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EOQ model for deteriorating items with exponential demand rate, price dependence and shortages under trapezoidal and neutrosophic fuzzy techniques

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ABSTRACT. In this study, we examined an Economic Order Quantity model for deteriorating items in the context of a completely backlogged exponential demand rate and price dependence. For global warming, we include the carbon tax in the total inventory cost of the proposed Economic Order Quantity model. This study aims to minimize the average total cost for the proposed Economic Order Quantity model. Using the shortage costs, we identify the different factors that impact the inventory cost. This research will examine inventory modeling in different fuzzy environments of uncertainty. In trapezoidal fuzzy environments, fuzzy values are considered inventory factors, including ordering costs, holding costs, and shortage costs. In a fuzzy neutrosophic method, fuzzy values are also considered inventory factors, including ordering costs, holding costs, and shortage costs. In this study, we developed the trapezoidal fuzzy Economic Order Quantity model and the neutrosophic fuzzy Economic Order Quantity model to optimize the average total cost. The trapezoidal and neutrosophic fuzzy Economic Order Quantity models are defuzzified by a trapezoidal intuitionistic defuzzification method and a neutrosophic trapezoidal defuzzification method, respectively. To identify the suitable neutrosophic fuzzy method, this study examines a comparison analysis between the neutrosophic fuzzy techniques using the proposed numerical example. The numerical example presents a comparative study between several fuzzy and crisp methods. Numerical calculations have also been done using the LINGO 21.0 software. The suggested model has been supported by graphical representation and sensitivity analysis. In the end, we come to a conclusion.

Keywords

Exponential demand; trapezoidal; neutrosophic; shortages; defuzzification.

1. Introduction

Fuzzy set theory is effective for the current scientific revolution in the disciplines of mathematical modeling, engineering, medical diagnosis, social sciences, and management. In his study article, Prof. L.A. Zadeh [19] demonstrated the impreciseness principle. This newsletter provides a brief overview of the distinction between fuzzy and crisp sets by examining membership classification and its formula. Smarandache [14] was the first to propose concepts of the neutrosophic set and neutrosophic logic, approaching them from the standpoint of nonstandard analysis.

Deli and Subas developed an approach to multi-attribute decision-making [4] that uses a ranking system for single-valued neutrosophic numbers. Suresh, Prakash, and Vengataasalam first developed a ranking system for neutrosophic trapezoidal fuzzy numbers [15].

Zimmermann [20] developed the fuzzy programming approach for multiobjective optimization problems by introducing the fuzzy set. The study conducted by Kundu and Chakrabarti [6] proposed an Economic Order Quantity (EOQ) Model that incorporates fuzzy demand and partial backlog. Dutta and Kumar's [5] study looked at a fuzzy inventory model that deals with managing deteriorating products that are in short supply, particularly when there is a backlog that is completely full.

A fuzzy inventory model was introduced in a study by M. Maragatham and P.K. Lakshmidevi [8] to handle the problem of decaying products with price-driven demand. D. Sharmila and R. Uthayakumar [13] created an inventory model in their work that makes use of fuzzy logic to manage shortages, exponential demand, and deteriorating products. Using a trapezoidal fuzzy number, Yao and Lee [18] created a fuzzy inventory for fuzzy order quantities with or without backorder. Trapezoidal fuzzy numbers were used by Bulancak and Kirkavak [2] for EOQ with backorder. Sen and Malakar [12]examined an EOQ model with scarcity, taking into account several factors including parabolic, trapezoidal, and triangular fuzzy numbers.

Backorders are any amounts of inventory that a business customer has requested but has not yet received because it is currently out of stock. Backorders are an effective component of an organization's inventory control assessment [9]. The number of backordered items and the turnaround time for these client requests might provide insight into how well the business handles its inventory. When products gradually lose value due to damage, degradation, or disintegration, this process is referred to as deterioration. This phenomenon is especially pertinent when it comes to materials that are meant for storage and possible future usage [16].

Therefore, deteriorating cannot be prevented in any type of company situation.

Many years before the sales period, the companies must establish a number of strategies with the goal of minimizing expenses and increasing profits [10]. Thus, in the current competitive market, information such as deterioration, shortages, and deviations in the manufacturing process's reliability is gaining attention.

Professionals in inventory management are now much more equipped to simulate carbon emissions and carbon tax policies in relation to supply chain management and production difficulties. The carbon tax's main objectives are cost containment and long-term industrial structure preservation. An inventory model developed by Wu and Chan [17] deteriorates when a carbon tax is implemented. To maximize the refill period, Paul et al. [11] looked at carbon pricing within a green inventory model. Maity et al. [7] recently developed an Economic Order model with a supply-rate-dependent carbon tax scheme. In this study, we develop an Economic Order Quantity model with deteriorating items and an exponential demand rate, which depends on time;demand also depends on the price of inventory. We try to minimize the carbon tax to minimize the average total cost of the proposed Economic Order Quantity model. The inventory model is transformed into two fuzzy inventory models. In the first fuzzy model, it is assumed that the trapezoidal fuzzy values represent holding costs, shortage costs, and purchasing costs. An efficient method to optimize the results for the number of inventory orders, the time, and the average total cost is presented. In addition, an effective algorithm is developed to identify the optimized result, and a numerical example is provided to demonstrate our methodology.

The structure of this paper is as follows:

- The fundamental definitions, as well as several assumptions and notations that are highly helpful in extending this suggested model, are provided in Sections 2 and 3.
- Section 4 provided a development of the crisp Economic Order Quantity model.
- Section 5 provided numerical analysis and sensitivity analysis for the Crisp model.
- We developed a trapezoidal fuzzy Economic Order Quantity model in Section 6. Section 6 provided a normalized trapezoidal defuzzification method and a generalized trapezoidal defuzzification method.
- We developed a neutrosophic fuzzy Economic Order Quantity model in Section 7.
- We provided a comparison analysis of neutrosophic techniques in Section 8. In this section, we developed another neutrosophic approach in the neutrosophic fuzzy economic order quantity model for comparison analysis.

- Section 9 provided a comparative study between crisp and different fuzzy environments. Section 10 provided the conclusion of this study. In addition, the limitations and future research work of this study are provided in Section 10.

2. Preliminary

In this section, we outline some basic definitions which will help clarify the concepts discussed later.

2.1. *Intuitionistic Fuzzy Set(IFS)*

Let I be a fixed known set. An intuitionistic fuzzy set (IFS) L in I is defined by $L = \{< x, \phi_L(x), \Delta_L(x) >: x \in I\}.\phi_L(x) : I \rightarrow [0,1]$ and $\Delta_L(x) : I \rightarrow [0,1]$ symbolize as membership function and nonmembership function [16],respectively where $0 \leq \phi_L(x) + \Delta_L(x) \leq 1$ for all $x \in I$.

2.2. *Trapezoidal Fuzzy Number(TFN)*

Let $L = [a, b, c, d]$ be a trapezoidal fuzzy number,and the membership function [1] is defined by $\phi_L(x) = \begin{cases} \frac{(x-a)}{(b-a)}; a \leq x \leq b \\ 1; b \leq x \leq c \\ \frac{(d-x)}{(d-c)}; c \leq x \leq d; \\ 0; Otherwise \end{cases}$ where $a, b, c, d \in \mathbb{R}$.

2.3. *Generalised Trapezoidal Fuzzy Number(GTFN)*

Let $L = [a, b, c, d; \omega]$ be a trapezoidal fuzzy number,and the membership function [1] is defined by $\phi_L(x) = \begin{cases} \frac{\omega(x-a)}{(b-a)}; a \leq x \leq b \\ \omega; b \leq x \leq c \\ \frac{\omega(d-x)}{(d-c)}; c \leq x \leq d; \\ 0; Otherwise \end{cases}$ where $a, b, c, d \in \mathbb{R}$ and $0 < \omega \leq 1$.

2.4. *Neutrosophic set*

Let U be a universal set. An indeterminacy I_L, a truth T_L, and a falsity-membership function F_L — [16] where $I_L(x)$ and $F_L(x)$ are real standard elements of $[0,1]$ — distinguish a neutrosophic set L in U. It is defined by $L = \{< x, I_L(x), T_L(x), F_L(x) >: x \in U, I_L(x), T_L(x), F_L(x) \in (0,1)\}$ and where $0 \leq I_L(x), T_L(x), F_L(x) \leq 3$.

2.5. *Single valued Trapezoidal Neutrosophic Number*

A particular set of neutrosophic type in $\mathbb{R}$ is a single-valued trapezoidal neutrosophic number $L_{neu} = (a_1, b_1, c_1, d_1; a_2, b_2, c_2, d_2; a_3, b_3, c_3, d_3)$ [3], whose truth, indeterminacy, and falsity membership function are defined by

$$T_L(x) = \begin{cases} \frac{(x-a_1)}{(b_1-a_1)}; a_1 \le x \le b_1 \\ 1; b_1 \le x \le c_1 \\ \frac{(d_1-x)}{(d_1-c_1)}; c_1 \le x \le d_1; \\ 0; Otherwise \end{cases}$$

$$I_L(x) = \begin{cases} \frac{(b_2-x)}{(b_2-a_2)}; a_2 \le x \le b_2 \\ 0; b_2 \le x \le c_2 \\ \frac{(x-c_2)}{(d_2-c_2)}; c_2 \le x \le d_2; \\ 1; Otherwise \end{cases}$$

$$F_L(x) = \begin{cases} \frac{(b_3-x)}{(b_3-a_3)}; a_3 \le x \le b_3 \\ 0; b_3 \le x \le c_3 \\ \frac{(x-c_3)}{(d_3-c_3)}; c_3 \le x \le d_3; \\ 1; Otherwise \end{cases}$$

respectively, where $0 \le T_L(x) + I_L(x) + F_L(x) \le 3$.

3. **Notations and Assumptions**

The following presumptions and notations are applied throughout this chapter in order to develop the suggested models.

3.1. *Notations*

Here, the following notations are utilised:

C_0 : The cost of placing each order

C_1 : The cost of holding each unit per unit of time

C_2 : The cost of purchasing one unit at a time

C_3 : Cost per unit of time shortage

θ : Function of deterioration $(0 < \theta < 1)$

D : Demand rate per unit of time at any given time t

T : Duration of the ordering cycle

Q : The quantity ordered per unit

C_5 : The specified amount of carbon emissions associated with inventory maintenance

C_6 : The variable amount of carbon emissions associated with inventory maintenance

C_4 : Government charge for per unit of carbon emission

$Z(t_1, T)$: Total cost of inventory per unit of time

$\bar{C}_0$: The fuzzy cost of placing each order

$\bar{C}_1$: The fuzzy cost of holding each unit per unit of time

$\bar{C}_3$: Fuzzy cost per unit of time shortage

$\bar{Z}(t_1, T)$: Fuzzy total cost of inventory per unit of time

3.2. *Assumptions*

The following presumptions guide the development of the suggested model.

(1) The demand rate increases exponentially with time t ($D(t) = ce^{at}$ $c > 0; a > 0$).

(2) No lead time exists.

(3) Complete backlogs and shortages are permitted.

(4) Deterioration is not replenished or mended during the cycle.

(5) The rate of replenishment never stops.

(6) A time-dependent exponential function characterizes the holding cost ($i.e. e^{at}$; $a > 0$).

(7) A linear time-dependent quadratic function characterizes the deterioration cost ($i.e. kt$; $k > 0$).

(8) The carbon tax is a linear function dependent on time and inventory.

4. **Crisp Model**

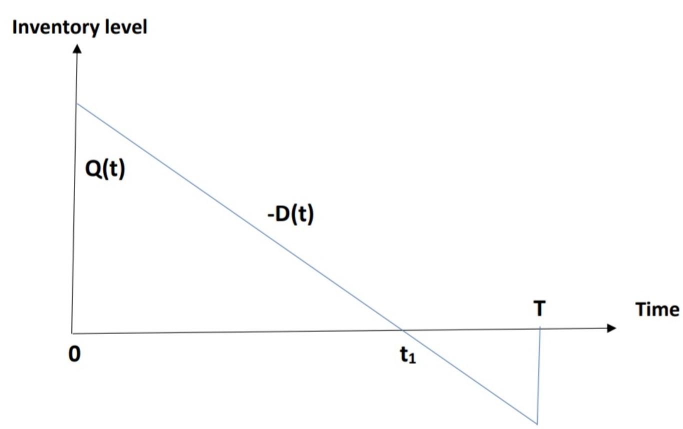

FIGURE 1. Graphical representation of the proposed model

The launch or purchase of the product depends on Q and the following fulfillment of backorders. The inventory level steadily declines over the course of $[0, t_1]$ and eventually drops to zero.

Shortages and a complete backlog may arise during certain periods. Let $I_1(t)$ and $I_2(t)$ represent the level of inventory available at time t, which is derived using the following equations:

$$\frac{dI_1}{dt} + \theta I_1 = -ce^{at} + hs; 0 \le t \le t_1 \tag{1}$$

with $I_1(0) = Q, I_1(t_1) = 0$

$$\frac{dI_2}{dt} = -ce^{at} + hs; t_1 \le t \le T \tag{2}$$

with $I_2(t_1) = 0, I_2(T) = -S$.

Solving (1) and (2), we get the solutions. The solutions are given by

$$I_1 = \frac{c}{(\theta + a)}[\frac{e^{(\theta+a)t_1}}{e^{\theta t}} - e^{at}] + \frac{hs}{\theta}[1 - e^{\theta(t_1 - t)}]. \tag{3}$$

$$I_2 = \frac{c}{a}(e^{at_1} - e^{at}) + \frac{hs}{\theta}(t - t_1). \tag{4}$$

$$Q = \frac{c}{(\theta + a)}[e^{(\theta+a)t_1} - 1] + \frac{hs}{\theta}[1 - e^{\theta t_1}]. \tag{5}$$

$$S = \frac{c}{a}(e^{aT} - e^{at_1}) - hs(T - t_1). \tag{6}$$

Holding cost $(HC) = C_1 \int_0^{t_1} e^{at} I_1(t)dt = C_1[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}]$.

(7)

Deterioration cost $(DC) = C_2 \int_0^{t_1} ktI_1(t)dt = C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1 e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1 e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}]$.

(8)

Shortage Cost $(SC) = C_3 S = C_3[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T - t_1)]$.

(9)

Considering [7], Carbon Tax $(CT) = C_4(C_5 + C_6 Q t_1) = C_4 C_5 + \frac{C_4 C_6 c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4 C_6 hs}{\theta}[1 - e^{\theta t_1}]$.

(10)

Total cost per unit of time $Z(t_1, T) = \frac{(C_0+HC+DC+SC+CT)}{T}$.

By solving the following equations, the ideal value of t_1 and T can be found to minimize the total cost function per time $Z(t_1, T)$: $\frac{\partial Z}{\partial t_1} = 0$ and $\frac{\partial Z}{\partial T} = 0$.

Now, $\frac{\partial Z}{\partial t_1} = \frac{1}{T}[C_1\{ce^{(\theta+a)t_1} - \frac{2ae^{2at_1}}{(\theta+a)(\theta-a)} - \frac{ce^{2at_1}}{(\theta+a)} + \frac{hse^{at_1}}{\theta} - \frac{hse^{\theta t_1}}{(\theta-a)} + \frac{hsae^{at_1}}{\theta(\theta-a)}\} + C_2\{(kce^{(\theta+a)t_1} - khse^{\theta t_1})(\frac{1}{\theta^2} - \frac{t_1 e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) + (\frac{kce^{(\theta+a)t_1}}{\theta+a} - \frac{khse^{\theta t_1}}{\theta})(\frac{e^{-\theta t_1}}{\theta} + t_1 e^{-\theta t_1} - \frac{e^{-\theta t_1}}{\theta)} - \frac{kct_1 e^{at_1}}{(\theta+a)} + \frac{khst_1}{\theta}\} + C_3(hs - ce^{at_1} + C_4 C_6(ce^{(\theta+a)t_1} - hse^{\theta t_1})] = 0$

(11)

and $\frac{\partial Z}{\partial T} = -\frac{1}{T^2}[C_0 + C_1[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C_3[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]] + \frac{C_3}{T}(ce^{aT} - hs) = 0.$

(12)

Using LINGO software and solving the following equations (11) and (12), we get the optimum value of t_1 and T to minimize $Z(t_1, T)$. Since $\frac{\partial^2 D}{\partial t_1^2} > 0$ and $\frac{\partial^2 D}{\partial T^2} > 0$ for this t_1 and T respectively.

5. **Numerical analysis for Crisp model**

Input value

In Table 1, we develop a numerical example for the proposed EOQ model.

Parameters	Value
C_0	Rs. 200 per unit order
C_1	Rs. 16 per year
C_2	Rs. 25 per year
C_3	Rs. 15 per year
C_4	Rs. 6 per year
C_5	Rs. 3 per year
C_6	Rs. 2 per year
θ	0.12 units per year
a	0.8 per unit
c	100 per year
h	2
s	Rs. 3 per unit per year
k	20

Table 1. Numerical input value for crisp model

Output value

In Table 2, we present the optimized output value for the above numerical example.

6. **Trapezoidal fuzzy model**

We examine the model in a trapezoidal fuzzy environment. It is difficult to define every parameter precisely because of uncertainty.

Let $C_0 = (\phi_1, \phi_2, \phi_3, \phi_4), C_1 = (\sigma_1, \sigma_2, \sigma_3, \sigma_4); C_3 = (\delta_1, \delta_2, \delta_3, \delta_4)$ be the trapezoidal fuzzy

Parameters	value
t_1	0.19
T	0.23
Q	19.92
$Z(t_1, T)$	1716.47

Table 2. Numerical output value for crisp model

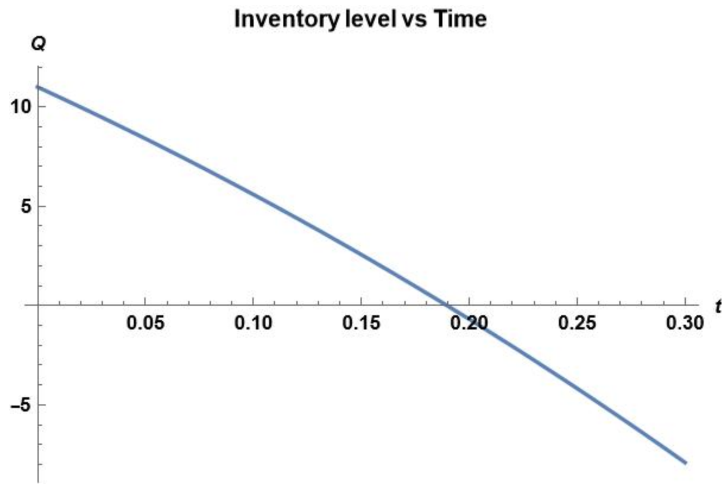

Figure 2. Graphical representation for the Crisp model

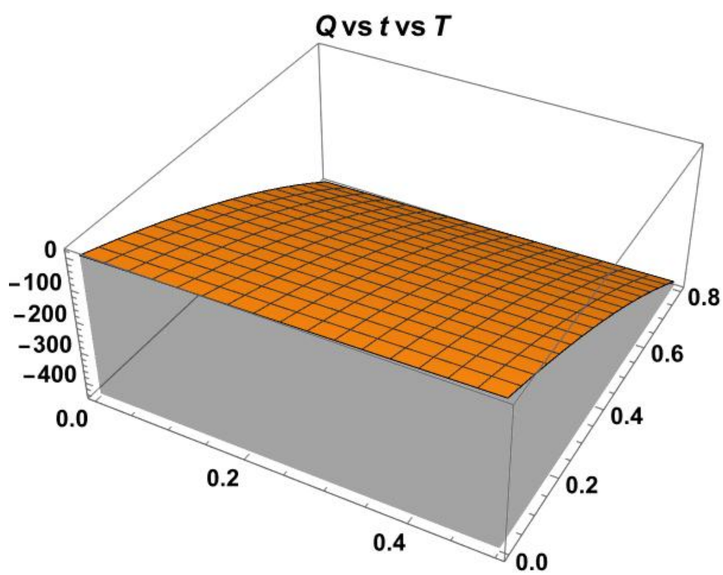

Figure 3. Inventory level vs Shortage starting time vs Total time

number. Then, the fuzzy total cost per unit time $\bar{Z}(t_1, T) = A, B, C, D$ where $A = \frac{1}{T}\Bigg[\phi_1 + \sigma_1[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] +$

Parameters	Change(%)	Q	t_1	T	Z
a	-20	19.62	0.19	0.27	1690.36
a	-10	19.78	0.19	0.25	1704.11
a	10	20.08	0.19	0.22	1727.58
a	20	20.23	0.19	0.21	1737.56
c	-20	15.71	0.19	0.36	1505.83
c	-10	17.81	0.19	0.3	1620.04
c	10	21.69	0.19	0.19	1787.76
c	20	22.96	0.18	0.18	1850.34
C_2	-20	21.55	0.21	0.21	1648.72
C_2	-10	20.81	0.2	0.21	1685.84
C_2	10	19.15	0.19	0.25	1741.49
C_2	20	18.45	0.18	0.26	1762.54
θ	-20	19.9	0.19	0.23	1715.69
θ	-10	19.91	0.19	0.23	1716.08
θ	10	19.93	0.19	0.23	1716.86
θ	20	19.94	0.19	0.23	1717.25
k	-20	21.55	0.21	0.21	1648.72
k	-10	20.81	0.2	0.21	1685.84
k	10	19.15	0.19	0.25	1741.49
k	20	18.45	0.18	0.26	1762.54
h	-20	20.16	0.19	0.23	1725.31
h	-10	20.04	0.19	0.23	1720.92
h	10	19.81	0.19	0.24	1711.96
h	20	19.69	0.19	0.24	1707.39
C_6	-20	20.33	0.21	0.21	1684.38
C_6	-10	20.12	0.19	0.22	1700.93
C_6	10	19.72	0.19	0.24	1731.14
C_6	20	19.53	0.19	0.25	1745.05

TABLE 3. Sensitivity analysis for crisp parameters for the crisp model changing from -20% to 20%

$$C_2[\{\tfrac{kce^{(\theta+a)t_1}}{(\theta+a)} - \tfrac{khse^{\theta t_1}}{\theta}\}(\tfrac{1}{\theta^2} - \tfrac{t_1e^{-\theta t_1}}{\theta} - \tfrac{e^{-\theta t_1}}{\theta^2}) - \tfrac{kc}{(\theta+a)}(\tfrac{t_1e^{at_1}}{a} + \tfrac{1}{a^2} - \tfrac{e^{at_1}}{a^2}) + \tfrac{khst_1^2}{2\theta}] + \delta_1[\tfrac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \tfrac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \tfrac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]\Bigg],$$

$$B = \frac{1}{T}\Bigg[\phi_2 + \sigma_2[\tfrac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\tfrac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \tfrac{c}{2a(\theta+a)}(e^{2at_1}-1) + \tfrac{hs}{\theta a}(e^{at_1}-1) - \tfrac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] +$$

$C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1 e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1 e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_2[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg]$,

$C = \frac{1}{T}\Bigg[\phi_3 + \sigma_3[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1 e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1 e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_3[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg]$,

and $D = \frac{1}{T}\Bigg[\phi_4 + \sigma_4[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1 e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1 e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_4[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg]$.

6.1. *Normalized defuzzification method*

Now, the α-cuts of the trapezoidal fuzzy numbers are given by $Z_L(\alpha) = A + (B-A)\alpha$ where

$A = \frac{1}{T}\Bigg[\phi_1 + \sigma_1[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1 e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1 e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_1[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg]$,

$B = \frac{1}{T}\Bigg[\phi_2 + \sigma_2[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1 e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1 e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_2[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg]$,

and $Z_R(\alpha) = D + (C-D)\alpha$ where $C = \frac{1}{T}\Bigg[\phi_3 + \sigma_3[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1 e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1 e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_3[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg]$,

$D = \frac{1}{T}\Bigg[\phi_4 + \sigma_4[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1 e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1 e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_4[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg]$.

The defuzzified value of the fuzzy number $D(Z(t_1,T))$ obtained by the centroid approach is provided by
$\frac{\int_0^1 \alpha \frac{Z_L(\alpha)+Z_R(\alpha)}{2}d\alpha}{\int_0^1 \alpha d\alpha} = \frac{A+2B+2C+D}{6}$.
By solving the following equations, the ideal value of t_1 and T can be found to minimize the total cost function per time $\bar{Z}(t_1,T)$: $\frac{\partial D(Z)}{\partial t_1} = 0$ and $\frac{\partial D(Z)}{\partial T} = 0$. Now, $\frac{\partial D}{\partial t_1} = \frac{1}{6T}[\sigma_1\{ce^{(\theta+a)t_1} - \frac{2ae^{2at_1}}{(\theta+a)(\theta-a)} - \frac{ce^{2at_1}}{(\theta+a)} + \frac{hse^{at_1}}{\theta} - \frac{hse^{\theta t_1}}{(\theta-a)} + \frac{hsae^{at_1}}{\theta(\theta-a)}\} + C_2\{(kce^{(\theta+a)t_1} - khse^{\theta t_1})(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) + (\frac{kce^{(\theta+a)t_1}}{\theta+a} - \frac{khse^{\theta t_1}}{\theta})(\frac{e^{-\theta t_1}}{\theta} + t_1e^{-\theta t_1} - \frac{e^{-\theta t_1}}{\theta)}) - \frac{kct_1e^{at_1}}{(\theta+a)} + \frac{khst_1}{\theta}\} + \delta_1(hs - ce^{at_1} + C_4C_6(ce^{(\theta+a)t_1} - hse^{\theta t_1})] + \frac{1}{3T}[\sigma_2\{ce^{(\theta+a)t_1} - \frac{2ae^{2at_1}}{(\theta+a)(\theta-a)} - \frac{ce^{2at_1}}{(\theta+a)} + \frac{hse^{at_1}}{\theta} - \frac{hse^{\theta t_1}}{(\theta-a)} + \frac{hsae^{at_1}}{\theta(\theta-a)}\} + C_2\{(kce^{(\theta+a)t_1} - khse^{\theta t_1})(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) + (\frac{kce^{(\theta+a)t_1}}{\theta+a} - \frac{khse^{\theta t_1}}{\theta})(\frac{e^{-\theta t_1}}{\theta} + t_1e^{-\theta t_1} - \frac{e^{-\theta t_1}}{\theta)}) - \frac{kct_1e^{at_1}}{(\theta+a)} + \frac{khst_1}{\theta}\} + \delta_2(hs - ce^{at_1} + C_4C_6(ce^{(\theta+a)t_1} - hse^{\theta t_1})] + \frac{1}{3T}[\sigma_3\{ce^{(\theta+a)t_1} - \frac{2ae^{2at_1}}{(\theta+a)(\theta-a)} - \frac{ce^{2at_1}}{(\theta+a)} + \frac{hse^{at_1}}{\theta} - \frac{hse^{\theta t_1}}{(\theta-a)} + \frac{hsae^{at_1}}{\theta(\theta-a)}\} + C_2\{(kce^{(\theta+a)t_1} - khse^{\theta t_1})(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) + (\frac{kce^{(\theta+a)t_1}}{\theta+a} - \frac{khse^{\theta t_1}}{\theta})(\frac{e^{-\theta t_1}}{\theta} + t_1e^{-\theta t_1} - \frac{e^{-\theta t_1}}{\theta)}) - \frac{kct_1e^{at_1}}{(\theta+a)} + \frac{khst_1}{\theta}\} + \delta_3(hs - ce^{at_1} + C_4C_6(ce^{(\theta+a)t_1} - hse^{\theta t_1})] + \frac{1}{6T}[\sigma_4\{ce^{(\theta+a)t_1} - \frac{2ae^{2at_1}}{(\theta+a)(\theta-a)} - \frac{ce^{2at_1}}{(\theta+a)} + \frac{hse^{at_1}}{\theta} - \frac{hse^{\theta t_1}}{(\theta-a)} + \frac{hsae^{at_1}}{\theta(\theta-a)}\} + C_2\{(kce^{(\theta+a)t_1} - khse^{\theta t_1})(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) + (\frac{kce^{(\theta+a)t_1}}{\theta+a} - \frac{khse^{\theta t_1}}{\theta})(\frac{e^{-\theta t_1}}{\theta} + t_1e^{-\theta t_1} - \frac{e^{-\theta t_1}}{\theta)}) - \frac{kct_1e^{at_1}}{(\theta+a)} + \frac{khst_1}{\theta}\} + \delta_4(hs - ce^{at_1} + C_4C_6(ce^{(\theta+a)t_1} - hse^{\theta t_1})] = 0$

(13)

and $\frac{\partial D}{\partial T} = -\frac{1}{6T^2}[\phi_1 + \sigma_1[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_1[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]] + \frac{\delta_1}{6T}(ce^{aT} - hs) - \frac{1}{3T^2}[\phi_2 + \sigma_2[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_2[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]] + \frac{\delta_2}{3T}(ce^{aT} - hs) - \frac{1}{3T^2}[\phi_3 + \sigma_3[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_3[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]] + \frac{\delta_3}{3T}(ce^{aT} - hs) - \frac{1}{6T^2}[\phi_4 + \sigma_4[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + +\delta_4[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]] + \frac{\delta_4}{6T}(ce^{aT} - hs) = 0$

(14)

Solving (13) and (14), we get the values of t_1 and t. Since $\frac{\partial^2 D}{\partial t_1^2} > 0$ and $\frac{\partial^2 D}{\partial T^2} > 0$ for this t_1 and T respectively, these values minimize $\bar{Z}$.

6.2. *Generalized defuzzification method*

Now, the α-cuts of the trapezoidal fuzzy numbers are given by $Z_L(\alpha) = A + \frac{(B-A)\alpha}{\omega}; 0 < \omega < 1$ where

$$A=\frac{1}{T}\Bigg[\phi_1+\sigma_1[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\}-\frac{c}{2a(\theta+a)}(e^{2at_1}-1)+\frac{hs}{\theta a}(e^{at_1}-1)-\frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}]+C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)}-\frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2}-\frac{t_1e^{-\theta t_1}}{\theta}-\frac{e^{-\theta t_1}}{\theta^2})-\frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a}+\frac{1}{a^2}-\frac{e^{at_1}}{a^2})+\frac{khst_1^2}{2\theta}]+\delta_1[\frac{c}{a}(e^{aT}-e^{at_1})-hs(T-t_1)]+C_4C_5+\frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1]+\frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]\Bigg],$$

$$B=\frac{1}{T}\Bigg[\phi_2+\sigma_2[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\}-\frac{c}{2a(\theta+a)}(e^{2at_1}-1)+\frac{hs}{\theta a}(e^{at_1}-1)-\frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}]+C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)}-\frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2}-\frac{t_1e^{-\theta t_1}}{\theta}-\frac{e^{-\theta t_1}}{\theta^2})-\frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a}+\frac{1}{a^2}-\frac{e^{at_1}}{a^2})+\frac{khst_1^2}{2\theta}]+\delta_2[\frac{c}{a}(e^{aT}-e^{at_1})-hs(T-t_1)]+C_4C_5+\frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1]+\frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]\Bigg],$$

and $Z_R(\alpha)=D+\frac{(C-D)\alpha}{\omega}; 0<\omega<1$ where $C=\frac{1}{T}\Bigg[\phi_3+\sigma_3[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\}-\frac{c}{2a(\theta+a)}(e^{2at_1}-1)+\frac{hs}{\theta a}(e^{at_1}-1)-\frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}]+C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)}-\frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2}-\frac{t_1e^{-\theta t_1}}{\theta}-\frac{e^{-\theta t_1}}{\theta^2})-\frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a}+\frac{1}{a^2}-\frac{e^{at_1}}{a^2})+\frac{khst_1^2}{2\theta}]+\delta_3[\frac{c}{a}(e^{aT}-e^{at_1})-hs(T-t_1)]+C_4C_5+\frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1]+\frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]\Bigg],$

$$D=\frac{1}{T}\Bigg[\phi_4+\sigma_4[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\}-\frac{c}{2a(\theta+a)}(e^{2at_1}-1)+\frac{hs}{\theta a}(e^{at_1}-1)-\frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}]+C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)}-\frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2}-\frac{t_1e^{-\theta t_1}}{\theta}-\frac{e^{-\theta t_1}}{\theta^2})-\frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a}+\frac{1}{a^2}-\frac{e^{at_1}}{a^2})+\frac{khst_1^2}{2\theta}]+\delta_4[\frac{c}{a}(e^{aT}-e^{at_1})-hs(T-t_1)]+C_4C_5+\frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1]+\frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]\Bigg].$$

The defuzzified value of the fuzzy number $D(Z(t_1,T))$ obtained by the centroid approach is provided by
$\frac{\int_0^1\alpha\frac{Z_L(\alpha)+Z_R(\alpha)}{2}d\alpha}{\int_0^1\alpha d\alpha}=\frac{A+D}{2}+\frac{B-A+C-D}{3\omega}=\frac{(3\omega-2)A+2B+2C+(3\omega-2)}{6\omega}.$
By solving the following equations, the ideal value of t_1 and T can be found to minimize the total cost function per time $\bar{Z}(t_1,T)$: $\frac{\partial D(Z)}{\partial t_1}=0$ and $\frac{\partial D(Z)}{\partial T}=0$.
Now, $\frac{\partial D}{\partial t_1}=\frac{(3\omega-2)}{6\omega T}[\sigma_1\{ce^{(\theta+a)t_1}-\frac{2ae^{2at_1}}{(\theta+a)(\theta-a)}-\frac{ce^{2at_1}}{(\theta+a)}+\frac{hse^{at_1}}{\theta}-\frac{hse^{\theta t_1}}{(\theta-a)}+\frac{hsae^{at_1}}{\theta(\theta-a)}\}+C_2\{(kce^{(\theta+a)t_1}-khse^{\theta t_1})(\frac{1}{\theta^2}-\frac{t_1e^{-\theta t_1}}{\theta}-\frac{e^{-\theta t_1}}{\theta^2})+(\frac{kce^{(\theta+a)t_1}}{\theta+a}-\frac{khse^{\theta t_1}}{\theta})(\frac{e^{-\theta t_1}}{\theta}+t_1e^{-\theta t_1}-\frac{e^{-\theta t_1}}{\theta)})-\frac{kct_1e^{at_1}}{(\theta+a)}+\frac{khst_1}{\theta}\}+\delta_1(hs-ce^{at_1}+C_4C_6(ce^{(\theta+a)t_1}-hse^{\theta t_1})]+\frac{1}{3\omega T}[\sigma_2\{ce^{(\theta+a)t_1}-\frac{2ae^{2at_1}}{(\theta+a)(\theta-a)}-\frac{ce^{2at_1}}{(\theta+a)}+\frac{hse^{at_1}}{\theta}-\frac{hse^{\theta t_1}}{(\theta-a)}+\frac{hsae^{at_1}}{\theta(\theta-a)}\}+C_2\{(kce^{(\theta+a)t_1}-khse^{\theta t_1})(\frac{1}{\theta^2}-\frac{t_1e^{-\theta t_1}}{\theta}-\frac{e^{-\theta t_1}}{\theta^2})+(\frac{kce^{(\theta+a)t_1}}{\theta+a}-\frac{khse^{\theta t_1}}{\theta})(\frac{e^{-\theta t_1}}{\theta}+t_1e^{-\theta t_1}-\frac{e^{-\theta t_1}}{\theta)})-\frac{kct_1e^{at_1}}{(\theta+a)}+\frac{khst_1}{\theta}\}+\delta_2(hs-ce^{at_1}+C_4C_6(ce^{(\theta+a)t_1}-hse^{\theta t_1})]+\frac{1}{3\omega T}[\sigma_3\{ce^{(\theta+a)t_1}-\frac{2ae^{2at_1}}{(\theta+a)(\theta-a)}-\frac{ce^{2at_1}}{(\theta+a)}+\frac{hse^{at_1}}{\theta}-\frac{hse^{\theta t_1}}{(\theta-a)}+\frac{hsae^{at_1}}{\theta(\theta-a)}\}+C_2\{(kce^{(\theta+a)t_1}-khse^{\theta t_1})(\frac{1}{\theta^2}-\frac{t_1e^{-\theta t_1}}{\theta}-\frac{e^{-\theta t_1}}{\theta^2})+(\frac{kce^{(\theta+a)t_1}}{\theta+a}-\frac{khse^{\theta t_1}}{\theta})(\frac{e^{-\theta t_1}}{\theta}+t_1e^{-\theta t_1}-\frac{e^{-\theta t_1}}{\theta)})-\frac{kct_1e^{at_1}}{(\theta+a)}+\frac{khst_1}{\theta}\}+\delta_3(hs-ce^{at_1}+C_4C_6(ce^{(\theta+a)t_1}-hse^{\theta t_1})]+\frac{(3\omega-2)}{6\omega T}[\sigma_4\{ce^{(\theta+a)t_1}-\frac{2ae^{2at_1}}{(\theta+a)(\theta-a)}-\frac{ce^{2at_1}}{(\theta+a)}+\frac{hse^{at_1}}{\theta}-\frac{hse^{\theta t_1}}{(\theta-a)}+\frac{hsae^{at_1}}{\theta(\theta-a)}\}+C_2\{(kce^{(\theta+a)t_1}-khse^{\theta t_1})(\frac{1}{\theta^2}-\frac{t_1e^{-\theta t_1}}{\theta}-\frac{e^{-\theta t_1}}{\theta^2})+(\frac{kce^{(\theta+a)t_1}}{\theta+a}-\frac{khse^{\theta t_1}}{\theta})(\frac{e^{-\theta t_1}}{\theta}+t_1e^{-\theta t_1}-\frac{e^{-\theta t_1}}{\theta)})-\frac{kct_1e^{at_1}}{(\theta+a)}+\frac{khst_1}{\theta}\}+$

$\delta_4(hs - ce^{at_1} + C_4C_6(ce^{(\theta+a)t_1} - hse^{\theta t_1})] = 0$

(15)

and $\frac{\partial D}{\partial T} = -\frac{(3\omega-2)}{6\omega T^2}[\phi_1 + \sigma_1[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_1[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T - t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]] + \frac{\delta_1}{6T}(ce^{aT} - hs) - \frac{1}{3\omega T^2}[\phi_2 + \sigma_2[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_2[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T - t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]] + \frac{\delta_2}{3T}(ce^{aT} - hs) - \frac{1}{3\omega T^2}[\phi_3 + \sigma_3[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \delta_3[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T - t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]] + \frac{\delta_3}{3T}(ce^{aT} - hs) - \frac{(3\omega-2)}{6\omega T^2}[\phi_4 + \sigma_4[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + +\delta_4[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T - t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]] + \frac{\delta_4}{6T}(ce^{aT} - hs) = 0$

(16)

Solving (15) and (16), we get the values of t_1 and t. Since $\frac{\partial^2 D}{\partial t_1^2} > 0$ and $\frac{\partial^2 D}{\partial T^2} > 0$ for this t_1 and T respectively, these values minimize $\bar{Z}$.

7. Neutrosophic fuzzy model

The inventory model with backorder in a neutrosophic environment is explained in this section. We use trapezoidal neutrosophic numbers to describe the inventory carrying cost, holding cost, and shortage cost since they are in trapezoidal neutrosophic numbers:

Let $\bar{C}_0 = (C_{01}, C_{02}, C_{03}, C_{04})(C'_{01}, C'_{02}, C'_{03}, C'_{04})(C''_{01}, C''_{02}, C''_{03}, C''_{04})$,
$\bar{C}_1 = (C_{11}, C_{12}, C_{13}, C_{14})(C'_{11}, C'_{12}, C'_{13}, C'_{14})(C''_{11}, C''_{12}, C''_{13}, C''_{14})$,
and $\bar{C}_3 = (C_{31}, C_{32}, C_{33}, C_{34})(C'_{31}, C'_{32}, C'_{33}, C'_{34})(C''_{31}, C''_{32}, C''_{33}, C''_{34})$.
Let $Z^* = (a_1, a_2, a_3, a_4)(b_1, b_2, b_3, b_4)(d_1, d_2, d_3, d_4)$ be the trapezoidal neutrosophic fuzzy number.

Taking into account [8], the neutrosophic fuzzy average total cost is defined by

$\bar{Z}(t_1, T) = \frac{1}{T}\Bigg[\bar{C}_0 + \bar{C}_1[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} - 1) + \frac{hs}{\theta a}(e^{at_1} - 1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1 - e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + \bar{C}_3[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T - t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg] = \frac{1}{T}\Bigg[(C_{01}, C_{02}, C_{03}, C_{04}, C'_{01}, C'_{02}, C'_{03}, C'_{04}, C''_{01}, C''_{02}, C''_{03}, C''_{04}) + (C_{11}, C_{12}, C_{13}, C_{14}, C'_{11}, C'_{12}, C'_{13}, C'_{14}, C''_{11}, C''_{12}, C''_{13}, C''_{14})[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1} -$

$1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + (C_{31}, C_{32}, C_{33}, C_{34}, C'_{31}, C'_{32}, C'_{33}, C'_{34}, C''_{31}, C''_{32}, C''_{33}, C''_{34})[\frac{c}{a}(e^{aT}-e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1] + \frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]\Bigg] = (a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4)(d_1,d_2,d_3,d_4)$

where $a_1 = \frac{1}{T}\Bigg[C_{01} + C_{11}[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C_{31}[\frac{c}{a}(e^{aT}-e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1] + \frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]\Bigg]$,

$a_2 = \Bigg[C_{02} + C_{12}[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C_{32}[\frac{c}{a}(e^{aT}-e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1] + \frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]\Bigg]$,

$a_3 = \frac{1}{T}\Bigg[C_{03} + C_{13}[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C_{33}[\frac{c}{a}(e^{aT}-e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1] + \frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]\Bigg]$

$a_4 = \frac{1}{T}\Bigg[C_{04} + C_{14}[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C_{34}[\frac{c}{a}(e^{aT}-e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1] + \frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]\Bigg]$,

$b_1 = \frac{1}{T}\Bigg[C'_{01} + C'_{11}[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C'_{31}[\frac{c}{a}(e^{aT}-e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1] + \frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]\Bigg]$,

$b_2 = \frac{1}{T}\Bigg[C'_{02} + C'_{12}[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C'_{32}[\frac{c}{a}(e^{aT}-e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1] + \frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]\Bigg]$,

$b_3 = \frac{1}{T}\Bigg[C'_{03} + C'_{13}[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] +$

$$C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C'_{33}[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg],$$

$$b_4 = \frac{1}{T}\Bigg[C'_{04}+C'_{14}[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C'_{34}[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg],$$

$$d_1 = \frac{1}{T}\Bigg[C''_{01}+C''_{11}[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C''_{31}[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg],$$

$$d_2 = \frac{1}{T}\Bigg[C''_{02}+C''_{12}[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C''_{32}[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg],$$

$$d_3 = \frac{1}{T}\Bigg[C''_{03}+C''_{13}[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C''_{33}[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg],$$

$$d_4 = \frac{1}{T}\Bigg[C''_{04}+C''_{14}[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\} - \frac{c}{2a(\theta+a)}(e^{2at_1}-1) + \frac{hs}{\theta a}(e^{at_1}-1) - \frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}] + C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)} - \frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) - \frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a} + \frac{1}{a^2} - \frac{e^{at_1}}{a^2}) + \frac{khst_1^2}{2\theta}] + C''_{34}[\frac{c}{a}(e^{aT} - e^{at_1}) - hs(T-t_1)] + C_4C_5 + \frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1} - 1] + \frac{C_4C_6hs}{\theta}[1 - e^{\theta t_1}]\Bigg],$$

Taking into account [3], the defuzzified value of $\bar{Z}(t_1, T)$ is given by
$D(\bar{Z}, 0) = \frac{a_1+a_2+a_3+a_4+b_1+b_2+b_3+b_4+d_1+d_2+d_3+d_4}{12}$.
Now,

$$\frac{\partial D(\bar{Z})}{\partial t_1} = \frac{1}{12T}[(C_{11} + C_{12} + C_{13} + C_{14} + C'_{11} + C'_{12} + C'_{13} + C'_{14} + C''_{11} + C''_{12} + C''_{13} + C''_{14})\{ce^{(\theta+a)t_1} - \frac{2ae^{2at_1}}{(\theta+a)(\theta-a)} - \frac{ce^{2at_1}}{(\theta+a)} + \frac{hse^{at_1}}{\theta} - \frac{hse^{\theta t_1}}{(\theta-a)} + \frac{hsae^{at_1}}{\theta(\theta-a)}\} + 12C_2\{(kce^{(\theta+a)t_1} - khse^{\theta t_1})(\frac{1}{\theta^2} - \frac{t_1e^{-\theta t_1}}{\theta} - \frac{e^{-\theta t_1}}{\theta^2}) + (\frac{kce^{(\theta+a)t_1}}{\theta+a} - \frac{khse^{\theta t_1}}{\theta})(\frac{e^{-\theta t_1}}{\theta} + t_1e^{-\theta t_1} - \frac{e^{-\theta t_1}}{\theta)}) - \frac{kct_1e^{at_1}}{(\theta+a)} + \frac{khst_1}{\theta}\} + (C_{31}+C_{32}+C_{33}+C_{34}+C'_{31}+C'_{32}+C'_{33}+C'_{34}+C''_{31}+C''_{32}+C''_{33}+C''_{34})(hs - ce^{at_1} + 12C_4C_6(ce^{(\theta+a)t_1} - 12hse^{\theta t_1})] = 0 \quad (17)$$

and $\frac{\partial D(\bar{Z})}{\partial T} = -\frac{1}{12T^2}[(C_{01}+C_{02}+C_{03}+C_{04}+C_{01}^{'}+C_{02}^{'}+C_{03}^{'}+C_{04}^{'}+C_{01}^{''}+C_{02}^{''}+C_{03}^{''}+C_{04}^{''})+(C_{11}+C_{12}+C_{13}+C_{14}+C_{11}^{'}+C_{12}^{'}+C_{13}^{'}+C_{14}^{'}+C_{11}^{''}+C_{12}^{''}+C_{13}^{''}+C_{14}^{''})[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\}-\frac{c}{2a(\theta+a)}(e^{2at_1}-1)+\frac{hs}{\theta a}(e^{at_1}-1)-\frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}]+12C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)}-\frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2}-\frac{t_1e^{-\theta t_1}}{\theta}-\frac{e^{-\theta t_1}}{\theta^2})-\frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a}+\frac{1}{a^2}-\frac{e^{at_1}}{a^2})+\frac{khst_1^2}{2\theta}]+(C_{31}+C_{32}+C_{33}+C_{34}+C_{31}^{'}+C_{32}^{'}+C_{33}^{'}+C_{34}^{'}+C_{31}^{''}+C_{32}^{''}+C_{33}^{''}+C_{34}^{''})[\frac{c}{a}(e^{aT}-e^{at_1})-hs(T-t_1)]+C_4C_5+\frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1]+\frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]]+\frac{(C_{31}+C_{32}+C_{33}+C_{34}+C_{31}^{'}+C_{32}^{'}+C_{33}^{'}+C_{34}^{'}+C_{31}^{''}+C_{32}^{''}+C_{33}^{''}+C_{34}^{''})}{12T}(ce^{aT}-hs)=0.$

(18)

Solving (17) and (18), we get the values of t_1 and t. Since $\frac{\partial^2 D}{\partial t_1^2}>0$ and $\frac{\partial^2 D}{\partial T^2}>0$ for this t_1 and T respectively, these values minimize $\bar{Z}$.

8. **Comparison analysis for the neutrosophic techniques**

To identify the suitable technique for the proposed crisp model, develop a another neutrosophic fuzzy methods.

Second approach for neutrosophic fuzzy model

We use trapezoidal neutrosophic numbers to describe the inventory carrying cost, holding cost, and shortage cost since they are in trapezoidal neutrosophic numbers:
Let $\bar{C}_0=(C_{01},C_{02},C_{03},C_{04})(C_{01}^{'},C_{02}^{'},C_{03}^{'},C_{04}^{'})(C_{01}^{''},C_{02}^{''},C_{03}^{''},C_{04}^{''})$,
$\bar{C}_1=(C_{11},C_{12},C_{13},C_{14})(C_{11}^{'},C_{12}^{'},C_{13}^{'},C_{14}^{'})(C_{11}^{''},C_{12}^{''},C_{13}^{''},C_{14}^{''})$,
and $\bar{C}_3=(C_{31},C_{32},C_{33},C_{34})(C_{31}^{'},C_{32}^{'},C_{33}^{'},C_{34}^{'})(C_{31}^{''},C_{32}^{''},C_{33}^{''},C_{34}^{''})$.
Let $Z^*=(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4)(d_1,d_2,d_3,d_4)$ be the trapezoidal neutrosophic fuzzy number. Now, by another approach, the defuzzified value of $\bar{Z}(t_1,T)$ is given by
$D(\bar{Z})=D(\bar{C}_0)+D(\bar{C}_1)[\frac{ce^{(\theta+a)t_1}}{(\theta+a)}\{\frac{1-e^{-(\theta-a)t_1}}{(\theta-a)}\}-\frac{c}{2a(\theta+a)}(e^{2at_1}-1)+\frac{hs}{\theta a}(e^{at_1}-1)-\frac{hse^{\theta t_1}}{\theta(\theta-a)}\{1-e^{-(\theta-a)t_1}\}]+C_2[\{\frac{kce^{(\theta+a)t_1}}{(\theta+a)}-\frac{khse^{\theta t_1}}{\theta}\}(\frac{1}{\theta^2}-\frac{t_1e^{-\theta t_1}}{\theta}-\frac{e^{-\theta t_1}}{\theta^2})-\frac{kc}{(\theta+a)}(\frac{t_1e^{at_1}}{a}+\frac{1}{a^2}-\frac{e^{at_1}}{a^2})+\frac{khst_1^2}{2\theta}]+D(\bar{C}_3)[\frac{c}{a}(e^{aT}-e^{at_1})-hs(T-t_1)]+C_4C_5+\frac{C_4C_6c}{(\theta+a)}[e^{(\theta+a)t_1}-1]+\frac{C_4C_6hs}{\theta}[1-e^{\theta t_1}]$,
where, based on [3], the defuzzified value of the inventory carrying cost $D(\bar{C}_0)=\frac{C_{01}+C_{02}+C_{03}+C_{04}+C_{01}^{'}+C_{02}^{'}+C_{03}^{'}+C_{04}^{'}+C_{01}^{''}+C_{02}^{''}+C_{03}^{''}+C_{04}^{''}}{12}$, based on [3], the defuzzified value of the holding cost $D(\bar{C}_1)=\frac{C_{11}+C_{12}+C_{13}+C_{14}+C_{11}^{'}+C_{12}^{'}+C_{13}^{'}+C_{14}^{'}+C_{11}^{''}+C_{12}^{''}+C_{13}^{''}+C_{14}^{''}}{12}$, and based on [3], the defuzzified value of the shortage cost $D(\bar{C}_3)=\frac{C_{31}+C_{32}+C_{33}+C_{34}+C_{31}^{'}+C_{32}^{'}+C_{33}^{'}+C_{34}^{'}+C_{31}^{''}+C_{32}^{''}+C_{33}^{''}+C_{34}^{''}}{12}$.

For a comparative analysis between the proposed neutrosophic approach and the above neutrosophic approach, we take the proposed numerical example and consider $\bar{C}_0=(100,160,250,320)(120,180,260,330)(80,130,220,280)$, $\bar{C}_1=$

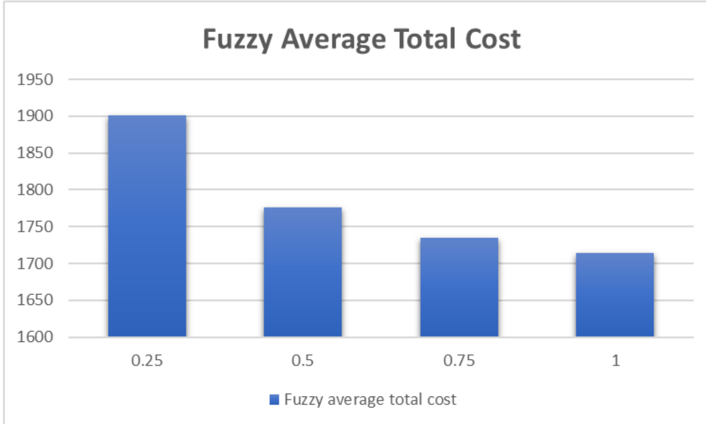

FIGURE 4. ω vs Fuzzy average total cost

$(9, 15, 20, 24)(12, 17, 21, 25)(7, 13, 18, 23)$, and $\bar{C}_3 = (7, 13, 17, 20)(8, 14, 18, 21)(5, 12, 16, 19)$. In the first proposed neutrosophic approach, the optimum average total cost is 1669.42 and in the next above neutrosophic approach, the optimum average total cost is 1706.76. The first proposed neutrosophic method gives the most optimum result. In addition, the savings of the first proposed approach is 2.74% and the savings of the other approach is 0.57%. Thus, the first proposed neutrosophic method is more appropriate for the Economic Order Quantity model.

9. Comparative study

Model	Optimize average total cost
Crisp model	1716.47
Normalized Trapezoidal fuzzy model	1713.91
Proposed Neutrosophic fuzzy model	1669.42
Second approach for Neutrosophic fuzzy model	1706.76

TABLE 4. Comparison table for all models

In Table 3, we see that the savings of the normalized trapezoidal fuzzy model is 0.15%, the savings of the neutrosophic fuzzy model is 2.74%.

In the figure, we observe that if the values of ω are decreasing, then the fuzzy average total costs increase.

10. Conclusion & discussion

10.1. *Conclusion*

In a totally backlogged state, we introduced two fuzzy inventory models for damaged items with shortages. Firstly, we introduced the trapezoidal fuzzy model. Secondly, we introduced the neutrosophic fuzzy model. The ordering cost, holding cost, deterioration cost, shortage cost, and carbon tax are included in the inventory model. Then we develop a trapezoidal fuzzy model and a neutrosophic fuzzy model. Next, we compare between crisp and different types of fuzzy models. We see that the neutrosophic fuzzy model gives a better result than the trapezoidal fuzzy model. The comparative study shows that the neutrosophic fuzzy model is more effective on the proposed EOQ model.

10.2. *Limitations and future researches*

There are some limitations on the databases and methodology used in our paper. We use a single numerical example to justify the crisp and fuzzy inventory models. We tried to develop the only neutrosophic fuzzy model using a deneutrification method. In the proposed neutrosophic fuzzy model, we use an approach to optimize the average total cost in the inventory models. In the proposed fuzzy models, we consider that the inventory carrying cost, the holding cost, and the shortage cost are the only fuzzy numbers for uncertainty. But in the comparative analysis section, we develop another technique to show that the first proposed approach is a better technique to optimize the average total cost in inventory models.

In the future, one can consider the other parameters as the fuzzy numbers for the uncertainty and use the other defuzzification methods in this proposed crisp model. In the future also, we will try to develop other defuzzification methods for the neutrosophic fuzzy model and a new approach for this type of model.

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Data Availability

There are no associated data.

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

Ethical approval

This article does not contain studies with human participants or animals performed by any of the authors.

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