Foundation of Appurtenance and Inclusion Equations for Constructing the Operations of Neutrosophic Numbers Needed in Neutrosophic Statistics (revised)

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Abstract

We introduce for the first time the appurtenance equation and inclusion equation, which help in understanding the operations with neutrosophic numbers within the frame of neutrosophic statistics. The way of solving them resembles the equations whose coefficients are sets (not single numbers).

Keywords: Real Neutrosophic Numbers; Neutrosophic Statistics;True Value; True Set-Value; Appurtenance Relationship; Appurtenance Equation; Inclusion Relationship; Inclusion Equation; Equality Equation with Set-Coefficients; NonAppurtenance Equation; NonInclusion Equation; NonEquality Equation; Operations with Neutrosophic Numbers

1. Introduction

In neutrosophic statistics, from the fact that the single true value v is into the indeterminate set I, it does not result that v is in the neutrosophic number $N = a + bI$ as well, but: $a + bv \in a + bI = N$. That's why the appurtenance relationship and equation must be introduced and studied.

Even more, if one has a set of true values V, from the fact that the set of true values V is included in I, it does not mean that V is included in $N = a + bI$ too,

but $a + bV \subset a + bI = N$ (or $a + bV \subseteq a + bI = N$). That's why the inclusion relationship and equation must be introduced.

In the same way as the "=" symbol is used for an equality relationship or an equality equation, we use the symbol "∈" {belong(s) to} for an *appurtenance (belongness) relationship* or *appurtenance (belongness) equation* of a number to a set, respectively the symbol $\subset$ (or $\subseteq$) {included in, or included in or equal to} for an *inclusion relationship* or *inclusion equation*.

We use in this paper the tautological denomination Equality Equation with Set-Coefficients (=), in order to distinguish it from the Appurtenance Equation (∈) and Inclusion Equation { $\subset$ or $\subseteq$ }.

Whatever operation we do on the left-hand side of an *appurtenance relationship* or *appurtenance equation* (respectively, *inclusion relationship* or *inclusion equation*), we must do the same on the right-hand side as well of the *appurtenance relationship* or *appurtenance equation* (respectively, *inclusion relationship* or *inclusion equation*).

In addition, we also present their complementary *NonAppurtenance Equation*, *NonInclusion Equation*, and the elementary *NonEquality Equation* respectively.

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2. Definition of the Real Neutrosophic Number

A *Real Neutrosophic Number* (N) has the form:

$N = a + bI$, where a, b are real numbers, "a" is called the determinate part of N, while "bI" is the indeterminate part of N, while I is a real subset, $I \subset \mathbb{R}$.

They are mostly used in *Neutrosophic Statistics*.

The neutrosophic numbers frequently occur in our real world, where one often has imprecise, unclear data to deal with.

For example, let's have a right triangular shape land whose legs are 5 and 6 kilometers respectively, then the length of its hypotenuse is $\sqrt{5^2 + 6^2} = \sqrt{61} = 7.81024967\ldots$ that we need to approximate with some required accuracy.

Another example, let's consider the real circular land of radius 10 km, compute its area $A = \pi \cdot 10^2 = 100\pi = 314.159265 \ldots \text{km}^2$, or to compute the volume or surface of a sphere, but π is a transcendental number (a number that is not the root of a non-zero polynomial with rational coefficients and of finite degree), having infinitely many decimals with no repeated pattern.

Similarly, the Euler's constant e = 2.71828182…is a transcendental number and occurs in many formulas.

In the same way when, from real world applications, one arises inexact results (radicals, exponential or logarithmic or trigonometric equations, differential equations, transcendental functions, etc.).

Examples of Real Neutrosophic Numbers

(i) $N_1 = 2 + 3I$, where $I_1 = [0, 1]$ is an interval.

Or

$N_1 = 2 + 3 \cdot [0, 1] = 2 + [3.0, 3.1] = 2 + [0, 3] = [2 + 0, 2 + 3] = [2, 5]$

(ii) Let $I_2 = \{0.6, 0.8, 0.9\}$, which is a finite discrete set of three elements.

Then:

$$N_2 = 2 + 3I_2 = 2 + 3 \cdot \{0.6, 0.8, 0.9\} = 2 + \{3 \cdot (0.6), 3 \cdot (0.8), 3 \cdot (0.9)\} = 2 + \{1.8, 2.4, 2.7\}$$
$$= \{2 + 1.8, 2 + 2.4, 2 + 2.7\} = \{3.8, 4.4, 4.7\} \subset [2, 5].$$

(iii) Let $I_3 = \left\{\frac{1}{n}, 1 \leq n \leq \infty, n \text{ is integer}\right\}$, which is an infinite discrete set.

Then:

$$N_3 = 2 + 3I_3 = \left\{2 + 3 \cdot \frac{1}{n}, 1 \leq n \leq \infty, n \text{ is integer}\right\} = \left\{2 + 3 \cdot \frac{1}{1}, 2 + 3 \cdot \frac{1}{2}, 2 + 3 \cdot \frac{1}{3}, \ldots, 2 + 3 \cdot \frac{1}{n}, \ldots\right\}$$
$$= \left\{5, 3.5, 3, \ldots, 2 + 3 \cdot \frac{1}{n}, \ldots\right\} \subset [2, 5].$$

3. Foundation of Appurtenance Relationship and Appurtenance Equation

The below theorems 1 and 2 allow us to do operations on both sides of an appurtenance relationship and appurtenance equation respectively.

3.1. Theorem 1

Let A and B be real sets.

If $a \in A$ and $b \in B$, then:

Addition

$a + b \in A + B$

Subtraction

$a - b \in A - B$

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Multiplication
$a \times b \in A \times B$

Division
$\frac{a}{b} \in \frac{A}{B}$

Power
$a^b \in A^B$

Proof
Let $\star$ be any of the above operation, then:

$A \star B = \{x \star y; \text{where } x \in A, y \in B\}$,

and the operation $\star$ is well-defined.

If we let $x = a \in A$, and $y = b \in B$ into the above definition, then:

$a \star b = A \star B$.

3.2. Theorem 2
Let A be a real set, $a \in A$, β be a real scalar, and m, n be (positive) integers. Then:

Scalar Multiplication of a Set

$\beta \cdot a \in \beta \cdot A$

Raising to a Power of a Set
$a^n \in A^n$

Root Index of a Set

$\sqrt[n]{a} \in \sqrt[n]{A}$

Negative Exponent of a Set
$a^{-n} \in A^{-n}$

Rational Exponent of a Set
$a^{\frac{m}{n}} \in A^{\frac{m}{n}}$

Proof
Similarly.

$$\beta \cdot A = \left\{\beta \cdot x, x \in A\right\} \quad (2)$$

Letting $x = a \in A$ into definition (2), it results that

$\beta \cdot a \in \beta \cdot A^{\cdot}$

For the next four appurtenance relationships, let p be any of the exponents, n, $\frac{1}{n}$, $-n$, or $\frac{m}{n}$, then:

$$A^p = \{x^p, x \in A\}. \quad (3)$$

Letting $x = a$ into the definition (3), it results that:

$a^p \in A^p$,

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for p being any of n, $\frac{1}{n}$, $-n$, or $\frac{m}{n}$.

3.3. Examples of Operations with Appurtenance Relationships
Let's take the true appurtenance:

0.67 ∊ (0, 1).

Scalar Addition of an Appurtenance Relationship
$(3 + 0.67) \in (3 + (0, 1))$

or $3.67 \in (3 + 0, 3 + 1)$

or $3.67 \in (3,4)$, which is true.

Scalar Multiplication of an Appurtenance Relationship
$2 \cdot (0.67) \in 2 \cdot (0, 1)$

or $1.34 \in (2 \cdot 0, 2 \cdot 1)$

or $1.34 \in (0, 2)$, which is true.

Power
$0.67^2 \in (0, 1)^2$

or $0.4489 \in (0^2, 1^2)$

or $00.4489 \in (0, 1)$, which is true.

Division by a Scalar of an Appurtenance Relationship
$\frac{0.67}{-2} \in \frac{(0,1)}{-2}$

or $-0.335 \in (-0.5, 0)$, which is true.

3.4. Appurtenance Equations

Its general form is defined as follows.

Let *R* be the set of real numbers, and *f* and *g* be real HyperFunctions { "hyper" stands for the fact that their domain and/or codomain are powersets *P(R)* },

$$f, g : P(R) \to P(R)$$

$$f(x) \in g(x)$$

All procedures done to solve a classical equation (but whose coefficients are sets, not single numbers) are similarly allowed to do for solving an *appurtenance equation*.

Because sometimes it is not clear what number, either *a* or *b* is bigger, we consider that:

$(a, b) \equiv (b, a)$ and $[a, b] \equiv [b, a]$.

3.5. Solution of an Appurtenance Equation

The solution of an appurtenance equation means a real set *S*, or $S \in P(R)$ the powerset of R, to whom the unknown *x* belongs to, or $x \in S$.

3.6. Example 1 of Appurtenance Equation
Solve for *x*.

$4 - 5x \in 1 + 2 \cdot (0.5, 0.8)$.

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Subtract 4 from both sides:

$(-5x) \in \left(-3 + 2(0.5, 0.8)\right)$.

We use parentheses ( ) in order to clearly distinguish between the left-hand side and the right-hand side of the appurtenance equation.

$(-5x) \in \left(-3 + (1, 1.6)\right)$

$(-5x) \in (-3 + 1, -3 + 1.6)$

Divide both sides by -5:

$\frac{(-5x)}{-5} \in \frac{(-3+1,-3+1.6)}{-5}$

$x \in \left(\frac{1.4}{5}, \frac{2}{5}\right)$

$x \in (0.28, 0.40)$.

There are infinitely many particular solutions of this appurtenance equation, i.e. all the numbers inside the open interval $(0.28, 0.40)$. We do not take the subsets of *(0.28, 0.40)* as particular solutions, since they are included ($\subset$ or $\subseteq$) in, not appurtenant ($\in$) to *(0.28, 0.40).*

Check the maximal solution of the appurtenance equation.

$4 - 5 \cdot x \in 1 + 2 \cdot (0.5, 0.8)$

$4 - 5 \cdot (0.28, 0.40) \overset{?}{\in} 1 + 2 \cdot (0.5, 0.8)$

where $\overset{?}{\in}$ means that we must check the appurtenance.

$4 - (1.4, 2.0) \overset{?}{\in} 1 + (1.0, 1.6)$

$(4 - 2.0, 4 - 1.4) \overset{?}{\in} (1 + 1.0, 1 + 1.6)$

or $(2, 2.6) \in (2, 2.6)$

actually we have an equality here above, which means that any number x, and any subset inside of the left-hand side interval, are solutions.

Therefore, this appurtenance equation has infinitely solutions, $x \in (0.28, 0.40)$.

Let's check some of them, a particular solution as a single number:

$x = 0.35 \in (0.28, 0.40)$

The appurtenance equation:

$4 - 5x \in 1 + 2 \cdot (0.5, 0.8)$ becomes, after substituting x,

$4 - 5 \cdot (0.35) \in 1 + 2(0.5, 0.8)$

$4 - (1.75) \in 1 + (1.0, 1.6)$

$2.25 \in (2.00, 2.60)$, which is true.

Let's check a particular solution-subset of (0.28, 0.40):

$x = (0.30, 0.34) \subset (0.28, 0.40)$.

The appurtenance equation: $4 - 5x \in 1 + 2 \cdot (0.5, 0.8)$ becomes:

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$4 - 5 \cdot (0.30, 0.34) \in (2.00, 2.60)$

$4 - (5 \cdot 0.30, 5 \cdot 0.34) \in (2.00, 2.60)$

$4 - (1.50, 1.70) \in (2.00, 2.60)$

$(4 - 1.70, 4 - 1.50) \in (2.00, 2.60)$

$(2.30, 2.50) \in (2.00, 2.60)$,

actually, the left-hand side is included into the right-hand side.

3.7. Example of Equality Equation with Set-Coefficients

The maximal solution-set $x = (0.28, 0.40)$ of this appurtenance equation becomes the set-solution of the following *equality equation with set-coefficients*:

$4 - 5x = 1 + 2 \cdot (0.5, 0.8)$.

This equation, whose one of the coefficients is a set, (0.5, 0.8),

is solved in the same way:

subtract *4* from both sides:

$-5x = -3 + 2 \cdot (0.5, 0.8)$,

then multiply and add the sets:

$-5x = -3 + (1.0, 1.6)$

$-5x = (-3 + 1.0, -3 + 1.6)$

$-5x = (-2, -1.4)$,

and divide by -5 to get:

$x = \left(\frac{-1.4}{-5}, \frac{-2}{-5}\right)$

$x = (0.28, 0.40)$, which is a set-solution.

4. Foundation of Inclusion Relationship and Inclusion Equation

Similarly for the *Inclusion* ($\subset$ or $\subseteq$) *Relationships* and *Inclusion Equation* as we did for Appurtenance Relationships and Appurtenance Equation respectively.

For the case where one has $\subseteq$, the below theorems will be the same, just using $\subseteq$ instead of $\subset$.

4.1. Inclusion Equations

Its general form is defined as follows.

Let *R* be the set of real numbers, and *f* and *g* be real HyperFunctions { "hyper" stands for the fact that their domain and/or codomain are powersets *P(R)* },

$$f, g : P(R) \to P(R)$$

Then, $f(x) \subset g(x)$ or $f(x) \subseteq g(x)$ are called inclusion equations.

4.2. Theorem 3

Let A and B be real sets, and A_1, B_1 also real sets, but:

$A_1 \subset$ A, and $B_1 \subset B$.

Then:

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Addition of Sets
$A_1 + B_1 \subset A + B$

Subtraction of Sets
$A_1 - B_1 \subset A - B$

Multiplication of Sets
$A_1 \times B_1 \subset A \times B$

Division of Sets
$\frac{A_1}{B_1} \subset \frac{A}{B}$

Power of Sets
${A_1}^{B_1} \subset A^B$

Proof
In the same way, let $\star$ be any of above operations $+, -, \times, \div$, ^ (power), then:

$$A \star B = \{x \star y; \text{where } x \in A, y \in B\} \quad (3)$$

and the operation $\star$ is well-defined.

We let $x = a_1 \in A_1 \subset A$, and $y = b_1 \in B_1 \subset B$ into the definition (3), then:

$a_1 \star b_1 \in A \star B$, for all $a_1 \in A_1$ and $b_1 \in B_1$, which means that:

$A_1 \star B_1 \subset A \star B$.

4.3. Theorem 4
Let A and A_1 be real sets, with $A_1 \subset A$, $\beta \neq 0$ a real number, and m, n positive integers.

Then:

Scalar Multiplication of a Set
$\beta \cdot A_1 \subset \beta \cdot A$

Raising to the Power n of a Set
$A_1^n \subset A^n$

Root Index n of a Set
$\sqrt[n]{A_1} \subset \sqrt[n]{A}$

Negative Exponent of a Set
$A_1^{-n} \subset A^{-n}$

Rational Exponent of a Set
$A_1^{\frac{m}{n}} \subset A^{\frac{m}{n}}$

Proof
Similarly to the previous theorem.

$$\beta \cdot A = \left\{\beta \cdot x, x \in A\right\}$$

and $\beta \cdot A_1 = \left\{\beta \cdot x, x \in A_1 \subset A\right\} \subset \left\{\beta \cdot x, x \in A\right\} = \beta \cdot A$.

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For the other inclusion relationships, we let again p be any of the exponents $n, \frac{1}{n}, -n, \frac{m}{n}$, then:

$A_1^p = \{x^p, x \in A_1\} \subset \{x^p, x \in A\} = A$, since $A_1 \subset A$,

where for any $x \in A$, and any p, all x^p operations are well-defined.

Analogously, these theorems 3 and 4 allow us to do many operations on both sides of an inclusion relationship or inclusion equation.

4.4. Examples of Inclusion Relationships

Let's consider the true inclusion relationship:

$(2, 3] \subset [0, 4]$

Let's add 1 in both sides:

$1 + (2, 3] \subset 1 + [0, 4]$

or $(1 + 2, 1 + 3] \subset [1 + 0, 1 + 4]$

$(3, 4] \subset [1, 5]$, which is true.

Let's add an interval $(-1, 5)$ on both sides:

$(2, 3] + (-1, 5) \subset [0, 4] + (-1, 5)$

$(2 - 1, 3 + 5) \subset (0 - 1, 4 + 5)$

$(1, 8) \subset (-1, 9)$, which is true.

Let's subtract 2 from both sides:

$(2, 3] - 2 \subset [0, 4] - 2$

$(2 - 2, 3 - 2] \subset [0 - 2, 4 - 2]$

$(0, 1] \subset [-2, 2]$, which is true.

Let's subtract a set [0.5, 0.6] from both sides.

$(2,3] - [0.5, 0.6] \subset [0, 4] - [0.5, 0.6]$

$(2 - 0.6, 3 - 0.5] \subset [0 - 0.6, 4 - 0.5]$

$(1.4, 2.5] \subset [-0.6, 3.5]$, which is true.

Let's multiply both sides by a positive (non-zero) number 7:

$7 \cdot (2, 3] \subset 7 \cdot [0, 4]$

$(7 \cdot 2, 7 \cdot 3] \subset [7 \cdot 0, 7 \cdot 4]$

$(14, 21] \subset [0, 28]$, which is true.

Let's multiply both sides by a negative (non-zero) number -5:

$-5 \cdot (2, 3] \subset -5 \cdot [0, 4]$

$(-5 \cdot 3, -5 \cdot 2] \subset [-5 \cdot 4, -5 \cdot 0]$

$(-15, -10] \subset [-20, 0]$, which is true.

Let's multiply both sides with a set $(-1, 1)$.

$(-1, 1) \cdot (2, 3] \subset (-1, 1) \cdot (0, 4]$

$(-3, 3) \subset (-4, 4)$, which is true.

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Let's raise to the second power both sides:

$(2,3]^2 \subset [0,4]^2$

$(2^2,3^2] \subset [0^2,4^2]$

$(4,9] \subset [0,16]$, which is true.

Let's divide by -5 both sides:

$\frac{(2,3]}{-5} \subset \frac{[0,4]}{-5}$

$\left[-\frac{3}{5},-\frac{2}{5}\right) \subset \left[-\frac{4}{5},-\frac{0}{5}\right]$

$[-0.6,-0.4) \subset [-0.8,0]$, which is true.

Let's divide each side by a real set [4, 5].

$\frac{(2,3]}{[4,5]} \subset \frac{[0,4]}{[4,5]}$

$\left(\frac{2}{5},\frac{3}{4}\right] \subset \left[\frac{0}{5},\frac{4}{4}\right]$

$(0.40,0.75] \subset [0,1]$, which is true.

4.5. Example 2 of Inclusion Equation

Solve for x.

$1 + x \cdot (1,2) \subset (0,5)$

$1 + (x,2x) \subset (0,5)$

$(x+1,2x+1) \subset (0,5)$

whence $0 < x+1 < 5$ or $-1 < x < 4$ or $x \subset (-1, 4)$,

and $0 < 2x+1 < 5$ or $-1 < 2x < 4$ or $-0.5 < x < 2$ or $x \subset (-0.5, 2)$,

whence $x \subset (-1,4) \cap (-0.5,2) = (-0.5,2)$.

So $x = (-0.5,2)$ is the maximal solution. All subsets of $(-0.5,2)$ are particular solutions – therefore one has infinitely many particular solutions.

Check it:

$1 + x(1,2) \subset (0,5)$

$1 + (-0.5,2) \cdot (1,2) \subset (0,5)$

$1 + (-1,2) \subset (0,5)$

$(1-1,1+2) \subset (0,5)$

$(0,3) \subset (0,5)$, which is true.

4.6. Another Example of Inclusion Equation

Solve for x.

$(4,5) + x \cdot [1,2] \subseteq [6,10]$

$(4,5) + [1 \cdot x, 2 \cdot x] \subseteq [6,10]$

$(4,5) + [x,2x] \subseteq [6,10]$

For $x \geq 0$, one gets:

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$(4 + x, 5 + 2x) \subseteq [6, 10]$

Hence:

$6 \leq 4 + x \leq 10,$

whence $2 \leq x \leq 6$

and $6 \leq 5 + 2x \leq 10$

or $1 \leq 2x \leq 5$

or $1.5 \leq x \leq 2.5$

thus, solution for $x \geq 0$ is $[2, 6] \cap [1.5, 2.5] = [2, 2.5]$.

For $x < 0$

$(4, 5) + x \cdot [1, 2] \subseteq [6, 10]$

$(4, 5) + [2x, x] \subseteq [6, 10]$

$(4 + 2x, 5 + x) \subseteq [6, 10]$

Whence

$6 \leq 4 + 2x \leq 10$, or $2 \leq 2x \leq 6$ or $1 \leq x \leq 3$

and $6 \leq 5 + x \leq 10$, or $1 \leq x \leq 5$.

But x must be negative in this case, therefore this situation doesn't produce any solution.

The maximum solution is $x = [2, 2.5]$.

Let's check the maximal solution.

$(4, 5) + x \cdot [1, 2] \subseteq [6, 10]$

Then:

$(4, 5) + [2, 2.5] \subseteq [6, 10]$

$(4 + 2, 5 + 2.5) \subseteq [6, 10]$

$(6, 7.5) \subseteq [6, 10]$, which is true.

As particular solutions are all subsets of the maximal solution [2, 2.5], therefore infinitely many.

Verifications.

Let $x = 2 \in [2, 2.5]$. We may also write x as a set, $x = [2, 2] \subseteq [2, 2.5]$.

$(4, 5) + x \cdot [1, 2] \subseteq [6, 10]$

$(4, 5) + 2 \cdot [1, 2] \subseteq [6, 10]$

$(4, 5) + [2, 4] \subseteq [6, 10]$

$(6, 9) \subseteq [6, 10]$, which is true.

Let $x = 2.3 \in [2, 2.5]$.

Then: $(4, 5) + 2.3[1.2] \subseteq [6, 10]$

$(4, 5) + [2.3, 4.6] \subseteq [6, 10]$

$(6.3, 9.6) \subseteq [6, 10]$, which is true.

Let $x = [2.1, 2.4) \subseteq [2, 2.5]$.

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Then $(4,5) + x \cdot [1,2] \subseteq [6,10]$

$(4,5) + [2.1, 2.4) \cdot [1,2] \subseteq [6,10]$

$(4,5) + [2.1, 4.8] \subseteq [6,10]$

$(6.1, 9.8) \subseteq [6,10]$, which is true.

In conclusion, this inclusion equation has one maximal solution and infinitely many particular solutions which are actually included into the maximal solution.

In order to deal with inclusion only, in this problem, since it is an *inclusion* equation, we take as solutions only the subsets of the maximal solution, since:

subset ⊆ *maximal_solution*,

not the single numbers, since:

number ∈ *maximal_solution* (not ⊆); this should better be adjusted as

[*number*, *number*] ⊆ *maximal_solution*, for example $[2.1, 2.1] \subseteq [2, 2.5]$.

4.7. Example of Inclusion Equation which has No Solution
Solve for x:

$(1,2) - 2x \subseteq (0, 0.5)$.

Hence:

$(1-2x, 2-2x) \subseteq (0, 0.5)$,

whence $0 \leq 1-2x \leq 0.5$ and $0 \leq 2-2x \leq 0.5$,

or $-1 \leq -2x \leq -0.5$ and $-2 \leq -2x \leq -1.5$,

or $0.25 \leq x \leq 0.50$ and $0.75 \leq x \leq 1$.

But $[0.25, 0.50] \cap [0.75, 1] = \emptyset$, therefore there is no solution x.

4.8. Inclusion Equation which has only One Solution
Solve for x:

$(1,2) - 2x \subseteq (0,1)$.

Hence: $(1-2x, 2-2x) \subseteq (0,1)$,

whence $0 \leq 1-2x \leq 1$, and $0 \leq 2-2x \leq 1$,

or $-1 \leq -2x \leq 0$, and $-2 \leq -2x \leq -1$,

or $0 \leq x \leq 0.5$, and $0.5 \leq x \leq 1$.

From $[0, 0.5] \cap [0.5, 1] = 0.5$, one gets that the only solution is $x = 0.5$ = [0.5, 0.5].

Let's check the inclusion solution:

$(1,2) - 2x \subseteq (0,1)$

$(1,2) - 2 \cdot 0.5 \subseteq (0,1)$

$(1,2) - 1 \subseteq (0,1)$

$(1-1, 2-1) \subseteq (0,1)$

$(0,1) \subseteq (0,1)$, which is true.

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5. First Method of Operating with Real Neutrosophic Numbers used in Neutrosophic Statistics

5.1. The true value v is a single number

A Neutrosophic Real Number has the form $N = a + bI$, where a and b are real numbers, while "I" is a real set. The determinate part of N is "a" and indeterminate (unclear) part of N is "bI".

Let's consider the real true value being the single value number v, that we are looking for in statistical problems where indeterminate, unclear, partially unknown data occur, where this single number v belongs to the real set I, or $v \in I$.

From the fact that the single true value v is in I, it does not result that v is in $a + bI = N$ as well,

but: $a + bv \in a + bI$.

Let $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ be two real neutrosophic numbers, where $a_1, b_1, a_2, b_2 \in R$ and I is a subset (not necessarily interval) of real numbers.

Let the true value, we are looking for in statistics, under indeterminate (unclear, vague) data, be $v \in I$. Then:

$$a_1 + b_1 v \in a_1 + b_1 I = N_1$$
$$a_2 + b_2 v \in a_2 + b_2 I = N_2$$

The previous Theorems 1 and 2 allow us to do straightforward operations with real neutrosophic numbers.

Addition of Real Neutrosophic Numbers

$$N_1 + N_2 = (a_1 + a_2) + (b_1 + b_2) I$$

Proof:

According to Theorem 1, since

$$a_1 + b_1 v \in a_1 + b_1 I$$

$$a_2 + b_2 v \in a_2 + b_2 I$$

we add, the left-hand sides, then the right-hand sides, referred to the appurtenance symbol ($\in$), and we get:

$$(a_1 + b_1 v) + (a_2 + b_2 v) \in (a_1 + b_1 I) + (a_2 + b_2 I) = N_1 + N_2$$

Therefore:

$$(a_1 + a_2) + (b_1 + b_2) v \in N_1 + N_2 .$$

By similar proofs we can do the next operations with real neutrosophic numbers.

Subtraction of Real Neutrosophic Numbers

$$N_1 - N_2 = (a_1 - a_2) + (b_1 - b_2) I$$

Scalar Multiplication of Real Neutrosophic Numbers

Let $\beta \neq 0$ be a real scalar. Then:

$$\beta \cdot N_1 = \beta \cdot (a_1 + b_1 I) = \beta \cdot a_1 + \beta \cdot b_1 I$$

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Multiplication of Real Neutrosophic Numbers

$$N_1 \cdot N_2 = (a_1 + b_1 I) \cdot (a_2 + b_2 I) = a_1 a_2 + (a_1 b_2 + a_2 b_1) I + b_1 b_2 I^2$$

Square of Real Neutrosophic Numbers

$$N^2 = (a + bI)^2 = a^2 + 2abI + b^2 I^2$$

n-Power of Real Neutrosophic Numbers

$N^n = (a+bI)^n = \sum_{k=0}^{n} C_n^k (a^{n-k} b^k I^k)$, for integer $n \geq 1$, where by definition $I^0 \overset{def}{=} [1,1] \equiv 1$.

n-Root of Real Neutrosophic Numbers

$$\sqrt[n]{N} = \sqrt[n]{a + bI}$$

Division

$$\frac{N_1}{N_2} = \frac{a_1 + b_1 I}{a_2 + b_2 I}$$

Remark

The above operations become easier when the indeterminacy *I = (0, 1), or (0, 1], [0, 1), [0, 1]* that are mostly used in applications, because $I^n = I$ for any real number *n > 0*.

5.2. The true value V is a set

Let's consider a set of true values *V*, that we are looking for in statistical problems where indeterminate, unclear, partially unknown data occur, where *V* is included in *I*, or $V \subset I$ (or $V \subseteq I$).
From the fact that the set of true values *V* is in *I*, it does not result that *V* is included in $a + bI = N$ as well, but: $a + bV \subset a + bI$.
Let $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ be two real neutrosophic numbers, where $a_1, b_1, a_2, b_2 \in R$ and *I* is a subset (not necessarily interval) of real numbers.
Let the set of true values, we are looking for in statistics, under indeterminate (unclear, vague) data, be $V \subset I$. Then:

$$a_1 + b_1 V \subset a_1 + b_1 I = N_1$$
$$a_2 + b_2 V \subset a_2 + b_2 I = N_2$$

Whence the **addition**:

$$(a_1 + b_1 V) + (a_2 + b_2 V) \subset (a_1 + b_1 I) + (a_2 + b_2 I) = N_1 + N_2 .$$

Similarly, for **subtraction**, **scalar multiplication**, **multiplication**, **division**, **nth-power**, **nth-root**, etc.

$$(a_1 + b_1 V) - (a_2 + b_2 V) \subset (a_1 + b_1 I) - (a_2 + b_2 I) = N_1 - N_2 .$$

$$\beta \cdot (a_1 + b_1 V) \subset \beta \cdot (a_1 + b_1 I) = \beta \cdot N_1 .$$

$$(a_1 + b_1 V) \cdot (a_2 + b_2 V) \subset (a_1 + b_1 I) \cdot (a_2 + b_2 I) = a_1 a_2 + (a_1 b_2 + a_2 b_1) I + b_1 b_2 I^2 =$$

$$= N_1 \cdot N_2 .$$

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$$\frac{a_1+b_1V}{a_2+b_2V}\subset\frac{N_1}{N_2}.$$

$$(a+bV)^n\subset N^n.$$

$$\sqrt[n]{a+bV}\subset\sqrt[n]{N}\text{ , etc.}$$

The previous Theorems 1, 2, 3, and 4 allow us to do straightforward operations with real neutrosophic numbers (for both cases: a single true value *v,* or a set of true values *V*).

6. Second Method of Operating with Neutrosophic Numbers

This method is to transform each real neutrosophic number into a real set:

$$N=a+bI=\{a+b\cdot x,x\in I\}$$

And do operations using sets as below.

In this case it is not necessarily to have the same indeterminate real set "I".

Firstly, we need to recall the operations with real sets.

6.1. Operations with Sets

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ the set of complex numbers, and $\mathcal{M}$ the set of other types of numbers.

Le A and B be two real or complex, or other type of number sets.

One or both may also be a scalar, because a scalar $\alpha\in\mathbb{R}$ may be written as a set, $[\alpha,\alpha]$.

Then $A\star B=\{a\star b,\text{where } a\star b \text{ is well defined}; a\in A, b\in B\}$, where $\star$ means any operations: addition, subtraction, scalar multiplication, multiplication, division, power, radical (root).

Afterwards, one computes *min/inf* and *max/sup* of $A\star B$.

In the next sections we are referring only to the sets of real numbers, since they are needed in *Neutrosophic Statistics*, but for the other types of sets the research is similar.

Addition of Sets

$A+B=\{a+b;a\in A,b\in B\}$

Examples:

$A=(2,3)$, $B=(0,1)$

$A+B=(2,3)+(0,1)=(2+0,3+1)$

$A+A=(2,3)+(2,3)=(2+2,3+3)=(4,6)=2\cdot(2,3)=2A$

The last one is similar to the addition of a number to itself, for example:

$5+5=2\cdot 5.$

Subtraction of Sets

$A-B=\{a-b;a\in A,b\in B\}$

Examples:

$A=(2,3)$, $B=(0,1)$

$A-B=(2,3)-(0,1)=(2-1,3-0)=(1,3)$

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$A - A = (2,3) - (2,3) = (2-3, 3-2) = (-1,1)$

Therefore:

$A - A \neq 0$ (zero) and $A - A \neq \emptyset$ (empty set),

contrarily to the subtraction of a number from itself (for example, 5 - 5 = 0).

Scalar Multiplication of Sets

Let the scalar $\beta \in R$, then:

$$\beta \cdot A = \{\beta \cdot a; a \in A\}.$$

Examples:

i). $\beta = 6$, $A = (2, 3)$, then: $\beta \cdot A = 6 \cdot (2,3) = (6 \cdot 2, 6 \cdot 3) = (12,18)$.

ii). $\beta = 0$, $A = (2, 3)$, then: $\beta \cdot A = 0 \cdot (2,3) = (0 \cdot 2, 0 \cdot 3) = (0,0) = \emptyset$ (empty set).

iii) $\beta = 0$, $B = [2, 3]$, then: $\beta \cdot B = 0 \cdot [2,3] = [0 \cdot 2, 0 \cdot 3] = [0,0] = \{0\}$ (a set that has only one element, 0).

Multiplication of Sets

$A \cdot B = \{a \cdot b; a \in A, b \in B\}$

Examples:

$A = (2,3)$, $B = (0,1)$

$A \cdot B = (2,3) \cdot (0,1) = (2 \cdot 0, 3 \cdot 1) = (0,3)$

$A \cdot A = (2,3) \cdot (2,3) = (2 \cdot 2, 3 \cdot 3) = (4,9) = A^2$.

Division of Sets

$A \div B = \frac{A}{B} = \{a \div b; a \div b \text{ is well defined}, a \in A, b \in B\}$

Examples:

$A = (2,3)$, $B = (0,1)$

i). For (A, B) intervals, one has $A \div B = \left(\frac{minA}{maxB}, \frac{maxA}{minB}\right) = \left(\frac{2}{1}, \frac{3}{0}\right) \to (2, +\infty)$ since the undefined $\frac{3}{0} \to +\infty$ (not $-\infty$, because B has only positive elements).

ii). Let $A = (2,3)$, and $C = (-1,0)$ be two intervals. Then:

$A \div C = \left(\frac{minA}{maxC}, \frac{maxA}{minC}\right) = \left(\frac{2}{0}, \frac{3}{-1}\right) = \left(\frac{2}{0}, -3\right) \to (-\infty, -3)$,

We take $\frac{2}{0}$ as $-\infty$, because the set C contains only negative elements.

iii). $A \div A = \left(\frac{minA}{maxA}, \frac{maxA}{minA}\right) = \left(\frac{2}{3}, \frac{3}{2}\right)$.

Therefore $A \div A \neq 1$, contrarily to the division of real numbers, where a non-zero number divided by itself is equal to 1, for example: $\frac{5}{5} = 1$.

iv). $B \div B = \left(\frac{minB}{maxB}, \frac{maxB}{minB}\right) = \left(\frac{0}{1}, \frac{1}{0}\right) \to (0, +\infty) \neq 1$

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Power and Root of Sets

Let r be a rational number, i.e. $r = \frac{m}{n}$, where m, n are integers, $n \neq 0$.

$A^r = \{a^r;\text{where } a^r \text{ is well defined}, a \in A\}$.

i). Positive integer power

$A = (2, 3),\ r = 4$

$A^4 = (2,3)^4 = (2^4, 3^4) = (16, 81)$.

Let $E = (-2, 3)$.

$E^2 = (-2,3) \cdot (-2,3) = (-2 \cdot 3, 3 \cdot 3) = (-6, 9) \neq ((-2)^2, 3^2) = (4, 9)$.

ii). Power zero

$A = (2, 3),\ r = 0$

$A^0 = (2,3)^0 = (2^0, 3^0) = (1, 1) = \emptyset$ (empty set).

Therefore, $A^0 \neq 1$, contrarily to the real numbers, where for example $7^0 = 1$.

Let $D = [2, 3]$, then $D^0 = [2^0, 3^0] = [1, 1] \equiv \{1\}$, as for real numbers, where for example $7^0 = 1$.

iii). Square Root

$\sqrt{A} = \sqrt{(2,3)} = (\sqrt{2}, \sqrt{3})$.

iv). Partial Square Root

Let the set $D = (-2, 3)$, then:

$\sqrt{D} = [\sqrt{0}, \sqrt{3}) = [0, \sqrt{3})$,

since in the set of real numbers one cannot compute square root of the negative numbers from the interval $(-2, 0)$.

We have only computed a partial square root of D.

v). Negative Power

$A = (2, 3), r = -2$.

$A^{-2} = (2,3)^{-2} = (2^{-2}, 3^{-2}) = \left(\frac{1}{4}, \frac{1}{9}\right) \equiv \left(\frac{1}{9}, \frac{1}{4}\right)$.

Now, the operations with the Real Neutrosophic Numbers follow the rules of operations with real sets presented above, because a Real Neutrosophic Number is equivalent to a real subset:

Let I be a real subset, $I \subset \mathbb{R}$.

$N = a + bI = \{a + b \cdot x, \text{where } x \in I\}$,

which is a real subset of the form as of I.

Actually N is the enlarged subset I.

If I is an interval of the form

$I = (c, d)$, or $[c, d)$, or $(c, d]$, or $[c, d]$,

then N will also be an interval of the same corresponding open/closed form.

If $I = \{c_1, c_2, \dots, c_n\}$ is a real discrete subset, of cardinal n, $1 \leq n \leq \infty$, then N will also be a real discrete subset of cardinal n.

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It is a union of several subsets, $I = I_1 \cup I_2 \cup \ldots \cup I_m$, then N will also be a union of corresponding subsets:

$N = N_1 \cup N_2 \cup \ldots \cup N_m = \cup_{k=1}^{m} N_k$,

where

$N_k = a + b \cdot I_k = \{a + bx, \text{where } x \in I_k\}$.

Second Method of Operations with Real Neutrosophic Numbers is the following.
Transform each real neutrosophic number into an equivalent real subset, especially when the indeterminacy (I) are not the same.

Examples:

$N_1 = 1 + 2I_1$, where $I_1 = \{02., 0.5, 0.8\}$

$N_2 = 3 - I_2$, where $I_2 = [0, 1)$

Then:

$N_1 = 1 + 2 \cdot \{0.2, 0.5, 0.8\} = 1 + \{0.4, 1.0, 1.6\} = \{1.4, 2.0, 2.6\}$

and

$N_2 = 3 - [0, 1) = (3 - 1, 3 - 0] = (2, 3]$

Addition of Real Neutrosophic Numbers

$$\begin{aligned} N_1 + N_2 = \{1.4, 2.0, 2.6\} + (2, 3] &= \{1.4 + (2, 3]\} \cup \{2.0 + (2, 3]\} \cup \{2.6 + (2, 3]\} \\ &= (1.4 + 2, 1.4 + 3] \cup (2.0 + 2, 2.0 + 3] \cup (2.6 + 2, 2.6 + 3] \\ &= (3.4, 4.4] \cup (4, 5] \cup (4.6, 5.6] = (3.4, 5.6]. \end{aligned}$$

Addition of a Scalar with a Neutrosophic Set

$0.9 + N_1 = \{1.4, 2.0, 2.6\} + 0.9 = \{1.4 + 0.9,\ 2.0 + 0.9,\ 2.6 + 0.9\}$

$= \{2.3, 2.9, 3.5\}$.

$0.9 + N_2 = 0.9 + (2, 3] = (0.9 + 2, 0.9 + 3] = (2.9, 3.9]$.

Subtraction of Real Neutrosophic Numbers

$$\begin{aligned} N_1 - N_2 = \{1.4, 2.0, 2.6\} - (2, 3] &= \{1.4 - (2, 3]\} \cup \{2.0 - (2, 3]\} \cup \{2.6 - (2, 3]\} \\ &= [1.4 - 3, 1.4 - 2) \cup [2.0 - 3, 2.0 - 2) \cup [2.6 - 3, 2.6 - 2) \\ &= [-1.6, -0.6) \cup [-1, 0) \cup [-0.4, 0.6) = [-1.6, 0.6). \end{aligned}$$

Multiplication of Real Neutrosophic Numbers

$$\begin{aligned} N_1 \cdot N_2 = \{1.4, 2.0, 2.6\} \cdot (2, 3] &= \{1.4 \cdot (2, 3]\} \cup \{2.0 \cdot (2, 3]\} \cup \{2.6 \cdot (2, 3]\} \\ &= (1.4 \cdot 2, 1.4 \cdot 3] \cup (2.0 \cdot 2, 2.0 \cdot 3] \cup (2.6 \cdot 2, 2.6 \cdot 3] = (2.8, 4.2] \cup (4, 6] \cup (5.2, 7.8] \\ &= (2.8, 7.8]. \end{aligned}$$

Multiplication of a Scalar with a Neutrosophic Number

$$4 \cdot N_1 = 4 \cdot \{1.4, 2.0, 2.6\} = \{4 \cdot (1.4), 4 \cdot (2.0), 4 \cdot (2.6)\}$$

$= \{5.6, 8.0, 10.4\}$.

Division of Real Neutrosophic Numbers

$$\begin{aligned} \frac{N_1}{N_2} = \frac{\{1.4, 2.0, 2.6\}}{(2, 3]} = \frac{1.4}{(2, 3]} \cup \frac{2.0}{(2, 3]} \cup \frac{2.6}{(2, 3]} &= \left(\frac{1.4}{3}, \frac{1.4}{2}\right] \cup \left(\frac{2.0}{3}, \frac{2.0}{2}\right] \cup \left(\frac{2.6}{3}, \frac{2.6}{2}\right] \\ &= (0.4\bar{6}, 0.7] \cup (0.\bar{6}, 1] \cup (0.8\bar{6}, 1.3] = (0.4\bar{6}, 1.3]. \end{aligned}$$

Another Example of Division of Neutrosophic Numbers

Let $N_1 = 2 - 3I_1$ where $I_1 = [4, 5]$, and $N_2 = 1 + 4I_2$ where $I_2 = \{-1, 3, 5\}$.

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Then:

$$N_1 = 2 - 3I_1 = 2 - 3\cdot[4,5] = 2 - [3\cdot 4, 3\cdot 5] = 2 - [12,15] =$$
$$= [2-15, 2-12] = [-13,-10]$$

$$N_2 = 1 + 4I_2 = 1 + 4\cdot\{-1,3,5\} = 1 + \{4\cdot(-1), 4\cdot 3, 4\cdot 5\} =$$
$$= 1 + \{-4,12,20\} = \{1+(-4), 1+12, 1+20\} = \{-3,13,21\}$$

$$\frac{N_1}{N_2} = \frac{[-13,-10]}{\{-3,13,21\}} = [\frac{-13}{-3}, \frac{-10}{-3}] \cup [\frac{-13}{13}, \frac{-10}{13}] \cup [\frac{-13}{21}, \frac{-10}{21}] =$$
$$= [\frac{10}{3}, \frac{13}{3}] \cup [\frac{-13}{13}, \frac{-10}{13}] \cup [\frac{-13}{21}, \frac{-10}{21}] =$$
$$= [\frac{-13}{13}, \frac{-10}{13}] \cup [\frac{-13}{21}, \frac{-10}{21}] \cup [\frac{10}{3}, \frac{13}{3}]$$

Division between a Real Neutrosophic Number and a Scalar

$$\frac{N_1}{4} = \frac{\{1.4,2.0,2.6\}}{4} = \left\{\frac{1.4}{4}, \frac{2.0}{4}, \frac{2.6}{4}\right\} = \{0.35, 0.50, 0.65\}.$$

$$\frac{4}{N_1} = \frac{4}{\{1.4,2.0,2.6\}} = \left\{\frac{4}{1.4}, \frac{4}{2.0}, \frac{4}{2.6}\right\} = \{2.857, 2.000, 1.538\}.$$

$$\frac{4}{N_2} = \frac{4}{(2,3]} = \left[\frac{4}{3}, \frac{4}{2}\right) \approx [1.333, 2.000)$$

$$\frac{N_2}{4} = \frac{(2,3]}{4} = \left[\frac{2}{4}, \frac{3}{4}\right) = [0.50, 0.75).$$

Power of Real Neutrosophic Numbers

$$N_1^{N_2} = \{1.4, 2.0, 2.6\}^{(2,3]} = 1.4^{(2,3]} \cup 2.0^{(2,3]} \cup 2.6^{(2,3]} = (1.4^2, 1.4^3] \cup (2.0^2, 2.0^3] \cup (2.6^2, 2.6^3]$$
$$= (1.960, 2.744] \cup (4, 8] \cup (6.760, 17.576] = (1.960, 2.744] \cup (4.000, 17.576].$$

*

$$N_2^{N_1} = (2,3]^{\{1.4,2.0,2.6\}} = \{(2^{1.4}, 3^{1.4}], (2^{2.0}, 3^{2.0}], (2^{2.6}, 3^{2.6}]\}$$
$$\approx \{(2.639, 4.656], (4.0, 9.0], (6.063, 17.399]\}$$
$$\equiv \{2.639, 4.656] \cup (4.0, 9.0] \cup (6.063, 17.399]$$
$$= (2.639, 17.399].$$

Power of a Neutrosophic Real Numbers to a Scalar

$(N_1)^4 = \{1.4, 2.0, 2.6\}^4 = \{1.4^4, 2.0^4, 2.6^4\} = \{3.8416, 16.000, 45.6976\}$;

and $4^{N_1} = 4^{\{1.4,2.0,2.6\}} = \{4^{1.4}, 4^{2.0}, 4^{2.6}\} \approx \{6.9644, 16.000, 36.7583\}$.

Real Root of a Neutrosophic Real Number

$$\sqrt{N_1} = \sqrt{\{1.4, 2.0, 2.6\}} = \{\sqrt{1.4}, \sqrt{2.0}, \sqrt{2.6}\} \approx \{1.183, 1.414, 1.612\}$$

$$\sqrt[3]{N_2} = \sqrt[3]{(2,3]} = (\sqrt[3]{2}, \sqrt[3]{3}) \simeq (1.260, 1.442].$$

Real Neutrosophic Root of a Neutrosophic Real Number

$$\sqrt[N_1]{N_2} = N_2^{\frac{1}{N_1}} = (2,3]^{\frac{1}{\{1.4,2.0,2.6\}}} = \left\{(2,3]^{\frac{1}{1.4}}, (2,3]^{\frac{1}{2.0}}, (2,3]^{\frac{1}{2.6}}\right\} = \left\{\left(2^{\frac{1}{1.4}}, 3^{\frac{1}{1.4}}\right], \left(2^{\frac{1}{2.0}}, 3^{\frac{1}{2.0}}\right], (2^{\frac{1}{2.6}}, 3^{\frac{1}{2.6}}]\right\}$$
$$\approx \{(1.641, 2.192], (1.414, 1.732], (1.306, 1.526]\} \equiv (1.306, 2.192]$$

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$$\sqrt[N_2]{N_1} = N_1^{\frac{1}{N_2}} = \{1.4, 2.0, 2.6\}^{\frac{1}{(2,3]}} = \{1.4, 2.0, 2.6\}^{[\frac{1}{3},\frac{1}{2})} = \left\{1.4^{[\frac{1}{3},\frac{1}{2})}, 2.0^{[\frac{1}{3},\frac{1}{2})}, 2.6^{[\frac{1}{3},\frac{1}{2})}\right\}$$
$$= \left\{\left[1.4^{\frac{1}{3}}, 1.4^{\frac{1}{2}}\right), \left[2.0^{\frac{1}{3}}, 2.0^{\frac{1}{2}}\right), \left[2.6^{\frac{1}{3}}, 2.6^{\frac{1}{2}}\right)\right\} \approx \{[1.119, 1.183), [1.260, 1.414), [1.375, 1.612)\}$$
$$\equiv [1.119, 1.183) \cup [1.260, 1.612).$$

7. Literal Neutrosophic Numbers

The Literal Neutrosophic Numbers (LNN) have the form:

$LNN = a + bI$, where a, b are real or complex numbers, and I = literal indeterminacy, where $I^2 = I$, and I/I = undefined.

Their Addition, Subtraction, Scalar Multiplication, Multiplication, Division, Power, Radical are straightforward. The Literal Neutrosophic Numbers are not used in Neutrosophic Statistics, but in Neutrosophic Algebraic Structures, that's why we do not present their operations herein.

8. NonAppurtenance Equation, NonInclusion Equation, and NonEquality Equation

They are complementarians of the Appurtenance Equation, Inclusion Equation, and Equality Equation respectively.

We present them as a curiosity, or as recreational mathematics.

(i) The Appurtenance Equation from previous Example 1 was:

$4 - 5x \in 1 + 2 \cdot (0.5, 0.8)$ whose solutions are all real numbers $x \in (0.28, 0.40)$.

Its corresponding **NonAppurtenance Equation** is:

$4 - 5x \notin 1 + 2 \cdot (0.5, 0.8)$ whose solutions are all real numbers $x \notin (0.28, 0.40)$, or all real numbers $x \in R - (0.28, 0.40)$.

(ii) The Inclusion Equation from previous Example 2 was:

$1 + x \cdot (1, 2) \subset (0, 5)$, whose maximal solution is $x = (-0.5, 2)$.

Its corresponding **NonInclusion Equation** is:

$1 + x \cdot (1, 2) \not\subset (0, 5)$, whose maximum solution is R - (- 0.5, 2).

(iii) An elementary Equality Equation

3x + 4 = 7, has the unique solution x = 1.

Its corresponding **NonEquality Equation** is:

$3x + 4 \neq 7$ has, of course, infinitely many solutions $x \in R - \{1\}$

9. Conclusion

In neutrosophic statistics, from the fact that the single true value v is in I, it does not result that v is in $a + bI = N$ as well, but: $a + bv \in a + bI$. That's why the appurtenance relationship and equation must be introduced and studied.

Even more, if one has a set of true values, from the fact that the set of true values V is included in I, it does not mean that V is included in $a + bI$ too, but $a + bV \subset a + bI$ (or $a + bV \subseteq a + bI$). That's why the inclusion relationship and equation must be introduced.

In the same way as the "=" symbol is used for an equality relationship or an equality equation, we use the symbol "∈" {belong(s) to} for an *appurtenance relationship* or *appurtenance equation* of a number to a set, respectively the symbol $\subset$ (or $\subseteq$) {included in, or included in or equal to} for an *inclusion relationship* or *inclusion equation*.

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We just introduced for the first time the Appurtenance Equation and Inclusion Equation, which help in understanding the operations with neutrosophic numbers within the frame of neutrosophic statistics. The way of solving them resembles the equations whose coefficients are sets (no single numbers).

In addition, we also presented their complementary NonAppurtenance Equation, NonInclusion Equation, and the elementary NonEquality Equation respectively.

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