

*Article***Refined Uncertain Set, Refined Nonstandard Uncertain Set,
Refined Uncertain Over/Under/Off-Set,
Refined Nonstandard Uncertain Over/Under/Off-Set**Florentin Smarandache^{1,*}¹ Math and Science Department, University of New Mexico, Gallup Campus, USA* Correspondence: smarand@unm.edu*Received:* April 28, 2026; *Accepted:* May 9, 2026; *Published:* May 16, 2026

Abstract: This short note introduces four generalized frameworks for modeling refined uncertainty: Refined Uncertain Set, Refined Nonstandard Uncertain Set, Refined Uncertain Over/Under/Off-Set, and Refined Nonstandard Uncertain Over/Under/Off-Set. These frameworks extend refined neutrosophic structures to a broader class of uncertainty representations, including fuzzy, intuitionistic fuzzy, picture fuzzy, q-rung orthopair fuzzy, spherical fuzzy, plithogenic, and related models. The proposed approach enables the decomposition of uncertainty components into refined subcomponents and further extends them into nonstandard and over/under/off domains. Analogous extensions may also be developed for logic, measure, probability, and statistics.

Keywords: Uncertain set theory; refined uncertainty; nonstandard analysis; neutrosophic set; refined neutrosophic logic; over-/under-/off-set; uncertainty formalism.

MSC 2020: 03E72 (Fuzzy set theory), 03B52 (Fuzzy logic), 28E10 (Fuzzy measure theory), 68T37 (Reasoning under uncertainty), 03H05 (Nonstandard models in mathematics).

1. Introduction and Motivation

Many uncertainty theories have been proposed to represent imprecise, incomplete, inconsistent, or indeterminate information, including fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, picture fuzzy sets, plithogenic sets, spherical fuzzy sets, and their various extensions. However, these theories often remain isolated and use different mathematical structures.

The notion of refinement, introduced in neutrosophic theory, allows uncertainty components to be decomposed into semantically meaningful subcomponents. Such decomposition permits a more nuanced representation of truth, indeterminacy, falsity, or other uncertainty factors.

The purpose of this note is to generalize the refined neutrosophic framework to a broader category called uncertain systems, thereby providing a common conceptual structure applicable to many existing uncertainty theories.

By uncertain, we refer to all fuzzy-based or uncertainty-based representations, including but not limited to: Fuzzy Sets, Intuitionistic Fuzzy Sets, Picture Fuzzy Sets (Ternary Fuzzy Sets), Neutrosophic Sets, Pythagorean Fuzzy Sets, Fermatean Fuzzy Sets, q-Rung Orthopair Fuzzy Sets, Spherical and n-Hyperspherical Fuzzy Sets, Refined Neutrosophic Sets, Plithogenic Sets, and related uncertainty extensions.

2. Refined Uncertain Structures

Definition 1. Refined Uncertain Set / Logic / Measure / Probability

A **Refined Uncertain Set** is an uncertain set whose uncertainty components are decomposed into multiple refined subcomponents.

Suppose an uncertainty representation consists of n subcomponents partitioned into:

$$T_1, T_2, \dots, T_p; I_1, I_2, \dots, I_r; F_1, F_2, \dots, F_s$$

where

$$p, r, s \geq 1, p + r + s = n, \text{ and } T_i, I_j, F_k \in [0, 1].$$

The refinement process decomposes uncertainty information into semantically meaningful categories. For example, truth may be refined into several kinds of evidence, indeterminacy into different sources of ambiguity, and falsity into different categories of contradiction or opposition.

This definition generalizes the concept of the **Refined Neutrosophic Set** [1] to broader uncertainty systems.

Definition 2. Refined Nonstandard Uncertain Set / Logic / Measure / Probability

A further generalization is the **Refined Nonstandard Uncertain Set**, defined within the framework of **Extended Nonstandard Analysis** [2].

In this case,

$$T_i, I_j, F_k \in]-0, 1+[$$

where the subcomponents may be: real numbers, hyperreal numbers, monads, or binads.

This extension enables modeling uncertainty beyond standard real-valued boundaries.

Definition 3. Refined Uncertain Over/Under/Off-Set / Logic / Measure / Probability

A **Refined Uncertain Over/Under/Off-Set** extends refined uncertainty beyond the classical interval $[0, 1]$.

The refined subcomponents satisfy:

$$T_i, I_j, F_k \in [\Omega, \Psi]$$

where

$$\Omega < 0 < 1 < \Psi.$$

Such generalization permits:

- **overset values** (> 1), representing over-membership or super-evidence,
- **underset values** (< 0), representing negative evidence or anti-membership,
- **offset structures**, where both under- and over-values coexist.

This extends the notion of neutrosophic over-/under-/off-set [3] to a general uncertainty framework.

Definition 4. Refined Nonstandard Uncertain Over/Under/Off-Set / Logic / Measure / Probability

Finally, the **Refined Nonstandard Uncertain Over/Under/Off-Set** combines refinement, nonstandard analysis, and over/under/off extensions.

The refined subcomponents satisfy:

$$T_i, I_j, F_k \in]\Omega^-, \Psi^+[$$

where

$$\Omega < 0 < 1 < \Psi,$$

and the components may be: real numbers, hyperreal numbers, monads, or binads.

This structure represents the most general refined uncertain framework introduced in this note.

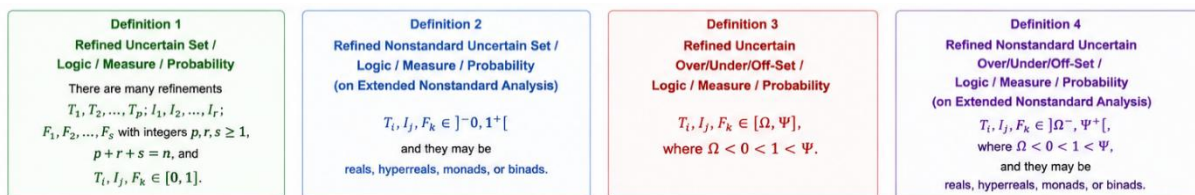


Figure 1: Definitions.

3. Illustrative Example

Consider an environmental assessment problem in which ecological sustainability is evaluated through refined criteria.

Truth-membership may be refined into:

$$T = (T_1, T_2) = (0.8, 0.7)$$

where:

- T_1 represents ecosystem integrity,
- T_2 represents biodiversity resilience.

Indeterminacy may be represented as:

$$I = (0.2)$$

due to incomplete field measurements.

Falsity may be refined into:

$$F = (F_1, F_2) = (0.1, 0.3)$$

where:

- F_1 measures pollution stress,
- F_2 measures habitat fragmentation.

In over/under/off extensions, some values may exceed the classical interval $[0, 1]$, allowing the representation of exceptional support or negative evidence.

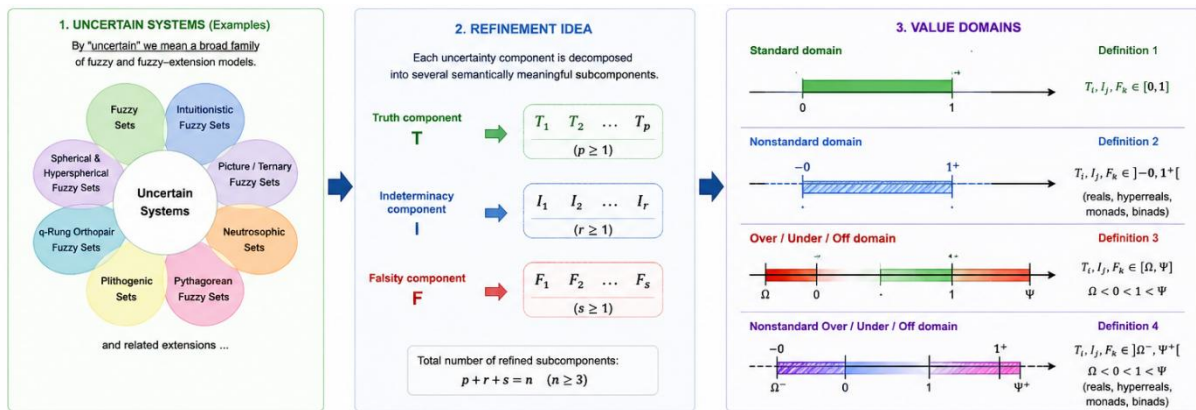


Figure 2: Refined Uncertain Framework.

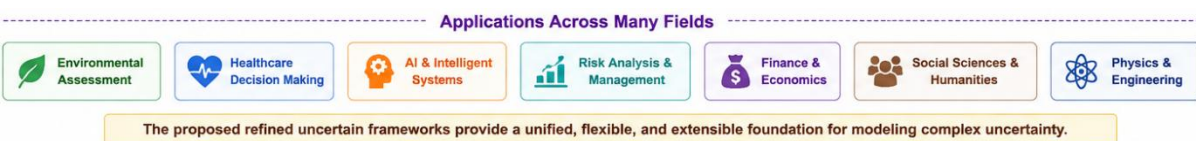


Figure 3: Applications.

4. Conclusion

This short note introduced four generalized frameworks for representing refined uncertainty: the **Refined Uncertain Set**, **Refined Nonstandard Uncertain Set**, **Refined Uncertain Over/Under/Off-Set**, and **Refined Nonstandard Uncertain Over/Under/Off-Set**. These structures extend the refined neutrosophic paradigm toward a broader class of uncertainty theories, including fuzzy, intuitionistic fuzzy, spherical fuzzy, plithogenic, q-rung orthopair fuzzy, and related systems.

The proposed framework enables the decomposition of uncertainty components into refined subcomponents, thereby allowing a more detailed and semantically meaningful representation of uncertain information. Furthermore, the incorporation of nonstandard analysis and over/under/off domains extends the modeling capability beyond classical intervals, permitting the representation of exceptional, incomplete, contradictory, or nonconventional uncertainty values.

By establishing a unified and extensible conceptual structure, the present work opens the possibility for developing corresponding generalized forms of uncertain logic, measure, probability, statistics, and decision-making models. Future research may focus on formal axiomatization, algebraic properties, aggregation operators, similarity and distance measures, and applications in artificial intelligence, environmental systems, engineering, social sciences, and complex decision environments.

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