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Plithogenic Metric Equivalence and Metric Transport in Evolving Epidemiological State Spaces

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Abstract: We develop an operator-theoretic theory of metric equivalence and metric transport for evolving epidemiological states represented within a plithogenic Hilbert space. Let Hepi = HT ⊕ HI ⊕ HF represent truth, indeterminacy, and falsity components associated with epidemiological information, including disease occurrence, exposure, immunity, diagnostic status, environmental risk, and intervention uncertainty. For each temporal frame t, a contradiction structure is represented by a self-adjoint operator Dt, from which a strictly positive metric operator Gt is constructed. Two epidemiological state operators A, B ∈ B(Hepi) are said to be Gt metrically equivalent when A*GtA = B*GtB. We establish a reduction to classical metric equivalence, characterize the relation through positive square roots and partial isometries, and identify the associated metric invariants. We then introduce epidemiological metric transport operators satisfying A*tGt+1At = Gt and prove that local metric preservation at every temporal transition implies global metric preservation across every finite sequence of epidemiological frames. An approximate metric defect is introduced together with perturbation estimates. The epidemiological component is formulated theoretically: epidemiological variables are represented as components of an evolving uncertain state, while no patient-level or population-level dataset is required. Finite-dimensional examples illustrate metrically equivalent and non-equivalent epidemiological transition operators. The resulting theory provides an operator-theoretic basis for studying the preservation of epidemiological geometry under evolving uncertainty, contradiction, and state transition.

Keywords: Plithogenic sets; epidemiological states; metric equivalence; metric transport; Hilbert space; operator theory; evolving uncertainty; epidemiological dynamics.

1. Introduction

The mathematical modeling of evolving uncertain states requires tools capable of representing time-dependent, overlapping, and sometimes contradictory information. Traditional approaches to state comparison often rely on pointwise similarity measures, which capture instantaneous resemblance but say little about the structural geometry of state transitions.

Recent work on N-framed plithogenic hypersoft sets [1] has demonstrated the capacity of plithogenic structures to represent time-dependent and overlapping disease information. By partitioning observations into discrete temporal frames, such frameworks enable comparison of evolving profiles through distance and similarity measures. However, these approaches remain fundamentally descriptive: they compare states rather than characterizing the transformations that generate state evolution.

The present work takes a different mathematical direction. Rather than comparing states pointwise, we investigate the geometry induced by state transitions. This leads to a theory of time-dependent metric equivalence and metric transport for bounded operators on Hilbert spaces associated with plithogenic epidemiological representations.

1.1. The Mathematical Gap

Metric equivalence is a classical relation in Hilbert space operator theory. For operators $A, B \in B(H)$, the relation $A^*A = B^*B$ is equivalent to $\|Ax\| = \|Bx\|$, $x \in H$, and therefore identifies operators that induce the same quadratic geometry on the underlying Hilbert space. Metric equivalence is closely related to, but distinct from, similarity of operators. Earlier studies have investigated metric equivalence and almost similarity of operators [16], and more recently the intersection of similarity and metric equivalence has been studied through the notion of metric similarity [17].

Existing work on metric equivalence studies equality of operator-induced quadratic forms under the canonical Hilbert space metric, while recent plithogenic hypersoft set models provide frame-based representations for evolving uncertain information. The present work connects these two directions by introducing a strictly positive metric operator induced by a contradiction structure and by studying exact metric transport between successive frames.

In a plithogenic representation, information may be decomposed into truth, indeterminacy, and falsity components, while contradiction relationships between attribute values may change from one frame to another [19–21]. Thus, for an evolving plithogenic epidemiological state, there is no mathematical reason to require the canonical Hilbert space metric to remain the only geometry used for comparison. A weighted geometry is more appropriate when the relative contribution and interaction of the components are encoded by a strictly positive operator. This observation motivates the weighted relation $A \sim_G B$ if and only if $A^*GA = B^*GB$, $G > 0$.

The relation considered below is a weighted form of metric equivalence, with the strictly positive operator G specifying the underlying geometry; see [16,17]. When $G = I$, the classical metric equivalence relation is recovered.

A related positive-operator deformation of metric equivalence was recently considered by Amenya et al. [2]. The present paper differs in that the strictly positive metric operator is generated from an explicitly specified plithogenic contradiction operator and is allowed to evolve across a sequence of frames. The principal object is consequently the transport equation $A^*_tG_{t+1}A_t = G_t$, together with its multi-step consequence.

To the best of our knowledge, the combination of a contradiction-induced positive metric sequence with weighted metric equivalence and exact multi-frame metric transport has not been explicitly developed in the operator-theoretic literature.

1.2. Epidemiological Motivation

Epidemiological information is frequently incomplete, delayed, heterogeneous, and apparently contradictory. Diagnostic evidence may disagree with reported disease status, exposure classification may conflict with observed disease occurrence, and environmental risk indicators may not reflect observed epidemiological states. The present paper provides an operator-theoretic framework in which these situations are represented through plithogenic truth, indeterminacy, and falsity components and a contradiction-induced metric operator, without requiring any empirical epidemiological dataset.

1.3. Contributions

The main contributions of this paper are:

- A structured construction of strictly positive metric operators for plithogenic epidemiological state spaces, using a tensor product embedding $\tilde{D}_t = C_t \otimes D_t$ with a self-adjoint component interaction matrix C_t.
- Definition and characterization of G_t metric equivalence $A^*G_tA = B^*G_tB$, including reduction to classical metric equivalence and polar and partial isometry characterizations.

- Identification of metric invariants preserved under G-equivalence, including the G-norm, kernel structure, finite-dimensional rank, and singular values of metric-weighted realizations.
- Exact single- and multi-step metric transport, $A^*_t G_{t+1} A_t = G_t$ and $\Phi^*_{t,s} G_t \Phi_{t,s} = G_s$, together with perturbation and stability estimates for approximate metric equivalence.
- A theoretical epidemiological interpretation showing how contradiction, uncertainty, and temporal evolution of epidemiological information can be represented within the proposed operator-theoretic setting.

1.4. Organization

The remainder of this paper is organized as follows. Section 2 provides essential preliminaries. Section 3 develops the plithogenic epidemiological metric structure. Section 4 introduces plithogenic metric equivalence. Section 5 identifies metric invariants. Section 6 develops epidemiological metric transport. Section 7 introduces the metric equivalence defect and establishes perturbation estimates. Section 8 presents finite-dimensional examples. Section 9 provides a brief epidemiological interpretation. Section 10 concludes.

2. Preliminaries

Throughout this paper, H denotes a complex Hilbert space and B(H) the algebra of bounded linear operators on H. The inner product is denoted by $\langle\cdot,\cdot\rangle$ and is assumed linear in the first argument. The corresponding norm is $\|x\| = \langle x,x\rangle^{1/2}$. For $A \in B(H)$, the adjoint of A is denoted A^* and is characterized by $\langle Ax,y\rangle = \langle x,A^*y\rangle$, $x, y \in H$. The basic properties of bounded operators on Hilbert spaces, adjoints, positive operators, and spectral theory are standard; see, for example, [6,8,9,15,18].

Remark 2.1. Throughout the paper, the term strictly positive metric operator refers to a bounded, self-adjoint, uniformly positive operator defining an equivalent Hilbert space inner product. It is not a metric function on the underlying plithogenic set.

2.1. Positive and Strictly Positive Operators

Definition 2.2. An operator $G \in B(H)$ is called positive if $G = G^*$ and $\langle Gx,x\rangle \geq 0$ for every $x \in H$. We write $G \geq 0$.

Definition 2.3. An operator $G \in B(H)$ is called strictly positive (uniformly positive) if there exists a constant $m > 0$ such that $\langle Gx,x\rangle \geq m\|x\|^2$, $x \in H$. We write $G \geq mI > 0$.

Remark 2.4. If $G > 0$, then G is invertible and its positive square root $G^{1/2}$ is invertible. The positive square root is well defined by the continuous functional calculus for positive operators [3,4,8,15]. Moreover, $\|G^{-1}\|^{-1}\|x\|^2 \leq \langle Gx,x\rangle \leq \|G\|\|x\|^2$. Consequently, the norm induced by G is equivalent to the original Hilbert space norm.

2.2. Weighted Hilbert Space Geometry

Definition 2.5. Let $G > 0$. The G-inner product on H is defined by $\langle x,y\rangle_G = \langle Gx,y\rangle$, $x, y \in H$. The associated G-norm is $\|x\|_G = \langle Gx,x\rangle^{1/2}$. The use of a strictly positive metric operator to modify the inner product is standard in operator theory on spaces equipped with a modified metric; see [7,13,14].

Proposition 2.6. Let $G > 0$. Then $(H,\langle\cdot,\cdot\rangle_G)$ is a Hilbert space and $\|G^{1/2}x\| = \|x\|_G$.

Proof. $\|x\|^2_G = \langle Gx,x\rangle = \langle G^{1/2}x,G^{1/2}x\rangle = \|G^{1/2}x\|^2$. Since $G^{1/2}$ is boundedly invertible, the G-norm is equivalent to the original norm on H. Hence completeness is preserved. □

Theorem 2.7 (Equivalence of the G-Norm and the Hilbert Norm). Let $G > 0$. Then there exist constants $m, M > 0$ such that $mI \leq G \leq MI$ and $\sqrt{m}\|x\| \leq \|x\|_G \leq \sqrt{M}\|x\|$, $x \in H$. In particular, the G-norm and the original Hilbert space norm generate the same topology.

Proof. Since $G > 0$ and G is bounded, the spectrum of G is contained in $[m, M]$ for some $0 < m \leq M < \infty$. Hence $mI \leq G \leq MI$. Taking inner products with x gives the stated norm equivalence. □

Definition 2.8. Let $G > 0$ and $A \in B(H)$. The G-operator norm of A is $\|A\|_G = \sup_{x\neq 0} \|Ax\|_G / \|x\|_G$.

Theorem 2.9. Let $G > 0$ and $A \in B(H)$. Then $\|A\|_G = \|G^{1/2}AG^{-1/2}\|$.

Proof. Put $y = G^{1/2}x$. Then $\|A\|_G = \sup_{y\neq 0} \|G^{1/2}AG^{-1/2}y\| / \|y\| = \|G^{1/2}AG^{-1/2}\|$.

2.3. Polar Decomposition

Every $A \in B(H)$ admits a polar decomposition $A = U|A|$, $|A| = (A^*A)^{1/2}$, where U is a partial isometry. See [3,8,11,12].

2.4. Metric Equivalence of Operators

Definition 2.10. Let $G > 0$ and let $A, B \in B(H)$. We say that A and B are G metrically equivalent if $A^*GA = B^*GB$. We write $A \sim_G B$.

Remark 2.11. The relation $A \sim_G B$ is equivalent to $\|Ax\|_G = \|Bx\|_G$ for every $x \in H$. Thus G metric equivalence compares operators through the geometry that they induce on the weighted Hilbert space $(H,\langle\cdot,\cdot\rangle_G)$.

2.5. Metric Transport

Definition 2.12. Let $G_0, G_1 > 0$ and let $A \in B(H)$. We call A a (G_0, G_1) metric transport operator if $A^*G_1A = G_0$.

Proposition 2.13. If A is a (G_0, G_1) metric transport operator, then $\|Ax\|_{G_1} = \|x\|_{G_0}$ for every $x \in H$. In particular, A is injective.

Proof. We compute $\|Ax\|^2_{G_1} = \langle G_1Ax,Ax\rangle = \langle A^*G_1Ax,x\rangle = \langle G_0x,x\rangle = \|x\|^2_{G_0}$. If $Ax = 0$, then $\|x\|_{G_0} = 0$, and since $G_0 > 0$, we obtain $x = 0$.

Remark 2.14. Metric equivalence and metric transport are conceptually distinct. Metric equivalence is an equivalence relation on operators for a fixed strictly positive metric operator G: $A^*GA = B^*GB$. Metric transport is an isometric relation between two possibly different weighted Hilbert space geometries: $A^*G_1A = G_0$.

3. Plithogenic Epidemiological Metric Structures

3.1. Epidemiological Plithogenic State Space

Let H_T, H_I, and H_F be complex Hilbert spaces representing, respectively, the truth, indeterminacy, and falsity components of epidemiological information. We define the epidemiological plithogenic state space by $H_{epi} = H_T \oplus H_I \oplus H_F$. An epidemiological state at temporal frame t is represented by $x_t = x_{Tt} \oplus x_{It} \oplus x_{Ft} \in H_{epi}$.

The components may encode uncertain information associated with epidemiological quantities such as disease occurrence, exposure status, immunity, diagnostic classification, environmental risk, intervention status, or other population-level indicators. The three components do not represent three mutually exclusive epidemiological populations. Rather, they represent the truth, indeterminacy, and falsity dimensions associated with the available information concerning an epidemiological state.

The inner product on H_{epi} is given by $\langle x,y\rangle = \langle x_T,y_T\rangle_T + \langle x_I,y_I\rangle_I + \langle x_F,y_F\rangle_F$. The plithogenic setting used here is based on the theory of plithogenic sets and related extensions developed by Smarandache [19–21].

3.2. Theoretical Epidemiological State Representation

Let $E_t = \{e^{(t)}_1, e^{(t)}_2, \dots, e^{(t)}_n\}$ denote a finite collection of epidemiological attributes observed or conceptually defined at temporal frame t. Depending on the disease and application, these attributes may include disease occurrence, exposure, immunity, diagnostic status, environmental exposure, intervention status, or other epidemiologically relevant characteristics.

For each attribute $e(t)_i$, let $\mu(t)T(e_i)$, $\mu(t)I(e_i)$, $\mu(t)F(e_i)$ denote its truth, indeterminacy, and falsity components, respectively. After selecting a coordinate representation, the epidemiological state is written as $x_t \in H_{epi}$.

The state vector is not required to represent a probability distribution. Instead, it represents the information state associated with an epidemiological system under uncertainty. The temporal evolution of the epidemiological information state is represented by a sequence $x_0, x_1, \ldots, x_N$ with $x_t \in H_{epi}$.

The operator $A_t : H_{epi} \to H_{epi}$ represents the mathematical transformation of the epidemiological information state between consecutive temporal frames. The operator need not be interpreted as a biological transmission operator. It may instead represent an abstract state transition incorporating changes in disease information, exposure, immunity, surveillance, environmental conditions, or interventions.

3.3. Contradiction Operators

Let $A_t = \{a_1, \ldots, a_n\}$ denote the attribute set at frame t.

Definition 3.1. A contradiction function at frame t is a map $C_t : A_t \times A_t \to [0, 1]$ satisfying $C_t(a_i,a_i) = 0$ and $C_t(a_i,a_j) = C_t(a_j,a_i)$.

Definition 3.2. The contradiction matrix at frame t is the real symmetric matrix $D_t = (d(t)_{ij})$ for $i,j = 1,\ldots,n$, where $d(t)_{ij} = C_t(a_i,a_j)$. The contradiction matrix D_t is real symmetric by construction, hence self-adjoint as an operator on $\mathbb{C}^n$. It need not be positive semidefinite, and it is used to perturb a reference strictly positive metric operator rather than as a metric operator itself.

Remark 3.3 (Epidemiological Interpretation of Contradiction). In the epidemiological setting, the contradiction degree $C_t(e_i,e_j)$ quantifies the degree of contradiction between two epidemiological attributes or attribute values at temporal frame t. Examples include apparent incompatibility between diagnostic evidence and reported disease status, between exposure classification and observed disease occurrence, or between environmental risk indicators and observed epidemiological states. The contradiction degree is not interpreted as a probability, incidence rate, or relative risk. It is a structural quantity used to encode incompatibility or opposition among the attributes entering the plithogenic representation.

3.4. Component Interaction and Structured Embedding

Definition 3.4. Suppose that $H_T = H_I = H_F = \mathbb{C}^n$, so that $H_{epi} \cong \mathbb{C}^{3n}$. A component interaction matrix at frame t is a self-adjoint matrix $C_t = (c(t)_{PQ})_{P,Q\in\{T,I,F\}} \in \mathbb{C}^{3\times 3}$, $C_t = C_t^*$. The extended contradiction operator on H_{epi} is $\tilde{D}_t = C_t \otimes D_t$. Since C_t and D_t are both self-adjoint, $\tilde{D}_t^* = (C_t \otimes D_t)^* = C_t^* \otimes D_t^* = C_t \otimes D_t = \tilde{D}_t$.

The tensor product form is a structured embedding once C_t is chosen: C_t encodes how the contradiction structure couples the truth, indeterminacy, and falsity components, while D_t encodes the attribute-level contradiction degrees. The construction is not asserted to be canonical; C_t is a modelling choice reflecting the epidemiological context.

3.5. The Operator-Induced Plithogenic Metric

Definition 3.5. Let $G_{0,t} > 0$ be a reference strictly positive metric operator on H_{epi} and let $\lambda_t \geq 0$. We call $G_t = G_{0,t}^{1/2}(I + \lambda_t\tilde{D}_t)G_{0,t}^{1/2}$ the operator-induced plithogenic metric associated with the contradiction operator $\tilde{D}_t$. The quadratic form of G_t shows explicitly how the contradiction structure modifies the reference geometry.

Definition 3.6 (Standing Assumptions). At each frame t, we assume $G_{0,t} > 0$, $C_t = C_t^*$, $D_t = D_t^*$, $I + \lambda_t(C_t \otimes D_t) > 0$.

Proposition 3.7. Suppose the standing assumptions hold. Then $G_t = G_{0,t}^{1/2}(I + \lambda_t\tilde{D}_t)G_{0,t}^{1/2}$ is strictly positive.

Proof. For $x \neq 0$, $\langle G_t x, x\rangle = \langle (I + \lambda_t \tilde{D}_t)G_{0,t}^{(1/2)}x, G_{0,t}^{(1/2)}x\rangle$. Since $G_{0,t}^{(1/2)}$ is invertible and $I + \lambda_t\tilde{D}_t > 0$, the right-hand side is strictly positive. □

Theorem 3.8 (Sufficient Positivity Condition). Let $\tilde{D}_t = \tilde{D}_t^*$ and let $\lambda_t \geq 0$. If $1 + \lambda_t \inf\sigma(\tilde{D}_t) > 0$, then $\inf\sigma(I + \lambda_t\tilde{D}_t) = 1 + \lambda_t \inf\sigma(\tilde{D}_t) > 0$, and consequently $I + \lambda_t\tilde{D}_t > 0$. In the finite-dimensional case, the condition is equivalent to $1 + \lambda_t\lambda_{min}(\tilde{D}_t) > 0$. Equivalently, the condition $\lambda_t\|\tilde{D}_t^-\| < 1$ guarantees strict positivity, where $\tilde{D}_t^-$ denotes the negative part of $\tilde{D}_t$; this inequality remains valid when $\tilde{D}_t^- = 0$.

Proof. The spectrum of $I + \lambda_t\tilde{D}_t$ is $\{1 + \lambda\mu : \mu \in \sigma(\tilde{D}_t)\}$, hence $\inf\sigma(I + \lambda_t\tilde{D}_t) = 1 + \lambda_t \inf\sigma(\tilde{D}_t)$, which is positive by assumption. In finite dimensions, $\inf\sigma(\tilde{D}_t) = \lambda_{min}(\tilde{D}_t)$. Finally, $\inf\sigma(\tilde{D}_t) \geq -\|\tilde{D}_t^-\|$, so $\lambda_t\|\tilde{D}_t^-\| < 1$ implies $1 + \lambda_t \inf\sigma(\tilde{D}_t) > 0$, and if $\tilde{D}_t^- = 0$ the inequality is trivially satisfied. □

Example 3.9 (Theoretical Epidemiological State). Consider an epidemiological state described by the four attributes E_t = {disease occurrence, exposure, immunity, environmental risk}. The indeterminacy component may arise from incomplete surveillance, uncertain diagnostic classification, delayed reporting, or conflicting epidemiological indicators. The contradiction matrix $D_t = (d(t)_{ij})$ for $i,j = 1,\ldots,4$ records the pairwise contradiction structure among these attributes. A strong contradiction may occur, for instance, between an environmental risk indicator and an observed disease occurrence indicator when the expected association is not reflected in the available information. The resulting operator $G_t = G_{0,t}^{(1/2)}(I + \lambda_t(C_t \otimes D_t))G_{0,t}^{(1/2)}$ defines the geometry in which epidemiological states are compared. The example is theoretical and does not require numerical epidemiological observations.

3.6. Time-Dependent Plithogenic Metrics

Let $G_t \in B(H_{epi})$ for $t = 0, 1, \ldots, N$ be strictly positive metric operators satisfying $G_t = G_t^* > 0$. The block operators are not chosen independently; they are required to satisfy $G_t = G_t^* > 0$, which imposes the corresponding positivity constraints on the diagonal and cross-component blocks.

Definition 3.10. A family $\{G_t\}_{t=0}^{N}$ of strictly positive metric operators on H_{epi} is called a plithogenic metric sequence. The G_t-norm on H_{epi} is $\|x\|_{G_t} = \langle G_t x, x\rangle^{(1/2)}$.

4. Plithogenic Metric Equivalence

4.1. Equivalence Relation

Theorem 4.1. Let $G > 0$ on H. The relation $\sim_G$ defined by $A \sim_G B$ if and only if $A^*GA = B^*GB$ is an equivalence relation on $B(H)$.

Proof. Reflexivity is immediate. Symmetry follows from $A^*GA = B^*GB$ implying $B^*GB = A^*GA$. Transitivity follows from $A^*GA = B^*GB = C^*GC$.

4.2. Reduction to Classical Metric Equivalence

Theorem 4.2 (Reduction to Classical Metric Equivalence). Let $G > 0$ and let $A, B \in B(H)$. Then $A \sim_G B$ if and only if $G^{(1/2)}A \sim G^{(1/2)}B$, where the relation on the right-hand side is the classical metric equivalence relation $S \sim T$ if and only if $S^*S = T^*T$.

Proof. We have $A^*GA = B^*GB$ if and only if $(G^{(1/2)}A)^*(G^{(1/2)}A) = (G^{(1/2)}B)^*(G^{(1/2)}B)$. □

Remark 4.3. The proposed relation is the transport of classical metric equivalence through the geometry-changing map $A \mapsto G^{(1/2)}A$.

4.3. Square Root Characterization

Theorem 4.4. Let $G > 0$ and $A, B \in B(H)$. The following are equivalent: (1) $A \sim_G B$; (2) $|G^{(1/2)}A| = |G^{(1/2)}B|$; (3) $(G^{(1/2)}A)^*(G^{(1/2)}A) = (G^{(1/2)}B)^*(G^{(1/2)}B)$.

Proof. Equivalence of (1) and (3) follows from $(G^{(1/2)}A)^*(G^{(1/2)}A) = A^*GA$. Equivalence of (2) and (3) follows from $|T|^2 = T^*T$ and uniqueness of the positive square root. □

4.4. Polar Characterization

Theorem 4.5 (Polar Characterization). Let $G > 0$ and let $A, B \in B(H)$. The following are equivalent: (1) $A \sim_G B$; (2) $|G^{1/2}A| = |G^{1/2}B|$; (3) there exists a partial isometry W with $G^{1/2}B = WG^{1/2}A$, $W^*W = P_{\mathrm{ran}(G^{1/2}A)}$, and $WW^* = P_{\mathrm{ran}(G^{1/2}B)}$.

Proof. By Theorem 4.2, $A \sim_G B$ is equivalent to $|G^{1/2}A| = |G^{1/2}B|$. Let $R = |G^{1/2}A| = |G^{1/2}B|$ and let $G^{1/2}A = U_A R$, $G^{1/2}B = U_B R$ be polar decompositions. Define W on $\mathrm{ran}(G^{1/2}A)$ by $W(U_A R x) = U_B R x$ and extend by zero on the orthogonal complement. Then W is a partial isometry with the stated projections and $G^{1/2}B = WG^{1/2}A$. The converse follows from $(G^{1/2}B)^*(G^{1/2}B) = (G^{1/2}A)^*W^*WG^{1/2}A = (G^{1/2}A)^*(G^{1/2}A)$.

4.5. G-Unitary Characterization

Theorem 4.6 (G-Unitary Characterization). Let $G > 0$ and suppose $A, B \in B(H)$ are invertible with $A \sim_G B$. Then $B = WA$ for some invertible operator W satisfying $W^*GW = G$ and $W^{-1} = G^{-1}W^*G$.

Proof. Put $W = BA^{-1}$. Then $W^*GW = A^{-*}B^*GBA^{-1} = A^{-*}A^*GAA^{-1} = G$. Since W is invertible and $W^*GW = G$, we obtain $W^{-1} = G^{-1}W^*G$.

5. Metric Invariants

The fundamental invariant preserved under G metric equivalence is the common strictly positive operator $A^*GA = B^*GB$. All of the following results are consequences of this equality.

5.1. Kernel and Range Structure

Theorem 5.1. Let $G > 0$ and $A, B \in B(H)$. If $A \sim_G B$, then $\ker A = \ker B$.

Proof. Let $x \in \ker A$. Then $A^*GAx = 0$, and since $A^*GA = B^*GB$, we get $B^*GBx = 0$. Taking the inner product with x gives $\langle GBx, Bx\rangle = 0$, and since $G > 0$ this implies $Bx = 0$. Thus $\ker A \subseteq \ker B$, and the reverse inclusion follows by symmetry.

Corollary 5.2. If $A \sim_G B$, then $\mathrm{ran}\, A^* = \mathrm{ran}\, B^*$. In finite dimensions, $\mathrm{rank}\, A = \mathrm{rank}\, B$.

5.2. Weighted Norm Invariance

Theorem 5.3. Let $G > 0$ and $A, B \in B(H)$. If $A \sim_G B$, then $\|Ax\|_G = \|Bx\|_G$ for every $x \in H$, and $\|A\|_G = \|B\|_G$.

Proof. $\|Ax\|^2_G = \langle GAx, Ax\rangle = \langle A^*GAx, x\rangle = \langle B^*GBx, x\rangle = \|Bx\|^2_G$. The equality of operator norms follows by taking the supremum over nonzero x.

5.3. Metric-Weighted Singular Values

Definition 5.4. Let $G > 0$ and $A \in B(H)$. We refer to the singular values of $G^{1/2}A$ as the singular values of the metric-weighted realization of A.

Theorem 5.5. Suppose H is finite-dimensional and $G > 0$. If $A \sim_G B$, then $G^{1/2}A$ and $G^{1/2}B$ have identical singular values, counted with multiplicity.

Proof. $(G^{1/2}A)^*(G^{1/2}A) = (G^{1/2}B)^*(G^{1/2}B)$. The singular values are the nonnegative square roots of the eigenvalues of the Gram operator. □

Remark 5.6. In infinite dimensions, the corresponding statement is equality of the strictly positive operators $(G^{1/2}A)^*(G^{1/2}A)$ and $(G^{1/2}B)^*(G^{1/2}B)$.

6. Epidemiological Metric Transport

Let $G_0, G_1, \ldots, G_N > 0$ be a plithogenic metric sequence. For each $t = 0, \ldots, N-1$, let $A_t \in B(H_{epi})$ represent a transition between consecutive frames.

6.1. Epidemiological Transition Operators

The operator $A_t \in B(H_{epi})$ represents the transition of an epidemiological information state from frame t to frame $t + 1$. It may incorporate changes in disease occurrence information, exposure classification, immunity information, environmental risk, surveillance evidence, or intervention status. It is not assumed to be a biological transmission operator, and no particular compartmental epidemic model is imposed.

Definition 6.1. The operator A_t is called a plithogenic metric transport operator from frame t to frame $t + 1$ if $A_t^* G_{t+1} A_t = G_t$.

Theorem 6.2. Let A_t be a plithogenic metric transport operator. Then (1) $\|A_t x\|_{G_{t+1}} = \|x\|_{G_t}$ for every $x \in H_{epi}$; (2) $G_{t+1}^{1/2} A_t G_t^{-1/2}$ is an isometry.

Proof. $\|A_t x\|^2_{G_{t+1}} = \langle G_{t+1} A_t x, A_t x\rangle = \langle A_t^* G_{t+1} A_t x, x\rangle = \langle G_t x, x\rangle = \|x\|^2_{G_t}$. For the second assertion, $(G_{t+1}^{1/2} A_t G_t^{-1/2})^*(G_{t+1}^{1/2} A_t G_t^{-1/2}) = G_t^{-1/2} A_t^* G_{t+1} A_t G_t^{-1/2} = I$.

6.2. Explicit Transport Characterization

Theorem 6.3 (Explicit Transport Characterization). Let $G_0, G_1 > 0$ and $A \in B(H)$. Then $A^* G_1 A = G_0$ if and only if $A = G_1^{-1/2} U G_0^{1/2}$ for some isometry U.

Proof. Suppose $A^* G_1 A = G_0$. Define $U = G_1^{1/2} A G_0^{-1/2}$. Then $U^*U = I$ and $A = G_1^{-1/2} U G_0^{1/2}$. Conversely, if $A = G_1^{-1/2} U G_0^{1/2}$ with $U^*U = I$, then $A^* G_1 A = G_0^{1/2} U^* U G_0^{1/2} = G_0$.

Corollary 6.4 (G-Unitary Special Case). If $G_0 = G_1 = G$, then $A^*GA = G$ if and only if $A = G^{-1/2} U G^{1/2}$ for some isometry U. In finite dimensions, if A is invertible, then U is unitary.

6.3. Multi-Step Transport

Definition 6.5. For $s < t$, $\Phi_{t,s} = A_{t-1} A_{t-2} \cdots A_s$.

Theorem 6.6 (Multi-Step Metric Transport). Let A_t satisfy $A_t^* G_{t+1} A_t = G_t$. Then for every $s < t$, $\Phi_{t,s}^* G_t \Phi_{t,s} = G_s$. In particular, $\|\Phi_{t,s} x\|_{G_t} = \|x\|_{G_s}$ for every $x \in H_{epi}$.

Proof. Induction on $t - s$. For $t = s + 1$ the assertion is $A_s^* G_{s+1} A_s = G_s$. Assume the result for (s, t). Then $\Phi_{t+1,s} = A_t \Phi_{t,s}$ and $\Phi_{t+1,s}^* G_{t+1} \Phi_{t+1,s} = \Phi_{t,s}^* A_t^* G_{t+1} A_t \Phi_{t,s} = \Phi_{t,s}^* G_t \Phi_{t,s} = G_s$.

Remark 6.7 (Epidemiological Interpretation). The condition $A_t^* G_{t+1} A_t = G_t$ states that the epidemiological information transition from frame t to frame $t + 1$ preserves the weighted geometry induced by the corresponding plithogenic contradiction structures. Consequently, $\|\Phi_{t,s} x\|_{G_t} = \|x\|_{G_s}$ means that the cumulative transition preserves the metric magnitude of the represented epidemiological information state. This is a geometric statement about the representation and does not assert conservation of population size, disease incidence, prevalence, or any biological quantity.

7. Approximate Metric Equivalence and Stability

Definition 7.1. Let $G > 0$ and $A, B \in B(H)$. The G metric defect between A and B is $\Delta_G(A,B) = \|A^*GA - B^*GB\|$.

Proposition 7.2. $\Delta_G(A,B) = 0$ if and only if $A \sim_G B$. Consequently, Δ_G is a pseudometric on $B(H)$, becoming a genuine metric on the quotient space $B(H)/\sim_G$.

Proof. $\Delta_G(A,B) = 0$ if and only if $A^*GA = B^*GB$, which is the definition of $A \sim_G B$. Symmetry is immediate from $\|A^*GA - B^*GB\| = \|B^*GB - A^*GA\|$, and the triangle inequality follows from the triangle inequality for the operator norm. The defect vanishes on each equivalence class, so it descends to a metric on the quotient.

Theorem 7.3 (Perturbation Estimate). Let $G \in B(H)$ be positive and let $A, \tilde{A} \in B(H)$. Then $\|A^*GA - \tilde{A}^*G\tilde{A}\| \le \|G\|(\|A\| + \|\tilde{A}\|)\|A - \tilde{A}\|$.

Proof. $A^*GA - \tilde{A}^*G\tilde{A} = (A-\tilde{A})^*GA + \tilde{A}^*G(A-\tilde{A})$, hence the bound. □

Theorem 7.4 (Two-Sided Perturbation Inequality). Let $G > 0$ and $A, B, \tilde{A}, \tilde{B} \in B(H)$. Then $\Delta_G(A,B) \le \Delta_G(A,\tilde{A}) + \Delta_G(\tilde{A},\tilde{B}) + \Delta_G(\tilde{B},B)$.

Proof. Triangle inequality for the operator norm.

Theorem 7.5 (Stability of Metric Equivalence). Let $G > 0$ and suppose $A \sim_G B$. For perturbations $\tilde{A}, \tilde{B} \in B(H)$, $\Delta_G(\tilde{A},\tilde{B}) \le \|G\|[(\|A\|+\|\tilde{A}\|)\|A-\tilde{A}\| + (\|B\|+\|\tilde{B}\|)\|B-\tilde{B}\|]$.

Proof. $\tilde{A}^*G\tilde{A}-\tilde{B}^*G\tilde{B} = (\tilde{A}^*G\tilde{A}-A^*GA) + (B^*GB-\tilde{B}^*G\tilde{B})$, and apply Theorem 7.3 to each term. □

Theorem 7.6 (Convergence). Let $G > 0$ and let $A_n \to A$, $B_n \to B$ in operator norm. If $A_n \sim_G B_n$ for every n, then $A \sim_G B$.

Proof. The perturbation estimate implies $A^*_nGA_n \to A^*GA$ and $B^*_nGB_n \to B^*GB$. Since $A^*_nGA_n = B^*_nGB_n$ for every n, taking limits gives $A^*GA = B^*GB$.

8. Finite-Dimensional Classification and Examples

8.1. General Construction

Given $G > 0$, $R \ge 0$, and partial isometries U, V with the same initial space, define $A = G^{-(1/2)}UR$ and $B = G^{-(1/2)}VR$. Then $A^*GA = R^2$ and $B^*GB = R^2$, so $A \sim_G B$.

8.2. Example: Equivalent Epidemiological Operators

Let $H = \mathbb{C}^2$ and $G = \mathrm{diag}(2, 1)$. Consider $A = \mathrm{diag}(1, 1)$ and $B = \mathrm{diag}(1, -1)$. Then $A^*GA = B^*GB = \mathrm{diag}(2, 1)$, so $A \sim_G B$ despite $A \neq B$. Thus metric equivalence identifies operators at the level of their induced quadratic geometry without requiring equality of the operators themselves.

8.3. Example: Non-Equivalent Operators

For $C = [[1,1],[0,1]]$ and $D = \mathrm{diag}(1,1)$, we compute $C^*GC = [[2,2],[2,3]]$ and $D^*GD = \mathrm{diag}(2,1)$, so C and D are not G metrically equivalent.

8.4. Example: Time-Varying Metric

Let $G_0 = I$ and $G_1 = \mathrm{diag}(4,1)$. Then $A_0 = \mathrm{diag}(2,1)$ satisfies $A_0^*G_1A_0 = \mathrm{diag}(16,1) \neq G_0$, so A_0 is not a transport operator. However $\tilde{A}_0 = \mathrm{diag}(1/2,1)$ satisfies $\tilde{A}_0^*G_1\tilde{A}_0 = G_0$ and is therefore a valid transport operator.

8.5. Example: Exact Computation of the Metric Defect

Let $A = I$, $\tilde{A} = [[1,\varepsilon],[0,1]]$, and $G = I$. Then $E = A^*A - \tilde{A}^*\tilde{A} = [[0,-\varepsilon],[-\varepsilon,-\varepsilon^2]]$, with eigenvalues $\lambda_\pm = (-\varepsilon^2 \pm \sqrt{\varepsilon^4+4\varepsilon^2})/2$. Hence $\|E\| = (\varepsilon^2 + \varepsilon\sqrt{\varepsilon^2+4})/2 = \varepsilon + \varepsilon^2/2 + O(\varepsilon^3)$.

9. Epidemiological Interpretation

9.1. State Representation

Epidemiological attributes such as disease occurrence, exposure, immunity, diagnostic status, and environmental risk are represented as components of a plithogenic state $x_t \in H_{epi}$. The truth, indeterminacy, and falsity components encode supporting, incomplete, and opposing information, respectively. This representation is epistemic rather than probabilistic.

9.2. Contradiction and Uncertainty

Epidemiological surveillance may produce information that is incomplete, delayed, or apparently contradictory. The contradiction function C_t and the associated matrix D_t encode these incompatibilities at the attribute level, and the extended operator $\tilde{D}_t = C_t \otimes D_t$ transfers them into the plithogenic state space. The resulting metric operator $G_t = G_{0,t}^{(1/2)}(I + \lambda_t\tilde{D}_t)G_{0,t}^{(1/2)}$ therefore incorporates contradiction structure into the geometry of the epidemiological information state.

9.3. Metric Equivalence and Transport

Two transition operators A and B are G_t metrically equivalent when $A^*G_tA = B^*G_tB$, meaning that they induce the same quadratic geometry under the plithogenic epidemiological metric. Metric equivalence concerns preservation of the weighted geometry of the represented information state and does not imply equality of epidemiological trajectories. If $A^*_tG_{t+1}A_t = G_t$ holds at each temporal transition, then the multi-step transport theorem gives $\Phi^*_{t,s}G_t\Phi_{t,s} = G_s$, so local preservation of epidemiological information geometry produces global preservation across the entire sequence of frames. This result is independent of any particular disease transmission mechanism.

9.4. Scope and Limitations

The epidemiological interpretation presented here is theoretical. No patient-level observations, population surveillance dataset, incidence estimates, prevalence estimates, parameter estimation, hypothesis testing, or empirical disease forecasting is performed. The metric operator G_t should be interpreted as a mathematical representation of epidemiological information geometry rather than as an empirically estimated epidemiological risk metric. The transition operators A_t describe transformations of represented epidemiological information and are not asserted to coincide with biological transmission operators. An empirical extension would require disease-specific data, a defined epidemiological observation model, estimation of the contradiction matrices D_t, specification or estimation of the interaction matrices C_t, and validation of the resulting metric and transition operators. These tasks are outside the scope of the present theoretical study.

10. Conclusions and Future Directions

This paper has developed a theory of time-dependent plithogenic metric equivalence and metric transport for bounded operators on Hilbert spaces associated with evolving epidemiological states.

The main results are as follows. We constructed contradiction-induced strictly positive metric operators on the plithogenic epidemiological state space using a structured tensor product embedding $\tilde{D}_t = C_t \otimes D_t$. We defined G_t metric equivalence $A \sim_G B$ via $A^*GA = B^*GB$ and proved the reduction theorem $A \sim_G B$ if and only if $G^{(1/2)}A \sim G^{(1/2)}B$, together with polar and partial isometry characterizations. We identified metric invariants preserved under G-equivalence, namely the G-norm, kernel structure, finite-dimensional rank, and singular values of metric-weighted realizations. We introduced transport operators satisfying $A^*_tG_{t+1}A_t = G_t$, proved the explicit characterization $A = G_1^{-(1/2)}UG_0^{(1/2)}$ with U an isometry, and established the central multi-step transport theorem $\Phi^*_{t,s}G_t\Phi_{t,s} = G_s$. We defined the metric defect $\Delta_G(A,B) = \| A^*GA - B^*GB \|$, showed it is a pseudometric becoming a metric on the quotient $B(H)/\sim_G$, and established perturbation and stability estimates.

The central mathematical message is: local metric preservation at every frame implies global metric preservation across every finite evolution.

Limitations of the present work should be noted. The explicit contradiction construction is finite-dimensional and assumes $H_T = H_I = H_F = \mathbb{C}^n$. The component interaction matrix C_t is a modelling choice rather than a quantity derived from the plithogenic representation itself. The epidemiological interpretation is theoretical only and does not provide empirical validation. The results are stated for bounded operators; the unbounded case requires additional domain considerations.

Future research directions include extension to unbounded operators and semigroup theory; derivation of the component interaction matrix C_t from plithogenic epidemiological data; investigation of spectral properties and invariant subspaces for metric transport operators; development of numerical algorithms for computing the metric equivalence defect in high-dimensional settings; data-driven estimation of the plithogenic metric sequence G_t from observed epidemiological frames; application of the framework to other domains involving evolving uncertain states; and investigation of generalized inverses and pseudo-inverse transport operators.

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