective. In particular, if two classical knots K_0, K_1 satisfy $\iota(K_0)$ ambient-hyperisotopic to $\iota(K_1)$, then K_0 is ambient-isotopic to K_1.

Proof. First, given a classical knot embedding $K \colon S^1 \hookrightarrow \mathbb{R}^3$, the assignment $\iota(K)(t) = \{K(t)\}$ is a continuous map $S^1 \to C(\mathbb{R}^3)$, and clearly $\{K(t)\}$ contains the core $K(t)$, so $\iota(K)$ is a HyperKnot.

Next, suppose $\iota(K_0)$ and $\iota(K_1)$ are ambient-hyperisotopic via $\Phi_t \colon C(\mathbb{R}^3) \to C(\mathbb{R}^3)$. Since each singleton $\{x\} \subset \mathbb{R}^3$ is identified in $C(\mathbb{R}^3)$, the family

$$\Psi_t(x) \;=\; \text{the unique point in } \Phi_t\bigl(\{x\}\bigr) \quad (x \in \mathbb{R}^3)$$

defines a continuous family of homeomorphisms of $\mathbb{R}^3$ with $\Psi_0 = \mathrm{id}$ and $\Psi_1 \circ K_0 = K_1$. Hence K_0 and K_1 are ambient-isotopic. □

Theorem 2.8 (Pointwise union of HyperKnots)**.** *Let K_H^1, K_H^2 be HyperKnots. Define*

$$(K_H^1 \odot K_H^2)(t) \;=\; K_H^1(t) \cup K_H^2(t).$$

Then $K_H^1 \odot K_H^2$ is again a HyperKnot.

Proof. (1) Each $K_H^i(t)$ is a nonempty compact subset of $\mathbb{R}^3$, so the union $K_H^1(t) \cup K_H^2(t)$ is nonempty and compact.

(2) Continuity in the Hausdorff metric follows from the estimate

$$d_H\big(K_H^1(t) \cup K_H^2(t),\, K_H^1(t_0) \cup K_H^2(t_0)\big) \leq d_H\big(K_H^1(t), K_H^1(t_0)\big) + d_H\big(K_H^2(t), K_H^2(t_0)\big),$$

which tends to zero as $t \to t_0$.

(3) Each K_H^i admits a smooth core embedding $K^i \colon S^1 \to \mathbb{R}^3$ with $\{K^i(t)\} \subset K_H^i(t)$. Choosing either core K^1 or K^2 gives a core for the union, so $K_H^1 \odot K_H^2$ satisfies the core-inclusion condition.

Therefore $K_H^1 \odot K_H^2$ is a HyperKnot. □

Theorem 2.9 (Hyperstructure properties of $\mathcal{HK}$)**.** *Let $\mathcal{HK}$ be the set of all HyperKnots. The binary operation $\odot \colon \mathcal{HK} \times \mathcal{HK} \to \mathcal{HK}$ defined by pointwise union is*

- Commutative*: $K_H^1 \odot K_H^2 = K_H^2 \odot K_H^1$.*
- Associative*: $(K_H^1 \odot K_H^2) \odot K_H^3 = K_H^1 \odot (K_H^2 \odot K_H^3)$.*

Hence $(\mathcal{HK}, \odot)$ is a commutative semihypergroup.

Proof. Both commutativity and associativity follow immediately from the corresponding properties of the set-union operation on compact subsets of $\mathbb{R}^3$. Specifically, for any $t \in S^1$,

$$K_H^1(t) \cup K_H^2(t) = K_H^2(t) \cup K_H^1(t),$$

and

$$\big(K_H^1(t) \cup K_H^2(t)\big) \cup K_H^3(t) = K_H^1(t) \cup \big(K_H^2(t) \cup K_H^3(t)\big).$$

Since pointwise union preserves continuity and the core-inclusion condition, $\odot$ makes $\mathcal{HK}$ into a commutative semihypergroup. □

2.2. *SuperHyperKnot*

We present the definition of a SuperHyperKnot as follows.

Definition 2.10 (n-SuperHyperKnot)**.** Let $n \geq 1$. Define recursively the k-th iterated hyperspace of $\mathbb{R}^3$:

$$\begin{cases} C^{(0)}(\mathbb{R}^3) = \mathbb{R}^3, \\ C^{(k)}(\mathbb{R}^3) = \big\{\, A \subseteq C^{(k-1)}(\mathbb{R}^3) \mid A \neq \emptyset,\ A \text{ is compact in the Hausdorff metric } d_H^{(k-1)} \big\}, \quad k \geq 1, \end{cases}$$

where $d_H^{(0)}$ is the Euclidean distance on $\mathbb{R}^3$, and for $k \geq 1$, $d_H^{(k)}$ is the induced Hausdorff metric on compact subsets of $C^{(k-1)}(\mathbb{R}^3)$.

An *n-SuperHyperKnot* is a continuous map

$$K^{(n)} \colon S^1 \;\to\; C^{(n)}(\mathbb{R}^3)$$

satisfying the *core inclusion condition*: there exists a classical knot embedding

$$K \colon S^1 \hookrightarrow \mathbb{R}^3 \quad \text{and selections} \quad A_k(t) \in C^{(k)}(\mathbb{R}^3) \quad (1 \le k \le n,\ t \in S^1)$$

with

$$A_n(t) = K^{(n)}(t), \quad A_{k-1}(t) \in A_k(t) \quad (1 \le k \le n), \quad A_0(t) = \{K(t)\}.$$

Two n-SuperHyperKnots $K_0^{(n)}, K_1^{(n)}$ are *ambient superhyperisotopic* if there exists a continuous family of homeomorphisms

$$\Phi_t^{(n)} \colon C^{(n)}(\mathbb{R}^3) \;\to\; C^{(n)}(\mathbb{R}^3), \quad t \in [0,1],$$

with $\Phi_0^{(n)} = \mathrm{id}$ and $\Phi_1^{(n)} \circ K_0^{(n)} = K_1^{(n)}$.

Example 2.11 (n-SuperHyperKnot via iterated arc-neighborhoods)**.** Let

$$K \colon S^1 \to \mathbb{R}^3, \quad K(\theta) = (\cos\theta, \sin\theta, 0)$$

be the standard unit circle. Fix a small $\varepsilon > 0$. We define a nested family of compact sets

$$A_0(\theta) = \{K(\theta)\}, \quad A_1(\theta) = \big\{\, K(\phi) \mid \phi \in [\theta - \varepsilon, \theta + \varepsilon] \big\},$$

and for $k = 2, \dots, n$,

$$A_k(\theta) = \big\{\, A_{k-1}(\phi) \mid \phi \in [\theta - \varepsilon, \theta + \varepsilon] \big\}.$$

Then:

- Each $A_k(\theta)$ is nonempty and compact in $C^{(k-1)}(\mathbb{R}^3)$ by continuity of $\phi \mapsto A_{k-1}(\phi)$ on the compact interval $[\theta - \varepsilon, \theta + \varepsilon]$.
- By construction $A_{k-1}(\theta) \in A_k(\theta)$ for all $1 \le k \le n$, and $A_0(\theta) = \{K(\theta)\}$ is the core.
- The map $K^{(n)} \colon S^1 \to C^{(n)}(\mathbb{R}^3)$ defined by

  $$K^{(n)}(\theta) = A_n(\theta)$$

  is continuous in the induced Hausdorff metric $d_H^{(n)}$, since each stage arises from a continuous image of a compact interval.

Therefore $K^{(n)}$ satisfies the core-inclusion condition and continuity, hence is an n-SuperHyperKnot.

Example 2.12 (n-SuperHyperKnot via iterated tubular neighborhoods)**.** Let

$$K \colon S^1 \to \mathbb{R}^3, \quad K(\theta) = (\cos\theta, \sin\theta, 0)$$

be the standard unit circle. Choose a radius $r > 0$ smaller than the reach of K and a small $\varepsilon > 0$. Define for each $\theta \in S^1$:

$$A_0(\theta) = \{K(\theta)\}, \quad A_1(\theta) = \{x \in \mathbb{R}^3 \mid \|x - K(\theta)\| \leq r\},$$

and for $2 \leq k \leq n$,

$$A_k(\theta) = \bigcup_{\phi \in [\theta-\varepsilon, \theta+\varepsilon]} A_{k-1}(\phi),$$

where intervals are taken modulo 2π. Then:

(1) *Nonempty compactness.* Each $A_k(\theta)$ is nonempty and compact in $C^{(k-1)}(\mathbb{R}^3)$ because it is a finite union of compact sets.

(2) *Nested inclusion.* By definition $A_{k-1}(\theta) \in A_k(\theta)$ for all $1 \leq k \leq n$, and $A_0(\theta) = \{K(\theta)\}$ is the core.

(3) *Continuity in the Hausdorff metric.* For each k,

$$d_H^{(k)}\big(A_k(\theta), A_k(\theta_0)\big) \;\leq\; \sup_{\phi \in [\theta-\varepsilon, \theta+\varepsilon]} d_H^{(k-1)}\big(A_{k-1}(\phi), A_{k-1}(\phi + \theta_0 - \theta)\big) \;\to\; 0 \quad (\theta \to \theta_0),$$

since $\phi \mapsto A_{k-1}(\phi)$ is continuous and the supremum over a small interval tends to zero.

Hence the map

$$K^{(n)} \colon S^1 \to C^{(n)}(\mathbb{R}^3), \quad K^{(n)}(\theta) = A_n(\theta)$$

is continuous and satisfies the core–inclusion condition. Therefore $K^{(n)}$ is an n-SuperHyperKnot.

Theorem 2.13 (Generalization of knots and HyperKnots)**.** *The assignments*

$$\iota_0 \colon \{\textit{classical knots}\} \to \{n\textit{-SuperHyperKnots}\}, \quad \iota_0\big(K\big)(t) = \{\{\cdots\{\{K(t)\}\}\cdots\}\}$$

(nested n times) and

$$\iota_1 \colon \{\textit{HyperKnots}\} \to \{n\textit{-SuperHyperKnots}\}, \quad \iota_1\big(K_H\big)(t) = \{\{\cdots\{\,K_H(t)\}\cdots\}\}$$

(nested $n-1$ times) are injective and respect ambient isotopy. Thus every classical knot and every HyperKnot embeds faithfully as an n-SuperHyperKnot.

Proof. Given a classical embedding K, define $\iota_0(K)(t)$ by $\{\,\{\cdots\{\{K(t)\}\}\cdots\}\} \subset C^{(n)}(\mathbb{R}^3)$. Continuity follows by iterated Hausdorff estimates:

$$d_H^{(k)}\big(\iota_0(K)(t), \iota_0(K)(t_0)\big) = d_H^{(k-1)}\big(\iota_0(K)(t), \iota_0(K)(t_0)\big) \to 0,$$

and ambient isotopies lift at each level by $\Phi_t^{(n)}(A) = \{\cdots\{\Psi_t(A)\}\cdots\}$. Injectivity holds since the unique core $\{K(t)\}$ recovers K. The argument for ι_1 is identical, treating $K_H(t)$ as the first level. $\square$

Theorem 2.14 (n-SuperHyperKnots form an n-Superhyperstructure)**.** *Let $\mathcal{SHK}_n = \{ K^{(n)} : S^1 \to C^{(n)}(\mathbb{R}^3)\}$. Define the binary* superhyperoperation

$$(K_1^{(n)} \star K_2^{(n)})(t) = K_1^{(n)}(t) \cup K_2^{(n)}(t) \subseteq C^{(n)}(\mathbb{R}^3).$$

Then $\big(\mathcal{SHK}_n, \star\big)$ is an n-Superhyperstructure: $\star$ is a well-defined map $\mathcal{SHK}_n \times \mathcal{SHK}_n \to \mathcal{P}\big(C^{(n)}(\mathbb{R}^3)\big)$, and $\mathcal{SHK}_n$ is closed under $\star$.

Proof. For any $K_1^{(n)}, K_2^{(n)}$, $K_1^{(n)}(t) \cup K_2^{(n)}(t)$ is nonempty compact in $C^{(n-1)}(\mathbb{R}^3)$, so lies in $C^{(n)}(\mathbb{R}^3)$. Continuity under $d_H^{(n)}$ follows from the union estimate

$$d_H^{(n)}\big(A \cup B,\ A_0 \cup B_0\big) \leq d_H^{(n)}(A, A_0) + d_H^{(n)}(B, B_0).$$

The core inclusion condition holds by choosing at each t one of the two nested core chains. Hence $\mathcal{SHK}_n$ is closed under $\star$ and defines an n-Superhyperstructure as in [24]. □

Theorem 2.15 (Projection to lower levels)**.** *For each $1 \leq m < n$, there is a natural "projection" map*

$$\pi_m^{(n)} : C^{(n)}(\mathbb{R}^3) \to C^{(m)}(\mathbb{R}^3), \qquad \pi_m^{(n)}(A) = A_m \quad \big(A \in C^{(n)}(\mathbb{R}^3)\big),$$

where A_m is any nonempty compact subset with $\{K(t)\} = A_0 \in A_1 \in \cdots \in A_n = A$. Then for any n-SuperHyperKnot $K^{(n)}$, the composition

$$K^{(m)} = \pi_m^{(n)} \circ K^{(n)}$$

is an m-SuperHyperKnot. Moreover, if $K_0^{(n)}, K_1^{(n)}$ are ambient superhyperisotopic via $\Phi_t^{(n)}$, then $\pi_m^{(n)} \circ K_0^{(n)}$ and $\pi_m^{(n)} \circ K_1^{(n)}$ are ambient superhyperisotopic in level m.

Proof. Since each $K^{(n)}(t) \in C^{(n)}(\mathbb{R}^3)$ admits a nested chain $\{K(t)\} = A_0(t) \in A_1(t) \in \cdots \in A_n(t)$, the map $\pi_m^{(n)}(A_n(t)) = A_m(t)$ is continuous in the Hausdorff metric $d_H^{(m)}$ and contains the core $\{K(t)\}$. Hence $K^{(m)}$ satisfies the definition of an m-SuperHyperKnot. If $\Phi_t^{(n)}$ is an ambient superhyperisotopy at level n, then by restriction $\Phi_t^{(m)} = \pi_m^{(n)} \circ \Phi_t^{(n)} \circ (\pi_m^{(n)})^{-1}$ defines a family of homeomorphisms on $C^{(m)}(\mathbb{R}^3)$, giving an ambient superhyperisotopy of $K_0^{(m)}$ and $K_1^{(m)}$. □

Theorem 2.16 (Core recovery)**.** *Every n-SuperHyperKnot $K^{(n)} : S^1 \to C^{(n)}(\mathbb{R}^3)$ determines uniquely a classical knot $\kappa : S^1 \to \mathbb{R}^3$ by the rule*

$$\kappa(t) = \text{the unique point in } \bigcap_{k=0}^{n} A_k(t),$$

where $A_k(t) \in C^{(k)}(\mathbb{R}^3)$ is any chain with $A_n(t) = K^{(n)}(t)$. Moreover, ambient superhyperisotopy of $K^{(n)}$ projects to ambient isotopy of κ.

Proof. By the core inclusion condition there is a nested sequence $\{K(t)\} = A_0(t) \in A_1(t) \in \cdots \in A_n(t)$. Since $A_0(t)$ is a singleton, the intersection $\bigcap_{k=0}^{n} A_k(t)$ equals $\{K(t)\}$. Continuity of κ follows from continuity of $K^{(n)}$ and the fact that intersections of nested compact sets vary continuously in $d_H^{(0)}$. If $K_0^{(n)}, K_1^{(n)}$ are related by $\Phi_t^{(n)}$, then their cores satisfy $\Psi_t(\{K_0(t)\}) = \{K_1(t)\}$ for Ψ_t the induced ambient isotopy on $\mathbb{R}^3$. □

Theorem 2.17 (Commutative semihypergroup structure). *Let $\mathcal{SHK}_n$ be the set of all n-SuperHyperKnots. Define*

$$(K_1^{(n)} \star K_2^{(n)})(t) = K_1^{(n)}(t) \cup K_2^{(n)}(t).$$

Then $(\mathcal{SHK}_n, \star)$ is a commutative semihypergroup:

(1) *$\star$ is well-defined: each union is nonempty compact in $C^{(n-1)}(\mathbb{R}^3)$.*

(2) *$\star$ is commutative and associative by properties of set union.*

(3) *The core inclusion condition holds since one may choose the core chain from either operand.*

Proof. (1) follows as in the HyperKnot case, using continuity of union in $d_H^{(n)}$. For (2), for all t,

$$K_1^{(n)}(t) \cup K_2^{(n)}(t) = K_2^{(n)}(t) \cup K_1^{(n)}(t),$$

and

$$(K_1^{(n)} \cup K_2^{(n)}) \cup K_3^{(n)} = K_1^{(n)} \cup (K_2^{(n)} \cup K_3^{(n)}).$$

For (3), if $A_k^i(t)$ are core chains for $K_i^{(n)}$, then $\{A_k^1(t) \cup A_k^2(t)\}_{k=0}^{n}$ is a valid core chain for $K_1^{(n)} \star K_2^{(n)}$. Thus $(\mathcal{SHK}_n, \star)$ satisfies all axioms of a commutative semihypergroup. □

Theorem 2.18 (Transitivity of Projections). *Let $0 \le \ell < m < n$. Then the projections*

$$\pi_m^{(n)} \colon C^{(n)}(\mathbb{R}^3) \to C^{(m)}(\mathbb{R}^3), \quad \pi_\ell^{(m)} \colon C^{(m)}(\mathbb{R}^3) \to C^{(\ell)}(\mathbb{R}^3),$$

satisfy

$$\pi_\ell^{(m)} \circ \pi_m^{(n)} \;=\; \pi_\ell^{(n)}.$$

Proof. By definition, for any $A \in C^{(n)}(\mathbb{R}^3)$ there is a nested chain $\{K(t)\} = A_0 \in A_1 \in \cdots \in A_n = A$. Then

$$\big(\pi_\ell^{(m)} \circ \pi_m^{(n)}\big)(A) = \pi_\ell^{(m)}\big(\pi_m^{(n)}(A)\big) = \pi_\ell^{(m)}(A_m) = A_\ell = \pi_\ell^{(n)}(A),$$

so the two compositions agree on every element. □

Theorem 2.19 (Projection of Core Injection). *Let ι_0 be the embedding of classical knots into n-SuperHyperKnots given by $\iota_0(K)(t) = \{\{\cdots\{\{K(t)\}\}\cdots\}\}$. Then*

$$\pi_0^{(n)}\big(\iota_0(K)(t)\big) = \{K(t)\}, \quad \forall\, t \in S^1,$$

and hence $\pi_0^{(n)} \circ \iota_0$ is the identity on classical knots.

Proof. By construction $\iota_0(K)(t)$ is an n-fold nested singleton whose level-0 element is exactly $\{K(t)\}$. Therefore $\pi_0^{(n)}$ extracts that singleton, recovering K pointwise. □

Theorem 2.20 (Classification Equivalence). *The "core" map*

$$\mathrm{core} \;:\; \{n\text{-}\mathit{SuperHyperKnots}\}/\!\sim\; \to\; \{\mathit{classical\ knots}\}/\!\approx, \quad [K^{(n)}] \mapsto [\kappa],$$

where $\kappa(t)$ is the unique point in $\bigcap_{k=0}^{n} A_k(t)$, is a bijection between superhyperisotopy classes of n-SuperHyperKnots and ambient-isotopy classes of classical knots.

Proof. Injectivity: If two n-SuperHyperKnots $K_0^{(n)}, K_1^{(n)}$ satisfy $\mathrm{core}(K_0^{(n)}) \approx \mathrm{core}(K_1^{(n)})$, then their classical cores are isotopic. By Theorem "Generalization of knots and HyperKnots", this lifts to a superhyperisotopy between the superhyperknots, so $[K_0^{(n)}] = [K_1^{(n)}]$.

Surjectivity: Given any classical knot K, the injection $\iota_0(K)$ is an n-SuperHyperKnot whose core is exactly K. Thus every classical knot type arises. □

Theorem 2.21 (Lifting of Classical Invariants). *Let I be any invariant of classical knots under ambient isotopy, i.e.*

$$I\colon \{\mathit{classical\ knots}\}/\!\approx\; \to\; \mathcal{X}.$$

Then

$$\widehat{I} \;=\; I \circ \pi_0^{(n)} \;:\; \{n\text{-}\mathit{SuperHyperKnots}\}/\!\sim\; \to\; \mathcal{X}$$

is invariant under ambient superhyperisotopy of n-SuperHyperKnots.

Proof. If $K_0^{(n)} \sim K_1^{(n)}$, then their projections $\pi_0^{(n)}(K_0^{(n)}), \pi_0^{(n)}(K_1^{(n)})$ are ambient-isotopic classical knots. Hence $\widehat{I}([K_0^{(n)}]) = I([\pi_0^{(n)} K_0^{(n)}]) = I([\pi_0^{(n)} K_1^{(n)}]) = \widehat{I}([K_1^{(n)}])$. □

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Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Authors' Contributions

The sole author designed, analysed, interpreted, and prepared the manuscript.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

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The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

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