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On Additive Analogues of Certain Arithmetic Functions

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1. The Smarandache, Pseudo-Smarandache, resp. Smarandache-simple functions are defined as ([7], [6])

$$S(n) = \min\{m \in \mathbb{N} : \ n|m!\}, \tag{1}$$

$$Z(n) = \min\left\{m \in \mathbb{N} : \ n|\frac{m(m+1)}{2}\right\}, \tag{2}$$

$$S_p(n) = \min\{m \in \mathbb{N} : \ p^n|m!\} \text{ for fixed primes } p. \tag{3}$$

The duals of S and Z have been studied e.g. in [2], [5], [6]:

$$S_*(n) = \max\{m \in \mathbb{N} : \ m!|n\}, \tag{4}$$

$$Z_*(n) = \max\left\{m \in \mathbb{N} : \ \frac{m(m+1)}{2}|n\right\}. \tag{5}$$

We note here that the dual of the Smarandache simple function can be defined in a similar manner, namely by

$$S_{p*}(n) = \max\{m \in \mathbb{N} : \ m!|p^n\} \tag{6}$$

This dual will be studied in a separate paper (in preparation).

2. The additive analogues of the functions S and S_* are real variable functions, and have been defined and studied in paper [3]. (See also our book [6], pp. 171-174). These functions have been recently further extended, by the use of Euler's gamma function, in place of the factorial (see [1]). We note that in what follows, we could define also the additive analogues functions by the use of Euler's gamma function. However, we shall apply the more transparent notation of a factorial of a positive integer.

The additive analogues of S and S_* from (1) and (4) have been introduced in [3] as follows:

$$S(x) = \min\{m \in \mathbb{N} : \ x \le m!\}, \quad S : (1, \infty) \to \mathbb{R}, \tag{7}$$

resp.

$$S_*(x) = \max\{m \in \mathbb{N} : \ m! \le x\}, \quad S_* : [1, \infty) \to \mathbb{R} \tag{8}$$

Besides of properties relating to continuity, differentiability, or Riemann integrability of these functions, we have proved the following results:

Theorem 1.

$$S_*(x) \sim \frac{\log x}{\log\log x} \quad (x \to \infty) \tag{9}$$

(the same for $S(x)$).

Theorem 2. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^\alpha} \tag{10}$$

is convergent for $\alpha > 1$ and divergent for $\alpha \le 1$ (the same for $S_*(n)$ replaced by $S(n)$).

3. The additive analogues of Z and Z_* from (2), resp. (4) will be defined as

$$Z(x) = \min\left\{m \in \mathbb{N}:\ x \le \frac{m(m+1)}{2}\right\}, \tag{11}$$

$$Z_*(x) = \max\left\{m \in \mathbb{N}:\ \frac{m(m+1)}{2} \le x\right\} \tag{12}$$

In (11) we will assume $x \in (0, +\infty)$, while in (12) $x \in [1, +\infty)$.

The two additive variants of $S_p(n)$ of (3) will be defined as

$$P(x) = S_p(x) = \min\{m \in \mathbb{N}:\ p^x \le m!\}; \tag{13}$$

(where in this case $p > 1$ is an arbitrary fixed real number)

$$P_*(x) = S_{p*}(x) = \max\{m \in \mathbb{N}:\ m! \le p^x\} \tag{14}$$

From the definitions follow at once that

$$Z(x) = k \ \Leftrightarrow\ x \in \left(\frac{(k-1)k}{2}, \frac{k(k+1)}{2}\right] \text{ for } k \ge 1 \tag{15}$$

$$Z_*(x) = k \ \Leftrightarrow\ x \in \left[\frac{k(k+1)}{2}, \frac{(k+1)(k+2)}{2}\right) \tag{16}$$

For $x \ge 1$ it is immediate that

$$Z_*(x) + 1 \ge Z(x) \ge Z_*(x) \tag{17}$$

Therefore, it is sufficient to study the function $Z_*(x)$.

The following theorems are easy consequences of the given definitions:

Theorem 3.

$$Z_*(x) \sim \frac{1}{2}\sqrt{8x+1} \quad (x \to \infty) \tag{18}$$

Theorem 4.

$$\sum_{n=1}^{\infty} \frac{1}{(Z_*(n))^\alpha} \text{ is convergent for } \alpha > 2 \tag{19}$$

and divergent for $\alpha \le 2$. The series $\sum_{n=1}^{\infty} \frac{1}{n(Z_*(n))^\alpha}$ is convergent for all $\alpha > 0$.

Proof. By (16) one can write $\frac{k(k+1)}{2} \le x < \frac{(k+1)(k+2)}{2}$, so $k^2 + k - 2x \le 0$ and $k^2 + 3k + 2 - 2x > 0$. Since the solutions of these quadratic equations are $k_{1,2} = \frac{-1 \pm \sqrt{8x+1}}{2}$, resp. $k_{3,4} = \frac{-3 \pm \sqrt{8x+1}}{2}$, and remarking that $\frac{\sqrt{8x+1}-3}{2} \ge 1 \Leftrightarrow x \ge 3$, we obtain that the solution of the above system of inequalities is:

$$\begin{cases} k \in \left[1, \frac{\sqrt{1+8x}-1}{2}\right] & \text{if } x \in [1,3); \\ k \in \left(\frac{\sqrt{1+8x}-3}{2}, \frac{\sqrt{1+8x}-1}{2}\right] & \text{if } x \in [3,+\infty) \end{cases} \tag{20}$$

So, for $x \ge 3$

$$\frac{\sqrt{1+8x}-3}{2} < Z_*(x) \le \frac{\sqrt{1+8x}-1}{2} \tag{21}$$

implying relation (18).

Theorem 4 now follows by (18) and the known fact that the generalized harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^\theta}$ is convergent only for $\theta > 1$.

The things are slightly more complicated in the case of functions P and P_*. Here it is sufficient to consider P_*, too.

First remark that

$$P_*(x) = m \ \Leftrightarrow\ x \in \left[\frac{\log m!}{\log p}, \frac{\log(m+1)!}{\log p}\right). \tag{22}$$

The following asymptotic results have been proved in [3] (Lemma 2) (see also [6], p. 172)

$$\log m! \sim m\log m, \quad \frac{m\log\log m!}{\log m!} \sim 1, \quad \frac{\log\log m!}{\log\log(m+1)!} \sim 1 \quad (m \to \infty) \tag{23}$$

By (22) one can write

$$\frac{m\log\log m!}{\log m!} - \frac{m}{\log m!}\log\log p \le \frac{m\log x}{\log m!} \le \frac{m\log\log(m+1)!}{\log m!} - (\log\log p)\frac{m}{\log m!},$$

giving $\frac{m\log x}{\log m!} \to 1$ $(m \to \infty)$, and by (23) one gets $\log x \sim \log m$. This means that:

Theorem 5.

$$\log P_*(x) \sim \log x \quad (x \to \infty) \tag{24}$$

The following theorem is a consequence of (24), and a convergence theorem established in [3]:

Theorem 6. *The series $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\log\log n}{\log P_*(n)}\right)^\alpha$ is convergent for $\alpha > 1$ and divergent for $\alpha \le 1$.*

Indeed, by (24) it is sufficient to study the series $\sum_{n \ge n_0} \frac{1}{n}\left(\frac{\log\log n}{\log n}\right)^\alpha$ (where $n_0 \in \mathbb{N}$ is a fixed positive integer). This series has been proved to be convergent for $\alpha > 1$ and divergent for $\alpha \le 1$ (see [6], p. 174).

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ON SOME SMARANDACHE PROBLEMS

Edited by M. Perez

1. PROPOSED PROBLEM

Let $n \geq 2$. As a generalization of the integer part of a number one defines the Inferior Smarandache Prime Part as: $ISPP(n)$ is the largest prime less than or equal to n. For example: $ISPP(9) = 7$ because $7 < 9 < 11$, also $ISPP(13) = 13$. Similarly the Superior Smarandache Prime Part is defined as: $SSPP(n)$ is smallest prime greater than or equal to n. For example: $SSPP(9) = 11$ because $7 < 9 < 11$, also $SSPP(13) = 13$. Questions:
1) Show that a number p is prime if and only if

$$ISPP(p) = SSPP(p).$$

2) Let $k > 0$ be a given integer. Solve the diophantine equation:

$$ISPP(x) + SSPP(x) = k.$$

Solution by Hans Gunter, Koln (Germany)

The Inferior Smarandache Prime Part, $ISPP(n)$, does not exist for $n < 2$.
1) The first question is obvious (Carlos Rivera).
2) The second question:

a) If $k = 2p$ and p =prime (i.e., k is the double of a prime), then the Smarandache diophantine equation

$$ISPP(x) + SSPP(x) = 2p$$

has one solution only: $x = p$ (Carlos Rivera).

b) If k is equal to the sum of two consecutive primes, $k = p(n) + p(n+1)$, where $p(m)$ is the m-th prime, then the above Smarandache diophantine equation has many solutions: all the integers between $p(n)$ and $p(n+1)$ [of course, the extremes $p(n)$ and $p(n+1)$ are excluded]. Except the case $k = 5 = 2+3$, when this equation has no solution. The sub-cases when this equation has one solution only is when $p(n)$ and $p(n+1)$ are twin primes, i.e. $p(n+1)-p(n) = 2$, and then the solution is $p(n)+1$. For example: $ISPP(x)+SSPP(x) = 24$ has the only solution $x = 12$ because $11 < 12 < 13$ and $24 = 11 + 13$ (Teresinha DaCosta).

Let's consider an example:

$$ISPP(x) + SSPP(x) = 100,$$

because 100=47+53 (two consecutive primes), then $x = 48, 49, 50, 51$, and 52 (all the integers between 47 and 53).

$$ISPP(48) + SSPP(48) = 47 + 53 = 100.$$

Another example:

$$ISPP(x) + SSPP(x) = 99$$

has no solution, because if $x = 47$ then

$$ISPP(47) + SSPP(47) = 47 + 47 < 99,$$

and if $x = 48$ then

$$ISPP(48) + SSPP(48) = 47 + 53 = 100 > 99.$$

If $x \leq 47$ then

$$ISPP(x) + SSPP(x) < 99,$$

while if $x \geq 48$ then

$$ISPP(x) + SSPP(x) > 99.$$

c) If k is not equal to the double of a prime, or k is not equal to the sum of two consecutive primes, then the above Smarandache diophantine equation has no solution.

A remark: We can consider the equation more general: Find the real number x (not necessarily integer number) such that

$$ISPP(x) + SSPP(x) = k,$$

where $k > 0$.

Example: Then if $k = 100$ then x is any real number in the open interval (47, 53), therefore infinitely many real solutions. While integer solutions are only five: 48, 49, 50, 51, 52.

A criterion of primality: The integers p and $p+2$ are twin primes if and only if the diophantine smarandacheian equation

$$ISPP(x) + SSPP(x) = 2p + 2$$

has only the solution $x = p + 1$.

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[2] T. Tabirca and S. Tabirca, "A New Equation For The Load Balance Scheduling Based on Smarandache f-Inferior Part Function", http://www.gallup.unm.edu/ smarandache/tabirca-sm-inf-part.pdf [The Smarandache f-Inferior Part Function is a greater generalization of ISPP.]

2. PROPOSED PROBLEM

Prove that in the infinite Smarandache Prime Base 1,2,3,5,7,11,13,... (defined as all prime numbers proceeded by 1) any positive integer can be uniquely written with only two digits: 0 and 1 (a linear combination of distinct primes and integer 1, whose coefficients are 0 and 1 only).

Unsolved question: What is the integer with the largest number of digits 1 in this base?

Solution by Maria T. Marcos, Manila, Philippines

For example: 12 is between 11 and 13 then 12=11+1 in SPB. or

$$12 = 1 \times 11 + 0 \times 7 + 0 \times 5 + 0 \times 3 + 0 \times 2 + 1 \times 1 = 100001$$

in SPB. Similarly as

$$402 = 4 \times 100 + 0 \times 10 + 4 \times 1 = 402$$

in base 10 (the infinite base 10 is:

$$1, 10, 100, 1000, 10000, 100000, \ldots).$$

$$0 = 0 \text{ in SPB}$$

$$1 = 1 \text{ in SPB}$$

$$2 = 1 \times 2 + 0 \times 1 = 10 \text{ in SPB}$$

$$3 = 1 \times 3 + 0 \times 2 + 0 \times 1 = 100 \text{ in SPB}$$

$$4 = 1 \times 3 + 0 \times 2 + 1 \times 1 = 101 \text{ in SPB}$$

$$5 = 3 + 2 = 1 \times 3 + 1 \times 2 + 0 \times 1 = 110 \text{ in SPB}$$

$$15 = 13 + 2 = 1 \times 13 + 0 \times 11 + 0 \times 7 + 0 \times 5 + 0 \times 3 + 1 \times 2 + 0 \times 1 = 1000010 \text{ in SPB}$$

This base is a particular case of the Smarandache general base - see [3].

Let's convert backwards: If 1001 is a number in the SPB, then this is in base ten:

$$1 \times 5 + 0 \times 3 + 0 \times 2 + 1 \times 1 = 5 + 0 + 0 + 1 = 6.$$

We do not get digits greater than 1 because of Chebyshev's theorem.

It is only a unique writing.

$10 = 7+3$, that is it. We do not decompose 3 anymore because 3 belongs to the Smarandache prime base.

$11 = 7 + 4 = 7 + 3 + 1$, because 4 did not belong to the SPB we had to decompose 4 as well.

11 has a unique representation: $11 = 7 + 3 + 1$.

The rule is:

- any number n is between $p(k)$ and $p(k+1)$ mandatory:

$$p(k) \leq n < p(k+1),$$

where $p(k)$ is the k-th prime; I mean any number is between two consecutive primes.

For another example:
27 is between 23 and 29, thus 27=23+4, but 4 is between 3 and 5 therefore 4=3+1, therefore 27=23+3+1 in the SPB (a unique representation).

Not allowed to say that $27 = 19 + 8$ because 27 is not between 19 and 29 but between 23 and 29.

The proof that all digits are 0 or 1 relies on the Chebyshev's theorem that between a number n and $2n$ there is at least a prime. Thus, between a prime q and $2q$ there is as least a prime. Thus $2p(k) > p(k+1)$ where $p(k)$ means the k-th prime.

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3. PROPOSED PROBLEM

Let p be a positive prime, and $S(n)$ the Smarandache Function, defined as the smallest integer such that $S(n)!$ be divisible by n. The factorial of m is the product of all integers from 1 to m. Prove that

$$S(p^p) = p^2.$$

Solution by Alecu Stuparu, 0945 Balcesti, Valcea, Romania

Because p is prime and $S(p^p)$ must be divisible by p, one gets that $S(p^p) = p$, or $2p$, or $3p$, etc.

More, $S(p^p)$ must be divisible by p^p, therefore

$$S(p^p) = p \cong p, \text{ or } p \cong (p+1), \text{ or } p \cong (p+2), \text{ etc.}$$

But the smallest one is $p \cong p$ [because $p \cong (p-1)!$ is not divisible by p^p, but by p^{p-1}]. Therefore

$$S(p^p) = p^2.$$

4. PROPOSED PROBLEM

Let $S3f(n)$ be the triple Smarandache function, i.e. the smallest integer m such that $m!!!$ is divisible by n. Here $m!!!$ is the triple factorial, i.e. $m!!! = m(m-3)(m-6)...$ the product of all such positive non-zero integers. For example $8!!! = 8(8-3)(8-6) = 8(5)(2) = 80$. $S3f(10) = 5$ because $5!!! = 5(5-3) = 5(2) = 10$, which is divisible by 10, and it is the smallest one with this property. $S3f(30) = 15, S3f(9) = 6. S3f(21) = 21$.

Question: Prove that if n is divisible by 3 then $S3f(n)$ is also divisible by 3.

Solution by K. L. Ramsharan, Madras, India

Let $S3f(n) = m$.

$S3f(n)!!! = m!!!$ has to be divisible by n according to the definition of this function, i.e. m has to be a multiple of 3, because n is a multiple of 3. In m is not a multiple of 3, then no factor of $m!!! = m(m-3)(m-6)...$ will be a multiple of 3, therefore $m!!!$ would not be divisible by n. Absurd.

5. PROPOSED PROBLEM

Let $Sdf(n)$ represent the Smarandache double factorial function, i.e. the smallest positive integer such that $Sdf(n)!!$ is divisible by n, where double factorial $m!! = 1 \times 3 \times 5 \times ... \times m$ if m is odd, and $m!! = 2 \times 4 \times 6 \times ... \times m$ if m is even. Solve the diophantine equation $Sdf(x) = p$, when p is prime. How many solutions are there?

Solution by Carlos Gustavo Moreira, Rio de Janeiro, Brazil

For the equation $Sdf(x) = p$ =prime, the number of solutions is $\geq 2^k$, where $k = (p-3)/2$. The general solution of the equation $Sdf(x) = p$ =prime is $p \times m$, where m is any divisor of $(p-2)!!$.

Let us consider the example for the Smarandache double factorial function $Sdf(x) = 17$. The solutions are $17 \times m$, where m is any divisor of $(17-2)!!$ which is equal to $3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 = (3^4) \times (5^2) \times 7 \times 11 \times 13$ which has $(4+1) \times (2+1) \times (1+1) \times (1+1) \times (1+1) = 120$ divisor, therefore 120 solutions $< 2^7 = 128$.

The number of solutions is not $2^7 = 128$ because some solutions were counted twice, for example: $17 \times 3 \times 5$ is the same as 17×15 or $17 \times 3 \times 15$ is the same as $17 \times 5 \times 9$.

Comment by Gilbert Johnson, Red Rock State Park. Church Rock, Box 1228, NM 87311, USA

How to determine the solutions and how to find a superior limit for the number of solutions.

Using the definition of Sdf, we find that: $p!!$ is divisible by x, and p is the smallest positive integer with this property. Because p is prime, x should be a multiple of p (otherwise p would not be the smallest positive integer with that property). $p!!$ is a multiple of x.

a) If $p = 2$, then $x = 2$.

b) If $p > 2$, then p is odd and $p!! = 1 \times 3 \times 5 \times ... \times p = Mx$ (multiple of x).

Solutions are formed by all combinations of p, times none, one, or more factors from 3, 5, ..., $p-2$.

Let $(p-3)/2 = k$ and rCs represent combinations of s elements taken by r.

So:

- for one factor: p, we have 1 solution: $x = p$; i.e. $0Ck$ solution;
- for two factors:

$$p \times 3, p \times 5, .., p \times (p-2),$$

we have k solutions:

$$x = p \times 3, p \times 5, ..., p \times (p-2);$$

i.e. $1Ck$ solutions;

- for three factors:

$$p \times 3 \times 5, p \times 3 \times 7, ..., p \times 3 \times (p-2); p \times 5 \times 7, ..., p \times 5 \times (p-2); ..., p \times (p-4) \times (p-2),$$

we have $2Ck$ solutions; etc. and so on: - for k factors:

$$p \times 3 \times 5 \times ... \times (p-2),$$

we have kCk solutions.

Thus, the general solution has the form:

$$x = p \times c_1 \times c_2 \times ... \times c_j,$$

with all c_j distinct integers and belonging to $\{3, 5, ..., p-2\}$, $0 \leq j \leq k$, and $k = (p-3)/2$. The smallest solution is $x = p$, the largest solution is $x = p!!$.

The total number of solutions is less than or equal to $0Ck + 1Ck + 2Ck + ... + kCk = 2k$, where $k = (p-3)/2$.

Therefore, the number of solutions of this equation is equal to the number of divisors of $(p-2)!!$.

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ON SOME PROBLEMS RELATED TO SMARANDACHE NOTIONS

Edited by M. Perez

1. Problem of Number Theory by L. Seagull, Glendale Community College

Let n be a composite integer > 4. Prove that in between n and $S(n)$ there exists at least a prime number.

Solution:

T.Yau proved that the Smarandache Function has the following property: $S(n) \leq \frac{n}{2}$ for any composite number n, because: if $n = pq$, with $p < q$ and $(p, q) = 1$, then

$$S(n) \max(S(p), S(q)) = S(q) \leq q = \frac{n}{p} \leq \frac{n}{2}.$$

Now, using Bertrand-Tchebichev's theorem, we get that in between $\frac{n}{2}$ and n there exists at least a prime number.

2. Proposed Problem by Antony Begay

Let $S(n)$ be the smallest integer number such that $S(n)!$ is divisible by n, where $m! = 1.2.3.\ldots.m$ (factoriel of m), and $S(1) = 1$ (Smarandache Function). Prove that if p is prime then $S(p) = p$. Calculate $S(42)$.

Solotion:

$S(p)$ cannot be less than p, because if $S(p) = n < p$ then $n! = 1.2.3.\ldots.n$ is not divisible by p (p being prime). Thus $S(p) \geq p$. But $p! = 1.2.3.\ldots.p$ is divisible by p, and is the smallest one with this property. Therefore $S(p) = p$.

$42 = 2.3.7, 7! = 1.2.3.4.5.6.7$ which is divisible by 2, by 3, and by 7. Thus $S(42) \leq 7$. But $S(42)$ can not be less than 7, because for example $6! = 1.2.3.4.5.6$ is not divisible by 7. Hence $S(42) = 7$.

3. Proposed Problem by Leonardo Motta

Let n be a square free integer, and p the largest prime which devides n. Show that $S(n) = p$, where $S(n)$ is the Smarandache Function, i.e. the smallest integer such that $S(n)!$ is divisible by n.

Solution:

Because n is a square free number, there is no prime q such that q^2 divides n. Thus n is a product of distinct prime numbers, each one to the first power only. For example 105 is square free because 105=3.5.7 , i.e. 105 is a product of distinct prime numbers, each of them to the power 1 only. While 945 is not a square free number because $945 = 3^3.5.7$, therefore 945 is divisible by 3^2 (which is 9, i.e. a square). Now, if we compute the Smarandache Function $S(105) = 7$ because $7!=1.2.3.4.5.6.7$ which is divisible by 3,5, and 7 in the same time, and 7 is smallest number with this property. But $S(945) = 9$, not 7. Therefore, if $n = a.b.\dots.p$, where all $a < b < \dots < p$ are distinct two by two primes, then $S(n) = \max(a, b, \dots, p = p$, because the factorial of p, the largest prime which divides n, includes the factors $a, b, \dots$ in its development: $p! = 1.\dots.a.\dots.b.\dots.p$.

4. Proposed Problem by Gilbert Johnson

Let $Sdf(n)$ be the Smarandache Double Factorial Function, i.e. the smallest integer such that $Sdf(n)!!$ is divisible by n, where $m!! = 1.3.5.\dots.m$ if m is odd and $m!! = 2.4.6.\dots.m$ if m is even. If n is an even square free number and p the largest prime which divides n, then $Sdf(n) = 2p$.

Solution:

Because n is even and square free, then $n = 2.a.b.\dots.p$ where all $2 < a < b < \dots < p$ are distinct primes two by two, occuring to the power 1 only. $Sdf(n)$ cannot be less that $2p$ because if it is $2p - k$, with $1 \leq k < 2p$, then $(2p - k)!!$ would not be divisible by p.

$$(2p)!! = 2.4.\dots.(2a).\dots.(2b).\dots.(2p)$$

is divisible by n and it is the smallest number with this property.

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GENERALIZED SMARANDACHE PALINDROME

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A Generalized Smarandache Palindrome is a number of the form: $a_1a_2...a_na_n...a_2a_1$ or $a_1a_2...a_{n-1}a_na_{n-1}...a_2a_1$, where all $a_1, a_2, ..., a_n$ are positive integers of various number of digits.

Examples:

a) 1235656312 is a GSP because we can group it as (12)(3)(56)(56)(3)(12), i.e. ABCCBA.

b) Of course, any integer can be consider a GSP because we may consider the entire number as equal to a_1, which is smarandachely palindromic; say $N = 176293$ is GSP because we may take $a_1 = 176293$ and thus $N = a_1$. But one disregards this trivial case.

Very interesting GSP are formed from smarandacheian sequences.

Let us consider this one:

$$11, 1221, 123321, ..., 12345678998765 4321,$$

$$1234567891010987654321, 12345678910111110987654321, ...$$

all of them are GSP.

It has been proven that 1234567891010987654321 is a prime (see

http://www.kottke.org/notes/0103.html,

and the Prime Curios site).

A question: How many other GSP are in the above sequence?

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ON 15-TH SMARANDACHE'S PROBLEM

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Introduction

The 15-th Smarandache's problem from [1] is the following: "**Smarandache's simple numbers:**

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 25, 26, 27, 29, 31, 33, \dots$$

A number n is called "Smarandache's simple number" if the product of its proper divisors is less than or equal to n. Generally speaking, n has the form $n = p$, or $n = p^2$, or $n = p^3$, or $n = pq$, where p and q are distinct primes".

Let us denote: by S - the sequence of all Smarandache's simple numbers and by s_n - the n-th term of S; by $\mathcal{P}$ - the sequence of all primes and by p_n - the n-th term of $\mathcal{P}$; by $\mathcal{P}^2$ - the sequence $\{p_n^2\}_{n=1}^{\infty}$; by $\mathcal{P}^3$ - the sequence $\{p_n^3\}_{n=1}^{\infty}$; by $\mathcal{PQ}$ - the sequence $\{p.q\}_{p,q\in\mathcal{P}}$, where $p < q$.

For an abitrary increasing sequence of natural numbers $C \equiv \{c_n\}_{n=1}^{\infty}$ we denote by $\pi_C(n)$ the number of terms of C, which are not greater that n. When $n < c_1$ we must put $\pi_C(n) = 0$.

In the present paper we find $\pi_S(n)$ in an explicit form and using this, we find the n-th term of S in explicit form, too.

1. $\pi_S(n)$-representation

First, we must note that instead of $\pi_{\mathcal{P}}(n)$ we shall use the well known denotation $\pi(n)$. Hence

$$\pi_{\mathcal{P}^2}(n) = \pi(\sqrt{n}), \quad \pi_{\mathcal{P}^3}(n) = \pi(\sqrt[3]{n}).$$

Thus, using the definition of S, we get

$$\pi_S(n) = \pi(n) + \pi(\sqrt{n}) + \pi(\sqrt[3]{n}) + \pi_{\mathcal{PQ}}(n). \tag{1}$$

Our first aim is to express $\pi_S(n)$ in an explicit form. For $\pi(n)$ some explicit formulae are proposed in [2]. Other explicit formulae for $\pi(n)$ are contained in [3]. One of them is known as Mináč's formula. It is given below

$$\pi(n) = \sum_{k=2}^{n}\left[\frac{(k-1)!+1}{k} - \left[\frac{(k-1)!}{k}\right]\right], \tag{2}$$

where $[.]$ denotes the function integer part. Therefore, the question about explicit formulae for functions $\pi(n), \pi(\sqrt{n}), \pi(\sqrt[3]{n})$ is solved successfully. It remains only to express $\pi_{\mathcal{PQ}}(n)$ in an explicit form.

Let $k \in \{1, 2, ..., \pi(\sqrt{n})\}$ be fixed. We consider all numbers of the kind $p_k.q$, where $q \in \mathcal{P}, q > p_k$ for which $p_k.q \le n$. The number of these numbers is $\pi(\frac{n}{p_k}) - \pi(p_k)$, or which is the same

$$\pi(\frac{n}{p_k}) - k. \tag{3}$$

When $k = 1, 2, ..., \pi(\sqrt{n})$, numbers $p_k.q$, that were defined above, describe all numbers of the kind $p.q$, where $p, q \in \mathcal{P}, p < q, p.q \le n$. But the number of the last numbers is equal to $\pi_{\mathcal{PQ}}(n)$. Hence

$$\pi_{\mathcal{PQ}}(n) = \sum_{k=1}^{\pi(\sqrt{n})}(\pi(\frac{n}{p_k}) - k), \tag{4}$$

because of (3). The equality (4), after a simple computation yields the formula

$$\pi_{\mathcal{PQ}}(n) = \sum_{k=1}^{\pi(\sqrt{n})}\pi(\frac{n}{p_k}) - \frac{\pi(\sqrt{n}).(\pi(\sqrt{n})+1)}{2}. \tag{5}$$

In [4] the identity

$$\sum_{k=1}^{\pi(b)}\pi(\frac{n}{p_k}) = \pi(\frac{n}{b}).\pi(b) + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\frac{n}{b})}\pi(\frac{n}{p_{\pi(\frac{n}{b})+k}}) \tag{6}$$

is proved, under the condition $b \ge 2$ (b is a real number). When $\pi(\frac{n}{2}) = \pi(\frac{n}{b})$, the right hand-side of (6) reduces to $\pi(\frac{n}{b}).\pi(b)$. In the case $b = \sqrt{n}$ and $n \ge 4$ equality (6) yields

$$\sum_{k=1}^{\pi(\sqrt{n})}\pi(\frac{n}{p_k}) = (\pi(\sqrt{n}))^2 + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\sqrt{n})}\pi(\frac{n}{p_{\pi(\sqrt{n})+k}}). \tag{7}$$

If we compare (5) with (7) we obtain for $n \ge 4$

$$\pi_{\mathcal{PQ}}(n) = \frac{\pi(\sqrt{n}).(\pi(\sqrt{n})-1)}{2} + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\sqrt{n})}\pi(\frac{n}{p_{\pi(\sqrt{n})+k}}). \tag{8}$$

Thus, we have two different explicit representations for $\pi_{\mathcal{PQ}}(n)$. These are formulae (5) and (8). We must note that the right hand-side of (8) reduces to $\frac{\pi(\sqrt{n}).(\pi(\sqrt{n})-1)}{2}$, when $\pi(\frac{n}{2}) = \pi(\sqrt{n})$.

Finally, we observe that (1) gives an explicit representation for $\pi_S(n)$, since we may use formula (2) for $\pi(n)$ (or other explicit formulae for $\pi(n)$) and (5), or (8) for $\pi_{\mathcal{PQ}}(n)$.

2. Explicit formulae for s_n

The following assertion decides the question about explicit representation of s_n.

Theorem: The n-th term s_n of S admits the following three different explicit representations:

$$s_n = \sum_{k=0}^{\theta(n)} [\frac{1}{1+[\frac{\pi_S(k)}{n}]}]; \tag{9}$$

$$s_n = -2\sum_{k=0}^{\theta(n)} \theta(-2[\frac{\pi_S(k)}{n}]); \tag{10}$$

$$s_n = \sum_{k=0}^{\theta(n)} \frac{1}{\Gamma(1-[\frac{\pi_S(k)}{n}])}, \tag{11}$$

where

$$\theta(n) \equiv [\frac{n^2+3n+4}{4}],\ n = 1,2,..., \tag{12}$$

ζ is Riemann's function zeta and Γ is Euler's function gamma.

Remark. We must note that in (9)-(11) $\pi_S(k)$ is given by (1), $\pi(k)$ is given by (2) (or by others formulae like (2)) and $\pi_{FQ}(n)$ is given by (5), or by (8). Therefore, formulae (9)-(11) are explicit.

Proof of the Theorem. In [2] the following three universal formulae are proposed, using $\pi_C(k)$ $(k = 0, 1, ...)$, which one could apply to represent c_n. They are the following

$$c_n = \sum_{k=0}^{\infty} [\frac{1}{1+[\frac{\pi_C(k)}{n}]}]; \tag{13}$$

$$c_n = -2\sum_{k=0}^{\infty} \zeta(-2[\frac{\pi_C(k)}{n}]); \tag{14}$$

$$c_n = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1-[\frac{\pi_C(k)}{n}])}. \tag{15}$$

In [5] is shown that the inequality

$$p_n \le \theta(n),\ n = 1, 2, ..., \tag{16}$$

holds. Hence

$$s_n = \theta(n), n = 1, 2, ..., \tag{17}$$

since we have obviously

$$s_n \le p_n,\ n = 1, 2, ... \tag{18}$$

Then to prove the Theorem it remains only to apply (13)-(15) in the case $C = S$, i.e., for $c_n = s_n$, putting there $\pi_S(k)$ instead of $\pi_C(k)$ and $\theta(n)$ instead of ∞.

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ON THE SECOND SMARANDACHE'S PROBLEM

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The second problem from [1] (see also 16-th problem from [2]) is the following:

Smarandache circular sequence:

$$\underbrace{1}_{1}, \underbrace{12, 21}_{2}, \underbrace{123, 231, 312}_{3}, \underbrace{1234, 2341, 3412, 4123}_{4},$$

$$\underbrace{12345, 23451, 34512, 45123, 51234}_{5}, \underbrace{123456, 234561, 345612, 456123, 561234, 612345}_{6}, \ldots$$

Let $]x[$ be the largest natural number strongly smaller than real (positive) number x. For example, $]7.1[= 7$, but $]7[= 6$.

Let $f(n)$ is the n-th member of the above sequence. We shall prove the following

Theorem: For every natural number n:

$$f(n) = \overline{s(s+1)...k12...(s-1)}, \tag{1}$$

where

$$k \equiv k(n) =]\frac{\sqrt{8n+1}-1}{2}[\tag{2}$$

and

$$s \equiv s(n) = n - \frac{k(k+1)}{2}. \tag{3}$$

Proof: When $n = 1$, then from (1) and (2) it follows that $k = 0$, $s = 1$ and from (3) – that $f(1) = 1$. Let us assume that the assertion is valid for some natural number n. Then for $n+1$ we have the following two possibilities:

1. $k(n+1) = k(n)$, i.e., k is the same as above. Then

$$s(n+1) = n+1 - \frac{k(n+1)(k(n+1)+1)}{2} = n+1 - \frac{k(n)(k(n)+1)}{2} = s(n)+1,$$

i.e.,

$$f(n+1) = \overline{(s+1)...k12...s}.$$

2. $k(n+1) = k(n)+1$. Then

$$s(n+1) = n+1 - \frac{k(n+1)(k(n+1)+1)}{2}. \tag{4}$$

On the other hand, it is seen directly, that in (2) number $\frac{\sqrt{8n+1}-1}{2}$ is an integer if and only if $n = \frac{m(m+1)}{2}$. Also, for every natural numbers n and $m \geq 1$ such that

$$\frac{(m-1)m}{2} < n < \frac{m(m+1)}{2} \tag{5}$$

it will be valid that

$$]\frac{\sqrt{8n+1}-1}{2}[=]\frac{\sqrt{\frac{m(m+1)}{2}+1}-1}{2}[= m.$$

Therefore, when $k(n+1) = k(n)+1$, then

$$n = \frac{m(m+1)}{2} + 1$$

and for it from (4) we obtain:

$$s(n+1) = 1,$$

i.e.,

$$f(n+1) = \overline{12...(n+1)}.$$

Therefore, the assertion is valid.

Let

$$S(n) = \sum_{i=1}^{n} f(i).$$

Then, we shall use again formulae (2) and (3). Therefore,

$$S(n) = \sum_{i=1}^{p} f(i) + \sum_{i=p+1}^{n} f(i),$$

where

$$p = \frac{m(m+1)}{2}.$$

It can be seen directly, that

$$\sum_{i=1}^{p} f(i) = \sum_{i=1}^{m} \overline{12...i} + \overline{23...i1} + \overline{i12...(i-1)} = \sum_{i=1}^{m} \frac{i(i+1)}{2}.\underbrace{11...1}_{i}$$

On the other hand, if $s = n - p$, then

$$\sum_{i=p+1}^{n} f(i) = \overline{12...(m+1)} + \overline{23...(m+1)1} + \overline{s(s+1)...m(m+1)12...(s-1)}$$

$$= \sum_{i=0}^{m+1} (\frac{(s+i)(s+i+1)}{2} - \frac{i(i+1)}{2}).10^{m-i}.$$

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