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Soil Microbial HyperNetworks and SuperHyperNetworks in Plant and Soil Sciences

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Abstract. Graph theory underpins many applications across science and engineering. Hypergraphs generalize graphs by allowing hyperedges to join any number of vertices, while superhypergraphs further extend this idea by layering iterated powersets to capture hierarchical, self-referential connections. A Soil Microbial Network is a graph where microbial taxa are represented as nodes, and edges indicate statistical associations or interactions among these taxa. In this paper, we provide a mathematical definition of the Soil Microbial Network and investigate its generalizations: the Soil Microbial HyperNetwork, formulated within the framework of HyperGraphs, and the Soil Microbial SuperHyperNetwork, formulated within the framework of SuperHyperGraphs.

Keywords: Superhypergraph, Hypergraph, Soil Microbial Network

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1. Preliminaries

We briefly recall the basic notions needed in the remainder of the paper. Throughout, all sets are assumed to be finite.

1.1. *Power Set and n-th Power Set*

The power set of a set S is the collection of all subsets of S, including both the empty set and S itself. The n-th power set of S is obtained by iteratively applying the power set operation n times, starting from S [1, 2]. The notion of the n-th power set was introduced by F. Smarandache. The formal definitions and concrete examples of this concept are given below.

Definition 1.1 (Universal Set)**.** Let U be a set containing all elements under consideration. Throughout, every set S is assumed to satisfy $S \subseteq U$.

Definition 1.2 (Base Set)**.** A *base set* S is any subset $S \subseteq U$ from which further constructions—such as powersets and hyperstructures—are formed.

Definition 1.3 (Power Set)**.** [3, 4] The *power set* of S, denoted $\mathcal{P}(S)$, is the collection of all subsets of S:

$$\mathcal{P}(S) \;=\; \{\, X \mid X \subseteq S\}.$$

Definition 1.4 (Iterated Power Set)**.** [5, 6] For each integer $n \geq 1$, define the n-fold iterated power set of S by

$$\mathcal{P}^1(S) = \mathcal{P}(S),$$
$$\mathcal{P}^{k+1}(S) = \mathcal{P}\big(\mathcal{P}^k(S)\big) \quad (k \geq 1).$$

Equivalently, one may write $P_n(S) = \mathcal{P}^n(S)$.

Definition 1.5 (Nonempty Iterated Power Set)**.** [5, 7] Define the nonempty iterated power set by

$$\mathcal{P}_1^*(S) = \mathcal{P}(S) \setminus \{\varnothing\},$$
$$\mathcal{P}_{k+1}^*(S) = \mathcal{P}^*\big(\mathcal{P}_k^*(S)\big) \quad (k \geq 1),$$

where $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\varnothing\}$ for any set X.

Example 1.6 (Plant organs; a concrete nonempty iterated power set for $|S| = 3$)**.** Let the plant-organ set be $S = \{\mathrm{Root}, \mathrm{Stem}, \mathrm{Leaf}\}$. Then the nonempty power set is

$$\mathcal{P}_1^*(S) = \big\{\{\mathrm{Root}\},\ \{\mathrm{Stem}\},\ \{\mathrm{Leaf}\},$$
$$\{\mathrm{Root}, \mathrm{Stem}\},\ \{\mathrm{Root}, \mathrm{Leaf}\},\ \{\mathrm{Stem}, \mathrm{Leaf}\},\ \{\mathrm{Root}, \mathrm{Stem}, \mathrm{Leaf}\}\big\},$$

so $|\mathcal{P}_1^*(S)| = 2^3 - 1 = 7$. By definition, $\mathcal{P}_2^*(S) = \mathcal{P}(\mathcal{P}_1^*(S)) \setminus \{\varnothing\}$ has $|\mathcal{P}_2^*(S)| = 2^7 - 1 = 127$. Two explicit elements of $\mathcal{P}_2^*(S)$ are, for instance,

$$A_1 = \big\{\{\mathrm{Root}\},\ \{\mathrm{Stem}\}\big\},$$

$$A_2 = \big\{\{\mathrm{Root}, \mathrm{Leaf}\},\ \{\mathrm{Stem}, \mathrm{Leaf}\},\ \{\mathrm{Root}, \mathrm{Stem}, \mathrm{Leaf}\}\big\}.$$

Here each member of A_1 and A_2 is a nonempty subset of S, hence $A_1, A_2 \subseteq \mathcal{P}_1^*(S)$ and both $A_1, A_2 \in \mathcal{P}_2^*(S)$. Interpretation in plant/soil science: elements of $\mathcal{P}_1^*(S)$ represent concrete organ-combinations (e.g., tissues sampled together), whereas elements of $\mathcal{P}_2^*(S)$ are *collections of such combinations* (e.g., sets of sampling schemes or treatment groups).

Example 1.7 (Soil microbial guilds; a concrete nonempty iterated power set for $|S| = 4$)**.** Let

$$S = \{\mathrm{Bact}, \mathrm{Fungi}, \mathrm{Arch}, \mathrm{Prot}\}$$

denote bacterial, fungal, archaeal, and protist guilds. Then

$$\begin{aligned}\mathcal{P}_1^*(S) = \{&\{\mathrm{Bact}\},\ \{\mathrm{Fungi}\},\ \{\mathrm{Arch}\},\ \{\mathrm{Prot}\},\\ &\{\mathrm{Bact}, \mathrm{Fungi}\},\ \{\mathrm{Bact}, \mathrm{Arch}\},\ \{\mathrm{Bact}, \mathrm{Prot}\},\ \{\mathrm{Fungi}, \mathrm{Arch}\},\ \{\mathrm{Fungi}, \mathrm{Prot}\},\ \{\mathrm{Arch}, \mathrm{Prot}\},\\ &\{\mathrm{Bact}, \mathrm{Fungi}, \mathrm{Arch}\},\ \{\mathrm{Bact}, \mathrm{Fungi}, \mathrm{Prot}\},\ \{\mathrm{Bact}, \mathrm{Arch}, \mathrm{Prot}\},\ \{\mathrm{Fungi}, \mathrm{Arch}, \mathrm{Prot}\},\\ &\{\mathrm{Bact}, \mathrm{Fungi}, \mathrm{Arch}, \mathrm{Prot}\}\},\end{aligned}$$

so $|\mathcal{P}_1^*(S)| = 2^4 - 1 = 15$ and $|\mathcal{P}_2^*(S)| = 2^{15} - 1 = 32767$. Two explicit elements of $\mathcal{P}_2^*(S)$ are

$$B_1 = \big\{\{\mathrm{Bact}\},\ \{\mathrm{Fungi}\},\ \{\mathrm{Arch}\}\big\},$$

$$B_2 = \big\{\{\mathrm{Bact}, \mathrm{Fungi}\},\ \{\mathrm{Bact}, \mathrm{Arch}\},\ \{\mathrm{Fungi}, \mathrm{Arch}\},\ \{\mathrm{Bact}, \mathrm{Fungi}, \mathrm{Arch}\}\big\}.$$

Again, every element listed inside B_1 and B_2 is a nonempty subset of S, hence $B_1, B_2 \subseteq \mathcal{P}_1^*(S)$, so $B_1, B_2 \in \mathcal{P}_2^*(S)$. Domain interpretation: members of $\mathcal{P}_1^*(S)$ are concrete microbial assemblages (e.g., $\{\mathrm{Bact}, \mathrm{Fungi}\}$ co-occurring), while members of $\mathcal{P}_2^*(S)$ are *sets of assemblages* (e.g., groups of communities considered jointly in a soil health index or meta-analysis).

As supplementary information, the Comparison of a set, its power set, and its iterated power set is presented in Table 1.

TABLE 1. Comparison of a set, its power set, and iterated power set

Aspect	Set S	Power set $\mathcal{P}(S)$	Iterated power set $\mathcal{P}^n(S)$
Object	Subset $S \subseteq U$	Collection of all subsets of S	Repeated powerset of S for $n \geq 1$
Typical element	Element $x \in S$	Subset $X \subseteq S$	Nested set $X_n \in \mathcal{P}^n(S)$ built from subsets
Cardinality (finite $\lvert S\rvert = m$)	$\lvert S\rvert = m$	$\lvert\mathcal{P}(S)\rvert = 2^m$	$\lvert\mathcal{P}^1(S)\rvert = 2^m$, $\lvert\mathcal{P}^2(S)\rvert = 2^{2^m}$, etc.
Level of abstraction	Base level: individual objects	First–order collections of objects	Higher–order collections of collections (hierarchical)

1.2. *Hypergraphs and SuperHypergraphs*

Hypergraphs generalize ordinary graphs (cf. [8]) by allowing each *hyperedge* to join an arbitrary nonempty subset of vertices, thereby modeling higher-order relations among elements [9–11]. A HyperGraph is an important research concept because, unlike a classical graph, it can represent complex and higher-order network structures. Moreover, several related extensions of HyperGraphs are known in the literature, including Directed HyperGraphs [12, 13], Fuzzy HyperGraphs [14, 15], Neutrosophic HyperGraphs [16, 17], and Subset-Vertex Graphs [18, 19]. A *SuperHyperGraph* further extends this idea by incorporating iterated powerset structures, enabling multi-layered, self-referential connections among hyperedges [20–22]. The notion of the SuperHyperGraph was introduced by F. Smarandache. Research on the applications of SuperHyperGraphs has been actively conducted in recent years due to their importance [23–25]. The definitions and concrete examples of this concept are provided below.

Definition 1.8 (Hypergraph)**.** [9, 26] Let V be a finite set of *vertices*. A *hypergraph* is a pair

$$H = \bigl(V,\ E\bigr), \qquad E \subseteq \mathcal{P}(V) \setminus \{\varnothing\},$$

where each element of E is called a *hyperedge*. No restriction is imposed on the size of a hyperedge.

Example 1.9 (University course enrollment as a hypergraph)**.** Consider four students

$$V = \{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3, \mathrm{S}_4\}.$$

Each course is taken by a group of students, so it is naturally modeled as a hyperedge. Suppose:

- Course A is taken by $\{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3\}$,
- Course B is taken by $\{\mathrm{S}_2, \mathrm{S}_3\}$,
- Course C is taken by $\{\mathrm{S}_3, \mathrm{S}_4\}$.

Then the hyperedge family is

$$E = \bigl\{\{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3\}, \{\mathrm{S}_2, \mathrm{S}_3\}, \{\mathrm{S}_3, \mathrm{S}_4\}\bigr\} \ \subseteq\ \mathcal{P}(V) \setminus \{\varnothing\}.$$

Thus

$$H = (V, E)$$

is a hypergraph, where each hyperedge represents the set of students taking the same course.

Definition 1.10 (n-SuperHyperGraph)**.** (cf. [20, 27]) Let V_0 be a finite nonempty *base set*. Define the iterated powersets by

$$\mathcal{P}^0(X) := X, \qquad \mathcal{P}^{k+1}(X) := \mathcal{P}\bigl(\mathcal{P}^k(X)\bigr) \quad (k \ge 0).$$

Fix $n \geq 1$. An *n-SuperHyperGraph* is a pair

$$\mathrm{SHG}^{(n)} = (V, E),$$

where the *n-supervertex set* V satisfies $\varnothing \neq V \subseteq \mathcal{P}^n(V_0)$, and the *n-superedge set* E is a nonempty family of nonempty subsets of V,

$$\varnothing \neq E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}.$$

Equivalently, each $e \in E$ is a finite nonempty set of n-supervertices. For $n = 0$, this reduces to an ordinary hypergraph on V_0.

Example 1.11 (Concrete finite instance for $n = 2$)**.** Let $V_0 = \{a, b, c\}$. Then

$$\mathcal{P}(V_0) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

and $\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big)$. Choose three 2-supervertices (each is a subset of $\mathcal{P}(V_0)$):

$$v_1 := \big\{\{a\}, \{b\}\big\}, \quad v_2 := \big\{\{a, c\}, \{b, c\}\big\}, \quad v_3 := \big\{\varnothing, \{a, b, c\}\big\}.$$

Set $V := \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0)$ and define superedges

$$e_1 := \{v_1, v_2\}, \qquad e_2 := \{v_2, v_3\}.$$

Then $E := \{e_1, e_2\} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$, and $\mathrm{SHG}^{(2)} = (V, E)$ is a 2-SuperHyperGraph with $|V| = 3$ and $|E| = 2$. Here every edge is a nonempty set of 2-supervertices, as required by Definition 1.10.

Example 1.12 (Plant organs as an n=1 SuperHyperGraph)**.** Let the base set be the plant organs [28, 29]

$$V_0 = \{\mathrm{Root}, \mathrm{Stem}, \mathrm{Leaf}\}.$$

Then $\mathcal{P}(V_0) = \{\varnothing, \{R\}, \{S\}, \{L\}, \{R, S\}, \{R, L\}, \{S, L\}, \{R, S, L\}\}$ (where R, S, L abbreviate Root, Stem, Leaf). Fix $n = 1$ and choose the 1-supervertex set

$$V = \big\{\{R\}, \{L\}, \{R, L\}, \{S\}\big\} \ \subseteq \ \mathcal{P}^1(V_0) = \mathcal{P}(V_0).$$

Define the 1-superedge family (each superedge is a nonempty set of 1-supervertices)

$$e_1 = \big\{\{R\}, \{L\}\big\}, \qquad e_2 = \big\{\{L\}, \{R, L\}\big\}, \qquad e_3 = \big\{\{S\}, \{R, S\}\big\} \cap \mathcal{P}(V) = \varnothing,$$

and take $E = \{e_1, e_2\} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$. Then $\mathrm{SHG}^{(1)} = (V, E)$ is an n=1 SuperHyperGraph per Definition 1.10. Domain reading: supervertices are organ-subsets (e.g. $\{R, L\}$ = roots and leaves sampled together); superedges join such subsets when a study design or functional relation requires them to be considered jointly.

Example 1.13 (Soil compartments as an n=2 SuperHyperGraph)**.** Let the base set be soil compartments

$$V_0 = \{\text{Rhizosphere}\,(R),\ \text{Humus}\,(H),\ \text{Mineral}\,(M)\}.$$

Then $\mathcal{P}(V_0) = \{\varnothing, \{R\}, \{H\}, \{M\}, \{R, H\}, \{R, M\}, \{H, M\}, \{R, H, M\}\}$ and $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$. Fix $n = 2$ and define the 2-supervertices (each is a subset of $\mathcal{P}(V_0)$):

$$v_1 = \big\{\{R\}, \{H\}\big\}, \qquad v_2 = \big\{\{R, M\}, \{H, M\}\big\}, \qquad v_3 = \big\{\{R, H, M\}\big\}, \qquad v_4 = \big\{\{R\}, \{R, H\}\big\}.$$

Set

$$V = \{v_1, v_2, v_3, v_4\} \ \subseteq\ \mathcal{P}^2(V_0).$$

Choose the 2-superedges

$$e_1 = \{v_1, v_2\}, \qquad e_2 = \{v_2, v_3\}, \qquad e_3 = \{v_1, v_4, v_3\}.$$

Then $E = \{e_1, e_2, e_3\} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$, so $\text{SHG}^{(2)} = (V, E)$ is an n=2 SuperHyperGraph by Definition 1.10. Domain reading: each 2-supervertex is a *collection of compartment-subsets* (e.g. v_2 bundles two horizons that include M); superedges join such bundles when a protocol, constraint, or hypothesis requires analyzing these collections together (e.g. cross-horizon microbial exchanges).

As supplementary information, Table 2 presents the Comparison of Graphs, Hypergraphs, and SuperHyperGraphs.

TABLE 2. Comparison of Graphs, Hypergraphs, and SuperHyperGraphs

Aspect	Graph	Hypergraph	SuperHyperGraph
Vertex set	V	V	$V \subseteq \mathcal{P}^n(V_0)$ (set of n-supervertices)
Edge / (hyper)edge	Edges $\{u, v\}$, $u \neq v \in V$	Hyperedges $e \subseteq V$, $e \neq \varnothing$	Superhyperedges $e \subseteq V$, $e \neq \varnothing$ (on supervertices)
Order of interactions	Pairwise relations	Higher–order relations among vertices	Higher–order relations among supervertices and levels
Underlying structure	Single layer on V	Single layer on V via $\mathcal{P}(V)$	Iterated powerset layers $\mathcal{P}^n(V_0)$
Specialization	Base model	All $\lvert e\rvert = 2$ gives a graph	For $n = 0$ and $V = V_0$ gives a hypergraph

1.3. *Soil Microbial Network*

A Soil Microbial Network is a graph where microbial taxa are nodes, and edges represent statistical associations or interactions among taxa (cf. [30–33]). The definitions and concrete examples of this concept are provided below.

Definition 1.14 (Soil Microbial Network). Let $T = \{1, \ldots, n\}$ be taxa and $S = \{1, \ldots, m\}$ samples. Given an abundance matrix $X = (x_{i,s}) \in \mathbb{R}^{n\times m}_{>0}$, define the centered log-ratio (CLR) transform

$$y_{i,s} = \log x_{i,s} - \frac{1}{n}\sum_{k=1}^{n} \log x_{k,s}, \qquad Y = (y_{i,s}) \in \mathbb{R}^{n\times m}.$$

Let $\hat{\Sigma}$ be the rowwise unbiased covariance of Y and $\hat{R} = (\hat{R}_{ij})$ the corresponding correlation matrix. For a threshold $\tau \in [0, 1)$ (optionally with FDR level $\alpha \in (0, 1)$), the Soil Microbial Network is the signed, weighted simple graph

$$G = (V, E, w, \sigma), \quad V = T, \quad E = \big\{\{i, j\} : i \neq j,\ |\hat{R}_{ij}| \geq \tau \text{ (and } p^{\mathrm{BH}}_{ij} \leq \alpha)\big\},$$

$$w(\{i, j\}) = |\hat{R}_{ij}| \in [\tau, 1], \qquad \sigma(\{i, j\}) = \operatorname{sign}(\hat{R}_{ij}) \in \{-1, +1\}.$$

CLR is sample-wise scale-invariant: if $x'_{i,s} = c_s x_{i,s}$ with $c_s > 0$, then

$$y'_{i,s} = \log(c_s x_{i,s}) - \frac{1}{n}\sum_{k} \log(c_s x_{k,s}) = \log x_{i,s} - \frac{1}{n}\sum_{k} \log x_{k,s} = y_{i,s}.$$

(Partial-dependence variant: replace $\hat{R}$ by $\tilde{R}_{ij} = -\hat{\Theta}_{ij}/\sqrt{\hat{\Theta}_{ii}\hat{\Theta}_{jj}}$ from a sparse precision $\hat{\Theta}$ via graphical lasso.)

Example 1.15 (Soil Microbial Network). Take $n = 3$, $m = 4$ with

$$X = \begin{pmatrix} 9 & 4 & 1 & 16 \\ 1 & 3 & 2 & 9 \\ 4 & 1 & 3 & 1 \end{pmatrix}.$$

CLR (natural log; rounded to 4 d.p.):

$$Y \approx \begin{pmatrix} 1.0027 & 0.5580 & -0.5973 & 1.1160 \\ -1.1945 & 0.2703 & 0.0959 & 0.5406 \\ 0.1918 & -0.8283 & 0.5014 & -1.6566 \end{pmatrix}.$$

Covariance and correlation:

$$\hat{\Sigma} \approx \begin{pmatrix} 0.61264 & -0.11711 & -0.49553 \\ -0.11711 & 0.59356 & -0.47646 \\ -0.49553 & -0.47646 & 0.97199 \end{pmatrix}, \quad \hat{R} \approx \begin{pmatrix} 1 & -0.1942 & -0.6422 \\ -0.1942 & 1 & -0.6273 \\ -0.6422 & -0.6273 & 1 \end{pmatrix}.$$

With $\tau = 0.5$,

$$E = \big\{\{1, 3\}, \{2, 3\}\big\}, \quad w(\{1, 3\}) = 0.6422, \quad w(\{2, 3\}) = 0.6273, \quad \sigma(\{1, 3\}) = \sigma(\{2, 3\}) = -1,$$

so G is a signed, weighted simple graph on three taxa with two negative edges.

2. Main Results

The main results of this paper are presented in this section. Specifically, we investigate the Soil Microbial HyperNetwork and the Soil Microbial SuperHyperNetwork, providing formal definitions and illustrative examples. For reference, a comparison of the Soil Microbial Network, Soil Microbial HyperNetwork, and Soil Microbial SuperHyperNetwork is summarized in Table 3.

TABLE 3. Comparison of Soil Microbial Network, Soil Microbial HyperNetwork, and Soil Microbial SuperHyperNetwork.

Aspect	**Soil Microbial Network**	**Soil Microbial HyperNetwork**	**Soil Microbial SuperHyperNetwork**
Vertices / supervertices	$V = T$ (individual taxa)	$V = T$ (individual taxa)	$V \subseteq \mathcal{P}^n(T)$ (supervertices built from taxa)
Edges / (hyper)edges	Edges $\{i, j\}$ with $s_{ij} \geq \tau$	Hyperedges $A \subseteq T$, $\lvert A\rvert \geq 2$, $s(A) \geq \tau$	Superhyperedges $A \subseteq V$, $\lvert A\rvert \geq 2$, $s^{(n)}(A) \geq \tau$
Interaction order	Pairwise associations	Multi-taxon associations in one layer	Multi-level associations among groups of taxa
Score used	Pairwise score $s_{ij} = \lvert\hat{R}_{ij}\rvert$	$s(A) = \min_{\{i,j\}\subseteq A} s_{ij}$	$s^{(n)}(A) = s(\mathrm{supp}_n(A))$
Mathematical structure	Weighted graph $G_\tau = (T, E_2)$	Weighted hypergraph $\mathcal{H}_\tau = (T, E, w)$	Weighted n-SuperHyperGraph $\mathcal{H}_\tau^{(n)} = (V, E, w)$
Relation to others	Base pairwise layer	Rank–2 hyperedges coincide with G_τ	Canonical embedding recovers G_τ; for $n = 0$, $V = T$ gives the HyperNetwork
Domain view	Pairwise co-occurrence / association network	Taxon consortia as single hyperedges	Hierarchical consortia and grouped communities across scales

2.1. *Soil Microbial HyperNetwork*

Soil Microbial HyperNetwork models microbial taxa as nodes with hyperedges linking multiple taxa simultaneously based on significant associations. The definitions and concrete examples of this concept are provided below.

Definition 2.1 (Pairwise association from CLR data)**.** Let $T = \{1, \ldots, n\}$ be the taxa set and $Y = (y_{i,s})$ the CLR–transformed abundance matrix defined in the Soil Microbial Network section. Let $\hat{R} = (\hat{R}_{ij})$ be the empirical correlation matrix computed rowwise from Y. Define the pairwise (unsigned) association score

$$s_{ij} := \lvert\hat{R}_{ij}\rvert \in [0, 1] \qquad (i \neq j), \qquad s_{ii} := 1.$$

Definition 2.2 (Soil Microbial HyperNetwork)**.** Fix a threshold $\tau \in [0, 1)$. For any finite nonempty $A \subseteq T$ with $|A| \geq 2$, define the multiway score

$$s(A) \;:=\; \min_{\{i,j\}\subseteq A} s_{ij} \;\in [0, 1].$$

The *Soil Microbial HyperNetwork* at level τ is the weighted hypergraph

$$\mathcal{H}_\tau \;:=\; (V, E, w), \qquad V := T, \quad E \;:=\; \big\{\, A \subseteq V : \; |A| \geq 2, \; s(A) \geq \tau \,\big\},$$

with hyperedge weights $w : E \to [\tau, 1]$ given by $w(A) := s(A)$. (Optionally, one may require all pairs $\{i, j\} \subseteq A$ to pass an FDR-corrected significance test in addition to $s(A) \geq \tau$.)

Remark 2.3. The aggregator $s(A) = \min_{\{i,j\}\subseteq A} s_{ij}$ is chosen so that $s(\{i, j\}) = s_{ij}$ holds identically and so that enlarging a set cannot increase its score: if $A \subseteq B$ then $s(B) \leq s(A)$. This monotonicity ensures that every hyperedge induces all its pairwise edges.

Example 2.4 (Instance continuing the network example)**.** Use the same CLR-based correlation matrix

$$\hat{R} \approx \begin{pmatrix} 1 & -0.1942 & -0.6422 \\ -0.1942 & 1 & -0.6273 \\ -0.6422 & -0.6273 & 1 \end{pmatrix}.$$

Thus $s_{12} = 0.1942$, $s_{13} = 0.6422$, $s_{23} = 0.6273$.

- With $\tau = 0.5$:

$$E_2 = \big\{\{1, 3\}, \{2, 3\}\big\}, \qquad s(\{1, 2, 3\}) = \min\{0.1942, 0.6422, 0.6273\} = 0.1942 < 0.5,$$

  so the hyperedge $\{1, 2, 3\}$ is not included. Hence

$$\mathcal{H}_{0.5} : \; E = \big\{\{1, 3\}, \{2, 3\}\big\}, \quad w(\{1, 3\}) = 0.6422, \; w(\{2, 3\}) = 0.6273.$$

  Here $\mathcal{H}_{0.5}$ coincides with the network at rank 2, as guaranteed by Theorem 2.9.

- With $\tau = 0.18$: All pairs are included, and since $s(\{1, 2, 3\}) = 0.1942 \geq 0.18$, the 3-way hyperedge appears:

$$E = \big\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\big\}, \quad w(\{1, 2, 3\}) = 0.1942.$$

  This captures a simultaneous association among the three taxa in addition to all pairwise links.

Example 2.5 (Four-taxa HyperNetwork at $\tau = 0.6$)**.** Let $T = \{A, B, C, D\}$. Suppose the pairwise association scores $s_{ij} \in [0, 1]$ are

$$S = \begin{pmatrix} 1 & 0.82 & 0.65 & 0.40 \\ 0.82 & 1 & 0.62 & 0.58 \\ 0.65 & 0.62 & 1 & 0.61 \\ 0.40 & 0.58 & 0.61 & 1 \end{pmatrix} \quad \text{(rows/columns ordered as } A, B, C, D).$$

By Definition 2.2, for any $A \subseteq T$ with $|A| \geq 2$, $s(A) := \min_{\{i,j\} \subseteq A} s_{ij}$. With $\tau = 0.6$,

$$E_2 = \big\{\{A, B\}, \{A, C\}, \{B, C\}, \{C, D\}\big\},$$

and among 3-sets,

$$s(\{A, B, C\}) = \min\{0.82, 0.65, 0.62\} = 0.62 \ (\geq \tau),$$

$$s(\{A, B, D\}) = \min\{0.82, 0.58, 0.40\} = 0.40 \ (< \tau),$$

$$s(\{A, C, D\}) = \min\{0.65, 0.40, 0.61\} = 0.40 \ (< \tau),$$

$$s(\{B, C, D\}) = \min\{0.62, 0.58, 0.61\} = 0.58 \ (< \tau).$$

Hence the Soil Microbial HyperNetwork is

$$\mathcal{H}_{0.6} = (V, E, w), \quad V = T, \quad E = \big\{\{A, B\}, \{A, C\}, \{B, C\}, \{C, D\}, \{A, B, C\}\big\},$$

with weights $w(\{i, j\}) = s_{ij}$ and $w(\{A, B, C\}) = 0.62$. No 4-way hyperedge appears since $\min_{\{i,j\} \subseteq T} s_{ij} = 0.40 < 0.6$.

Example 2.6 (Five-taxa clustered HyperNetwork at $\tau = 0.8$)**.** Let $T = \{1, 2, 3, 4, 5\}$. Suppose the scores are

$$S = \begin{pmatrix} 1 & 0.91 & 0.88 & 0.55 & 0.52 \\ 0.91 & 1 & 0.86 & 0.50 & 0.49 \\ 0.88 & 0.86 & 1 & 0.48 & 0.51 \\ 0.55 & 0.50 & 0.48 & 1 & 0.83 \\ 0.52 & 0.49 & 0.51 & 0.83 & 1 \end{pmatrix}.$$

With $\tau = 0.8$,

$$E_2 = \big\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}\big\},$$

and for 3-sets,

$$s(\{1, 2, 3\}) = \min\{0.91, 0.88, 0.86\} = 0.86 \ (\geq \tau), \qquad s(\{1, 2, 4\}) \leq 0.55 < \tau, \ \ldots$$

so the only 3-way hyperedge is $\{1, 2, 3\}$. No 4- or 5-way hyperedge exists because some cross-cluster pairs fall below τ. Therefore,

$$\mathcal{H}_{0.8} = (V, E, w), \quad V = T, \quad E = \big\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{1, 2, 3\}\big\},$$

with weights $w(\{i, j\}) = s_{ij}$ and $w(\{1, 2, 3\}) = 0.86$. This captures a tight triad $\{1, 2, 3\}$ and a strong pair $\{4, 5\}$, with weak cross-links excluded by the threshold.

The theorems related to this concept are presented below.

Theorem 2.7 (Hypergraph property)**.** *Let $\mathcal{H}_\tau = (V, E, w)$ be as in Definition 2.2. Then (V, E) is a (finite) hypergraph in the sense of Definition 1.8, and w makes it a weighted hypergraph.*

Proof. By construction, $V = T$ is finite. Every $A \in E$ satisfies $A \subseteq V$ and $|A| \geq 2$; hence $A \in \mathcal{P}(V)\backslash\{\varnothing\}$. Therefore $E \subseteq \mathcal{P}(V)\backslash\{\varnothing\}$, i.e., (V, E) is a hypergraph. Finally, $w : E \to [\tau, 1]$ assigns a nonnegative weight to each hyperedge, giving a weighted hypergraph structure. □

Definition 2.8 (Soil Microbial Network at level τ)**.** Define the (pairwise) Soil Microbial Network

$$G_\tau \ := \ (V, E_2), \qquad V := T, \quad E_2 := \big\{\{i,j\} \subseteq V : \ i \neq j, \ s_{ij} \geq \tau\big\},$$

with optional signs and weights as in the earlier definition.

Theorem 2.9 (Generalizes the Soil Microbial Network)**.** *For $\mathcal{H}_\tau = (V, E, w)$ and $G_\tau = (V, E_2)$ at the same threshold τ, the following hold.*

(i) Rank-2 consistency: $\{i, j\} \in E$ *if and only if* $\{i, j\} \in E_2$.

(ii) Downward closure: *If $A \in E$ and $\{i, j\} \subseteq A$, then $\{i, j\} \in E_2$.*

In particular, the 2-edge layer of $\mathcal{H}_\tau$ coincides exactly with the Soil Microbial Network G_τ.

Proof. (i) For any 2-element set $A = \{i, j\}$, we have $s(A) = \min_{\{u,v\}\subseteq A} s_{uv} = s_{ij}$. Hence $A \in E \iff s(A) \geq \tau \iff s_{ij} \geq \tau \iff \{i, j\} \in E_2$.
(ii) If $A \in E$, then $s(A) = \min_{\{u,v\}\subseteq A} s_{uv} \geq \tau$. For any $\{i, j\} \subseteq A$ we have $s_{ij} \geq s(A) \geq \tau$, so $\{i, j\} \in E_2$. □

2.2. *Soil Microbial SuperHyperNetwork*

Soil Microbial SuperHyperNetwork extends this by allowing multi-layered supervertices and superhyperedges, capturing hierarchical, higher-order microbial interactions. The definitions and concrete examples of this concept are provided below.

Definition 2.10 (Base pairwise and multiway scores on taxa)**.** Let $T = \{1, \ldots, n\}$ be a finite nonempty set of taxa. From CLR–transformed abundances (as in the Soil Microbial Network section), let $\hat{R} = (\hat{R}_{ij})$ be the empirical correlation matrix and set the pairwise association score $s_{ij} := |\hat{R}_{ij}| \in [0, 1]$ for $i \neq j$ (and $s_{ii} := 1$). For any finite nonempty $B \subseteq T$ with $|B| \geq 2$, define the multiway score

$$s(B) \ := \ \min_{\{i,j\}\subseteq B} s_{ij} \ \in [0, 1].$$

Definition 2.11 (Support (flattening) from level n to taxa)**.** For each integer $n \geq 0$, define the *support map* $\mathrm{supp}_n : \mathcal{P}^n(T) \to \mathcal{P}(T)$ recursively by

$$\mathrm{supp}_0(i) := \{i\} \quad (i \in T), \qquad \mathrm{supp}_{k+1}(X) := \bigcup_{x\in X} \mathrm{supp}_k(x) \quad (X \in \mathcal{P}^{k+1}(T)).$$

For a finite $A \subseteq \mathcal{P}^n(T)$, write $\mathrm{supp}_n(A) := \bigcup_{v\in A} \mathrm{supp}_n(v) \subseteq T$.

Definition 2.12 (Soil Microbial SuperHyperNetwork at depth n)**.** Fix $n \geq 1$ and a threshold $\tau \in [0,1)$. Choose a nonempty *supervertex set* $V \subseteq \mathcal{P}^n(T)$. Define, for any finite $A \subseteq V$ with $|A| \geq 2$,

$$s^{(n)}(A) \ := \ s\big(\mathrm{supp}_n(A)\big) \ = \ \min_{\{i,j\} \subseteq \mathrm{supp}_n(A)} s_{ij} \ \in [0,1].$$

The *Soil Microbial SuperHyperNetwork* (at level τ) is the weighted superhypergraph

$$\mathcal{H}_\tau^{(n)} \ := \ (V, E, w), \qquad E \ := \ \big\{ \, A \subseteq V : \ |A| \geq 2, \ s^{(n)}(A) \geq \tau \, \big\},$$

with weights $w : E \to [\tau, 1]$ given by $w(A) := s^{(n)}(A)$.

Remark 2.13. (i) Monotonicity: If $A \subseteq B \subseteq V$, then $\mathrm{supp}_n(B) \supseteq \mathrm{supp}_n(A)$, hence $s^{(n)}(B) \leq s^{(n)}(A)$. In particular, enlarging a hyperedge cannot increase its score.
(ii) Rank-2 consistency: For $|A| = 2$, $s^{(n)}(A) = \min_{\{i,j\} \subseteq \mathrm{supp}_n(A)} s_{ij}$, which reduces to the pairwise score on the union of the underlying taxa of the two supervertices.

Example 2.14 (Depth $n = 1$ (supervertices are subsets of taxa))**.** Let $T = \{1, 2, 3\}$ with pair scores $s_{12} = 0.1942, \ s_{13} = 0.6422, \ s_{23} = 0.6273$ (from the numerical instance in the Network section). Take

$$V = \big\{\{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\big\} \subseteq \mathcal{P}(T), \qquad \tau = 0.6.$$

For two supervertices, $\mathrm{supp}_1(\{a\}) = \{a\}$ and $\mathrm{supp}_1(\{a,b\}) = \{a,b\}$. Hence

$$s^{(1)}\big(\{\{1\},\{3\}\}\big) = s(\{1,3\}) = 0.6422 \ (\geq \tau), \quad s^{(1)}\big(\{\{2\},\{3\}\}\big) = s(\{2,3\}) = 0.6273 \ (\geq \tau),$$

so these 2-hyperedges are present. However, $s^{(1)}\big(\{\{1,3\},\{2,3\}\}\big) = s(\{1,2,3\}) = \min\{0.1942, 0.6422, 0.6273\} = 0.1942 < \tau$, so $\{\{1,3\},\{2,3\}\} \notin E$. This example shows both generalization (pairs of embedded singletons reproduce the Soil Microbial Network) and genuine supervertex interactions.

Example 2.15 (Depth $n = 2$ (supervertices are sets of subsets))**.** Let $T = \{1, 2, 3\}$ with the same s_{ij} as above and $\tau = 0.6$. Define level-2 supervertices

$$v_1 := \big\{\{1\},\{3\}\big\}, \qquad v_2 := \big\{\{2\},\{3\}\big\}, \qquad v_3 := \big\{\{1\}\big\} \ \in \mathcal{P}^2(T),$$

and set $V := \{v_1, v_2, v_3\}$. By Definition 2.11, $\mathrm{supp}_2(v_1) = \{1,3\}$, $\mathrm{supp}_2(v_2) = \{2,3\}$, $\mathrm{supp}_2(v_3) = \{1\}$. Then

$$s^{(2)}(\{v_1, v_3\}) = s(\{1,3\}) = 0.6422 \ (\geq \tau), \qquad s^{(2)}(\{v_2, v_3\}) = s(\{1,2,3\}) = 0.1942 \ (< \tau),$$

and $s^{(2)}(\{v_1, v_2\}) = s(\{1,2,3\}) = 0.1942 < \tau$. Thus E contains $\{v_1, v_3\}$ but not $\{v_1, v_2\}$ nor $\{v_2, v_3\}$. This illustrates how multi-level grouping (depth 2) is evaluated via the flattened underlying taxa and remains consistent with the pairwise base scores.

The theorems related to this concept are presented below.

Theorem 2.16 (SuperHyperGraph property)**.** *Let $\mathcal{H}_\tau^{(n)} = (V, E, w)$ be as in Definition 2.12. Then, with base set $V_0 := T$, the pair (V, E) is an n-SuperHyperGraph in the sense of Definition 1.10.*

Proof. By construction $V \subseteq \mathcal{P}^n(T) = \mathcal{P}^n(V_0)$ and $V \neq \varnothing$. Moreover $E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$ because every $A \in E$ is a finite subset of V with $|A| \geq 2$. Hence (V, E) satisfies Definition 1.10, and w equips it with weights. □

Theorem 2.17 (Generalizes the Soil Microbial HyperNetwork and Network)**.** *Let $\mathcal{H}_\tau^{(n)} = (V, E, w)$ be as above and write $G_\tau = (T, E_2)$ for the Soil Microbial Network at threshold τ (pairwise edges $\{i, j\}$ with $s_{ij} \geq \tau$). Then:*

(i) ($n = 0$ reduction) If $n = 0$ and $V = T$, then

$$E = \{ B \subseteq T : |B| \geq 2, s(B) \geq \tau \},$$

which is exactly the Soil Microbial HyperNetwork (with weight $w(B) = s(B)$).

(ii) (Canonical embedding) For $n \geq 1$, define the embedding $\eta_n : T \to \mathcal{P}^n(T)$ by $\eta_0(i) = i$ and $\eta_{k+1}(i) = \{\eta_k(i)\}$. If $V = \eta_n[T]$, then the set of 2-element hyperedges of $\mathcal{H}_\tau^{(n)}$ is

$$\{\{\eta_n(i), \eta_n(j)\} : s_{ij} \geq \tau\},$$

which coincides with the edge set E_2 of the Soil Microbial Network after inverse projection by η_n.

(iii) (Downward closure to pairs) For any $n \geq 0$, if $A \in E$ and $\{u, v\} \subseteq A$, then $\{\mathrm{supp}_n(u), \mathrm{supp}_n(v)\} \subseteq \mathrm{supp}_n(A)$ implies $s^{(n)}(\{u, v\}) \geq s^{(n)}(A) \geq \tau$. Thus every hyperedge induces all of its pairwise 2-faces, and these 2-faces correspond to edges in G_τ on the underlying taxa.

Proof. (i) If $n = 0$ and $V = T$, then for $A \subseteq V$ we have $\mathrm{supp}_0(A) = A$ by Definition 2.11. Hence $s^{(0)}(A) = s(A)$ and the condition $s^{(0)}(A) \geq \tau$ is exactly the hyperedge rule of the Soil Microbial HyperNetwork; weights also match.

(ii) For the embedding η_n, we prove by induction that $\mathrm{supp}_n(\eta_n(i)) = \{i\}$ for all $i \in T$. The claim is trivial for $n = 0$. If it holds for $n = k$, then $\mathrm{supp}_{k+1}(\eta_{k+1}(i)) = \mathrm{supp}_{k+1}(\{\eta_k(i)\}) = \mathrm{supp}_k(\eta_k(i)) = \{i\}$. Therefore, for any $i \neq j$,

$$s^{(n)}(\{\eta_n(i), \eta_n(j)\}) = s(\mathrm{supp}_n(\{\eta_n(i), \eta_n(j)\})) = s(\{i, j\}) = s_{ij}.$$

Thus $\{\eta_n(i), \eta_n(j)\} \in E$ iff $s_{ij} \geq \tau$, as claimed.

(iii) If $A \in E$, then by Definition 2.12 we have $s^{(n)}(A) = \min_{\{i,j\} \subseteq \mathrm{supp}_n(A)} s_{ij} \geq \tau$. For any $\{u, v\} \subseteq A$, $\mathrm{supp}_n(\{u, v\}) \subseteq \mathrm{supp}_n(A)$ and hence $s^{(n)}(\{u, v\}) \geq s^{(n)}(A) \geq \tau$. □

Theorem 2.18 (Monotonicity in the threshold)**.** *Let T be a finite taxa set with base scores as in Definition* 2.10*, fix $n \geq 1$, and let $V \subseteq \mathcal{P}^n(T)$ be nonempty. For $\tau_1, \tau_2 \in [0, 1)$ with $\tau_1 \leq \tau_2$, let*

$$\mathcal{H}^{(n)}_{\tau_k} = (V, E_{\tau_k}, w_{\tau_k}) \qquad (k = 1, 2)$$

be the Soil Microbial SuperHyperNetworks from Definition 2.12*. Then*

$$E_{\tau_2} \subseteq E_{\tau_1}, \qquad w_{\tau_1}(A) = w_{\tau_2}(A) = s^{(n)}(A) \quad \text{for all } A \in E_{\tau_2}.$$

Proof. By Definition 2.12,

$$E_{\tau_k} = \{A \subseteq V : \ |A| \geq 2, \ s^{(n)}(A) \geq \tau_k\} \quad (k = 1, 2).$$

If $A \in E_{\tau_2}$, then $s^{(n)}(A) \geq \tau_2 \geq \tau_1$, hence $A \in E_{\tau_1}$ and so $E_{\tau_2} \subseteq E_{\tau_1}$. By construction $w_{\tau_k}(A) = s^{(n)}(A)$ for every $A \in E_{\tau_k}$, so for $A \in E_{\tau_2}$ we have $w_{\tau_1}(A) = w_{\tau_2}(A) = s^{(n)}(A)$. □

Theorem 2.19 (Downward closure under inclusion)**.** *Let $\mathcal{H}^{(n)}_\tau = (V, E, w)$ be a Soil Microbial SuperHyperNetwork as in Definition* 2.12*, with $n \geq 1$ and $\tau \in [0, 1)$. If $A \in E$ and $B \subseteq A$ with $|B| \geq 2$, then $B \in E$, and*

$$s^{(n)}(B) \ \geq \ s^{(n)}(A).$$

Proof. By Definition 2.11,

$$\text{supp}_n(B) \subseteq \text{supp}_n(A).$$

Hence, by Definition 2.10,

$$s^{(n)}(B) = s\big(\text{supp}_n(B)\big) = \min_{\{i,j\} \subseteq \text{supp}_n(B)} s_{ij} \ \geq \ \min_{\{i,j\} \subseteq \text{supp}_n(A)} s_{ij} = s^{(n)}(A).$$

If $A \in E$, then $s^{(n)}(A) \geq \tau$, so $s^{(n)}(B) \geq s^{(n)}(A) \geq \tau$. Since $|B| \geq 2$, it follows from Definition 2.12 that $B \in E$. □

Theorem 2.20 (Each superhyperedge induces a clique on taxa)**.** *Let $\mathcal{H}^{(n)}_\tau = (V, E, w)$ be a Soil Microbial SuperHyperNetwork at threshold τ, and let $G_\tau = (T, E_2)$ be the Soil Microbial Network at the same τ (pairwise edges $\{i, j\}$ with $s_{ij} \geq \tau$). For any $A \in E$, the induced subgraph of G_τ on* $\text{supp}_n(A)$ *is a complete graph. Equivalently,*

$$\{i, j\} \subseteq \text{supp}_n(A) \quad \Longrightarrow \quad \{i, j\} \in E_2.$$

Proof. Fix $A \in E$ and two distinct taxa $i, j \in \text{supp}_n(A)$. By Definition 2.10 and Definition 2.12,

$$s^{(n)}(A) = s\big(\text{supp}_n(A)\big) = \min_{\{u,v\} \subseteq \text{supp}_n(A)} s_{uv} \ \leq \ s_{ij}.$$

Since $A \in E$, we have $s^{(n)}(A) \geq \tau$, whence $s_{ij} \geq s^{(n)}(A) \geq \tau$. By the definition of G_τ, this means $\{i, j\} \in E_2$. As this holds for every unordered pair $\{i, j\} \subseteq \text{supp}_n(A)$, the induced subgraph on $\text{supp}_n(A)$ is complete. □

Theorem 2.21 (Isomorphism invariance under taxa relabeling)**.** *Let T and T' be finite taxa sets, and let s_{ij} and s'_{uv} be base scores on T and T' as in Definition 2.10. Suppose $\varphi : T \to T'$ is a bijection with*

$$s'_{\varphi(i)\varphi(j)} \;=\; s_{ij} \quad \textit{for all } i, j \in T.$$

For each $n \geq 0$, define $\Phi_n : \mathcal{P}^n(T) \to \mathcal{P}^n(T')$ recursively by

$$\Phi_0(i) := \varphi(i), \qquad \Phi_{k+1}(X) := \big\{\Phi_k(x) : x \in X\big\} \quad (X \in \mathcal{P}^{k+1}(T)).$$

Fix $n \geq 1$, let $V \subseteq \mathcal{P}^n(T)$ be nonempty, and put $V' := \Phi_n[V] \subseteq \mathcal{P}^n(T')$. For a threshold $\tau \in [0, 1)$, let

$$\mathcal{H}^{(n)}_\tau = (V, E, w), \qquad \mathcal{H}'^{(n)}_\tau = (V', E', w')$$

be the corresponding Soil Microbial SuperHyperNetworks constructed from s_{ij} and s'_{uv}. Then Φ_n induces an isomorphism of weighted superhypergraphs between $\mathcal{H}^{(n)}_\tau$ and $\mathcal{H}'^{(n)}_\tau$.

Proof. First note that $\Phi_n : V \to V'$ is a bijection by construction. We claim that for every $v \in V$,

$$\mathrm{supp}'_n\big(\Phi_n(v)\big) = \varphi\big[\mathrm{supp}_n(v)\big],$$

where supp_n and supp'_n are the support maps on T and T' (Definition 2.11), and $\varphi[\cdot]$ denotes image. This is proved by induction on n. For $n = 0$ it is just the definition of Φ_0 and supp_0. If the statement holds for $n = k$, then for $X \in \mathcal{P}^{k+1}(T)$

$$\mathrm{supp}'_{k+1}\big(\Phi_{k+1}(X)\big) = \bigcup_{x' \in \Phi_{k+1}(X)} \mathrm{supp}'_k(x')$$

$$= \bigcup_{x \in X} \mathrm{supp}'_k\big(\Phi_k(x)\big) = \bigcup_{x \in X} \varphi\big[\mathrm{supp}_k(x)\big] = \varphi\big[\mathrm{supp}_{k+1}(X)\big].$$

Now let $A \subseteq V$ with $|A| \geq 2$. Then

$$\mathrm{supp}'_n\big(\Phi_n(A)\big) = \bigcup_{v \in A} \mathrm{supp}'_n\big(\Phi_n(v)\big) = \bigcup_{v \in A} \varphi\big[\mathrm{supp}_n(v)\big] = \varphi\big[\mathrm{supp}_n(A)\big].$$

Using the assumption on the scores,

$$s'^{(n)}\big(\Phi_n(A)\big) = \min_{\{u', v'\} \subseteq \mathrm{supp}'_n(\Phi_n(A))} s'_{u'v'} = \min_{\{i, j\} \subseteq \mathrm{supp}_n(A)} s_{ij} = s^{(n)}(A).$$

Hence $s'^{(n)}(\Phi_n(A)) = s^{(n)}(A)$ for all finite $A \subseteq V$. In particular,

$$A \in E \iff s^{(n)}(A) \geq \tau \iff s'^{(n)}\big(\Phi_n(A)\big) \geq \tau \iff \Phi_n(A) \in E',$$

so Φ_n induces a bijection $E \to E'$. Finally, the weights satisfy

$$w'(\Phi_n(A)) = s'^{(n)}\big(\Phi_n(A)\big) = s^{(n)}(A) = w(A),$$

so the isomorphism preserves weights. □

Theorem 2.22 (Recovery of the Soil Microbial Network via canonical embedding)**.** *Let T be a finite taxa set with base scores s_{ij} and fix $n \geq 1$ and $\tau \in [0, 1)$. Define the canonical embedding $\eta_n : T \to \mathcal{P}^n(T)$ recursively by*

$$\eta_0(i) := i, \qquad \eta_{k+1}(i) := \{\eta_k(i)\} \quad (k \geq 0).$$

Let $V \subseteq \mathcal{P}^n(T)$ satisfy $\eta_n[T] \subseteq V$, and let $\mathcal{H}_\tau^{(n)} = (V, E, w)$ be the corresponding Soil Microbial SuperHyperNetwork. Let $G_\tau = (T, E_2)$ be the Soil Microbial Network at the same threshold τ. Then, for all distinct $i, j \in T$,

$$\{\eta_n(i), \eta_n(j)\} \in E \quad \Longleftrightarrow \quad \{i, j\} \in E_2.$$

In particular, the 2-hyperedges of $\mathcal{H}_\tau^{(n)}$ whose vertices lie in $\eta_n[T]$ are in one-to-one correspondence with the edges of G_τ.

Proof. We first show that $\mathrm{supp}_n(\eta_n(i)) = \{i\}$ for all $i \in T$. This is proved by induction on n. For $n = 0$, $\mathrm{supp}_0(i) = \{i\}$ by definition. Assume $\mathrm{supp}_k(\eta_k(i)) = \{i\}$. Then

$$\mathrm{supp}_{k+1}(\eta_{k+1}(i)) = \mathrm{supp}_{k+1}(\{\eta_k(i)\}) = \bigcup_{x \in \{\eta_k(i)\}} \mathrm{supp}_k(x) = \mathrm{supp}_k(\eta_k(i)) = \{i\}.$$

Thus the claim holds for all n.

Now fix distinct $i, j \in T$ and consider $A := \{\eta_n(i), \eta_n(j)\} \subseteq V$. By Definition 2.11,

$$\mathrm{supp}_n(A) = \mathrm{supp}_n\big(\{\eta_n(i), \eta_n(j)\}\big) = \mathrm{supp}_n(\eta_n(i)) \cup \mathrm{supp}_n(\eta_n(j)) = \{i\} \cup \{j\} = \{i, j\}.$$

Hence, by Definition 2.10,

$$s^{(n)}(A) = s\big(\mathrm{supp}_n(A)\big) = s(\{i, j\}) = \min_{\{u,v\} \subseteq \{i,j\}} s_{uv} = s_{ij}.$$

Therefore,

$$\{\eta_n(i), \eta_n(j)\} \in E \iff s^{(n)}(A) \geq \tau \iff s_{ij} \geq \tau \iff \{i, j\} \in E_2,$$

which proves the desired equivalence and the stated bijection between these 2-hyperedges and the edges of G_τ. □

3. Conclusion

In this paper, we provided a mathematical definition of the Soil Microbial Network and investigated its generalizations: the Soil Microbial HyperNetwork, using the framework of HyperGraphs, and the Soil Microbial SuperHyperNetwork, using the framework of SuperHyperGraphs. In the future, we hope to study extensions of these concepts by employing Fuzzy Sets [34], Soft Sets [35], HyperFuzzy Sets [36], Functorial Sets [37], SuperHyperFuzzy Sets [38], Neutrosophic Sets [39–41], QuadriPartitioned Neutrosophic Sets [42], and Plithogenic Sets [43, 44]. Moreover, since SuperHyperStructures other than SuperHyperGraphs, such as

SuperHyperAlgebra [6], are also known, we expect that future research will develop extensions based on these structures and explore our concepts from perspectives beyond the purely graph-theoretic one.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany this paper.

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