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Introducing Kolmogorov−Smirnov Tests under Uncertainty: An Application to Radioactive Data

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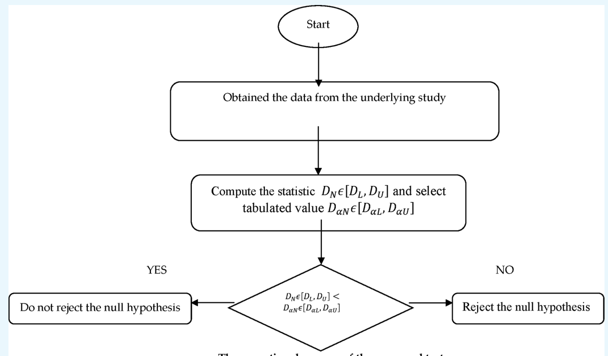
ABSTRACT: The Kolmogorov−Smirnov (K−S) tests based on the assumptions of determined observations in the sample have been popularly applied for the analysis of the data. The existing K−S tests for one sample and two samples cannot be applied when the data contains neutrosophic observations measured from the complex system or under uncertainty. In this paper, we propose the generalization of the existing K−S tests under the neutrosophic statistics. The proposed tests are known as neutrosophic Kolmogorov−Smirnov (NK−S) tests. We present the necessary measures and procedures to perform the proposed tests. An example and advantages of the proposed NK−S tests are given in the paper.



The operational process of the proposed test

1. INTRODUCTION

The statistical methods/techniques have been commonly used in all fields for the analysis of the data, estimation, and forecasting purposes. The data obtained from the system always follows some statistical distribution, which is unknown in advance. Usually, it is assumed that the data follows the normal distribution. However, in practice, it is not always necessary that the data in hand follows the normal distribution. Therefore, statisticians designed several tests to test some hypotheses about the distribution of the data under investigation; see ref 1. As mentioned by Massey,[1] "Attempts have been made to find test statistics whose sampling distribution does not depend upon either the explicit form of or the value of certain parameters in, the distribution of the population. Such tests have been called non-parametric or distribution-free tests." The Kolmogorov−Smirnov (K−S) test is an alternative non-parametric test, which uses the cumulative distribution to decide about the specific distribution of the data. The K−S test is found to be efficient for goodness of fit purposes. Many authors worked on the K−S test; see, for example, refs 1−8.

The K−S test under classical statistics is applied when all observations in the data are determined, precise, and sure. However, in real situations, it may happen that the data cannot be represented by statistical terms or the data may be in an interval or imprecise data. For example, the ecology data, soil data, ocean data, and censored data may be fuzzy data rather than exact data. Therefore, several authors developed the K−S test for the analysis of fuzzy data; see, for example, refs 9−16.

The fuzzy logic is a special case of neutrosophic logic. The neutrosophic logic is considered the measure of indeterminacy in addition to the fuzzy logic; see ref 17. More applications of the neutrosophic logic can be seen in refs 18−23. The neutrosophic statistics was developed by Smarandache[24] using the neutrosophic logic. The neutrosophic statistics is an extension of classical statistics, which considers the measure of indeterminacy. The neutrosophic statistics is applied when the observations in the data are neutrosophic numbers. Chen et al.[25,26] discussed the advantages of methods based on neutrosophic numbers. Previous work[27,28] introduced several basic concepts for the neutrosophic statistics. Recently, another previous work[29] proposed the neutrosophic ANOVA test.

The existing K−S test under classical statistics and a fuzzy approach cannot be applied when the measure of indeterminacy is needed. By exploring the literature on classical statistics and the fuzzy approach, we did not find any work on the K−S test under the neutrosophic statistics. In this paper, we propose neutrosophic Kolmogorov−Smirnov (NK−S) tests for a single sample and two samples. It is expected that the proposed NK−S tests will effectively analyze the imprecise, vague, and uncertain data compared to the existing K−S test under classical statistics.

2. RESULTS

For a radioactive source model, radioactive engineering is interested in testing the assumption that the count rate per second follows a neutrosophic Poisson distribution. The count rate per second has the neutrosophic mean [5,7] counts per second. To test the assumption, radioactive engineering collected a large number of count data. The neutrosophic Poisson distribution from ref 18 is given by

$$F_{0N}(x_n) = \frac{\exp(-\lambda_N)\lambda_N^{x_n}}{x_n!};\; F_{0N}(x_{nN})\epsilon[F_{0L}(x_{nL}),\, F_{0N}(x_{nU})]$$

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Table 1. Necessary Computations for the NK−S Test

no.	X_{iN}	$\mathrm{Cu}(X_{iN})$	$S_{nN}(x_{nN})$	$F_{0N}(x_n)$	D_N
1	[1,4]	[1,4]	[0.0128,0.0435]	[0.0404,0.1730]	[0.0276,0.1295]
2	[1,4]	[2,8]	[0.0256,0.0870]	[0.0404,0.1730]	[0.0148,0.0860]
3	[3,5]	[5,13]	[0.0641,0.1413]	[0.2650,0.3007]	[0.2009,0.1594]
4	[3,5]	[8,18]	[0.1026,0.1957]	[0.2650,0.3007]	[0.1625,0.1051]
5	[4,5]	[12,23]	[0.1538,0.2500]	[0.4405,0.3007]	[0.2866,0.0507]
6	[5,6]	[17,29]	[0.2179,0.3152]	[0.6160,0.4497]	[0.3980,0.1345]
7	[5,6]	[22,35]	[0.2821,0.3804]	[0.6160,0.4497]	[0.3339,0.0693]
8	[6,6]	[28,41]	[0.3590,0.4457]	[0.7622,0.4497]	[0.4032,0.0041]
9	[6,6]	[34,47]	[0.4359,0.5109]	[0.7622,0.4497]	[0.3263,0.0612]
10	[6,7]	[40,54]	[0.5128,0.5870]	[0.7622,0.5987]	[0.2494,0.0118]
11	[8,8]	[48,62]	[0.6154,0.6739]	[0.9319,0.7291]	[0.3165,0.0552]
12	[8,9]	[56,71]	[0.7179,0.7717]	[0.9319,0.8305]	[0.2141,0.0588]
13	[10,9]	[66,80]	[0.8462,0.8696]	[0.9863,0.8305]	[0.1402,0.0391]
14	[12,12]	[78,92]	[1.0000,1.0000]	[0.9980,0.9730]	[0.0020,0.0270]

and $S_{nN}(x_{nN}) = \frac{\mathrm{Cu}(X_{iN})}{[78,\ 92]}$ where $\mathrm{Cu}(X_{iN})$ is the neutrosophic commutative values, $\lambda_N \epsilon [5,7]$ and $n_N \epsilon [78,92]$.

The neutrosophic count data and neutrosophic statistics are shown in Table 1. Suppose the level of significance for this test is 0.01. The critical neutrosophic value from ref 32 is $D_{0.01,14} = [1.63/\sqrt{78}, 1.63/\sqrt{92}] = [0.1845,0.1699]$. According to eq 4, the statistic in the indeterminacy interval can be written as 0.1594 + 0.4032I; $I_N \epsilon [0,0.6046]$. The neutrosophic statistic from Table 1 is $D_N \epsilon [0.4032,0.1594]$. Note here that the lower value of the indeterminacy interval denotes the determined part. By comparing the values of D_N with $D_{0.01,14}$, we note that the determinate part follows the Poisson distribution, but the indeterminate part of the data does not follow the Poisson distribution.

3. DISCUSSION

In this section, we compare the performance of the proposed NK−S test over the K−S test under classical statistics. According to refs 25 and 26, a method that provides the results in the indeterminacy interval when the data have the neutrosophic numbers is said to be more adequate and effective than the method that provides the results in the determined form. To compare the proposed NK−S test with the existing NK test, we will use the same data that are given in Table 1. Note here that the data given in Table 1 reduces to the determined part under classical statistics if no observations of uncertainty are recorded. For example, for sample 1, the first value, which is 1, represents the indeterminate part of the indeterminacy interval. The second value of this sample represents the determinate part of the interval. From Table 1, we note that the proposed test provides the results in the indeterminacy interval rather than the determined values. Using eq 4, the values of the statistic in the indeterminacy form can be written as 0.1594 + 0.4032I; $I_N \epsilon [0,0.6046]$. Note here that the proposed test provides a good measure of indeterminacy. At a level of significance 0.01, the probability that the null hypothesis will be accepted is 0.99, the probability of rejecting the null hypothesis when it is true is 0.01, and the probability of indeterminacy is 0.60. For example, in the statistic $D_N \epsilon [0.4032,0.1594]$, the value $D_L = 0.1594$ presents the determined part under the classical statistics, and the value $D_U = 0.4032$ shows the indeterminate part under the uncertainty. By comparing both tests, we note that $D_L < 0.1845$, which shows that the existing NK test indicates that the sample belongs to the Poisson distribution. However, the indeterminate part shows that under uncertainty, the sample does not come from the Poisson distribution. From this comparison, we conclude that the values of the statistic D_N can be from 0.1594 to 0.4032 under uncertainty. Hence, the theory of the proposed NK−S test concurs with the theories of refs 25 and 26.

4. CONCLUDING REMARKS

In this paper, we presented the modifications of the Kolmogorov−Smirnov (K−S) test under the neutrosophic statistics. We proposed the neutrosophic Kolmogorov−Smirnov (NK−S) tests, which are the generalization of the K−S tests. The proposed NK−S test under the neutrosophic statistical interval method is more adequate, informative, and effective to be applied when the data have neutrosophic numbers. The proposed test provides the results in the indeterminacy interval, which is desirable under uncertainty or when the data is measured from the complex system. We presented an example and found that the proposed test is better than the existing K−S test. We recommend applying the proposed NK−S tests for the analysis of the data in biomedical sciences, big data analysis, engineering, and statistics. More properties using the simulation data and/or the development of software for the analysis of the proposed NK−S tests can be considered for future research.

5. COMPUTATIONAL METHODS

Assume that $X_N = a_N + b_N I_N$ be a neutrosophic number (NN) where a_N is the determinate part and $b_N I_N$; $I_N \epsilon [I_L, I_U]$ is the indeterminate part of the NN. Let $X_N = X + I_N X$; $X_N \epsilon [X_L, X_U]$ be a random variable based on the NN where X_L and X_U are lower and upper values of the indeterminacy interval. Note here that the NN and $X_N \epsilon [X_L, X_U]$ reduce to a number and a variable under classical statistics if $I_L = 0$ or $X_L = X_U$, respectively. The neutrosophic variable $X_N \epsilon [X_L, X_U]$ presented the NNs in a sample selected from the population having imprecise, uncertain, and indeterminate values or parameters. More details about neutrosophic statistics can be seen in ref 17. The main aim is to propose the K−S tests under the neutrosophic statistics to determine the specific distribution of the data in the presence of neutrosophy.

5.1. Neutrosophic Kolmogorov−Smirnov Tests. The Kolmogorov−Smirnov (K−S) test was originally derived by Kolmogorov[30] and Smirnov[31] and has been used in non-

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parametric testing of the hypothesis. In classical statistics, the K−S test has been commonly used to test whether the sample under study belongs to a specific distribution or not. In other words, the K−S test is applied to decide whether the observed distribution significantly differs from the specified population distribution.[32] The existing K−S test is applied under the assumption that all observations/parameters in the observed sample and in the population are determined and precise. The data that came from complex systems such as the ocean, the human brain data, and power grid or under uncertainty may not have all determined observations. In these situations, the K−S test under classical statistics cannot be applied for testing whether the data belong to a specific distribution. We modify the existing K−S test under classical statistics using the neutrosophic statistics. The proposed neutrosophic Kolmogorov−Smirnov (NK−S) test is the generalization of the existing K−S test proposed by Kolmogorov[30] and Smirnov.[31] The proposed NK−S test will be applicable under the following assumptions:

1. The data consists of uncertain, imprecise, and indeterminate values.
2. The two neutrosophic samples should be mutually independent.

The K−S test can be applied independent of the cumulative distribution function. Woodruff et al.[33] used it for the Weibull distribution. Papadopolous and Qiao[34] and Frey[35] presented the K−S test for the Poisson distribution.

Suppose that X_{1N}, X_{2N}, ..., X_{nN} be a neutrosophic random sample from a neutrosophic population having a neutrosophic cumulative frequency distribution function, say $F_{0N}(x_n)$. By following ref 1, the null hypothesis that the neutrosophic sample came from the specified neutrosophic distribution is rejected if the neutrosophic cumulative frequency distribution function is not close to the specified neutrosophic distribution function. Suppose now that $F_{0N}(x_n)$; $F_{0N}(x_{nN})\epsilon[F_{0L}(x_{nL}), F_{0N}(x_{nU})]$ and $S_{nN}(x_{nN})$; $S_{nN}(x_{nN})\epsilon[S_{nL}(x_{nL}), S_{nU}(x_{nU})]$ be the neutrosophic population cumulative distribution function and the observed neutrosophic sample distribution function, respectively. Then, the neutrosophic maximum difference statistic based on $F_{0N}(x_n)$; $F_{0N}(x_{nN})\epsilon[F_{0L}(x_{nL}), F_{0N}(x_{nU})]$ and $S_{nN}(x_{nN})\epsilon[S_{nL}(x_{nL}), S_{nU}(x_{nU})]$ is given by

$$D_N = |F_{0N}(x_n) - S_{nN}(x_n)|;\ D_N\epsilon[D_L, D_U];\ F_{0N}(x_{nN})\epsilon[F_{0L}(x_{nL}), F_{0N}(x_{nU})];\ S_{nN}(x_{nN})\epsilon[S_{nL}(x_{nL}), S_{nU}(x_{nU})] \quad (1)$$

The proposed test in the indeterminacy interval can be written as

$$D_N = A_N + B_N I_N;\ I_N\epsilon[I_L, I_U];\ D_N\epsilon[D_L, D_U] \quad (2)$$

Note here that A_N and $B_N I_N$ are the determined and indeterminate parts of the test. The proposed test reduces to the tests in refs 30 and 31 if no indeterminacy is found in the data. Also, note here that the proposed NK−S test reduces to the tests in refs 30 and 31 when $D_L = D_U$. The neutrosophic null hypothesis that the sample came from the neutrosophic specified population is accepted if $D_N\epsilon[D_L, D_U] > D_{\alpha N}$ where $D_{\alpha N}\epsilon[D_{\alpha L}, D_{\alpha U}]$ is a neutrosophic critical value and can be selected from ref 32.

5.2. NK−S Test for Comparing Two Populations. Kolmogorov[30] and Smirnov[31] also extended the K−S test for comparing two populations. Like the K−S test for a single population, this test is also based on the assumption that the observations/parameters of two populations should be determined and precise. In this section, we present the NK−S test for comparing two neutrosophic populations. Let X_{1N}, X_{2N}, ..., $X_{n_{1N}}$ and Y_{1N}, Y_{2N}, ..., $Y_{n_{2N}}$ be two neutrosophic independent samples of sizes $n_{1N}\epsilon[n_{1L}, n_{1U}]$ and $n_{2N}\epsilon[n_{1L}, n_{1U}]$ from a specified population, respectively. Let $S_{n_{1N}}(x_{n_{1N}})\epsilon[S_{nL}(x_{n_{1L}}), S_{nU}(x_{n_{1U}})]$ and $S_{n_{2N}}(y_{n_{2N}})\epsilon[S_{nL}(y_{n_{2L}}), S_{nU}(y_{n_{2U}})]$ be neutrosophic sample cumulative distribution functions. Then, the neutrosophic maximum difference statistic based on $S_{n_{1N}}(x_{n_{1N}})\epsilon[S_{nL}(x_{n_{1L}}), S_{nU}(x_{n_{1U}})]$ and $S_{n_{2N}}(y_{n_{2N}})\epsilon[S_{nL}(y_{n_{2L}}), S_{nU}(y_{n_{2U}})]$ is given by

$$D_N = |S_{n_{2N}}(y_{n_{2N}}) - S_{n_{1N}}(x_{n_{1N}})|;\ D_N\epsilon[D_L, D_U];\ S_{n_{1N}}(x_{n_{1N}})\epsilon[S_{nL}(x_{n_{1L}}), S_{nU}(x_{n_{1U}})],\ S_{n_{2N}}(y_{n_{2N}})\epsilon[S_{nL}(y_{n_{2L}}), S_{nU}(y_{n_{2U}})] \quad (3)$$

The proposed test for two populations in the form of indeterminacy can be written as

$$D_N = C_N + E_N I_N;\ I_N\epsilon[I_L, I_U];\ D_N\epsilon[D_L, D_U] \quad (4)$$

Note here that C_N and $E_N I_N$ are the determined and indeterminate parts of the test. The proposed test reduces to the tests in refs 30 and 31 if no indeterminacy is found in the data. Note also here that the proposed NK−S test reduces to the tests in refs 30 and 31 when $S_{nL}(x_{n_{1L}}) = S_{nU}(x_{n_{1U}})$ and $S_{nL}(y_{n_{2L}}) = S_{nU}(y_{n_{2U}})$. The neutrosophic null hypothesis that two samples came from the same neutrosophic specified population is accepted if $D_N\epsilon[D_L, D_U] > D_{\alpha N}$ where $D_{\alpha N}\epsilon[D_{\alpha L}, D_{\alpha U}]$ is a neutrosophic critical value.

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Notes

The author declares no competing financial interest.

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