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Reduction Formulae of 2-refined Neutrosophic Integrals and their Applications

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Abstract. The main aim of this manuscript is to establish for the first time the reduction formulae of 2-refined neutrosophic indefinite integrals (shortly, RF2RNII) which have double indeterminacy I_1 and I_2 respectably. We then verify reduction formulae with appropriate examples and also apply reduction formulae to evaluate definite integrals in 2-refined neutrosophic environment. The reduction formula of 2-refined trigonometric functions in 2-refined neutrosophic integrals (shortly, RF2RNI) is a useful mathematical technique for calculating and deriving a simplified version of integrals. We also classify these reduction formulae by investigating various cases of 2-refined neutrosophic integrals (shortly,2RNI).

Keywords: Neutrosophic indefinite integral, neutrosophic definite integral, 2-refined neutrosophic indefinite integral (2RNII), 2-refined neutrosophic definite integral (2RNDI) and reduction formulae of 2-refined neutrosophic indefinite integral (RF2RNII).

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1. **Introduction and literature review**

It was observed from historical background that classical logic is a nineteenth and twentieth century innovation. Thereafter fuzzy logic [31] opened up a new research area in the branch of mathematical science since 1965. Smarandache [25] initiated a new type of logic, known as neutrosophic logic [25]. In the light of this new logic he formulated a mathematical model of uncertainty,ambiguity,inconsistency, undefined, unknown, contradiction and so on. This novel concept of neutrosophy opened up a new dimension of philosophy due to Smarandache (see [26], [27], [28]) which has wider of applications in various fields like multi criterion decision making problem, pattern recognition, medical diagnosis and classification problems especially problems with more than one decision makers (see [17] , [16], [13], [18], [24]).This new theory will also be a useful tool in decision and ranking problems such as robot selection,green supliers selections, solid waste landfill site section problems etc.He presented refined neutrosophic number [29] in the form: $(a,b_1I_1, b_2I_2,b_3I_3,....., b_nI_n)$, where $a,b_1, b_2,b_3,....., b_n \in$ R or C.In the modern age, science and technology had been significantly dealing with intricate phenomena and processes for which there is inadequate information. In such circumstances, these models are invented for dealing with different types of systems that have uncertain, vague, imprecise, incomplete and redundant components. Many of these models, like Fuzzy model [31], Intuitionistic fuzzy model [12],Neutrosophic model [25] and many more are built on the extensions of standard set theoretic models.

Zadeh [31] proposed the concept of fuzzy set (FS) when he handled vague, imprecise and uncertain data set. As a generalisation of fuzzy set, Atanassov [12] created intuitionistic fuzzy set (IFS) in 1986. His theory thereafter became widely acknowledged as an essential resource in real life applications. In 1995, neutrosophic set (in short,NS) was proposed by Smarandache [26] as for generalization of intuitionistic fuzzy set.This theoretical framework demonstrates itself to be a strong tool for addressing the complex web of unclear and conflicting data that permeates our everyday environment. He also defined the notion of standard form of neutrosophic real number and the condition for division of two neutrosophic real numbers to exist.

Neutrophic logic [26], neutrosophic vector space [1], neutrosophic topological space [22], neutrosophic group theory [23], neutrosophic ring theory [2], and dombi neutrosophic graph [20] are only a few of the many academics who have contributed to the field of neutrosophic theory. In 2015, Agboola [3] introduced the notion of refined neutrosophic algebraic structures. The refined neutrosophic ring (I) was studied by Adeleke et al. [5], where (I) was splited into two indeterminacies I_1 and I_2 so that I_1 (stands for contradiction(T) and false(F)) and I_2 (stands for ignorance(T) or False(F)). From which logically it implies that: $I_1 \times I_1=I_1$; $I_2 \times I_2=I_2$; $I_1 \times I_2=I_1$; $I_2 \times I_1=I_1$. In networking problem and shortest path problems, Chokraborty [14,15] proposed pentagonal neutrosophic numbers. In 2021 and 2022, Alhasan (see [4], [6], [7], [8]) developed various method of integration such as by parts method, definite integrals and partial fraction methods in the neutrosophic environment with single indeterminacy (I). Alhasan, Y. A. et al. [9] initiated the indefinite integrals of trigonometric functions in refined neutrosophic model. Subsequently Manshath et al. [21] studied the concept of neutrosophic integrals by reduction formula and partial fraction methods for indefinite integrals and very recently Yesar et al [11] proposed the differential and integrals of 2-refined hyperbolic functions. Lots of work had also been studied in the field of neutrosophic logic in statistics and others ([19], [30]).

Inspiring upon the work cited above we are for the first time focusing our research on reduction formulae of 2-refined neutrosophic integrals and their applications.

1.1. Research gaps and motivation

(i) Neutrosophic differentials and neutrosophic indefinite integrals have almost been established. However, a very little study has been addressed on neutrosophic 2-refined indefinite integrals involving double indeterminacy.
(ii) We noticed that no papers are found in literature which uses reduction formula for 2-refined neutrosophic indefinite integrals. This motivates us to introduce reduction formulae for 2-refined neutrosophic integral approach to describe double indeterminacy model in neutrosophic framework

1.2. Structure of the paper

The present paper is organized in the following manner: 1st section provides the introduction which represents the literature review of neutrosophic logic and calculus. Second Section focuses into common definitions and preliminaries. Third Section describes the derivation of reduction formulae of 2-refined neutrosophic indefinite integrals (shortly, RF2RNII) and classify them by investigating various cases of 2-refined neutrosophic integrals. In section four, we verify some of the reduction formulae with appropriate examples and apply reduction formulae to evaluate 2-refined neutrosophic definite integrals in neutrosophic environment of double indeterminacy.

2. Mathematical prelimineries

This section consists of some common notations and definitions which have been involved in the course of the paper.
Definition 2.1. Neutrosophic set: [26] Let X be a Universe of discourse. Then the neutrosophic set is defined by N= {(x, τ(x), λ(x), η(x)), x$\in$X, where, τ, λ, η $\in$ [0,1], indicating the degrees of truth, indeterminacy and falsehood respectively that satisfy $0 \leq \inf(\tau) + \inf(\lambda) + \inf(\eta) \leq \sup(\tau) + \sup(\lambda) + \sup(\eta) \leq 3$}
Definition 2.2.Neutrosophic real numbers: [27] Suppose that w is a neutrosophic number, then it takes the followinng form w =a+bI, where a,b are real coefficients, and I represents indeterminacy such that 0.I=I.0=0 and I.n=n.I=I for all n$\in$N.
Suppose that U=a+bI and V=c+dI be two neutrosophic real numbers. To find $\frac{a+bI}{c+dI}$ we assume that $\frac{a+bI}{c+dI}$ = x+yI $\Rightarrow$ a+bI= (x+yI)(c+dI) $\Rightarrow$ a+bI= cx+(cy+xd+yd)I
Comparing the coefficients we get, a=cx and b=xd+y(c+d). For unique solution we must have c(c+d)$\neq$0. Hence, c$\neq$0 and c $\neq$-d are the conditions for the division of two neutrosophic

real numbers to exist. Then, $\frac{a+bI}{c+dI}$=$\frac{a}{c}$ + $\frac{cb-ad}{c(c+d)}$I.

Definition 2.3. Neutrosophic indefinite integral: [27] We just extend the classical definition of anti-derivative. The neutrosophic antiderivative of neutrosophic function f(x) is the neutrosophic function g(x) such that g$'$(x)=f(x).

Definition 2.4. Neutrosophic indefinite integral: [4] Let f: D$_f$⊆R→ R$_f$ ∪ {I}, to evaluate the integral $\int f(x)\,dx$. For that put x=g(u) ⇒ dx=g$'(u)$du. By substituting we get, $\int f(x)\,dx$=$\int f(u)g'(u)\,du$. Then we can directly integrate it.

Theorem 2.5. [4] If $\int f(x,I)\,dx$=ϕ(x,I) Then $\int f((a+bI)x+c+dI)\,dx$ =$(\frac{1}{a}-[\frac{b}{a(a+b)}]I)$ × ϕ((a+bI)x+c+dI) + C, where C is indeterminate real constant, a≠0, a≠–b and b,c,d are real numbers, while I is indeterminacy.

Definition 2.6. 2-refined neutrosophic indefinite integral: [10] Let f: R(I$_1, I_2$) → R(I$_1, I_2$), to evaluate the integral $\int f(x,I_1,I_2)\,dx$. Put x=g(u) ⇒ dx=g$'(u)$du. By substituting we get, $\int f(x,I_1,I_2)\,dx$=$\int f(u)g'(u)\,du$. Then we can directly integrate it.

Theorem

2.7. [10]If $\int f(x,I_1,I_2)\,dx$=$\phi(x,I_1,I_2)$, then $\int f((a+bI_1+cI_2)x+s+tI_1+uI_2)\,dx$ =$(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$ × $\phi((a+bI_1+cI_2)x+s+tI_1+uI_2)$ + C, where C is indeterminate real constant (i.e. constant of the form p+qI$_1+rI_2$, where p,q,r are real numbers), a≠0, a≠–b , a≠–b–c and b,c,d are real numbers, while I$_1 and I_2$ are two indeterminacy.

2.8. Some standard 2-refined neutrosophic indefinite integrals: [10]

(i) $\int((a+bI_1+cI_2)x+p+qI_1+rI_2)^n\,dx$

$=(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)\times\frac{((a+bI_1+cI_2)x+p+qI_1+rI_2)^{n+1}}{(n+1)}$ + C (n≠–1)

(ii) $\int e^{((a+bI_1+cI_2)x+p+qI_1+rI_2)}\,dx$

$=(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)\times e^{((a+bI_1+cI_2)x+p+qI_1+rI_2)}$ +C

(iii) $\int\frac{1}{((a+bI_1+cI_2)x+p+qI_1+rI_2)}\,dx$

$=(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)\times\ln|((a+bI_1+cI_2)x+p+qI_1+rI_2)|$ + C

(iv) $\int sin((a+bI_1+cI_2)x+p+qI_1+rI_2)\,dx$

$=-(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$ × cos((a+b$I_1+cI_2)x+p+qI_1+rI_2$) + C

(v) $\int cos((a+bI_1+cI_2)x+p+qI_1+rI_2)\,dx$

$=(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$ × sin((a+b$I_1+cI_2)x+p+qI_1+rI_2$) + C

(vi) $\int sec^2((a+bI_1+cI_2)x+p+qI_1+rI_2)\,dx$

$=(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$ × tan((a+b$I_1+cI_2)x+p+qI_1+rI_2$) + C

(vii) $\int cosec^2((a+bI_1+cI_2)x+p+qI_1+rI_2)\,dx$

$=-(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$ × cot((a+b$I_1+cI_2)x+p+qI_1+rI_2$) + C

3. Refined reduction formula of neutrosophic indefinite integral

It is observed that the reduction formula for refined trigonometric, algebraic and exponential functions in refined neutrosophic integrals is a useful mathematical technique for calculating and deriving a simplified version of integrals. In this section we derive the reduction formulae of 2-refined neutrosophic indefinite integrals of neutrosophic trigonometric functions which involves double indeterminacies I_1 and I_2 respactbly. We also classify them by investigating various special cases.

3.1: Obtain the reduction formula for the 2-refined neutrosophic integral $\int((a+bI_1+cI_2)x+p+qI_1+rI_2)^n e^{(s+tI_1+uI_2)((a+bI_1+cI_2)x+p+qI_1+rI_2)}\,dx$.

Solution: Let $J_n=\int((a+bI_1+cI_2)x+p+qI_1+rI_2)^n e^{(s+tI_1+uI_2)((a+bI_1+cI_2)x+p+qI_1+rI_2)}\,dx$

We put $v=(a+bI_1+cI_2)x+p+qI_1+rI_2$, $k=s+tI_1+uI_2$, then $dv=(a+bI_1+cI_2)dx$

$\Rightarrow \frac{1}{a+bI_1+cI_2}dv=dx$

Then $J_n=\frac{1}{(a+bI_1+cI_2)}\int v^n e^{kv}\,dv$

Integratig by parts we have,

$J_n=\frac{1}{(a+bI_1+cI_2)}\left[\frac{1}{k}v^n e^{kv}-\frac{1}{k}\int v^{n-1}e^{kv}\,dv\right]$

$=\frac{1}{k(a+bI_1+cI_2)}\left[v^n e^{kv}-n\int v^{n-1}e^{kv}\,dv\right]$

$=\frac{1}{(s+tI_1+uI_2)(a+bI_1+cI_2)}\left[v^n e^{kv}-n(a+bI_1+cI_2)\times\frac{1}{(a+bI_1+cI_2)}\int v^{n-1}e^{kv}\,dv\right]$

$=\frac{1}{(as+(at+bt+ct+bs+bu)I_1+(au+cu+cs)I_2)}\left[v^n e^{kv}-n(a+bI_1+cI_2)J_{n-1}\right]$

$=\frac{1}{(as+(at+bt+ct+bs+bu)I_1+(au+cu+cs)I_2)}\left[((a+bI_1+cI_2)x+p+qI_1+rI_2)^n\right.$
$\left.\times e^{(s+tI_1+uI_2)((a+bI_1+cI_2)x+p+qI_1+rI_2)}-n(a+bI_1+cI_2)J_{n-1}\right]$

Hence, $J_n=\left(\frac{1}{as}-\left[\frac{(at+bt+ct+bs+bu)}{(as+au+cu+cs)(as+at+bt+ct+bs+bu+au+cu+cs)}\right]I_1+\left[\frac{au+cu+cs}{as(as+au+cu+cs)}\right]I_2\right)$
$\times\left[((a+bI_1+cI_2)x+p+qI_1+rI_2)^{n-1}\, e^{(s+tI_1+uI_2)((a+bI_1+cI_2)x+p+qI_1+rI_2)}-n(a+bI_1+cI_2)J_{n-1}\right]$–**(1)**

Equation (1) is the required reduction formula for the the given 2-refined neutrosophic integral. □

Remark 3.1(a): 2-Refined neutrosophic integral is the generalization of neutrosophic integral and neutrosophic integral is the generalization of real integral which can be seen from the following cases:

Case-I: In **(1)**, if we put $p+qI_1+rI_2=0$, then the 2-refined neutrosophic integral reduces to the reduction formula

$J_n=\int((a+bI_1+cI_2)x)^n e^{(s+tI_1+uI_2)((a+bI_1+cI_2)x)}\,dx$

$=\left(\frac{1}{as}-\left[\frac{(at+bt+ct+bs+bu)}{(as+au+cu+cs)(as+at+bt+ct+bs+bu+au+cu+cs)}\right]I_1+\left[\frac{au+cu+cs}{as(as+au+cu+cs)}\right]I_2\right)$
$\times\left[((a+bI_1+cI_2)x)^{n-1}\, e^{(s+tI_1+uI_2)((a+bI_1+cI_2)x)}-n(a+bI_1+cI_2)J_{n-1}\right]$———————**1(a)**

Case-II: In **(1)**, if we put $p+qI_1+rI_2=0$, $s+tI_1+uI_2=1$,(i.e. s=1,t=u=0), then the 2-refined neutrosophic integral reduces to the reduction formula

$J_n=\int((a+bI_1+cI_2)x)^n e^{(a+bI_1+cI_2)x}\,dx$

$=\left(\frac{1}{a}-\left[\frac{b}{(a+c)(a+b+c)}\right]I_1+\left[\frac{c}{a(a+c)}\right]I_2\right)\times\left[((a+bI_1+cI_2)x)^{n-1}\, e^{(a+bI_1+cI_2)x}-n(a+bI_1+cI_2)J_{n-1}\right]$-**1(b)**

Case-III: In **(1)**, if we put p+qI_1+rI_2=0,s+tI_1+uI_2=1, c=0, I_1=I(indeterminacy), then the 2-refined neutrosophic integral reduces to the neutrosophic reduction formula

$J_n=\int((a+bI)x)^n e^{(a+bI)x}\,dx=\frac{1}{a+bI}\times[((a+bI)x)^{n-1}\,e^{(a+bI)x}-nJ_{n-1}]$

$=(\frac{1}{a}-[\frac{b}{a(a+b)}]I)\times[((a+bI)x)^{n-1}\,e^{(a+bI)x}-n(a+bI)J_{n-1}]$———————————**1(c)**

Case-IV: In **(1)**, if we put p+qI_1+rI_2=0,s+tI_1+uI_2=1, a=1, b=c=0, then the 2-refined neutrosophic integral reduces to real integral having reduction formula

$J_n=\int x^n e^x\,dx=[x^{n-1}\,e^x-nJ_{n-1}]$, ——————————————————**-1(d)**

3.2: Obtain the 2-refined neutrosophic reduction formula for the following integrals

(a) $\int tan^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

(b) $\int cot^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$**,where n is a positive integer**.

Solution:(a) Let $J_n=\int tan^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

We put v=[(a+bI_1+cI_2)x+p+qI_1+rI_2], then dv=(a+bI_1+cI_2)dx $\Rightarrow$ $\frac{1}{(a+bI_1+cI_2)}$dv=dx

Then $J_n=\frac{1}{(a+bI_1+cI_2)}\int tan^n v\,dv=\frac{1}{(a+bI_1+cI_2)}\int tan^{n-2}v tan^2 v\,dv$

$=\frac{1}{(a+bI_1+cI_2)}\int tan^{n-2}v(sec^2 v-1)\,dv$

$=\frac{1}{(a+bI_1+cI_2)}\int tan^{n-2}v sec^2 v\,dv-\frac{1}{(a+bI_1+cI_2)}\int tan^{n-2}v\,dv$

$J_n=\frac{1}{(a+bI_1+cI_2)}\int t^{n-2}\,dt-J_{n-2}$ [Take tanv=t $\Rightarrow$ sec^2vdv=dt]

$J_n=\frac{1}{(a+bI_1+cI_2)}\;\frac{tan^{n-1}v}{(n-1)}-J_{n-2}$

$J_n=(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$

$\times\;\frac{tan^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(n-1)}-J_{n-2}$——————————————**2(a)**

Special cases: Case-I: In **2(a)**, if we put p+qI_1+rI_2=0, c=0 and I_1=I(indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_n=\int tan^n[(a+bI)x]\,dx=(\frac{1}{a}-[\frac{b}{a(a+b)}]I)\times\frac{tan^{n-1}[(a+bI)x]}{(n-1)}-J_{n-2}$——————**2(a)(i)**

Case-II: In **2(a)**, if we put p+qI_1+rI_2=0, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n=\int tan^n ax\,dx=\frac{tan^{n-1}ax}{a(n-1)}-J_{n-2}$——————————————————**2(a)(ii)**

Case-III: In **2(a)**, if we put p+qI_1+rI_2=0, a=1, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n=\int tan^n x\,dx=\frac{tan^{n-1}x}{(n-1)}-J_{n-2}$——————————————————**2(a)(iii)**

(b) Let $J_n=\int cot^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

We put v=[(a+b$I_1+cI_2)x+p+qI_1+rI_2$], then dv=(a+bI_1+cI_2)dx $\Rightarrow$ $\frac{1}{(a+bI_1+cI_2)}$dv=dx

Then $J_n=\frac{1}{(a+bI_1+cI_2)}\int cot^n v\,dv=\frac{1}{(a+bI_1+cI_2)}\int cot^{n-2}v cot^2 v\,dv$

$=\frac{1}{(a+bI_1+cI_2)}\int cot^{n-2}v(cosec^2 v-1)\,dv$

$=\frac{1}{(a+bI_1+cI_2)}\int cot^{n-2}v cosec^2 v\,dv-\frac{1}{(a+bI_1+cI_2)}\int cot^{n-2}v\,dv$

$J_n=-\frac{1}{(a+bI_1+cI_2)}\int u^{n-2}\,du-J_{n-2}$ [Take cotv=u $\Rightarrow$ –$cosec^2$vdv=du]

$J_n = -\frac{1}{(a+bI_1+cI_2)} \frac{cot^{n-1}v}{(n-1)} - J_{n-2}$

$J_n = -(\frac{1}{a} - [\frac{b}{(a+c)(a+b+c)}]I_1 + [\frac{c}{a(a+c)}]I_2)$

$\times \frac{cot^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(n-1)} - J_{n-2}$ ———————————— **2(b)**

Special cases: Case-I: In **2(b)**, if we put $p+qI_1+rI_2=0$, c=0 and I_1=I(indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by $J_n=\int cot^n[(a+bI)x]\,dx = -(\frac{1}{a} - [\frac{b}{a(a+b)}]I) \times \frac{cot^{n-1}[(a+bI)x]}{(n-1)} - J_{n-2}$ ———— **2(b)(i)**

Case-II: In **2(b)**, if we put $p+qI_1+rI_2=0$, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n=\int cot^n ax\,dx = -\frac{cot^{n-1}ax}{a(n-1)} - J_{n-2}$ ———————————— **2(b)(ii)**

Case-III: In **2(b)**, if we put $p+qI_1+rI_2=0$, a=1, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n=\int cot^n x\,dx = -\frac{cot^{n-1}x}{(n-1)} - J_{n-2}$ ———————————— **2(b)(iii)**

3.3: Prove that the reduction formula for the 2-refined neutrosophic integral $\int e^{(a+bI_1+cI_2)x} cos^n(p+qI_1+rI_2)x\,dx$ is given by

$\mathbf{J}_n = \frac{e^{(a+bI_1+cI_2)x}cos^{n-1}(p+qI_1+rI_2)x[(a+bI_1+cI_2)cos(p+qI_1+rI_2)x+n(p+qI_1+rI_2)sin(p+qI_1+rI_2)x]}{n^2(p+qI_1+rI_2)^2+(a+bI_1+cI_2)^2} + \frac{n(n-1)(p+qI_1+rI_2)^2}{(n^2(p+qI_1+rI_2)^2+(a+bI_1+cI_2)^2)} \mathbf{J}_{n-2}$.

Proof: Let $J_n=\int e^{(a+bI_1+cI_2)x} cos^n(p+qI_1+rI_2)x\,dx$

We put $u=a+bI_1+cI_2$ and $v=p+qI_1+rI_2$

$J_n=\int e^{ux} cos^n vx\,dx$

Applying 2-refined neutrosophic integration by parts we have,

Then $J_n = \frac{e^{ux}cos^n vx}{u} + \frac{nv}{u} \times \int e^{ux} cos^{n-1}vx \times sinvx\,dx$

$= \frac{e^{ux}cos^n vx}{u} + \frac{nv}{u} \times [\frac{e^{ux}cos^{n-1}vx \times sinvx}{u} -$

$\frac{1}{u} \int e^{ux}[(n-1)cos^{n-2}vx(-sinvx).v.sinvx + cos^{n-1}vxcosvx \cdot v]\,dx]$

$= \frac{e^{ux}cos^n vx}{u} + \frac{nv}{u} \times [\frac{e^{ux}cos^{n-1}vx \times sinvx}{u} - \frac{1}{u} \int e^{ux}(n-1)cos^{n-2}vx(-sinvx).v.sinvx\,dx$

$-\frac{1}{u} \int e^{ux}cos^{n-1}vxcosvx \cdot v\,dx]$

$= \frac{e^{ux}cos^n vx}{u} + \frac{nve^{ux}cos^{n-1}vx \times sinvx}{u^2} + \frac{nv^2}{u^2} \int e^{ux} \times (n-1)cos^{n-2}vx \times sin^2 vx\,dx - \frac{nv^2}{u^2}$

$\int e^{ux}cos^n vx\,dx$

$= \frac{e^{ux}cos^n vx}{u} + \frac{nve^{ux}cos^{n-1}vx \times sinvx}{u^2} + \frac{n(n-1)v^2}{u^2} \int e^{ux} \times cos^{n-2}vx(1-cos^2 vx)\,dx - \frac{nv^2}{u^2}$

$\int e^{ux}cos^n vx\,dx$

$= \frac{e^{ux}cos^n vx}{u} + \frac{nve^{ux}cos^{n-1}vx \times sinvx}{u^2} + \frac{n(n-1)v^2}{u^2} \int e^{ux}.cos^{n-2}vx\,dx - \frac{n^2v^2}{u^2} \int e^{ux}cos^n vx\,dx$

$J_n = \frac{e^{ux}cos^n vx}{u} + \frac{nve^{ux}cos^{n-1}vx \times sinvx}{u^2} + \frac{n(n-1)v^2}{u^2} J_{n-2} - \frac{n^2v^2}{u^2} J_n$

$(1+\frac{n^2v^2}{u^2}) J_n = \frac{e^{ux}cos^n vx}{u} + \frac{nve^{ux}cos^{n-1}vx \times sinvx}{u^2} + \frac{n(n-1)v^2}{u^2} J_{n-2}$

$(\frac{n^2v^2+u^2}{u^2}) J_n$

$= \frac{e^{ux}cos^{n-1}vx}{u^2}$ [ucosvx +nvsinvx]+ $\frac{n(n-1)v^2}{u^2} J_{n-2}$

Hence, $J_n = \frac{e^{(a+bI_1+cI_2)x}cos^{n-1}(p+qI_1+rI_2)x[(a+bI_1+cI_2)cos(p+qI_1+rI_2)x+n(p+qI_1+rI_2)sin(p+qI_1+rI_2)x]}{n^2(p+qI_1+rI_2)^2+(a+bI_1+cI_2)^2}$

$+ \frac{n(n-1)(p+qI_1+rI_2)^2}{(n^2(p+qI_1+rI_2)^2+(a+bI_1+cI_2)^2)} J_{n-2}$, which is the required reduction formula.——— **(3)** □

Special cases: Case-I: In **(3)**, if we put c=r=0 and I_1=I (indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_n = \int e^{(a+bI)x}cos^n(p+qI)x\,dx = \frac{e^{(a+bI)x}cos^{n-1}(p+qI)x[(a+bI)cos(p+qI)x+n(p+qI)sin(p+qI)x]}{n^2(p+qI)^2+(a+bI)^2} + \frac{n(n-1)(p+qI)^2}{(n^2(p+qI)^2+(a+bI)^2)} J_{n-2}$, ———————————————— **3(a)**

Case-II: In **(3)**, if we put b=q=c=r=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int e^{ax}cos^n px\,dx = \frac{e^{ax}cos^{n-1}px[acospx+npsinpx]}{n^2p^2+a^2} + \frac{n(n-1)p^2}{(n^2p^2+a^2)} J_{n-2}$, ——————————-**3(b)**

Case-III: In **(3)**, if we put a=1,b=q=c=r=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int e^{x}cos^n px\,dx = \frac{e^{x}cos^{n-1}px[cospx+npsinpx]}{n^2p^2+1^2} + \frac{n(n-1)p^2}{(n^2p^2+1^2)} J_{n-2}$, ——————————————**3(c)**

Case-IV: In **(3)**, if we put p=1, b=q=c=r=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int e^{ax}cos^n x\,dx = \frac{e^{ax}cos^{n-1}x[acosx+npsinx]}{n^2+a^2} + \frac{n(n-1)}{(n^2+a^2)} J_{n-2}$, ———————————————**3(d)**

Case-V: In **(3)**, if we put a=p=1, b=q=c=r=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int e^{x}cos^n x\,dx = \frac{e^{x}cos^{n-1}x[cosx+nsinx]}{n^2+1^2} + \frac{n(n-1)}{(n^2+1^2)} J_{n-2}$, ———————————————-**3(e)**

Case-VI:

In **(3)**, if we put ($a+bI_1+cI_2$)=1, then the 2-refined neutrosophic integral reduces to $J_n = \int e^{x}cos^n(p+qI_1+rI_2)x\,dx = \frac{e^{x}cos^{n-1}(p+qI_1+rI_2)x[cos(p+qI_1+rI_2)x+n(p+qI_1+rI_2)sin(p+qI_1+rI_2)x]}{n^2(p+qI_1+rI_2)^2+1^2} + \frac{n(n-1)(p+qI_1+rI_2)^2}{(n^2(p+qI_1+rI_2)^2+1^2)} J_{n-2}$,——————————————**3(f)**

Case-VII: In **(3)**, if we put ($a+bI_1+cI_2$)=1,r=0,I_1=I (indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_n = \int e^{x}cos^n(p+qI)x\,dx = \frac{e^{x}cos^{n-1}(p+qI)x[cos(p+qI)x+n(p+qI)sin(p+qI)x]}{n^2(p+qI)^2+1^2} + \frac{n(n-1)(p+qI)^2}{(n^2(p+qI)^2+1^2)} J_{n-2}$,—————————————————**3(g)**

Case-VIII: In **(3)**, if we put ($a+bI_1+cI_2$)=1,q=r=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int e^{x}cos^n px\,dx = \frac{e^{x}cos^{n-1}px[cos(p+qI)x+n(p+qI)sinpx]}{n^2p^2+1^2} + \frac{n(n-1)p^2}{(n^2p^2+1^2)} J_{n-2}$,——**3(h)**

Case-IX: In **(3)**, if we put ($p+qI_1+rI_2$)=1, then the 2-refined neutrosophic integral reduces to $J_n = \int e^{(a+bI_1+cI_2)x}cos^n x\,dx = \frac{e^{(a+bI_1+cI_2)x}cos^{n-1}x[(a+bI_1+cI_2)cosx+nsinx]}{n^2+(a+bI_1+cI_2)^2} + \frac{n(n-1)}{(n^2+(a+bI_1+cI_2)^2)} J_{n-2}$,———————————————-**3(i)**

Case-X: In **(3)**, if we put ($p+qI_1+rI_2$)=1, c=0,I_1=I (indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_n = \int e^{(a+bI)x}cos^n x\,dx = \frac{e^{(a+bI)x}cos^{n-1}x[(a+bI)cosx+nsinx]}{n^2+(a+bI)^2} + \frac{n(n-1)}{(n^2+(a+bI)^2)} J_{n-2}$,——————**3(j)**

Case-XI: In **(3)**, if we put ($p+qI_1+rI_2$)=1, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int e^{ax} cos^n x\, dx = \frac{e^{ax} cos^{n-1} x[acosx + nsinx]}{n^2+a^2} + \frac{n(n-1)}{(n^2+a^2)}$ J_{n-2}, ——————————————— **3(k)**

Theorem 3.4: Prove that the reduction formula for the 2-refined neutrosophic integral $\int e^{(a+bI_1+cI_2)x} sin^n(p+qI_1+rI_2)x\, dx$ is given by

$\mathbf{J}_n = \frac{e^{(a+bI_1+cI_2)x} sin^{n-1}(p+qI_1+rI_2)x[(a+bI_1+cI_2)sin(p+qI_1+rI_2)x - n(p+qI_1+rI_2)cos(p+qI_1+rI_2)x]}{n^2(p+qI_1+rI_2)^2+(a+bI_1+cI_2)^2}$

$+ \frac{n(n-1)(p+qI_1+rI_2)^2}{(n^2(p+qI_1+rI_2)^2+(a+bI_1+cI_2)^2)}$ $\mathbf{J}_{n-2}$.

Proof: Let $J_n = \int e^{(a+bI_1+cI_2)x} sin^n(p+qI_1+rI_2)x\, dx$

We assume that $u = a+bI_1+cI_2$ and $v = p+qI_1+rI_2$

Then $J_n = \int e^{ux} sin^n vx\, dx$

Applying 2-refined neutrosophic integration by parts we have,

$J_n = \frac{e^{ux} sin^n vx}{u} - \frac{nv}{u} \times \int e^{ux} sin^{n-1} vx \times cosvx\, dx$

$= \frac{e^{ux} sin^n vx}{u} - \frac{nv}{u} \times [\frac{e^{ux} sin^{n-1} vx \times cosvx}{u} - \frac{1}{u} \int e^{ux} \times (n-1) sin^{n-2} vx.cosvx.v.cosvx\, dx$

$-\frac{1}{u} \int e^{ux} sin^{n-1} vx(-sinux) \cdot v\, dx]$

$= \frac{e^{ux} sin^n vx}{u} - \frac{nve^{ux} sin^{n-1} vx \times cosvx}{u^2} + \frac{n(n-1)v^2}{u^2} \int e^{ux} sin^{n-2} vx \times cos^2 vx\, dx - \frac{nv^2}{u^2} \int e^{ux} sin^n vx\, dx$

$= \frac{e^{ux} sin^n vx}{u} - \frac{nve^{ux} sin^{n-1} vx \times cosvx}{u^2} + \frac{n(n-1)v^2}{u^2} \int e^{ux} sin^{n-2} vx \times (1 - sin^2 vx)\, dx - \frac{nv^2}{u^2}$
$\int e^{ux} sin^n vx\, dx$

$= \frac{e^{ux} sin^n vx}{u} - \frac{nve^{ux} sin^{n-1} vx \times cosvx}{u^2} + \frac{n(n-1)v^2}{u^2} \int e^{ux} sin^{n-2} vx\, dx - \frac{n^2v^2}{u^2} \int e^{ux} sin^n vx\, dx$

$J_n = \frac{e^{ux} sin^n vx}{u} - \frac{nve^{ux} sin^{n-1} vx \times cosvx}{u^2} + \frac{n(n-1)v^2}{u^2}$ $J_{n-2} - \frac{n^2v^2}{u^2}$ J_n

$(1 + \frac{n^2v^2}{u^2})$ $J_n = \frac{e^{ux} sin^n vx}{u} - \frac{nve^{ux} sin^{n-1} vx \times cosvx}{u^2} + \frac{n(n-1)v^2}{u^2}$ J_{n-2}

$(\frac{n^2v^2+u^2}{u^2})$ $J_n = \frac{e^{ux} sin^{n-1} vx}{u^2}$ [usinvx −nvcosvx]+ $\frac{n(n-1)v^2}{u^2}$ J_{n-2}

Hence, $J_n = \frac{e^{(a+bI_1+cI_2)x} sin^{n-1}(p+qI_1+rI_2)x[(a+bI_1+cI_2)sin(p+qI_1+rI_2)x - n(p+qI_1+rI_2)cos(p+qI_1+rI_2)x]}{n^2(p+qI_1+rI_2)^2+(a+bI_1+cI_2)^2}$

$+ \frac{n(n-1)(p+qI_1+rI_2)^2}{(n^2(p+qI_1+rI_2)^2+(a+bI_1+cI_2)^2)}$ J_{n-2}, which is the required reduction formula.——— **(4)** □

Special cases: Case-I: In **(4)**, if we put c=r=0 and I_1=I (indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_n = \int e^{(a+bI)x} sin^n(p+qI)x\, dx = \frac{e^{(a+bI)x} sin^{n-1}(p+qI)x[(a+bI)sin(p+qI)x - n(p+qI)cos(p+qI)x]}{n^2(p+qI)^2+(a+bI)^2} +$
$\frac{n(n-1)(p+qI)^2}{(n^2(p+qI)^2+(a+bI)^2)}$ J_{n-2}, ——————————————————————— **-4(a)**

Case-II: In **(4)**, if we put b=c=q=r=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int e^{ax} sin^n px\, dx = \frac{e^{ax} sin^{n-1} px[asinpx - npcospx]}{n^2p^2+a^2} + \frac{n(n-1)p^2}{(n^2p^2+a^2)}$ J_{n-2}, ——————————— **4(b)**

Case-III: In **(4)**, if we put a=1, b=c=q=r=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int e^{x} sin^n px\, dx = \frac{e^{x} sin^{n-1} px[sinpx - npcospx]}{n^2p^2+1^2} + \frac{n(n-1)p^2}{(n^2p^2+1^2)}$ J_{n-2}, ——————————— **4(c)**

Case-IV: In **(4)**, if we put p=1, b=c=q=r=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int e^{ax} sin^n x\, dx = \frac{e^{ax} sin^{n-1} x[asinx - ncosx]}{n^2+a^2} + \frac{n(n-1)}{(n^2+a^2)}$ J_{n-2}, ——————————— **4(d)**

Case-V: In **(4)**, if we put a=p=1, b=c=q=r=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int e^x sin^n x\, dx = \frac{e^x sin^{n-1}x[sinx - ncosx]}{n^2+1^2} + \frac{n(n-1)}{(n^2+1^2)} J_{n-2}$,——————————**4(e)**

Case-VI:

In **(4)**, if we put $(a+bI_1+cI_2)=1$, then the 2-refined neutrosophic integral reduces to $J_n = \int e^x sin^n(p+qI_1+rI_2)x\, dx = \frac{e^x sin^{n-1}(p+qI_1+rI_2)x sin(p+qI_1+rI_2)x - n(p+qI_1+rI_2)cos(p+qI_1+rI_2)x]}{n^2(p+qI_1+rI_2)^2+1^2} + \frac{n(n-1)(p+qI_1+rI_2)^2}{n^2(p+qI_1+rI_2)^2+1^2} J_{n-2}$,——————————-**4(f)**

Case-VII: In **(4)**, if we put $(a+bI_1+cI_2)=1, r=0, I_1=I$ (indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_n = \int e^x sin^n(p+qI)x\, dx = \frac{e^x sin^{n-1}(p+qI)x[sin(p+qI)x - n(p+qI)cos(p+qI)x]}{n^2(p+qI)^2+1^2} + \frac{n(n-1)(p+qI)^2}{n^2(p+qI)^2+1^2} J_{n-2}$,............................**4(g)**

Case-VIII: In **(4)**, if we put $(a+bI_1+cI_2)=1, q=r=0$, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int e^x sin^n px\, dx = \frac{e^x sin^{n-1}px[sinpx - npcospx]}{n^2p^2+1^2} + \frac{n(n-1)p^2}{n^2p^2+1^2} J_{n-2}$,——————————**4(h)**

Case-IX: In **(4)**, if we put $(p+qI_1+rI_2)=1$, then the 2-refined neutrosophic integral reduces to $J_n = \int e^{(a+bI_1+cI_2)x} sin^n x\, dx = \frac{e^{(a+bI_1+cI_2)x} sin^{n-1}x[(a+bI_1+cI_2)sinx - ncosx]}{n^2+(a+bI_1+cI_2)^2} + \frac{n(n-1)}{(n^2+(a+bI_1+cI_2)^2)} J_{n-2}$,——————————-**4(i)**

Case-X: In **(4)**, if we put $(p+qI_1+rI_2)=1, c=0, I_1=I$ (indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_n = \int e^{(a+bI)x} sin^n x\, dx = \frac{e^{(a+bI)x} sin^{n-1}x[(a+bI)sinx - ncosx]}{n^2+(a+bI)^2} + \frac{n(n-1)}{(n^2+(a+bI)^2)} J_{n-2}$,——————-**4(j)**

Case-XI: In **(4)**, if we put $(p+qI_1+rI_2)=1, b=c=0$, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int e^{ax} sin^n x\, dx = \frac{e^{ax} sin^{n-1}x[asinx - ncosx]}{n^2+a^2} + \frac{n(n-1)}{(n^2+a^2)} J_{n-2}$,——————————-**4(k)**

3.5: Prove that the reduction formula for the 2-refined neutrosophic integral $\int sin^m[(a+bI_1+cI_2)x+p+qI_1+rI_2]cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\, dx$**, where m,n being positive integers, is given by** $\mathbf{J}_{m,n} = (\frac{1}{a} - [\frac{ab}{a(a+b)(a+b+c)}]I_1 + [\frac{c}{a(a+c)}]I_2) \times \frac{cos^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]sin^{m+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(m+n)} + (\frac{n-1}{m+n})\, \mathbf{J}_{m,n-2}$**, (m+n≠0) and** $\mathbf{J}_{m,n} = -(\frac{1}{a} - [\frac{ab}{a(a+b)(a+b+c)}]I_1 + [\frac{c}{a(a+c)}]I_2)$

$\times \frac{sin^{m-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]cos^{n+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(m+n)} + (\frac{m-1}{m+n})\, \mathbf{J}_{m-2,n}$**, (m+n≠0)**

Proof: Let

$J_{m,n} = \int sin^m[(a+bI_1+cI_2)x+p+qI_1+rI_2]cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\, dx$

We put $v=[(a+bI_1+cI_2)x+p+qI_1+rI_2]$, $A=a+bI_1+cI_2$, then $dv=(a+bI_1+cI_2)dx \Rightarrow \frac{1}{A}dv=dx$

Then, $J_{m,n} = \frac{1}{A} \int sin^m v cos^n v\, dx$, $(A\neq 0)$

$= \frac{1}{A} \int cos^{n-1}v(sin^m v cosv)\, dx$

Using neutrosophic integration by parts we have,

$J_{m,n} = \frac{1}{A}$ [$cos^{n-1}v \int sin^m v cosv\, dx - \int\{(n-1)cos^{n-2}v(-sinv) \times \int(sin^m v cosv)dx\}\, dx$], $(A\neq 0)$

$= \frac{1}{A}$ [$\frac{cos^{n-1}v sin^{m+1}v}{m+1} + (n-1)\int\{cos^{n-2}v \times sinv \times \frac{sin^{m+1}v}{m+1}\}dx$] [Taking sinv=z, cosvdv=dz, m≠–1]

$=\frac{1}{A}$ [$\frac{cos^{n-1}vsin^{m+1}v}{m+1}$ $+\frac{n-1}{m+1}$ $\int cos^{n-2}v \times sin^{m+2}v\,dx$], (m≠–1

$=\frac{1}{A}$ [$\frac{cos^{n-1}vsin^{m+1}v}{m+1}$ $+\frac{n-1}{m+1}$ $\int cos^{n-2}v \times sin^{2}v \times sin^{m}v\,dx$], (m≠–1)

$=\frac{1}{A}$ [$\frac{cos^{n-1}vsin^{m+1}v}{m+1}$ $+\frac{n-1}{m+1}$ $\int cos^{n-2}v \times (1-cos^{2}v) \times sin^{m}v\,dx$], (m≠–1)

$=\frac{1}{A}$ [$\frac{cos^{n-1}vsin^{m+1}v}{m+1}$ $+\frac{n-1}{m+1}$ $\int cos^{n-2}v \times sin^{m}v\,dx$ $-$ $\frac{n-1}{m+1}$ $\int cos^{n}v \times sin^{m}v\,dx$], (m≠–1)

$=\frac{1}{A}$ $\frac{cos^{n-1}vsin^{m+1}v}{m+1}$ $+\frac{n-1}{(m+1)A}$ $\int cos^{n-2}v \times sin^{m}v\,dx$ $-$ $\frac{n-1}{(m+1)A}$ $\int cos^{n}v \times sin^{m}v\,dx$, (m≠–1)

$J_{m,n}=\frac{1}{A}$ $\frac{cos^{n-1}vsin^{m+1}v}{m+1}$ + $(\frac{n-1}{m+1})$ $J_{m,n-2}$ – $(\frac{n-1}{m+1})$ $J_{m,n}$, (m≠–1)

$(1+\frac{n-1}{m+1})J_{m,n}=\frac{1}{A}$ $\frac{cos^{n-1}vsin^{m+1}v}{m+1}$ + $(\frac{n-1}{m+1})$ $J_{m,n-2}$, (m≠–1)

$(\frac{m+n}{m+1})J_{m,n}=$ $\frac{cos^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]sin^{m+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(a+bI_1+cI_2)(m+1)}$ + $(\frac{n-1}{m+1})$ $J_{m,n-2}$, (m≠–1)

$J_{m,n}=$ $(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$

$\times$ $\frac{cos^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]sin^{m+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(m+n)}$ + $(\frac{n-1}{m+n})$ $J_{m,n-2}$, (m+n≠0)

which is the required reduction formula.————————————————————**(5)** □

Again we have, $J_{m,n}=\frac{1}{A}$ $\int sin^{m}vcos^{n}v\,dx$

$=\frac{1}{A}$ $\int sin^{m-1}v(cos^{n}vsinv)\,dx$

Using neutrosophic integration by parts we have,

$J_{m,n}=\frac{1}{A}$ [$\sin^{m-1}v$ $\int cos^{n}vsinv\,dx$–$\int\{(m-1)sin^{m-2}v(cosv) \times \int(cos^{n}vsinv)dx\}\,dx$]

$=\frac{1}{A}$ [$-\frac{sin^{m-1}vcos^{n+1}v}{n+1}$ +(m-1)$\int\{sin^{m-2}v \times cosv \times \frac{cos^{n+1}v}{n+1}\}dx$] [Assuming cosv=z and -sinvdv=dz]

$=\frac{1}{A}$ [$-\frac{sin^{m-1}vcos^{n+1}v}{n+1}$ $+\frac{m-1}{n+1}$ $\int sin^{m-2}v \times cos^{n+2}v\,dx$] (n≠–1)

$=\frac{1}{A}$ [$-\frac{sin^{m-1}vcos^{n+1}v}{n+1}$ $+\frac{m-1}{n+1}$ $\int sin^{m-2}v \times cos^{2}v \times cos^{n}v\,dx$] (n≠–1)

$=\frac{1}{A}$ [$-\frac{sin^{m-1}vcos^{n+1}v}{n+1}$ $+\frac{m-1}{n+1}$ $\int sin^{m-2}v \times (1-sin^{2}v) \times cos^{n}v\,dx$] (n≠–1)

$=\frac{1}{A}$ [$-\frac{sin^{m-1}vcos^{n+1}v}{n+1}$ $+\frac{m-1}{n+1}$ $\int sin^{m-2}v \times cos^{n}v\,dx$ $-$ $\frac{m-1}{n+1}$ $\int sin^{m}v \times cos^{n}v\,dx$] (n≠–1)

$=-\frac{1}{A}$ $\frac{sin^{m-1}vcos^{n+1}v}{n+1}$ $+\frac{m-1}{(n+1)A}$ $\int sin^{m-2}v \times cos^{n}v\,dx$ $-$ $\frac{m-1}{(n+1)A}$ $\int sin^{m}v \times cos^{n}v\,dx$, (n≠–1)

$J_{m,n}=-\frac{1}{A}$ $\frac{sin^{m-1}vcos^{n+1}v}{n+1}$ + $(\frac{m-1}{n+1})$ $J_{m-2,n}$ – $(\frac{m-1}{n+1})$ $J_{m,n}$ (n≠–1)

$(1+\frac{m-1}{n+1})J_{m,n}=-\frac{1}{A}$ $\frac{sin^{m-1}vcos^{n+1}v}{n+1}$ + $(\frac{m-1}{n+1})$ $J_{m-2,n}$ (n≠–1)

$(\frac{m+n}{n+1})J_{m,n}=$ $-\frac{sin^{m-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]cos^{n+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(a+bI_1+cI_2)(n+1)}$ + $(\frac{m-1}{n+1})$ $J_{m-2,n}$, (n≠–1)

$J_{m,n}=-(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$

$\times\frac{sin^{m-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]cos^{n+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(m+n)}$ + $(\frac{m-1}{m+n})$ $J_{m-2,n}$, (m+n≠0)——**(6)**

This is the required reduction formula. □

Special cases: Case-I: In **(5) and (6)**, if we put p+qI_1 + rI_2=0,c=0 and I_1=I(indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_{m,n}=\int sin^{m}(a+bI)xcos^{n}(a+bI)x\,dx$ $=(\frac{1}{a}-[\frac{b}{a(a+b)}]I)$ $\times$

$\frac{cos^{n-1}(a+bI)xsin^{m+1}(a+bI)x}{(m+n)}$ + $(\frac{n-1}{m+n})$ $J_{m,n-2}$, (m+n≠0)———————————————————**5(a)**

$J_{m,n}=\int sin^m(a+bI)xcos^n(a+bI)x\,dx=$

$-\frac{sin^{m-1}(a+bI)xcos^{n+1}(a+bI)x}{(a+bI)(m+n)}$ + $(\frac{m-1}{m+n})$ $J_{(m-2,n)}$, (m+n≠0)——————————————————**6(a)**

Case-II: In **(5) and (6)**, if we put p+qI_1 + rI_2=0,b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_{m,n}=\int sin^m ax \times cos^n ax\,dx=\frac{cos^{n-1}axsin^{m+1}ax}{a(m+n)}$ + $(\frac{n-1}{m+n})$ $J_{m,n-2}$, (m+n≠0)——————**5(b)**

$J_{m,n}=\int sin^m ax \times cos^n ax\,dx=-\frac{sin^{m-1}axcos^{n+1}ax}{a(m+n)}$ + $(\frac{m-1}{m+n})$ $J_{(m-2,n)}$, (m+n≠0)—————**6(b)**

Case-III: In **(5) and (6)**, if we put p+qI_1 + rI_2=0,a=1, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_{m,n}=\int sin^m x \times cos^n x\,dx=\frac{cos^{n-1}xsin^{m+1}x}{(m+n)}$ + $(\frac{n-1}{m+n})$ $J_{m,n-2}$, (m+n≠0)————————**5(c)**

$J_{m,n}=\int sin^m x \times cos^n x\,dx=-\frac{sin^{m-1}xcos^{n+1}x}{a(m+n)}$ + $(\frac{m-1}{m+n})$ $J_{m-2,n}$ (m+n≠0)——————**6(c)**

3.6: Obtain the reduction formula for the 2-refined neutrosophic integral $\int \frac{sin^m[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]}\,dx$, where m, n (n≠1) being positive integers.

Solution: Let $J_{m,n}=\int \frac{sin^m[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]}\,dx$, where m, n (n≠1) being positive integers.

⇒ $J_{m,n}$ $=\int sin^m\{[(a+bI_1+cI_2)x+p+qI_1+rI_2]\}cos^{-n}\{[(a+bI_1+cI_2)x+p+qI_1+rI_2]\}\,dx$——————————————————————————————————**-(7)**

Let us consider $H_{p,q}$ $=\int sin^p\{[(a+bI_1+cI_2)x+p+qI_1+rI_2]\}cos^q\{[(a+bI_1+cI_2)x+p+qI_1+rI_2]\}\,dx$, where m=p and -n=q.

Then $J_{m,n}$ =$H_{m,-n}$————————————————————————————**(8)**

We put v=[(a+bI_1+$cI_2)x+p+qI_1+rI_2$], A=a+bI_1+cI_2, then dv=(a+bI_1+cI_2)dx ⇒ $\frac{1}{A}$dv=dx

Then, $H_{p,q}=\frac{1}{A}\int sin^p vcos^q v\,dx$

Using equation **(5)**, we have,

$H_{p,q}=\frac{cos^{q-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]sin^{p+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(a+bI_1+cI_2)(p+q)}$ + $(\frac{q-1}{p+q})$ $H_{p,q-2}$, (p+q≠0)

Replacing q by q+2 we get,

$H_{p,q+2}=\frac{cos^{q+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]sin^{p+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(a+bI_1+cI_2)(p+q+2)}$ + $(\frac{q+1}{p+q+2})$ $H_{p,q}$, (p+q+2≠0)

Then $H_{p,q}=-\frac{cos^{q+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]sin^{p+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(a+bI_1+cI_2)(q+1)}$

+ $(\frac{p+q+2}{q+1})$ $H_{p,q+2}$, (q+1≠0)——————————————————————**(9)**

Replacing p by m and q by -n in **(9)**, we have,

$H_{m,-n}=\frac{sin^{m+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(n-1)(a+bI_1+cI_2)cos^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}-(\frac{m-n+2}{n-1})$ $H_{m,-(n-2)}$, (n≠1)————**-(10)**

Now using the relation **(8)** in **(10)**, we get

$J_{m,n}=(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$

$\times\frac{sin^{m+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(n-1)cos^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}-(\frac{m-n+2}{n-1})$ $J_{m,(n-2)}$, (n≠1)———————————————**(11)**

Equation **(11)** represents the required reduction formula for the given integral. □

Special cases: Case-I: In **(11)**, if we put p+qI_1+rI_2=0, c=0 and I_1=I(indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_{m,n}=\int \frac{sin^m(a+bI)x}{cos^n(a+bI)x}\,dx=(\frac{1}{a}-[\frac{b}{a(a+b)}]I)\ \frac{sin^{m+1}(a+bI)x}{(n-1)cos^{n-1}(a+bI)x}-(\frac{m-n+2}{n-1})\ J_{m,(n-2)}$, (n≠1)——**-11(a)**

Case-II: In **(11)**, if we put p+qI_1+rI_2=0,b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_{m,n}=\int \frac{sin^m ax}{cos^n ax}\,dx=\frac{sin^{m+1}ax}{a(n-1)cos^{n-1}ax}-(\frac{m-n+2}{n-1})\ J_{m,(n-2)}$, (n≠1) ————————————**11(b)**

Case-III: In **(11)**, if we put p+qI_1+rI_2=0,a=1, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_{m,n}=\int \frac{sin^m x}{cos^n x}\,dx=\frac{sin^{m+1}x}{(n-1)cos^{n-1}x}-(\frac{m-n+2}{n-1})\ J_{m,(n-2)}$, (n≠1) ————————————**-11(c)**

3.7: Derive the reduction formula for the 2-refined neutrosophic integral $\int \frac{cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{sin^m[(a+bI_1+cI_2)x+p+qI_1+rI_2]}\,dx$, where m, n being positive integers.

Solution: Let $J_{m,n}=\int \frac{cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{sin^m[(a+bI_1+cI_2)x+p+qI_1+rI_2]}\,dx$, where m, n being positive integers.

$\Rightarrow J_{m,n}=\int sin^{-m}\{[(a+bI_1+cI_2)x+p+qI_1+rI_2]\}cos^n\{[(a+bI_1+cI_2)x+p+qI_1+rI_2]\}\,dx$————————————————————————**-(12)**

Let us consider $H_{p,q}=\int sin^p\{[(a+bI_1+cI_2)x+p+qI_1+rI_2]\}cos^q\{[(a+bI_1+cI_2)x+p+qI_1+rI_2]\}\,dx$————————————————————————**(13)**

(where -m=p and n=q)

Then $J_{m,n}=H_{(-m,n)}$————————————**(14)**

Using the reduction formula (6) in (13), we have,

$H_{p,q}=-\frac{sin^{p-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]cos^{q+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(a+bI_1+cI_2)(p+q)}+(\frac{p-1}{p+q})\ H_{p-2,q}$, (p+q≠0)

$\Rightarrow H_{p+2,q}=-\frac{sin^{p+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]cos^{q+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(a+bI_1+cI_2)(p+q+2)}+(\frac{p+1}{p+q+2})\ H_{p,q}$, (p+q+2≠0)

[Replacing p by p+2]

$\Rightarrow H_{p,q}=\frac{sin^{p+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]cos^{q+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(a+bI_1+cI_2)(p+1)}+(\frac{p+q+2}{p+1})\ H_{p+2,q}$, (p+1≠0)

Replacing p by -m and q by n in the above equation we have,

$\Rightarrow H_{-m,n}=-\frac{cos^{n+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(m-1)(a+bI_1+cI_2)sin^{m-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}+(\frac{m-n-2}{m-1})\ H_{-(m-2),n}$, (m≠1)

$\Rightarrow J_{m,n}=-(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$

$\times\frac{cos^{n+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(m-1)sin^{m-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}+(\frac{m-n-2}{m-1})\ J_{(m-2),n}$, (m≠1) [Using relation **(14)**]—**(15)**

Equation **(15)** represents the required reduction formula for the given integral. □

Special cases: Case-I: In **(15)**, if we put p+qI_1+rI_2=0,c=0 and I_1=I(indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_{m,n}=\int \frac{cos^n(a+bI)x}{sin^m(a+bI)x}\,dx=-(\frac{1}{a}-[\frac{b}{a(a+b)}]I)\ \frac{cos^{n+1}(a+bI)x}{(m-1)sin^{m-1}(a+bI)x}+(\frac{m-n-2}{m-1})\ J_{(m-2),n}$ (m≠1)—**15(a)**

Case-II: In **(15)**, if we put p+qI_1+rI_2=0, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_{m,n}=\int \frac{cos^n ax}{sin^m ax}\,dx=-\frac{cos^{n+1}ax}{a(m-1)sin^{m-1}ax}+(\frac{m-n-2}{m-1})\,J_{(m-2),n}$, (m≠1) ———————— **15(b)**

Case-III: In **(15)**, if we put p+qI_1+rI_2=0, a=1, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_{m,n}=\int \frac{cos^n x}{sin^m x}\,dx=-\frac{cos^{n+1}x}{(m-1)sin^{m-1}x}+(\frac{m-n-2}{m-1})\,J_{(m-2),n}$, (m≠1)———————— **15(c)**

3.8: Derive the reduction formula for the 2-refined neutrosophic integral $\int \frac{1}{sin^m[(a+bI_1+cI_2)x+p+qI_1+rI_2]\times cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]}\,dx$, where m, n being positive integers.

Solution: Let $J_{m,n}=\int \frac{1}{sin^m[(a+bI_1+cI_2)x+p+qI_1+rI_2]\times cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]}\,dx$, where m, n being positive integers.

$\Rightarrow J_{m,n}=\int sin^{-m}\{[(a+bI_1+cI_2)x+p+qI_1+rI_2]\}cos^{-n}\{[(a+bI_1+cI_2)x+p+qI_1+rI_2]\}\,dx$———————————— **(16)**

Let us consider $H_{p,q}$

$=\int sin^p\{[(a+bI_1+cI_2)x+p+qI_1+rI_2]\}cos^q\{[(a+bI_1+cI_2)x+p+qI_1+rI_2]\}\,dx$,——— **(17)**

(where -m=p and -n=q)

Then $J_{m,n}=H_{(-m,-n)}$———————————-**(18)**

Using the formula **(9)** of Art. 3.6, we have,

$H_{p,q}=\frac{sin^{p+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]cos^{q+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(a+bI_1+cI_2)(q+1)}+(\frac{p+q+2}{q+1})\,H_{p,q+2}$, (q+1≠0)

Replacing p by -m and q by -n in the above equation we have,

$\Rightarrow H_{-m,-n}=-\frac{1}{(m-1)(a+bI_1+cI_2)sin^{m-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]cos^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}$
$+(\frac{m+n-2}{m-1})\,H_{-m,-(n-2)}$, (m≠1)

$\Rightarrow J_{m,n}=-(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$
$\times\frac{1}{(m-1)(a+bI_1+cI_2)sin^{m-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]cos^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}+(\frac{m+n-2}{m-1})\,J_{m,n-2}$,
(m≠1) [Using relation **(18)**]———————————————**(19)**

Equation **(19)** represents the required reduction formula for the given integral. □

Special cases: Case-I: In **(19)**, if we put p+qI_1+rI_2=0, c=0 and I_1=I(indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_{m,n}=\int \frac{1}{sin^m(a+bI)x\times cos^n(a+bI)x}\,dx=-(\frac{1}{a}-[\frac{b}{a(a+b)}]I)\times\frac{1}{(m-1)sin^{m-1}(a+bI)xcos^{n-1}(a+bI)x}+(\frac{m+n-2}{m-1})\,J_{m,n-2}$, (m≠1)——————————————**19(a)**

Case-II: In **(19)**, if we put p+qI_1+rI_2=0, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_{m,n}=\int \frac{1}{sin^m ax\times cos^n ax}\,dx=-\frac{1}{a(m-1)sin^{m-1}axcos^{n-1}ax}+(\frac{m+n-2}{m-1})\,J_{m,n-2}$, (m≠1)————-**19(b)**

Case-III: In **(19)**, if we put p+qI_1+rI_2=0, a=1, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_{m,n}=\int \frac{1}{sin^m x\times cos^n x}\,dx=-\frac{1}{(m-1)sin^{m-1}xcos^{n-1}x}+(\frac{m+n-2}{m-1})\,J_{m,n-2}$, (m≠1)—————**19(c)**

3.9: Derive the reduction formula for the 2-refined neutrosophic integral $\int Sin^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$ where n is a positive integer.

Solution: Let $J_n=\int Sin^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

We put $v=[(a+bI_1+cI_2)x+p+qI_1+rI_2]$, then $dv=(a+bI_1+cI_2)dx \Rightarrow \frac{1}{(a+bI_1+cI_2)}dv=dx$

Then $J_n=\frac{1}{(a+bI_1+cI_2)}\int Sin^n v\,dv=\frac{1}{(a+bI_1+cI_2)}\int Sin^{n-1}v sinv\,dv$

$=\frac{1}{(a+bI_1+cI_2)}\times[\text{Sin}^{n-1}\text{v (-cosv)} +(\text{n-1})\int Sin^{n-2}v cos^2v\,dv]$

$=\frac{1}{(a+bI_1+cI_2)}\times[-\text{Sin}^{n-1}\text{v cosv} +(\text{n-1})\int Sin^{n-2}v(1-sin^2v)\,dv]$

$=\frac{1}{(a+bI_1+cI_2)}\times[-\text{Sin}^{n-1}\text{v cosv} +(\text{n-1})\int Sin^{n-2}v\,dv-(\text{n-1})\int Sin^n v\,dv]$

$=\frac{1}{(a+bI_1+cI_2)}\times[-\text{Sin}^{n-1}\text{v cosv}] + \frac{n-1}{(a+bI_1+cI_2)}\int Sin^{n-2}v\,dv-\frac{n-1}{(a+bI_1+cI_2)}\int Sin^n v\,dv]$

$\Rightarrow J_n=\frac{1}{(a+bI_1+cI_2)}\times[-\text{Sin}^{n-1}\text{v cosv}] + (n-1)J_{n-2}-(n-1)J_n$

$\Rightarrow J_n=\frac{1}{(a+bI_1+cI_2)}\times[-\text{Sin}^{n-1}\text{v cosv}] + (n-1)J_{n-2}-(n-1)J_n$

$\Rightarrow (1+n-1)J_n=\frac{1}{(a+bI_1+cI_2)}\times[-\text{Sin}^{n-1}\text{v cosv}] + (n-1)J_{n-2}$

$\Rightarrow nJ_n=\frac{1}{(a+bI_1+cI_2)}\times[-\text{Sin}^{n-1}\text{v cosv}] + (n-1)J_{n-2}$

$\Rightarrow J_n=-(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$
$\times\frac{1}{n}[\text{Sin}^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]\cos[(a+bI_1+cI_2)x+p+qI_1+rI_2]]$
$+\frac{n-1}{n}J_{n-2}$ ——————————————————**(20)**

Equation (20) is the required reduction formula □

Special cases: Case-I: In **(20)**, if we put $p+qI_1+rI_2=0$, c=0 and I_1=I(indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_n=\int Sin^n[(a+bI)x]\,dx=\frac{-1}{n}(\frac{1}{a}-[\frac{b}{a(a+b)}]I)[\text{Sin}^{n-1}[(a+bI)x]\cos[(a+bI)x]]+\frac{n-1}{n}J_{n-2}$–**20(a)**

Case-II: In **(20)**, if we put $p+qI_1+rI_2=0$, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n=\int Sin^n ax\,dx=\frac{-Sin^{n-1}axcosax}{na}+\frac{n-1}{n}J_{n-2}$ ——————————**20(b)**

Case-III: In **(20)**, if we put $p+qI_1+rI_2=0$, a=1, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n=\int Sin^n x\,dx=\frac{-Sin^{n-1}xcosx}{n}+\frac{n-1}{n}J_{n-2}$ ——————————**20(c)**

3.10: Obtain the reduction formula for the 2-refined neutrosophic integral $\int cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

Solution: Let $J_n=\int cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

We put $v=(a+bI_1+cI_2)x+p+qI_1+rI_2$, then $dv=(a+bI_1+cI_2)dx \Rightarrow \frac{1}{(a+bI_1+cI_2)}dv=dx$

Then $J_n=\frac{1}{(a+bI_1+cI_2)}\int cos^n v\,dv=\frac{1}{(a+bI_1+cI_2)}\int cos^{n-1}v cosv\,dv$

$=\frac{1}{(a+bI_1+cI_2)}\times[\text{cos}^{n-1}\text{v (sinv)} +(\text{n-1})\int cos^{n-2}v sin^2v\,dv]$

$=\frac{1}{(a+bI_1+cI_2)}\times[\text{cos}^{n-1}\text{v sinv} +(\text{n-1})\int cos^{n-2}v(1-cos^2v)\,dv]$

$=\frac{1}{(a+bI_1+cI_2)}\times[\text{cos}^{n-1}\text{v sinv} +(\text{n-1})\int cos^{n-2}v\,dv-(\text{n-1})\int cos^n v\,dv]$

$=\frac{1}{(a+bI_1+cI_2)}\times[\text{cos}^{n-1}\text{v sinv}] + \frac{n-1}{(a+bI_1+cI_2)}\int cos^{n-2}v\,dv-\frac{n-1}{(a+bI_1+cI_2)}\int cos^n v\,dv]$

$\Rightarrow$ J$_n$=$\frac{1}{(a+bI_1+cI_2)}$ $\times$ [cos^{n-1}v sinv] + (n–1)J$_{n-2}$ –(n–1)J$_n$
$\Rightarrow$ J$_n$=$\frac{1}{(a+bI_1+cI_2)}$ $\times$ [cos^{n-1}v sinv] + (n–1)J$_{n-2}$ –(n–1)J$_n$
$\Rightarrow$ (1+n-1)J$_n$=$\frac{1}{(a+bI_1+cI_2)}$ $\times$ [cos^{n-1}v sinv] + (n–1)J$_{n-2}$
$\Rightarrow$ nJ$_n$=$\frac{1}{(a+bI_1+cI_2)}$ $\times$ [cos^{n-1}v sinv] + (n–1)J$_{n-2}$
$\Rightarrow$ J$_n$=$(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$
$\times\frac{1}{n}$ [cos^{n-1}[(a+bI$_1$+cI$_2$)x+p+qI$_1$+rI$_2$] sin[(a+bI$_1$+cI$_2$)x+p+qI$_1$+rI$_2$]]+$\frac{n-1}{n}$ J$_{n-2}$———-**(21)**

Equation (21) is the required reduction formula. □

Special cases: Case-I: In **(21)**, if we put p+qI$_1$+rI_2=0, c=0 and I$_1$=I(indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

J$_n$=$\int cos^n[(a+bI)x]\,dx$=$\frac{1}{n}$ $(\frac{1}{a}-[\frac{b}{a(a+b)}]I)$ [cos^{n-1}[(a+bI)x] sin[(a+bI)x]] + $\frac{n-1}{n}$ J$_{n-2}$——**21(a)**

Case-II: In **(21)**, if we put p+qI$_1+rI_2$=0, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

J$_n$=$\int cos^n ax\,dx$=$\frac{cos^{n-1}axsinax}{na}$ + $\frac{n-1}{n}$ J$_{n-2}$————————————————**21(b)**

Case-III: In **(21)**, if we put p+qI$_1+rI_2$=0, a=1, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

J$_n$=$\int cos^n x\,dx$=$\frac{cos^{n-1}xsinx}{n}$ + $\frac{n-1}{n}$ J$_{n-2}$————————————————**21(c)**

3.11: Obtain the reduction formula for the 2-refined neutrosophic integral $\int sec^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

Solution: Let J$_n$=$\int sec^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$
We put v=(a+bI$_1$+cI$_2$)x+p+qI$_1$+rI$_2$, then dv=(a+bI$_1$+cI$_2$)dx $\Rightarrow$ $\frac{1}{(a+bI_1+cI_2)}$dv=dx
J$_n$=$\frac{1}{(a+bI_1+cI_2)}\int sec^n v\,dv$= $\frac{1}{(a+bI_1+cI_2)}\int sec^{n-2}vsec^2v\,dv$
Using integrating by parts with sec^{2}v as 2nd function, we have
=$\frac{1}{(a+bI_1+cI_2)}$ [sec^{n-2}vtanv–(n-2)$\int sec^{n-3}vsecvtan^2v\,dv$]
=$\frac{1}{(a+bI_1+cI_2)}$ [sec^{n-2}vtanv–(n-2)$\int sec^{n-2}v(sec^2v-1)\,dv$]
=$\frac{1}{(a+bI_1+cI_2)}$ [sec^{n-2}vtanv–(n-2)$\int sec^n v\,dv$+ (n-2)$\int sec^{n-2}v\,dv$]
$\Rightarrow$ J$_n$=$\frac{1}{(a+bI_1+cI_2)}$ $\times$ sec^{n-2}vtanv–$\frac{n-2}{(a+bI_1+cI_2)}\int sec^n v\,dv$ + $\frac{n-2}{(a+bI_1+cI_2)}$ J$\int sec^{n-2}v\,dv$
$\Rightarrow$ J$_n$=$\frac{1}{(a+bI_1+cI_2)}$ $\times$ sec^{n-2}vtanv–(n-2)J$_n$ + (n-2)J$_{n-2}$
$\Rightarrow$ (n-1)J$_n$=$\frac{1}{(a+bI_1+cI_2)}$ $\times$ sec^{n-2}vtanv+ (n-2)J$_{n-2}$
$\Rightarrow$ J$_n$=$(\frac{1}{a}-[\frac{b}{(a+c)(a+b+c)}]I_1+[\frac{c}{a(a+c)}]I_2)$
$\times\frac{sec^{n-2}[(a+bI_1+cI_2)x+p+qI_1+rI_2]tan[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{n-1}$+ $\frac{n-2}{n-1}$J$_{n-2}$————————**(22)**

Equation (22) is the required reduction formula. □

Special cases: Case-I: In **(22)**, if we put p+qI$_1$+rI_2=0, c=0 and I$_1$=I(indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

J$_n$=$\int sec^n[(a+bI)x]\,dx$=$(\frac{1}{a}-[\frac{b}{a(a+b)}]I)$ $\frac{sec^{n-2}[(a+bI)x]tan[(a+bI)x]}{(n-1)}$+ $\frac{n-2}{n-1}$J$_{n-2}$————————**22(a)**

Case-II: In **(22)**, if we put p+q$I_1 + rI_2$=0, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int sec^n ax\, dx = \frac{sec^{n-2}ax.tanax}{a(n-1)} + \frac{n-2}{n-1} J_{n-2}$ ——— **22(b)**

Case-III: In **(22)**, if we put p+q$I_1 + rI_2$=0, a=1, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int sec^n x\, dx = \frac{sec^{n-2}x.tanx}{n-1} + \frac{n-2}{n-1} J_{n-2}$ ——— **22(c)**

3.12: Derive the reduction formula for the 2-refined neutrosophic integral $\int cosec^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\, dx$

Solution: Let $J_n = \int cosec^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\, dx$

We put v=[(a+bI_1+cI_2)x+p+qI_1+rI_2], then dv=(a+bI_1+cI_2)dx ⇒ $\frac{1}{(a+bI_1+cI_2)}$dv=dx

$J_n = \frac{1}{(a+bI_1+cI_2)} \int cosec^n v\, dv = \frac{1}{(a+bI_1+cI_2)} \int cosec^{n-2} v cosec^2 v\, dv$

Using integrating by parts with $cosec^2$v as 2nd function, we have

$= \frac{1}{(a+bI_1+cI_2)} [-cosec^{n-2}vcotv + (n-2)\int cosec^{n-3} v cosecv cot^2 v\, dv]$

$= \frac{1}{(a+bI_1+cI_2)} [-cosec^{n-2}vcotv + (n-2)\int cosec^{n-2} v (cosec^2 v - 1)\, dv]$

$= \frac{1}{(a+bI_1+cI_2)} [-cosec^{n-2}vcotv + (n-2)\int cosec^{n} v\, dv - (n-2)\int cosec^{n-2} v\, dv]$

$\Rightarrow J_n = -\frac{1}{(a+bI_1+cI_2)} \times cosec^{n-2}vcotv + \frac{n-2}{(a+bI_1+cI_2)} \int cosec^n v\, dv - \frac{n-2}{(a+bI_1+cI_2)} J\int cosec^{n-2} v\, dv$

$\Rightarrow J_n = -\frac{1}{(a+bI_1+cI_2)} \times cosec^{n-2}vcotv + (n-2)J_n - (n-2)J_{n-2}$

$\Rightarrow -(n-3)J_n = -\frac{1}{(a+bI_1+cI_2)} \times cosec^{n-2}vcotv - (n-2)J_{n-2}$

$\Rightarrow J_n = (\frac{1}{a} - [\frac{b}{(a+c)(a+b+c)}]I_1 + [\frac{c}{a(a+c)}]I_2)$

$\times \frac{cosec^{n-2}[(a+bI_1+cI_2)x+p+qI_1+rI_2]cot[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{n-3} + \frac{n-2}{n-3} J_{n-2}$ ——— **(23)**

Equation (23) is the required reduction formula. □

Special cases: Case-I: In **(23)**, if we put p+q$I_1 + rI_2$=0, c=0 and I_1=I(indeterminacy), then the 2-refined neutrosophic integral reduces to neutrosophic integral whose reduction formula is given by

$J_n = \int cosec^n[(a+bI)x]\, dx = (\frac{1}{a} - [\frac{b}{a(a+b)}]I) \frac{cosec^{n-2}[(a+bI)x]cot[(a+bI)x]}{(n-3)} + \frac{n-2}{n-3} J_{n-2}$ ——— **23(a)**

Case-II: In **(23)**, if we put p+q$I_1 + rI_2$=0, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int cosec^n ax\, dx = \frac{cosec^{n-2}ax.cotax}{a(n-3)} + \frac{n-2}{n-3} J_{n-2}$ ——— **23(b)**

Case-III: In **(23)**, if we put p+q$I_1 + rI_2$=0, a=1, b=c=0, then the 2-refined neutrosophic integral reduces to real integral whose reduction formula is given by

$J_n = \int cosec^n x\, dx = \frac{cosec^{n-2}x.cotx}{(n-3)} + \frac{n-2}{n-3} J_{n-2}$ ——— **23(c)**

4. Some applications in 2-refined neutrosophic definite integrals

In this section, we apply reduction formulae in 2-refined neutrosophic definite integrals to justify the results found in the previous section.

Problem 4.1. Using integration by parts evaluate the 2-refined neutrosophic integral
$\int((2-3I_1+5I_2)x+1-2I_1-3I_2)^2\times e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}\,dx$ **and also verify the result with the help of 2-refined neutrosophic reduction formula**.

Solution: Let $J_2=\int((2-3I_1+5I_2)x+1-2I_1-3I_2)^2e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}\,dx$

We put $v=(2–3I_1+5I_2)x+1–2I_1–3I_2$, $k=1+I_1+I_2$, then $dv=(2–3I_1+5I_2)dx$

$\Rightarrow \frac{1}{(2-3I_1+5I_2)}dv=dx$

Then $J_2=\frac{1}{(2-3I_1+5I_2)}\int v^2e^{kv}\,dv$

Integratig by parts we have,

$J_2=\frac{1}{(2-3I_1+5I_2)}\,[\frac{1}{k}\,v^2\,e^{kv}–\frac{1}{k}\int 2ve^{kv}\,dv]$

$=\frac{1}{(1+I_1+I_2)(2-3I_1+5I_2)}\,[\,v^2\,e^{kv}–2\int ve^{kv}\,dv]$

$=\frac{1}{(1+I_1+I_2)(2-3I_1+5I_2)}\,[\,v^2\,e^{kv}–2\{\,\frac{1}{k}\,ve^{kv}–\frac{1}{k}\int e^{kv}\,dv\}]$

$=\frac{1}{(1+I_1+I_2)(2-3I_1+5I_2)}\,[\,v^2\,e^{kv}–2\{\,\frac{1}{k}\,ve^{kv}–\frac{1}{k^2}\,e^{kv}\}]$

$=\frac{1}{(1+I_1+I_2)^3(2-3I_1+5I_2)}\,[k^2\,v^2\,e^{kv}–2kve^{kv}+2e^{kv}]$

$=\frac{1}{(1+19I_1+7I_2)(2-3I_1+5I_2)}\,[(1+I_1+I_2)^2\,v^2\,e^{kv}–2(1+I_1+I_2)ve^{kv}+2e^{kv}]$

$=(\frac{1}{2}-\frac{13}{1512}I_1+\frac{27}{56}I_2)\,e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}\,[(1+5I_1+3I_2)((2–3I_1+5I_2)x+1–2I_1–3I_2)^2$
$–2(1+I_1+I_2)((2–3I_1+5I_2)x+1–2I_1–3I_2)+2]$————————————**(4.1)** □

Again, using reduction formula (1), we have,

$J_2=\frac{1}{(1+I_1+I_2)(2-3I_1+5I_2)}$
$\times\,[((2\text{-}3I_1+5I_2)x+1\text{-}2I_1\text{-}3I_2)^2\,e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}–2(2\text{-}3I_1+5I_2)J_1]$

$J_1=\frac{1}{(1+I_1+I_2)(2-3I_1+5I_2)}$
$\times\,[((2\text{-}3I_1+5I_2)x+1\text{-}2I_1\text{-}3I_2)\,e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}–(2\text{-}3I_1+5I_2)J_0]$

$J_0=\int e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}\,dx$

$=\frac{1}{(1+I_1+I_2)(2-3I_1+5I_2)}\,e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}$

Putting the value of J_0 in J_1 we have,

$J_1=\frac{1}{(1+I_1+I_2)(2-3I_1+5I_2)}\times[((2\text{-}3I_1+5I_2)x+1\text{-}2I_1\text{-}3I_2)\,e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}$
$-\frac{1}{(1+I_1+I_2)}\,e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}]$

$=\frac{1}{(1+I_1+I_2)^2(2-3I_1+5I_2)}$
$\times\,[((2\text{-}3I_1+5I_2)x+1\text{-}2I_1\text{-}3I_2)\,(1+I_1+I_2)–1]\,e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}$

Putting the value of J_1 in J_2, we find that

$J_2=\frac{1}{(1+I_1+I_2)(2-3I_1+5I_2)}$
$\times\,[((2\text{-}3I_1+5I_2)x+1\text{-}2I_1\text{-}3I_2)^2\,e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}–\frac{2}{(1+I_1+I_2)^2}$
$\times\,[((2\text{-}3I_1+5I_2)x+1\text{-}2I_1\text{-}3I_2)\,(1+I_1+I_2)–1]\,e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}]$

$J_2=\frac{e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}}{(1+I_1+I_2)^3(2-3I_1+5I_2)}$
$\times\,[((2\text{-}3I_1+5I_2)x+1\text{-}2I_1\text{-}3I_2)^2(1+I_1+I_2)^2–2(1+I_1+I_2)\,((2\text{-}3I_1+5I_2)x+1\text{-}2I_1\text{-}3I_2)+2]$

$J_2=\frac{e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}}{(1+19I_1+7I_2)(2-3I_1+5I_2)}$

$\times$ [((2-3I_1+5I_2)x +1-2I_1-3I_2)$^2(1+5I_1+3I_2)$ −2(1+I_1+I_2) ((2-3I_1+5I_2)x +1-2I_1-3I_2) +2]

$J_2=(\frac{1}{2}-\frac{13}{1512}I_1+\frac{27}{56}I_2)\ e^{(1+I_1+I_2)((2-3I_1+5I_2)x+1-2I_1-3I_2)}$

$\times$ [((2-3I_1+5I_2)x +1-2I_1-3I_2)$^2(1+5I_1+3I_2)$ −2(1+I_1+I_2) ((2-3I_1+5I_2)x +1-2I_1-3I_2) +2]—**(4.2)**

Hence from **(4.1)** and **(4.2)** the result is verified. □

Theorem 4.2: If $J_{m,n}=\int_{\frac{-(p+qI_1+rI_2}{(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-rI_2}{2(a+bI_1+cI_2)}} sin^m[(a+bI_1+cI_2)x+p+qI_1+rI_2]\times cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$**,**
then prove that (m+n)$J_{m,n}$**−(n−1)** $J_{m,n-2}$**=0, (where m,n being positive integers).**

Proof: Let $J_{m,n}$

$=\int_{\frac{-(p+qI_1+rI_2}{(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-rI_2}{2(a+bI_1+cI_2)}} sin^m[(a+bI_1+cI_2)x+p+qI_1+rI_2]\times cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

By the reduction formula (5),we have,

$\int sin^m[(a+bI_1+cI_2)x+p+qI_1+rI_2]\times cos^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

$=\frac{cos^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]sin^{m+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(a+bI_1+cI_2)(m+n)}+$

$\frac{n-1}{(a+bI_1+cI_2)(m+n)}$

$\int sin^m[(a+bI_1+cI_2)x+p+qI_1+rI_2]\times cos^{n-2}[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx,$

(m+n≠0)

Then, $J_{m,n}=$

$$\left[\frac{cos^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]sin^{m+1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(a+bI_1+cI_2)(m+n)}\right]_{\frac{-(p+qI_1+rI_2)}{(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-I_2}{2(a+bI_1+cI_2)}}$$

$+(\frac{n-1}{m+n})\ J_{m,n-2}$

$\Rightarrow J_{m,n}-(\frac{n-1}{m+n})\ J_{m,n-2}=\left[\frac{cos^{n-1}(0)sin^{m+1}(\frac{\pi}{2})-cos^{n-1}(0)sin^{m+1}(\frac{\pi}{2})}{(a+bI_1+cI_2)(m+n)}\right]$

$\Rightarrow J_{m,n}-(\frac{n-1}{m+n})\ J_{m,n-2}=0$

$\Rightarrow (m+n)J_{m,n}-(n-1)\ J_{m,n-2}=0$

Theorem 4.3: If $J_n=\int_{\frac{-(p+qI_1+rI_2}{(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-rI_2}{4(a+bI_1+cI_2)}} tan^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$**, then prove that (n−1)(a+b**I_1 **+** cI_2**)**J_n**+**J_{n-2}**=1 (where, n is a positive integer).Hence find the value of the 2-refined neutrosophic definite integral** $\int_{\frac{-(4+5I_1-6I_2)}{(1+2I_1+3I_2)}}^{\frac{\pi-4-5I_1+6I_2}{4(1+2I_1+3I_2)}} tan^6[(1+2I_1+3I_2)x+4+5I_1-6I_2]\,dx.$

Proof: Part-I: Let $J_n=\int_{\frac{-(p+qI_1+rI_2)}{(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-rI_2}{4(a+bI_1+cI_2)}} tan^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

By the reduction formula 2(a),we have,

$\int tan^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

$=\frac{1}{(a+bI_1+cI_2)}\ \frac{tan^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(n-1)}-\int tan^{n-2}[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

$$\Rightarrow J_n=\frac{1}{(a+bI_1+cI_2)}\left[\frac{tan^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(n-1)}\right]_{\frac{-(p+qI_1+rI_2)}{(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-I_2}{4(a+bI_1+cI_2)}}$$

$-\int_{\frac{-(p+qI_1+rI_2)}{(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-I_2}{4(a+bI_1+cI_2)}} tan^{n-2}[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

$\Rightarrow$ J$_n$= $\frac{1}{(a+bI_1+cI_2)}$ $\left[\dfrac{tan^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(n-1)}\right]_{\frac{-(p+qI_1+rI_2)}{(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-I_2}{4(a+bI_1+cI_2)}}$ –J$_{n-2}$

$\Rightarrow$ J$_n$= $\frac{1}{(a+bI_1+cI_2)}$ $\left[\dfrac{tan^{n-1}(\frac{\pi}{4})-tan^{n-1}(0)}{(n-1)}\right]$–J$_{n-2}$

$\Rightarrow$ J$_n$= $\frac{1}{(a+bI_1+cI_2)}$ $\left[\dfrac{1-0}{(n-1)}\right]$–J$_{n-2}$

$\Rightarrow$ (n-1)(a+bI$_1+cI_2$)J$_n+J_{n-2}$=1

Part-II: We have, J$_6$=$\int_{\frac{-(4+5I_1-6I_2)}{(1+2I_1+3I_2)}}^{\frac{\pi-4-5I_1+6I_2}{4(1+2I_1+3I_2)}} tan^6[(1+2I_1+3I_2)x+4+5I_1-6I_2]\,dx$

=$\frac{1}{(1+2I_1+3I_2)}$ $\frac{1}{5}$–J$_4$ =$\frac{1}{(1+2I_1+3I_2)}$ $\frac{1}{5}$–[$\frac{1}{(1+2I_1+3I_2)}$ $\frac{1}{3}$–J$_2$] =$\frac{1}{(1+2I_1+3I_2)}$ $\frac{1}{5}$–$\frac{1}{(1+2I_1+3I_2)}$ $\frac{1}{3}$+[$\frac{1}{(1+2I_1+3I_2)}$ $\frac{1}{1}$–J$_0$]

=$\frac{1}{(1+2I_1+3I_2)}$ $\frac{1}{5}$–$\frac{1}{(1+2I_1+3I_2)}$ $\frac{1}{3}$+$\frac{1}{(1+2I_1+3I_2)}$–$\int_{\frac{-(4+5I_1-6I_2)}{(1+2I_1+3I_2)}}^{\frac{\pi-4-5I_1+6I_2}{4(1+2I_1+3I_2)}} dx$

=$\frac{1}{(1+2I_1+3I_2)}$ $\frac{1}{5}$–$\frac{1}{(1+2I_1+3I_2)}$ $\frac{1}{3}$+$\frac{1}{(1+2I_1+3I_2)}$–$\frac{\pi+12+15I_1-18I_2}{4(1+2I_1+3I_2)}$

=$\frac{1}{(1+2I_1+3I_2)}$ [$\frac{1}{5}$– $\frac{1}{3}$+1–$\frac{\pi+12+15I_1-18I_2}{4}$]

=$\frac{1}{(1+2I_1+3I_2)}$ [$\frac{1}{5}$– $\frac{1}{3}$+1–$\frac{\pi+12+15I_1-18I_2}{4}$]

=(1-$\frac{1}{12}$I$_1$+$\frac{3}{4}$I$_2$) [$\frac{13}{15}$–$\frac{\pi+12+15I_1-18I_2}{4}$] □

Theorem 4.4: If J$_n$=$\int_{\frac{\pi-p-qI_1-rI_2}{2(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-rI_2}{4(a+bI_1+cI_2)}} cot^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$,then prove that (n–1)(a+bI$_1+cI_2$)J$_n$+J$_{n-2}$=–1 (where n is a positive integer).Hence evaluate the 2-refined neutrosophic definite integral $\int_{\frac{-(1-5I_1+3I_2)}{(2-I_1+5I_2)}}^{\frac{\pi-1+5I_1+3I_2}{4(2-I_1+5I_2)}} cot^8[(2-I_1+5I_2)x+1-5I_1+3I_2]\,dx$.

Proof: Part-I: Let J$_n$=$\int_{\frac{\pi-p-qI_1-rI_2}{2(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-rI_2}{4(a+bI_1+cI_2)}} cot^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

By the reduction formula 2(a),we have,

$\int cot^n[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

$= -\frac{1}{(a+bI_1+cI_2)}$ $\frac{cot^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(n-1)}$ $-\int cot^{n-2}[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

$\Rightarrow$ J$_n$= $-\frac{1}{(a+bI_1+cI_2)}$ $\left[\dfrac{cot^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(n-1)}\right]_{\frac{\pi-p-qI_1-rI_2}{2(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-I_2}{4(a+bI_1+cI_2)}}$

$-\int_{\frac{\pi-p-qI_1-rI_2}{2(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-I_2}{4(a+bI_1+cI_2)}} cot^{n-2}[(a+bI_1+cI_2)x+p+qI_1+rI_2]\,dx$

$\Rightarrow$ J$_n$= $\frac{1}{(a+bI_1+cI_2)}$ $\left[\dfrac{cot^{n-1}[(a+bI_1+cI_2)x+p+qI_1+rI_2]}{(n-1)}\right]_{\frac{\pi-p-qI_1-rI_2}{4(a+bI_1+cI_2)}}^{\frac{\pi-p-qI_1-I_2}{2(a+bI_1+cI_2)}}$ –J$_{n-2}$

$\Rightarrow$ J$_n$= $\frac{1}{(a+bI_1+cI_2)}$ $\left[\dfrac{cot^{n-1}(\frac{\pi}{2})-cot^{n-1}(\frac{\pi}{4})}{(n-1)}\right]$–J$_{n-2}$

$\Rightarrow$ J$_n$= $\frac{1}{(a+bI_1+cI_2)}$ $\left[\dfrac{-1}{(n-1)}\right]$–J$_{n-2}$

$\Rightarrow$ (n-1)(a+bI$_1+cI_2$)J$_n+J_{n-2}$=–1

Part-II: We have, $J_8=\int_{\frac{-(1-5I_1+3I_2)}{(2-I_1+5I_2)}}^{\frac{\pi-1+5I_1+3I_2}{4(2-I_1+5I_2)}} cot^8[(2-I_1+5I_2)x+1-5I_1+3I_2]\,dx$

$=\frac{1}{(2-I_1+5I_2)}\ \frac{(-1)}{7}\ -J_6 = -\frac{1}{(2-I_1+5I_2)}\ \frac{1}{7}-[\frac{1}{(2-I_1+5I_2)}\ \frac{(-1)}{5}-J_4]$

$= -\frac{1}{(2-I_1+5I_2)}\ \frac{1}{7}+\frac{1}{(2-I_1+5I_2)}\ \frac{1}{5}+[\frac{1}{(2-I_1+5I_2)}\ \frac{(-1)}{3}-J_2]$

$= -\frac{1}{(2-I_1+5I_2)}\ \frac{1}{7}+\frac{1}{(2-I_1+5I_2)}\ \frac{1}{5}-\frac{1}{(2-I_1+5I_2)}\ \frac{1}{3}-[\frac{-1}{(2-I_1+5I_2)}-\int_{\frac{-(4+5I_1-6I_2)}{(2-I_1+5I_2)}}^{\frac{\pi-4-5I_1+6I_2}{4(2-I_1+5I_2)}} dx]$

$= -\frac{1}{(2-I_1+5I_2)}\ \frac{1}{7}+\frac{1}{(2-I_1+5I_2)}\ \frac{1}{5}-\frac{1}{(2-I_1+5I_2)}\ \frac{1}{3}+\frac{1}{(2-I_1+5I_2)}+\int_{\frac{-(4+5I_1-6I_2)}{(2-I_1+5I_2)}}^{\frac{\pi-4-5I_1+6I_2}{4(2-I_1+5I_2)}} dx$

$= -\frac{1}{(2-I_1+5I_2)}\ \frac{1}{7}+\frac{1}{(2-I_1+5I_2)}\ \frac{1}{5}-\frac{1}{(2-I_1+5I_2)}\ \frac{1}{3}+\frac{1}{(2-I_1+5I_2)}+\frac{\pi+12+15I_1-18I_2}{4(2-I_1+5I_2)}$

$= -\frac{1}{(2-I_1+5I_2)}\ [\frac{1}{7}-\ \frac{1}{5}+\ \frac{1}{3}-1-\frac{\pi+12+15I_1-18I_2}{4}]$

$= -(\frac{1}{2}+\frac{1}{42}I_1+\frac{5}{14}I_2)\ [\frac{-76}{105}-\frac{(\pi+12+15I_1-18I_2)}{4}] = (\frac{1}{2}+\frac{1}{6}I_1+\frac{5}{14}I_2)\ \frac{(105\pi+1564+1575I_1-1890I_2)}{420}$ □

5. Conclusions

The crucial concept of integration plays a vital role in daily life, particularly when calculating areas with unfamiliar shapes. As a result, we looked at the integrals of real variable functions but the reduction formulae for 2-refined neutrosophic integrals (RF2RNII) provide significant role to serve the integrals of neutrosophic functions of double indeterminacy.The originality of the present paper is that we establish the reduction formulae of 2-refined neutrosophic integrals (RF2RNII) for the first time. In this article we apply substitution method and method of integration by parts to establish reduction formulae of 2-refined neutrosophic indefinite integrals (RF2RNII) with suitable examples in a neutrosophic framework. We also apply reduction formulae directly to evaluate 2-refined neutrosophic definite integrals. The reduction formula for 2-refined trigonometric functions in 2-refined neutrosophic integrals is a useful mathematical technique for calculating and deriving a simplified version of 2-refined neutrosophic integrals. These reduction formulae can be used to solve differential equations involving indeterminacy and uncertainty, the area of any irregular closed surfaces and the summation of a series involving indeterminacy and uncertainty. Our future research plane includes some more work on 2-refined neutrosophic reduction formulae, 2-refined neutrosophic multiple integrals, changing the order of integration of 2-refined neutrosophic double integrals etc.

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