Neutrosophic Sets and Systems, Vol. 98, 2026
University of New Mexico

Horizontal and Vertical Generalized n-Fold Algebra: Formal Construction and Applications in Multi-Dimensional Modeling

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Abstract: Two-Fold Algebra (TFA) was recently developed to bridge classical algebraic operations with fuzzy and fuzzy-extension (especially neutrosophic) components, allowing for the simultaneous modeling of objects and their associated uncertainty descriptors. However, as real-world systems increasingly demand the integration of multiple, independent qualification dimensions—such as risk, sustainability, and reliability, the binary nature of TFA becomes a limiting factor. This paper introduces two generalized frameworks: the Horizontal and respectively Vertical Generalization n-Fold Algebra (n-FA), and from 2-valued to m-values operations, $m \geq 2$. We formally define the n-FA structure as a coupling of a classical backbone (*1*) with (*n-1*) independent or interdependent component sub-laws. We provide rigorous systematic construction, explore various specializations (including fuzzy and intuitionistic-fuzzy cases), and derive the essential algebraic properties—such as closure, associativity, and monotonicity—required for coherent multi-component operations. Finally, we demonstrate the versatility of n-FA through numerical examples in supply-chain risk and multi-criteria decision-making, establishing it as a robust mathematical language for complex, high-dimensional uncertainty modeling.

Keywords: n-Fold Algebra, Horizontal n-Fold Algebra, Vertical n-Fold Algebra, Fuzzy, Fuzzy-Extensions, Neutrosophic Logic, Algebraic Structures, Uncertainty Modeling, Multi-Criteria Decision Making (MCDM), Fuzzy Sets, Hybrid Algebraic Laws, Information Fusion, Neutrosophic Two-Fold Algebra, Decision Support Systems.

1. Introduction

The evolution of uncertainty-aware modeling has seen a steady progression from Zadeh's fuzzy sets [1] to Atanassov's intuitionistic fuzzy sets [6], and finally to neutrosophic logic [3]. While these frameworks excel at qualifying a single algebraic operation with degrees of truth, indeterminacy, or falsehood, they often treat these descriptors as a monolithic "label" attached to an element. In early year 2024, Smarandache introduced the Two-Fold Algebra (TFA) [4, 5]. Unlike previous iterations, TFA treats the classical element and its uncertainty descriptor as a coupled pair that evolves through a dual-component law. This allows the structural transformation of the base object to occur in tandem with the evolution of its qualitative attributes.

However, complex real-world systems—particularly in the domains of global supply chains and multi-criteria decision-making—frequently involve more than two layers of qualification. A project manager must simultaneously track performance, financial risk, environmental sustainability, and regulatory compliance. These dimensions are often governed by distinct algebraic properties; for instance, "Risk" may aggregate via a probabilistic sum [2], while "Sustainability" might be governed by a bottleneck (minimum) principle.

To address this need for high-dimensional qualitative modeling, this paper extends the two-fold construction to a generalized n-Fold Algebra (n-FA). We define a structure where n independent component algebras coexist with a classical backbone, allowing for a modular and extensible algebraic framework.

2. Two-Fold Algebra (TFA)

2.1 Formal Definition

Let A be a non-empty set of *classical* elements. Define a binary operation

$$\#: A \times A \longrightarrow A.$$

Let each element $x \in A$ be equipped with a neutrosophic triple

$$x\ (T, I, F), T, I, F \in [0,1],$$

representing truth-membership, indeterminacy-membership, and falsity-membership, respectively.

Define a second binary operation

$$*: [0,1]^3 \times [0,1]^3 \longrightarrow [0,1]^3,$$

acting component-wise (or via a prescribed aggregation rule).

The Two-Fold Law Δ on the enriched set

$$\tilde{A} = \{\, x(T, I, F) \mid x \in A,\ T, I, F \in [0,1] \,\}$$

is given by

$$x_1(T_1, I_1, F_1)\ \Delta\ x_2(T_2, I_2, F_2) = (x_1 \# x_2)[(T_1, I_1, F_1) * (T_2, I_2, F_2)].$$

Thus, the classical outcome $x_1 \# x_2$ is qualified by the aggregated neutrosophic component.

2.2 Specializations

Table 1. Comparison of Component Sets and Operations in Two-Fold Algebra Variants.

Variant	*Component set*	*Typical component operation* $*$
Fuzzy Two-Fold	Single membership degree $t \in [0,1]$	Minimum, product, or any t-norm
Intuitionistic-Fuzzy Two-Fold	Pair (t, i) with $t + i \le 1$	Minimum for t, maximum for i
Neutrosophic Two-Fold	Triple (T, I, F) with no sum constraint	Component-wise t-norm/t-conorm, probabilistic sum, etc.

2.3 Illustrative Numerical Example

Consider two projects P_1 and P_2 with neutrosophic evaluations

Project	***T***	***I***	***F***
$\boldsymbol{P_1}$	0.8	0.1	0.1
$\boldsymbol{P_2}$	0.7	0.2	0.1

Choose # as "project merger" (producing a combined project P). Select $*$ as:

$$T_{\text{new}} = \min\,(T_1, T_2)$$
$$I_{\text{new}} = \max\,(I_1, I_2)$$
$$F_{\text{new}} = \max\,(F_1, F_2)$$

Then

$$P_1 \Delta P_2 = P(0.7, 0.2, 0.1).$$

The resulting project inherits the merged base object and the aggregated uncertainty profile.

3. Horizontal Generalization to n-Fold Algebra (n-FA)

3.1 Motivation

When a system requires more than two independent qualification dimensions—e.g., performance, risk, sustainability, and regulatory compliance—a higher-order folding is necessary. The n-Fold Algebra provides a systematic way to attach $k = n - 1$ component algebras to a classical backbone.

3.2 Structure

Let

- A be a set of classical elements with binary operation $\#: A \times A \to A$.
- For each $j \in \{1, \dots, k\}$ (where $k = n - 1$), let C_j denote a component space (e.g., $[0, 1]$, $[0, 1]^2$, $[0, 1]^3$, …).
- Define binary component operations $*_j: C_j \times C_j \to C_j$.

An element of the n-Fold algebra is a tuple

$$x(c_1, c_2, \dots, c_k), x \in A, c_j \in C_j.$$

The n-Fold Law Λ is

$$x_1(c_1^{(1)}, \dots, c_k^{(1)}) \wedge x_2(c_1^{(2)}, \dots, c_k^{(2)}) = (x_1 \# x_2)\ (c_1^{(1)} *_1 c_1^{(2)}, \dots, c_k^{(1)} *_k c_k^{(2)}).$$

Thus, each component algebra evolves independently (or with prescribed dependence) while the classical part evolves under $\#$.

3.3 Notational Compactness

Denote the vector of component operations as

$$* = (*_1, \dots, *_k), \mathbf{c}^{(i)} = (c_1^{(i)}, \dots, c_k^{(i)}).$$

Then

$$x_1(\mathbf{c}^{(1)}) \wedge x_2(\mathbf{c}^{(2)}) = (x_1 \# x_2)(\mathbf{c}^{(1)} * \mathbf{c}^{(2)}).$$

4. Concrete Instances

4.1 Horizontal Three-Fold Algebra

Components:

1. Classical element x.
2. First component c_1 (e.g., truth-membership T).
3. Second component c_2 (e.g., liquidity or risk profile).

Operations: $\#$, $*_1$, $*_2$.

Example (Financial Portfolio)

- Classical: Asset class (Stocks, Bonds).
- c_1: Expected return profile (T).
- c_2: Liquidity profile (L).

Applying Λ yields a new portfolio with combined assets, aggregated expected return (via a t-norm) and aggregated liquidity (via a t-conorm).

4.2 Horizontal Four-Fold Algebra

Components: Classical, Performance, Risk, Sustainability.

Example (Supply-Chain Risk)

- #: Concatenation of two logistics nodes.

- $*_1$: Performance (speed, cost) combined by weighted average.
- $*_2$: Risk (security, disruption probability) combined by probabilistic sum.
- $*_3$: Sustainability (carbon footprint) combined by minimum (most stringent standard dominates).

Resulting node reflects the merged route together with three orthogonal qualification profiles.

4.3 Generic Horizontal n-Fold Example

Consider a **multi-attribute vendor selection** problem with five criteria: cost, quality, delivery reliability, environmental impact, and social responsibility.

- Classical element: Vendor identifier.
- Component spaces:
  - C_1 – Cost score $[0, 1]$.
  - C_2 – Quality score $[0, 1]$.
  - C_3 – Reliability score $[0, 1]$.
  - C_4 – Environmental impact score $[0, 1]$.
  - C_5 – Social responsibility score $[0, 1]$.

Choosing appropriate $*_j$ (e.g., product for cost, minimum for quality, probabilistic sum for reliability) yields a **Five-Fold Algebra** that aggregates vendors pairwise, preserving all five assessment dimensions.

5. Algebraic Properties of Horizontal n-Fold Algebra

To ensure the *n*-Fold structure behaves coherently, each sub-law should satisfy a set of desirable properties. The *Table below* summarizes typical requirements.

Table 2. Necessary Properties for Coherent Multi-Component Operations.

Sub-law	Property	Rationale
Classical $\#$	Closure, Associativity, Identity, (optional) Commutativity	Guarantees that repeated combinations stay within A and are order-independent when needed.
Component $*_j$	Closure, Associativity, Commutativity, Monotonicity, Boundary conditions (e.g., 0 and 1 act as neutral/extremal elements)	Enables consistent aggregation of uncertainty measures; monotonicity ensures that improving a component never degrades the result.
Inter-dependence	Optional distributivity of $\#$ over $*_j$ or vice-versa	In some applications the classical operation may affect component aggregation (e.g., mixing liquids changes purity calculation).

When sub-laws are **independent**, the *n*-Fold Law reduces to a simple Cartesian product of the individual algebras. When **dependent**, additional constraints (e.g., $\#$ distributing over $*_j$) must be imposed to preserve algebraic coherence.

6. Applications of Horizontal n-Fold Algebra

Table 3. Cross-Disciplinary Applications and Structural Configurations of *n*-Fold Algebra.

Domain	Why *n*-Fold Algebra fits	Sample configuration
Chemistry (Mixture Modelling)	Simultaneous tracking of base substance and multiple quality attributes (purity, concentration uncertainty, impurity)	3-Fold (classical substance, purity, uncertainty)
Decision-Making (MADM)	Multiple criteria (performance, risk, sustainability) require separate aggregation rules	4-Fold (asset, performance, risk, sustainability)
Supply-Chain Management	Physical routing combined with cost, security, and environmental profiles	4-Fold (node, cost, risk, sustainability)
Financial Engineering	Portfolio construction with return, volatility, liquidity, regulatory compliance	5-Fold (asset, return, volatility, liquidity, compliance)
Artificial Intelligence (Hybrid Reasoning)	Fusion of symbolic reasoning (classical) with probabilistic, fuzzy, and neutrosophic belief layers	*n*-Fold with arbitrary component algebras

In each case, the Horizontal *n*-Fold formalism provides a clear mathematical skeleton that separates the *structural* combination from the *qualitative* combination, facilitating modular design and analysis.

7. Vertical Generalization to *n*-Fold Algebra (Vertical *n*-FA)

With similar and extended notations as before, one has:

$$\begin{pmatrix} & c_1^{(1)} \\ x_1 & \vdots \\ & c_k^{(1)} \end{pmatrix} \boxed{\Lambda} \begin{pmatrix} & c_1^{(2)} \\ x_2 & \vdots \\ & c_k^{(2)} \end{pmatrix} = (\mathrm{x}_1 \# \mathrm{x}_2) \begin{pmatrix} c_1^{(1)} \\ \vdots \\ c_k^{(1)} \end{pmatrix} \circledast \begin{pmatrix} c_1^{(2)} \\ \vdots \\ c_k^{(2)} \end{pmatrix}$$

7.1. Example of Vertical 3-fold Algebra

$$\mathrm{x}_1 \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix} \boxed{\Lambda} \mathrm{x}_2 \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix} = (\mathrm{x}_1 \cdot \mathrm{x}_2) \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix} \circledast \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix}$$

where $x_1 \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix}$ means that the membership degree of x_1 is 0.6, obtained with a confidence of 0.5; similarly for $\mathrm{x}_2 \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix}$; whence $x_1 \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix} = x_{1_{0.6(0.5)}} = x_{1_{0.3}}$ and $\mathrm{x}_2 \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix} = x_{2_{0.7(0.1)}} = x_{2_{0.07}}$.

Let, $\mathrm{x}_1 = 20$; $\mathrm{x}_2 = 30$; then $(20 \cdot 30)$.

Let's take the multiplication of x_1 with x_2 as operation; and optimistic view (max) of the membership:

$$\mathrm{x}_{1_{0.3}} \cdot \mathrm{x}_{2_{0.07}} = (20 \cdot 30)_{(\max\{0.3, 0.07\})} = 600_{0.3}.$$

7.2. Extension to m-valued operations of Vertical Generalization of n-fold Algebra

$$\boxed{\Lambda}_m \left(\begin{pmatrix} & c_1^{(1)} \\ x_1 & \vdots \\ & c_k^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} & c_1^{(m)} \\ x_m & \vdots \\ & c_k^{(m)} \end{pmatrix} \right) = (x_1 \# \dots \# x_m) \circledast_m \begin{pmatrix} c_1^{(1)} & & c_1^{(m)} \\ \vdots & , \dots , & \vdots \\ c_k^{(1)} & & c_k^{(m)} \end{pmatrix}$$

8. Conclusion and Future Work

The transition from Two-Fold to n-Fold Algebra offers a principled pathway to model systems where several independent qualification dimensions coexist with a classical backbone. By defining a family of component algebras $\{(C_j, *_j)\}_{j=1}^{k}$ and coupling them through a shared classical operation $\#$, the n-Fold framework preserves algebraic rigor while remaining flexible enough for diverse applications.

Future research directions include:

- **Category-theoretic characterization** of n-Fold Algebras, exploring functorial relationships between component algebras.
- **Automated synthesis** of suitable component operations from empirical data (e.g., learning optimal t-norms for a given domain).
- **Software libraries** implementing generic n-Fold operations, enabling rapid prototyping in decision-support systems.
- **Investigation of dependence patterns** (partial, total) among sub-laws and their impact on overall system behavior.

By extending the algebraic toolbox in this manner, researchers and practitioners gain a powerful, extensible language for handling multi-dimensional uncertainty across mathematics, engineering, and the social sciences.

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Received: July 30, 2025. Accepted: Jan 2, 2026