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Foundations of Neutrosophic N-Semirings Theory and Structural Properties

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Abstract: In this paper the concept of neutrosophic n-semirings $(S\cup I, *_1, *_2, *_3, \ldots, *_n, *_{n+1})$ has been introduced. The substructure of n-semirings $(S\cup I, *_1, *_2, *_3, \ldots, *_n, *_{n+1})$ has been defined and some useful results have been proved. Moreover, in order to familiarize the readers with these concepts some worthy examples have been coined. The left, right and two sided ideals of neutrosophic n-semirings have been paid a special heed. Finally we have turned our discussion towards the compatible and congruence relations and some remarkable properties have also been considered. Furthermore, we vividly have given examples to express the rationality of each notion popularized in this paper.

1. Introduction

To deal the vagueness in various real life problems in 1965, Lotfi, A. Zadeh [19] introduced the fuzzy set theory which is basically the generalization of crisp set. The concept of fuzzy sets involves the degree of membership for the objects to become the members of the set which lies between 0 and 1.In 1975, Zadeh [18] defined the the interval valued fuzzy set which is the extension of fuzzy set. Later on Atanassove [9] in 1986, gave the idea of intuitionistic fuzzy set which is the generalization of fuzzy set. In 1988, Atanassove [8] extended the idea of intuitionistic fuzzy set by defining interval valued intuitionistic fuzzy set. In 1995 F.Smarandache [6] introduced the concept of neutrosophic set which is the generalization of classic set, fuzzy set and intuitionistic fuzzy set and is the powerful tool to deal the indeterminate and inconsistent information.

After the pioneering work of Smarandache, different researchers defined different structures on the grounds of neutrosophic set and have done some useful work about these structures.

This paper is structured as under. In section 2 we have recalled the basic definitions which were needed for the next section 3. In section 3 we have given the definitions of neutrosophic n-semirings, neutrosophic sub n-semirings, left, right and two sided ideals, compatible relations, congruence relations in neutrosophic n-semirings and each concept has been helped out for understanding by several vivid examples.

Keywords: Neutrosophic n-semirings, neutrosophic sub n-semirings, ideals and congruence, relations in neutrosophic structures.

2. Preliminaries

In this section we have recalled some basic definitions and have given some relevant examples which were needed to be clear with the concepts brought in the next section.

Definition 2.1 [19]. Let X be the universe of discourse and I be the unit interval [0, 1] then a function $\theta : X \longrightarrow [0,1]$ is called fuzzy set.

Definition 2.2 [18]. Let X be the non-empty set and I be the unit interval then a function $\theta_1 : X \longrightarrow [I]$ from X into the interval numbers in unit interval is called interval valued fuzzy set in X.

Definition 2.3 [9]. Let X be the non-empty set then by intuitionistic fuzzy set θ in X we mean

$\theta(x) = \langle \alpha_\theta(x), \beta_\theta(x) \rangle$ for all $x \in X$ where $\alpha_\theta(x)$ is the degree of membership and $\beta_\theta(x)$ is the degree of non-membership of an element of X in θ with conditions $0 \leq \alpha_\theta(x) \leq 1$, $0 \leq \beta_\theta(x) \leq 1$ and $0 \leq \alpha_\theta(x) + \beta_\theta(x) \leq 1$ for all $x \in X$.

Definition 2.4 [8]. Let X be the non-empty set. An interval valued intuitionistic fuzzy set θ in X is a set having the following form `$\theta(x) = \langle \alpha_\theta(x), \beta_\theta(x) \rangle$ for all $x \in X$ where $\alpha_\theta(x) = [\alpha^-_\theta(x), \alpha^+_\theta(x)]$ and $\beta_\theta(x) = [\beta^-_\theta(x), \beta^+_\theta(x)]$ are the interval valued fuzzy sets and $\alpha_\theta(x)$ is known as the degree of membership and $\beta_\theta(x)$ is called degree of non-membership of the elements of X in θ.

Definition 2.5 [17]. Let X be the universe of discourse then a set $\Omega = \{< x, \alpha_T(x), \alpha_I(x), \alpha_F(x) > : x \in X\}$ is termed as a neutrosophic set where $\alpha_T, \alpha_I, \alpha_F : X \longrightarrow]0^-, 1^+[$ and the membership function α_T, α_I and α_F are the truth, indeterminacy and falsity membership degrees respectively.

Definition 2.6 [10]. A semiring $(S, *_1, *_2)$ is an algebraic structure in which $(S, *_1)$ and $(S, *_2)$ are semigroups such that $*_2$ is distributive over $*_1$.

Example.2.7 Let N ={1, 2, 3, ….} then under usual addition and multiplication N becomes semiring.

Definition 2.8 [16]. An algebraic structure $(S, *_1, *_2, *_3)$ is called a bisemiring if $(S, *_1)$, $(S, *_2)$ and $(S, *_3)$ are semigroups such that ∀ u, v and w $\in$ S, $u *_2 (v *_1 w) = (u *_2 v) *_1 (u *_2 w)$,$(v *_1 w) *_2 u = (v *_2 u) *_1 (w *_2 u)$,$u *_3 (v *_2 w) = (u *_3 v) *_2 (u *_3 w)$ and $(v *_2 w) *_3 u = (v *_3 u) *_2 (w *_3 u)$.

Example 2.9 Let N be the set of natural numbers. Then $(N, *_1, *_2, *_3)$ is a bisemirings where $*_1$ gives minimum and $*_2$ gives the maximum while $*_3$ is the usual addition.

Definition 2.10 [12]..An algebraic structure $(S, *_1, *_2, *_3, \dots, *_n, *_{n+1})$ is called an n-semiring if $(S, *_1, *_2)$, $(S, *_2, *_3)$,…,$(S, *_{n-1}, *_n)$, $(S, *_n, *_{n+1})$ are semirings, i.e. $(S, *_1)$, $(S, *_2)$,…,$(S, *_n)$, $(S, *_{n+1})$ are semigroups, such that for all s, w, z ∈ S

$s *_2 (w *_1 z) = (s *_2 w) *_1 (s *_2 z)$, $s *_3 (w *_2 z) = (s *_3 w) *_2 (s *_3 z), \ldots, s *_n (w *_{n-1} z) = (s *_n w) *_{n-1} (s *_n z)$, $s *_{n+1} (w *_n z) = (s *_{n+1} w) *_n (s *_{n+1} z)$.

And $(w *_1 z) *_2 s = (w *_2 s) *_1 (z *_2 s)$, $(w *_2 z) *_3 s = (w *_3 s) *_2 (z *_3 s), \ldots, (w *_{n-1} z) *_n s = (w *_n s) *_{n-1} (z *_n s)$, $(w *_n z) *_{n+1} s = (w *_{n+1} s) *_n (z *_{n+1} s)$.

Example 2.11 Let $Z_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. Let $*_1$ = min (a, b), $*_2$ = max (a, b), $*_3$ = addition (mod 5) and $*_4$ = multiplication (mod 5). Then $(Z_5, *_1, *_2, *_3, *_4)$ is a 3-semiring.

Now here, we define neutrosophic n-semirings, neutrosophic sub-n-semirings and give some examples.

3. Neutrosophic-N-Semirings

Here, we define neutrosophic n-semirings, discussed some properties and results and have given some examples.

Definition 3.1 An algebraic structure $(S \cup I, *_1, *_2, *_3, \ldots, *_n, *_{n+1})$ is called neutrosophic-n-semiring if $*_1, *_2, *_3, \ldots, *_n$, and $*_{n+1}$ are the closed and associative binary operations and $*_2$ is distributive over $*_1$ and $*_3$ is distributive over $*_2, \ldots, *_{n+1}$ is distributive over $*_n$ where S is n-semiring with respect to $*_1, *_2, *_3, \ldots, *_n, *_{n+1}$ and I is the neutrosophic element ($I = I^2$) and $< S \cup I > = \{ a + bI : a, b \in S \}$.

Example 3.2 Let $S = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ and $a *_1 b = \min(a, b)$, $a *_2 b = \max(a, b)$, $a *_3 b = r$ (addition mod 5) and $a *_4 b = r$ (multiplication mod 5) . Then S is the 3-semiring under the operations given above. Then $< S \cup I > = \{0 + 0I, 0 + 1I, 0 + 2I, 0 + 3I, 0 + 4I, 1 + 0I, 1 + 1I, 1 + 2I, 1 + 3I, 1 + 4I, 2 + 0I, 2 + 1I, 2 + 2I, 2 + 3I, 2 + 4I, 3 + 0I, 3 + 1I, 3 + 2I, 3 + 3I, 3 + 4I, 4 + 0I, 4 + 1I, 4 + 2I, 4 + 3I, 4 + 4I \}$ is the neutrosophic-3-semiring under the same operations as that of S i.e. for all $a + bI, c + dI \in S \cup I$, $(a + bI) *_1 (c + dI) = (a *_1 c)+(b *_1 d)I$, $(a + bI) *_2 (c + dI) = (a *_2 c)+(b *_2 d)I$ and $(a + bI) *_3 (c + dI) = (a *_3 c)+(b *_3 d)I$ and $(a + bI) *_4 (c + dI) = (a *_4 c)+(b *_4 d)I$.

Example 3.3 Let $N = \{1,2,3,\ldots\}$ and $a *_1 b = \min(a, b)$, $a *_2 b = \max(a, b)$, $a *_3 b = a + b$, $a *_4 b = ab$. Then $N \cup I = \{ a + bI ; a, b \in N \}$ with respect to $*_1, *_2, *_3$ and $*_4$ is the neutrosophic-3-semiring where for all $a + bI, c + dI \in N \cup I$, $(a + bI) *_1 (c + dI) = (a *_1 c) + (b *_1 d)I$, $(a + bI) *_2 (c + dI) = (a *_2 c)+(b *_2 d)I$, $(a + bI) *_3 (c + dI) = (a *_3 c)+(b *_3 d)I$ and $(a + bI) *_4 (c + dI) = (a *_4 c) + (b *_4 d)I$.

Definition 3.4 A proper subset P of a neutrosophic n-semiring $(S \cup I, *_1, *_2, *_3, \ldots, *_n, *_{n+1})$ is called neutrosophic sub-n-semiring if P itself is a neutrosophic-n-semiring under the same binary operations as that of $S \cup I$.

Example 3.5 Let $W = \{0,1,2,3,\ldots\}$ and $*_1, *_2, *_3$ and $*_4$ are the operations taken in the above Example 3.3 Then $W \cup I = \{ a + bI; a, b \in W \}$ with respect to $*_1, *_2, *_3$ and $*_4$ is the neutrosophic-3-semiring where for all $a + bI, c + dI \in W \cup I$ $(a + bI) *_1 (c + dI) = (a *_1 c)+(b *_1 d)I$, $(a + bI) *_2 (c + dI) = (a *_2 c)+(b *_2 d)I$, $(a + bI) *_3 (c + dI) = (a *_3 c)+(b *_3 d)I$ and $(a + bI) *_4 (c + dI) = (a *_4 c)+(b *_4 d)I$. The set $N \cup I = \{ a + bI ; a, b \in N\}$ is the subset of $W \cup I = \{ a + bI ; a, b \in N \}$ so, becomes the neutrosophic sub-3-semiring of $W \cup I$ under the operations of $W \cup I$.

Lemma 3.6 A subset P of neutrosophic n-semiring $(S\cup I, *_1, *_2, *_3, \ldots, *_n, *_{n+1})$ is neutrosophic sub-n-semiring if and only if it satisfies the following:
$(a + bI) *_1 (c + dI) \in P$, $(a + bI) *_2 (c + dI) \in P$ and
$(a + bI) *_3 (c + dI) \in P$,..,$(a + bI) *_n (c + dI)$ and $(a + bI) *_{n+1} (c + dI), \in P\ \forall\ (a + bI), (c + dI) \in P$.

Proof. Let $(P, *_1, *_2, *_3, \ldots, *_{n+1})$ is a neutrosophic sub-n-semiring then by closure property

$\forall\ (a + bI), (c + dI) \in P$, $(a + bI) *_1 (c + dI) \in P$, $(a + bI) *_2 (c + dI) \in P$, $(a + bI) *_3 (c + dI), \ldots,$ $(a + bI) *_n (c + dI)$ and $(a + bI) *_{n+1} (c + dI) \in P$.

Conversely, let $\forall\ (a + bI), (c + dI) \in P$,
$(a + bI) *_1 (c + dI) \in P$ $(a + bI) *_2 (c + dI) \in P$ and
$(a + bI) *_3 (c + dI) \in P, \ldots, (a + bI) *_n (c + dI)$ and $(a + bI) *_{n+1} (c + dI) \in P$.

Then closure property is satisfied obviously. And since elements of P are of neutrosophic-n-semiring $(S\cup I, *_1, *_2, *_3, \ldots, *_n, *_{n+1})$ and operations are also same as that of $S\cup I$ and since associative and distributive laws are satisfied therein so, both laws are satisfied for P with respect to $*_1, *_2, *_3, \ldots, *_n$ and $*_{n+1}$.Hence this makes $(S\cup I, *_1, *_2, *_3, \ldots, *_n, *_{n+1})$ into neutrosophic sub-n-semiring.

Proposition 3.7 The intersection of any number of neutrosophic sub-n-semirings is either empty or neutrosophic sub-n-semiring.

Proof. Let $\{B_i ; i\in J\}$ is the collection of neutrosophic sub-n-semirings of $(S\cup I, *_1, *_2, *_3, \ldots, *_n, *_{n+1})$. Suppose $\bigcap B_i \neq \phi$ then we show that the subset $\bigcap B_i$ of $S\cup I$ is the neutrosophic sub-n-semiring. Let $(a + bI)$ and $(c + dI) \in \bigcap B_i$ this implies that $(a + bI)$ and $(c + dI) \in B_i\ \forall\ i\in J$. This implies that $(a + bI) *_1 (c + dI)$, $(a + bI) *_2 (c + dI)$, $(a + bI) *_3 (c + dI), \ldots, (a + bI) *_n (c + dI)$ and $(a + bI) *_{n+1} (c + dI) \in B_i\ \forall\ i\in J$ because each B_i is neutrosophic sub-n-semiring of $(S\cup I, *_1, *_2, *_3, \ldots, *_n, *_{n+1})$. So, this implies that $(a + bI) *_1 (c + dI)$, $(a + bI) *_2 (c + dI)$, and $(a + bI) *_3 (c + dI), \ldots, (a + bI) *_n (c + dI)$ and $(a + bI) *_{n+1} (c + dI) \in \bigcap B_i\ \forall\ i\in J$. Thus by the Lemma 3.10. $\bigcap B_i$ is the neutrosophic sub-n-semiring.

Remark 3.8 Union of neutrosophic sub n-semirings may or may not be neutrosophic n-semiring. Let us see an example.

Example 3.9 If $X = \{a, b, c, d, e\}$ then $(P(X), \cup, \cap, *_1, *_2)$ is a n-semiring where $A *_1 B = A$ and $A *_2 B = B$ Now let $P(X) = S$, then $< S\cup I > = \{A + BI : A , B \in P(X)\}$is the neutrosophic n-semiring. Suppose $S_1 = \{\{\ \}, \{a\}, \{a, b\}, X\}$ and $S_2 = \{\{\ \}, \{c\}, \{b, c\}, X\}$ are the subsets of S. Then $< S_1\cup I > = \{C + DI : C , D \in S_1\}$ and $< S_2\cup I > = \{E + FI : E , F \in S_2\}$. In tabular form:

$< S_1\cup I > = \{\{\ \} + \{\ \}I , \{\ \} + \{a\}I, \{\ \} + \{a, b\}I, \{\ \} + X I , \{a\} + \{\ \}I , \{a\} + \{a\}I, \{a\} + \{a, b\}I, \{a\} + X I, \{a, b\} + \{a, b\}I , \{a, b\} + \{a\}I, \{a, b\} + \{a, b\}I, \{a, b\} + X I, X + \{\ \}I , X + \{a\}I, X + \{a, b\}I, X + X I\}$ and $< S_2\cup I > = \{\{\ \} + \{\ \}I , \{\ \} + \{c\}I, \{\ \} + \{b, c\}I, \{\ \} + X I , \{c\} + \{\ \}I , \{c\} + \{c\}I, \{c\} + \{b, c\}I, \{c\} + X I, \{b, c\} + \{b, c\}I , \{b, c\} + \{c\}I, \{b, c\} + \{b, c\}I, \{b, c\} + X I, X + \{\ \}I , X + \{c\}I, X + \{b, c\}I, X + X I\}$ are the neutrosophic sub-n-semirings but their union $< S_1\cup I > \cup < S_2\cup I > = \{\{\ \} + \{\ \}I , \{\ \} + \{a\}I, \{\ \} + \{a, b\}I, \{\ \} + X I , \{a\} + \{$

}I , { a} + {a}I, { a} + {a, b }I, {a } + X I, { a, b} + {a, b }I , {a, b } + {a}I, {a, b } + {a, b }I, {a, b } + X I, X + { }I , X + {a}I, X + {a, b }I, X + X I , { } + { }I , { } + {c}I, { } + {b, c }I, { } + X I , { c } + { }I , { c} + {c}I, { c} + {b, c }I, {c } + X I, { b, c} + {b, c }I , {b, c } + {c}I, {b, c } + {b, c }I, {b, c } + X I, X + { }I , X + {c}I, X + {b, c }I, X + X I } is not the neutrosophicsub-n-semiring because the operation of union is not closed i.e. { a, b} + {a, b }I ∪{ b, c} + {b, c }I = ({ a, b}∪{ b, c}) + ({ a, b}∪{ b, c}) I = {a, b, c} + {a, b, c}I $\notin$ < S_1∪I > ∪ < S_2∪I >

Now we will show that under what condition union of neutrosophic sub-n-semirings is again a neutrosophic sub-n-semiring.

Proposition 3.10 Let { B_k : k∈ K } be the family of neutrosophic sub-n-semirings. If $B_k \subseteq B_{k+1}$ then $\bigcup B_k$ is neutrosophic sub-n-semiring.

Proof. Let a + bI, c + dI ∈ $\bigcup B_k$ ⟹ that there exist some p and q ∈ K such that a + bI ∈ B_p and c + dI ∈ B_q . Let q > p then a + bI, c + dI ∈ B_q . Now since B_q is neutrosophic sub-n-semiring so (a + bI) $*_1$ (c + dI), (a + bI) $*_2$ (c + dI), (a + bI) $*_3$ (c + dI),…, (a + bI) $*_n$ (c + dI), (a + bI) $*_{n+1}$ (c + dI) ∈ B_q so, this implies that (a + bI) $*_1$ (c + dI), (a + bI) $*_2$ (c + dI), (a + bI) $*_3$ (c + dI),…, (a + bI) $*_n$ (c + dI), (a + bI) $*_{n+1}$ (c + dI) ∈$\bigcup B_k$ thus by the Lemma 3.10 $\bigcup B_k$ is neutrosophic sub-n-semiring.

Corollary 3.11 Union of two neutrosophic sub-n-semirings N_1 and N_2 of N is again neutrosophic sub-n-semiring of N if either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$.

Proof. Let $N_1 \subseteq N_2$ then obviously $N_1 \bigcup N_2 = N_2$ so, it follows that $N_1 \bigcup N_2$ is a neutrosophic sub-n-semiring. Again let $N_2 \subseteq N_1$ then clearly $N_1 \bigcup N_2 = N_1$. Now as N_1 is neutrosophic sub-n-semiring so, $N_1 \bigcup N_2$ is, by transitive property.

3.2 Ideals in Neutrosophic-N-Semirings

In this section, we define left ideals in neutrosophic n-semirings and discuss some properties.

Definition 3.2.1 Let J be the non-empty subset of neutrosophic-n-semiring S , then J is called left ideal of S if ∀ ($a_1 + b_1$I), ($a_2 + b_2$I) ∈ J, ($a_1 + b_1$I) $*_1$ ($a_2 + b_2$I) ∈ J , (a + bI) $*_2$ (c + dI) and (a + bI) $*_3$ (c + dI) , …, (a + bI) $*_n$ (c + dI) and (a + bI) $*_{n+1}$ (c + dI), ∈ J ∀ (a + bI) ∈ S and (c + dI) ∈ J.

Example 3.2.2 Let N ={1, 2, 3,….} is the set of natural numbers where a $*_1$ b = a + b, a $*_2$ b = ab, and $*_3$, $*_4$ are defined as (a + bI) $*_3$ (c + dI) = 2(ac) + 2(bd)I and (a + bI) $*_4$ (c + dI) = 3(ac) + 3(bd)I then N∪I ={a + bI :a, b ∈ N }becomes the neutrosophic-n-semiring and for set S = {2, 3, 4, ….} the structure S∪I ={c + dI : c, d ∈ S } becomes the left ideal because ∀ e + f I ∈ N , a + bI ∈ S and c + dI ∈ S, (a + bI) $*_1$ (c + dI) ∈ S and (e + f I) $*_2$ (c + dI), (e + f I) $*_3$ (c + dI) and (e + f I) $*_3$ (c + dI) ∈ S .

Theorem 3.2.3 The intersection of any number of left ideals of neutrosophic-n-semiring N is again a left ideal.

Proof. We suppose that $\{J_k : k \in K\}$ be the family of left ideals of neutrosophic n-semiring N. We have to show that $\bigcap J_k$ is the left ideal of N. Let $(a + bI)$, $(c + dI) \in \bigcap J_k$ and $(e + fI) \in N$ this implies that $(a + bI), (c + dI) \in J_k$ $\forall k \in K$, and $(e + fI) \in N$ this implies that $(a + bI) *_1 (c + dI)$, $(e + fI) *_2 (a + bI), \ldots, (e + fI) *_n (a + bI)$ and $(e + fI) *_{n+1} (a + bI) \in J_k$ $\forall k \in K$ because each J_k is the left ideal so, this shows that $(e + fI) *_2 (a + bI), \ldots, (e + fI) *_n (a + bI)$ and $(e + fI) *_{n+1} (a + bI) \in \bigcap J_k$ $\forall k \in K$ so from hence we say that $\bigcap J_k$ is the left ideal.

Note : The case is same for right ideals and ideals of neutrosophic n-semirings.

Remark 3.2.4 Union of left (right) ideals of neutrosophic n-semiring may or may not be the left ideal again.

In order to be clearer with the preceding remark we give an example.

Example 3.2.5 The set $N \cup I = \{a + bI ; a, b \in N\}$ under the operations $(a + bI) *_1 (c + dI) = \gcd(a, c) + (\gcd(b, d))I$, $(a + bI) *_2 (c + dI) = \mathrm{lcm}(a, c) + (\mathrm{lcm}(b, d))I$, $(a + bI) *_3 (c + dI) = (c + dI)$ and $(a + bI) *_4 (c + dI) = (ac) + (bd)I$ is the neutrosophic-3-semiring and $2N \cup I = \{a + bI ; a, b \in 2N\}$ and $3N \cup I = \{a + bI : a, b \in 3N\}$ are the left ideals of $N \cup I$ but $(2N \cup I) \cup (3N \cup I)$ is not the left ideal of $N \cup I$ because $2 + 2I$ and $9 + 9I \in (2N \cup I) \cup (3N \cup I)$ and $(2 + 2I) *_1 (9 + 9I) = \gcd(2, 9) + (\gcd(2, 9))I = 1 + 1I$ which does not belong to $(2N \cup I) \cup (3N \cup I)$, hence because of failure of closure property $(2N \cup I) \cup (3N \cup I)$ does not become ideal again.

Now we write the condition for union of left ideals of neutrosophic n-semirings to become ideal so, here we go.

Theorem 3.2.6 Let $\{J_k : k \in K\}$ be the family of left ideals of neutrosophic n-semiring N. Then $\bigcup J_k$ is left ideal if $J_k \subseteq J_{k+1}$ for all $k \in K$.

Proof. Let $(a + bI), (c + dI) \in \bigcup J_k$ and $(e + fI) \in N$ then there exist p and $q \in K$ such that $(a + bI) \in J_p$ and $(c + dI) \in J_q$. Let $q > p$ then $(a + bI), (c + dI) \in J_q$. Now as J_q is an ideal so it follows that $(a + bI) *_1 (c + dI) \in J_q$ and $(e + fI) *_2 (a + bI), (e + fI) *_3 (a + bI), \ldots, (e + fI) *_{n+1} (a + bI), \in J_q$. Thus $(e + fI) *_2 (a + bI), (e + fI) *_3 (a + bI), \ldots, (e + fI) *_{n+1} (a + bI) \in \bigcup J_k$. So, it follows that $\bigcup J_k$ is the left ideal. In similar fashion we can show that $\bigcup J_k$ is right ideal. This completes the proof.

Theorem 3.2.7 Every left ideal of a neutrosophic n-semiring N is neutrosophic sub n-semiring of N.

.**Proof**. Let J be the left ideal of N and for all $(a + bI), (c + dI) \in J$, $(a + bI) *_1 (c + dI) \in J$. Since $J \subseteq N$ so, $(a + bI) \in N$. Now as J is the left ideal of N so, $(a + bI) *_2 (c + dI)$, $(a + bI) *_3 (c + dI), \ldots, (a + bI) *_n (c + dI)$, $(a + bI) *_{n+1} (c + dI) \in J$ and associative and distributive laws are surely satisfied because $J \subseteq N$ and N is neutrosophic n-semiring. Similar result can be proved for every right ideal .

Remark 3.2.8 Converse of the above theorem is not true in general and it can be seen from following example.

Example 3.2.9 Let N = {1, 2, 3,...} be the set of natural numbers. Then N∪I ={a + bI :a, b ∈N } under the operations defined as (a + bI) $*_1$ (c + dI) = min(a, c) + (min(b, d))I , (a + bI) $*_2$ (c + dI) = max(a, c) + (max(b, d))I , (a + bI) $*_3$ (c + dI) = (a + c) + (b + d) and (a + bI) $*_4$ (c + dI) = (ac) + (bd)I becomes the neutrosophic-3-semiring and 2N∪I ={a + bI :a, b ∈ 2N }being the subset of N∪I becomes the neutrosophic sub-3-semiring. But 2N∪I ={a + bI :a, b ∈2N } is not a left ideal because if we take 5 + 5I from N∪I and 4 + 4I from 2N∪I and apply operation $*_2$ upon the selected members we will see that resultant element is not of 2N∪I i.e (5 + 5I) $*_2$ (4 + 4I) = max(5, 4) + (max(5, 4))I = 5 + 5I ∉ 2N∪I. Hence it is clear now that converse part of above theorem is not true in general.

3.3 Congruences in Neutrosophic N-Semirings

Here, we give the definition of congruence relations and pay attention to some important results.

Definition 3.3.1 Let N∪I ={a + bI :a, b ∈N } be the neutrosophic-n-semiring under the operations $*_1$, $*_2$, $*_3$, … , $*_n$, $*_{n+1}$. A relation ρ on N∪I is called left compatible if for all (a+bI), (c + dI) and (e+ f I) ∈ N∪I and ((a + bI), (c + dI)) ∈ ρ , ((e + f I) $*_1$ (a + bI), (e + f I) $*_1$ (c + dI)) ∈ ρ , ((e + f I) $*_2$ (a + bI), (e + f I) $*_2$ (c + dI)) , ((e + f I) $*_3$ (a + bI), (e + f I) $*_3$ (c + dI)) , ((e + f I) $*_n$ (a + bI), (e + f I) $*_n$ (c + dI)) and ((e + f I) $*_{n+1}$ (a + bI), (e + f I) $*_{n+1}$ (c + dI)) ∈ ρ .

The relation is called right compatible if for all (a+bI), (c + dI) and (e + f I) ∈ N∪I and ((a + bI), (c + dI)) ∈ ρ , ((a + bI) $*_1$ (e + f I), (c + dI) $*_1$ (e + f I)) ∈ ρ , ((a + bI) $*_2$ (e + f I), (c + dI) $*_2$ (e + f I)) , ((a + bI) $*_3$ (e + f I), (c + dI) $*_3$ (e + f I)),…, ((a + bI) $*_n$ (e + f I), (c + dI) $*_n$ (e + f I)) and ((a + bI) $*_{n+1}$ (e + f I), (c + dI) $*_{n+1}$ (e + f I)) ∈ ρ .

It is called compatible relation if for all (a + bI), (c + dI), (e + f I) and (g + hI) ∈ N∪I and ((a + bI), (c + dI)) and ((e + f I), (g + hI)) ∈ ρ , ((a + bI) $*_1$ (e + f I) , (c + dI) $*_1$ (g + hI)) ∈ ρ , ((a + bI) $*_2$ (e + f I) , (c + dI) $*_2$ (g + hI)) , ((a + bI) $*_3$ (e + f I) , (c + dI) $*_3$ (g + hI)) ,…, ((a + bI) $*_n$ (e + f I) , (c + dI) $*_n$ (g + hI)) and ((a + bI) $*_{n+1}$ (e + f I) , (c + dI) $*_{n+1}$ (g + hI)) ∈ ρ .

A left (right) compatible equivalence relation is called left (right) congruence relation. A compatible equivalence relation is called congruence relation.

To understand the above concept we consider the following example.

Example 3.3.2 Consider the neutrosophic-n-semiring (Z_5 ∪ I , $*_1$, $*_2$, $*_3$, $*_4$) where for all (a + bI), (c + dI) ∈ Z_5∪I , (a + bI) $*_1$ (c + dI) = min ((a + bI), (c + dI)) , (a + bI) $*_2$ (c + dI) = max ((a + bI), (c + dI)), (a + bI) $*_3$ (c + dI) = (a + c) + (b + d) I (addition mod 5) and (a + bI) $*_4$ (c + dI) = (ac) + (bd) I (multiplication mod 5)and consider ρ = {((a + bI), (c + dI)) : (a + bI) = (c + dI))} then this relation is left compatible(left congruence), right compatible(right congruence) and compatible(congruence) and one can verify it easily.

Proposition 3.3.3. A relation ρ on a neutrosophic-n-semiring N is congruence relation if and only if it is both left and right congruence relation.

Proof. Let ρ be a congruence relation on a neutrosophic-n-semiring N. Let (a + bI), (c + dI) and (e+ f I) ∈ N and ((a + bI), (c + dI)) ∈ . Then ((e + f I) $*_1$ (a + bI), (e + f I) $*_1$ (c + dI)) , ((e+ f I) $*_2$ (a + bI), (e + f I) $*_2$ (c + dI)) , ((e + f I) $*_3$ (a + bI), (e + f I) $*_3$ (c + dI)),…, ((e + f I) $*_n$ (a + bI), (e + f I) $*_n$ (c + dI)), ((e + f I) $*_{n+1}$ (a + bI), (e + f I) $*_{n+1}$ (c + dI)) ∈ ρ. This shows that ρ is a left congruence relation and similarly it can be shown that ρ is a right congruence relation.

Conversely, assume that ρ is both left and right congruence relation. Let (a + bI), (c + dI), (e + f I) and (g + hI) ∈ N such that ((a + bI), (c + dI)) and ((e + f I), (g + hI)) ∈ ρ . Therefore ((a + bI) $*_1$ (e + f I) , (c + dI) $*_1$ (e + f I)) ∈ ρ , ((a + bI) $*_2$ (e + f I) , (c + dI) $*_2$ (e + f I)) , ((a + bI) $*_3$ (e + f I) , (c + dI) $*_3$ (e + f I)) ,…, ((a + bI) $*_n$ (e + f I) , (c + dI) $*_n$ (e + f I)) and ((a + bI) $*_{n+1}$ (e + f I) , (c + dI) $*_{n+1}$ (e + f I)) ∈ ρ as ρ is right compatible. Also ((c + dI) $*_1$ (e + f I) , (c + dI) $*_1$ (g + hI)) ∈ ρ , ((c + dI) $*_2$ (e + f I) , (c + dI) $*_2$ (g + hI)) , ((c + dI) $*_3$ (e + f I) , (c + dI) $*_3$ (g + hI)),…, ((c + dI) $*_n$ (e + f I) , (c + dI) $*_n$ (g + hI)) , ((c + dI) $*_{n+1}$ (e + f I) , (c + dI) $*_{n+1}$ (g + hI)) ∈ ρ because ρ is left compatible relation. Therefore ((a + bI) $*_1$ (e + f I) , (c + dI) $*_1$ (g + hI)) , ((a + bI) $*_2$ (e + f I) , (c + dI) $*_2$ (g + hI)) , ((a + bI) $*_3$ (e + f I) , (c + dI) $*_3$ (g + hI)) ,…, ((a + bI) $*_n$ (e + f I) , (c + dI) $*_n$ (g + hI)) and ((a + bI) $*_{n+1}$ (e + f I) , (c + dI) $*_{n+1}$ (g + hI)) ∈ ρ. Thus ρ is a congruence relation.

4. Conclusions. In this paper we have defined the neutrosophic n-semirings and then some examples have been coined to understand the notions. Moreover, we have defined their substructures and added some examples. Left (right) ideals have also been defined in this paper with examples. Compatible relations, equivalence relations and congruence relations have been defined too, with examples. Furthermore, for newly given concepts, different results and properties have been considered.

5. Future work. In future, the ideas can be extended to other structures like neutrosophic n- rings, neutrosophic n-bipolar semirings, neutrosophic neutrosophic n-bipolar rings. These notions with n-operations can be studied for multipolar structures neutrosophic semirings and rings and other extensions of neutrosophic sets. Moreover, all the results which have been studied in this manuscript can be carried out for above mentioned structures.

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