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Structured Neutrosophic Sets with Entropy-Derived Indeterminacy: Injectivity, Stability, and Aggregation Guarantees for Variability-Sensitive MCDM

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ABSTRACT

In neutrosophic multi-criteria decision-making, indeterminacy has been universally encoded as a scalar value — a design choice that systematically suppresses distributional information and undermines ranking reliability. This paper shows that this scalar encoding leads to three different pathologies in the structure: many-to-one representational collapse, ranking instability when the input is perturbed by limited perturbations, and aggregation inconsistency in environments with variations. Being non-dimensional, it is shown that this is a functional deficit, so that a generic expansion of the representation gap by an interval valued (IVW) extension and an orthopair (OPW) extension of the representation cannot solve this gap. To examine the cause, we propose that indeterminacy serves as the principal function of Shannon entropy, statistical dispersion, and inter-response inconsistency, which led to the Entropy-Anchored Structured Neutrosophic Sets (EA-SNS), and then decompose these components into eleven structurally independent components (hendecagonal representation). This basis is shown to be functionally independent using a full-rank Jacobian analysis. Theorems are completely provided with proofs: (i) boundedness and strict monotonicity of the indeterminacy functional; (ii) structural injectivity, which allows to exclude -

formally - many-to-one collapse; (iii) score function monotonicity preserving preference order; (iv) an explicit Lipschitz perturbation stability bound. Structure-aware aggregation operators include witnesses of idempotency, commutativity, and boundary preservation. Empirical results, using two real educational datasets and three controlled benchmarks, compared to five competing neutrosophic methods, are obtained with a mean discrimination gain of 34.7% above the mean discrimination gain of the standard SVNS in all cases, compared to the expert-validated ground truth.

Keywords: Entropy-Driven Indeterminacy; Functionally Decomposed Uncertainty; Hendecagonal Neutrosophic Representation; Multi-Criteria Decision Making; Perturbation-Stable Ranking; Structure-Preserving Aggregation

1. INTRODUCTION

How should distributional uncertainty be represented in multi-criteria decision-making? Neutrosophic set theory offers three membership components — truth, falsity, and indeterminacy — enabling richer modeling than classical fuzzy approaches. However, after 30 years of development, one crucial assumption has never been questioned: that indeterminacy is a scalar (Siddhya, 2025; Abdelaziz et al., 2023). A degree of indeterminacy is given for each proposition, and the dimension of real-world uncertainty is reduced to a single value, whether the proposition is a bounded real number, an interval endpoint, or a component of an orthopair (Arora & Naithani, 2024; Rogulj et al., 2021). That collapse is the central problem this paper addresses.

The real-world importance of scalar indeterminacy becomes evident in situations that require consideration of variability. Two classrooms were assessed for instructional effectiveness, and they had the same mean scores on the assessment and the same average response difference (inconsistency). One class shows relatively low dispersion and a high level of agreement, with responses tightly clustered, while the other shows responses that are spread out. For any current work on the neutrosophic one, the same indeterminant values will appear in both classrooms. The encoding will not preserve the information that distinguishes true from false news disagreements. What makes true disagreements news vs. volatile disagreements is not encoded.

This representational failure is structural. The many-to-one correspondence between the 2D configuration of the transformations and the scalar indeterminacy generates three entangled pathologies. As regards information loss due to variability, it dissipates during encoding (Irvanizam et al., 2021; Abdel-Basset et al., 2022; Adalı et al., 2023). The instability is explained by the small-scale sensitivity of the input to the output ranks in the absence of structural constraints on the surface of the scores (Yaacob et al., 2024; Ye et al., 2023). These distortions are further exacerbated by aggregation inconsistency, as operators combine 'indistinguishable representations' (Deng & Wang, 2021; Jana et al., 2021; Martina & Deepa, 2023). Unfortunately, none of the existing extensions, interval-valued, q-rung orthopair, and hesitant neutrosophic, can cure such pathologies, as none correlates the value of indeterminacy with the properties of the induction of the input.

The present paper presents Entropy-Anchored Structured Neutrosophic Sets (EA-SNS), where the indeterminacy is a principled function of three measurable distributional attributes: Shannon entropy, statistical dispersion, and inter-response inconsistency, and is decomposed into eleven orthogonal parts of indeterminacy, each representing an independent source of uncertainty variability. The framework is strung together by four theorems, each of which is proved in full. The aggregation operators include formal proofs that they are idempotent, commutative, and boundary-preserving, which are necessary properties to ensure that any decision aggregation framework can be trusted. A step-by-step approach to following up is explained in a fully worked numerical example. Wilcoxon significance testing was carried out throughout for the five competitive baselines and for the five datasets used in the empirical validation.

Decomposition of indeterminacy as a construct of data, not a scalar value; 2. Functionally independent uncertainty decomposition (with Jacobian rank analysis) with conservation of distributional structure; Theoretical guarantees, such as those discussed above in this paper, are given to a formal level with boundedness, structural injectivity (repels many-to-one collapse), monotonic ranking behavior, and Lipschitz-bounded perturbation stability; Introduce structure-aware aggregation operators that are idempotent and commutative, and preserve the edges, with proofs. Very comprehensive empirical tests have been conducted on both real and synthetic datasets, validating the system and demonstrating significant improvements in discrimination, robustness, and ranking stability compared to existing neutrosophic approaches.

The contribution reimagines indeterminacy, from an assigned scalar to a derived, decomposable, and structurally faithful object, to formally establish, with four theorems and empirical evidence over five datasets, its new decision-theoretic properties not explored in previous formulations. This claim is backed by some structural and empirical arguments: no scalar-encoding frameworks meet structural injectivity (Theorem 2) and are guaranteed by a Lipschitz bound on perturbations (Theorem 4); and the results in Section 9 show that, on average, the discrimination indices of a framework are 34.7% higher.

The rest of the paper is organized as follows. Section 2 reviews the current state of neutrosophic and fuzzy decision frameworks, formally describes the structural deficit, and situates the contributions of this work. This section summarizes the mathematical background needed later. The framework of EA-SNS, including the data-driven indeterminacy functional, the hendecagonal decomposition, the aggregation operators, and the entropy-modulated-ranking function, is introduced in Section 4. Section 5 contains four formal theorems along with complete formal proofs. The entire 10-step MCDM algorithm is detailed in Section 6. Section 7 contains an extensively elaborated example of computations. The absence of references in sections 8 and 9 indicates the experimental design and empirical results for five datasets and five competitive baselines. The findings and limitations of the study are discussed in Section 10. Section 11 concludes.

(C1) Indeterminacy is refunctionalized as a data-driven structure (construct) which emerges from a measurable property (entropy, dispersion, inconsistency) and is decomposable by it; this is an arbitrary scalar-assignable structure, which has been used everywhere by other formulations of neutrosophic distributions. (C2) With respect to the end of the time interval, the uncertainty can be decomposed into eleven structurally independent components (Proposition 1), which are functionally independent as well. (Formal proof) structural injectivity (Theorem 2), excluding a significant number of many-to-one representational collapses; not possible to construct formally using any scalar-encoding framework. Explicit Lipschitz perturbation stability bound (Theorem 4), which is the first analytic rank-stability guarantee for neutrosophic MCDM. Works of the structure-aware aggregation operators (EA-SNWA, EA-SNWG) with proofs of idempotency, commutativity, boundary preservation, and simplex closure. This is the performance of the empirical validation across 5 datasets, combined with 5 competitive baselines: Mean discrimination gain = 0.347, PSC = 0.973, and Spearman rank correlation = 0.991, with p-values < 0.001 for each dataset.

2. LITERATURE REVIEW AND IDENTIFIED RESEARCH GAP

2.1 Evolution of Neutrosophic and Fuzzy Decision Frameworks

The neutrosophic set theory by Smarandache has extended earlier fuzzy set theories, which were limited to the standard unit interval, by introducing truth, indeterminacy, and falsity as independent membership functions defined on non-standard intervals, making the computation of these functions feasible for applications of MCDM (Rahimi et al., 2021; Unver et al., 2022). Turskis et al. (2022) restricted this to the standard unit interval to enable computational tractability in MCDM applications. Interval-valued neutrosophic sets were an interpolation much richer in formal content but shallow in representation because each membership component was given an intervalous representation [a,b] or [a,b] where a and b were again in [0,1] (Peng et al., 2023; Muthulakshmi et al., 2025; Nabeeh, 2023).

The q-rung orthopair generalisation relaxed the constraint $T + F \leq 1$ to $Tq + Fq \leq 1$, and the expansion of the admissibility came with benefits, since it turned out that the accuracy of the classifications could be improved using the Fermatean fuzzy multicriteria model (Nafei et al., 2021; Peng et al., 2024). The improvement is due to domain enlargement, not to the decomposition of domain uncertainty — as this paper makes precise (Kumar et al., 2022). In addition, hesitant neutrosophic sets provided an additional type of membership degree, in the form of finite sets, and could express expert disagreement directly without relying on the input structure (Fujita, 2026; Gao, 2025; Garg et al., 2022).

Entropy-based is one of the parallel development streams. In the study by Thakur et al. (2021), intuitionistic fuzzy entropy was incorporated into the COPRAS weighting method, demonstrating that information-theoretic weighting enhances rank stability. Ismail et al. (2025) further elaborated on this idea in the context of financial performance appraisal under neutrosophic uncertainty. These contributions consider entropy as an external weighting factor once the basic representation is given. The indeterminacy structure itself is not affected by entropy. It is not entropy that enters the indeterminacy structure — that makes the difference.

Hybrid fuzzy-neutrosophic architectures have proliferated since 2022. As stated earlier, research has been undertaken to develop objective weighting approaches for Pythagorean neutrosophic MCDM; Kamari et al. (2025a, 2025b) proposed one such approach. These are kinds of systems that can expand the representational range without challenging the assumption of scalar encoding. The theoretical floor does not vary across the formulations tested.

2.2 The Structural Deficit: Formal Characterisation

The central limitation across all reviewed frameworks is functional, not dimensional. Scalar-encoding methods implement a mapping I: $X \rightarrow [0,1]$ that is many-to-one with respect to distributional structure. Two input configurations $x, y \in X$ may possess entirely distinct entropy profiles, dispersion magnitudes, and consistency patterns while satisfying $I(x) = I(y)$ under any existing formulation. This is not a degenerate edge case. It is the structural norm for input spaces with heterogeneous variability, and it is analytically irresolvable within the scalar paradigm (Farid et al., 2023).

The consequences compound through the decision pipeline. Aggregation operators applied to indistinguishable representations produce indistinguishable aggregate scores (Fauzi et al., 2025). The score surface becomes flat precisely in regions where dispersion-sensitive alternatives must be distinguished (Alahmadi et al., 2023). Perturbation analysis of classical neutrosophic TOPSIS implementations confirms this: rank reversals occur under perturbations as small as 5% of the input range in high-variability environments, as reported in Section 9.3 of the present study.

Dimensional expansion cannot resolve a functional inadequacy. Interval-valued sets introduce range uncertainty about the indeterminacy value but do not condition that value on the generative distribution. Q-rung orthopair models expand the feasibility region but retain magnitude-based assignment (Ali et al., 2024). Entropy-augmented models apply information-theoretic corrections post-hoc. The research gap is therefore one of inadequate structural correspondence between input geometry and representational geometry — a gap that demands a representational rearchitecture rather than a parametric extension.

2.3 Positioning of the Present Contribution

This article addresses this lacuna by redefining indeterminacy as an intermediate, measurable function of the input distribution and of the structural independence among its decomposition elements. The dimensionality of the representation is not given by convention, but by the rank of the descriptor Jacobian, a mathematically-based convention. The V1-V2 mapping that underlies ranking instability or failure in all previous formulations is replaced by a principled V1-V2 mapping that relates the distributional structure to the representational structure.

2.3.1 Motivating Intuitive Example: The Two-Classroom Problem

Consider two classrooms, each evaluated on instructional effectiveness by 30 students using a 1–7 Likert scale. Classroom A produces responses {4, 4, 4, 5, 5, 4, 4, 5, 4, 4, ...} — tightly clustered near the mean of 4.3, with low dispersion and high agreement. Classroom B produces responses {1, 7, 1, 7, 2, 6, 7, 1, 6, 2, ...} — bimodal and polarised, yet also averaging to 4.3. Under any existing neutrosophic formulation, both classrooms receive identical indeterminacy values, which are determined solely by the mean. A decision-maker comparing these two classrooms under such a model cannot distinguish a reliable consensus from a volatile disagreement. This is not a pathological edge case: In any survey-based MCDM application, alternatives whose response distributions are indeed structurally different will regularly have comparable central tendencies. In practice, the Many-to-One collapse described formally in Section 2.2 is the norm rather than the exception. It is a direct point of difference that EA-SNS obtains when considering the score difference, namely $H(\Theta) = 3.06$ (diffuse, high entropy uncertainty) vs $H(\Theta) = 1.21$ (concentrated, low entropy uncertainty), leading to a score separation of 0.556, which is structurally grounded discrimination that is not achievable with scalar methods.

2.3.2 Theoretical and Empirical Justification for Eleven Dimensions

The choice of eleven uncertainty descriptors is neither arbitrary nor universal; it is theoretically grounded and empirically validated. The theoretical framework is founded on the necessity of functional independence: each descriptor must refer to a different and non-redundant source of distributional uncertainty. It is the formal requirement that $J_\Phi(x)$ become a square matrix with full row rank, as described in Proposition 1. Conversely, the question of "why eleven" becomes "what is the minimum number of descriptors that can be used functionally and are still enough to characterise the relevant uncertainty geometry of survey-based data?"

The eleven descriptors encompass four theoretical classes of distributional information: (i) measures of overall uncertainty spread (according to Shannon entropy $\tilde{H}$, Rényi entropy H_2, as well as local entropy gradient LEG); (ii) measures of dispersion that capture the magnitude of the spread (coefficient of variation CV, normalised range $\tilde{\kappa}$); (iii) measures of distributional asymmetry and tail behaviour (skewness |Skew|, kurtosis Kurt, distributional asymmetry index Asym, tail-weight ratio TWR); and (iv) measures of the structure of the distribution (bimodality coefficient BMC, IQR ratio). They offer access to uncertainty geometry that encompasses all possible geometries resulting from bounded, discrete response data, without making any distributional assumptions. It is not possible to represent any descriptor in the set as a linear combination of the others, as demonstrated by symbolic differentiation and verified by pairwise Pearson correlation coefficients, which did not exceed 0.31 across any of the validation sets.

Empirical evidence to support this is given directly in the ablation study in Table 6 (Section 9.7). For all three evaluation methods (DI, PSC and GTA), deleting descriptors one by one monotonically reduces performance, with each deletion resulting in measurable degradation. There is no significant improvement (ΔDI = 0.002, ΔPSC = 0.001) between $k = 11$ and $k = 12$, and an additional ($k = 12$) descriptor is only moderately correlated with BMC ($r = 0.44$), suggesting that $k = 12$ is redundant. This $k = 11$ configuration is thus the empirical sufficiency boundary: increasing the number of descriptors is no longer useful, and decreasing it unequivocally worsens performance.

The eleven-dimensional representation is not a freely chosen design but a minimum representation that captures the validation geometry, validated both theoretically (Jacobian rank analysis) and empirically (Ablation study).

2.3.3 Novelty Positioning: Three Differentiating Contributions

EA-SNS proposes new ideas in a field where many extensions of neutrosophic sets have been suggested, and those novelties are quite clear and separable from previous works within three specific axes:

(a) Functional Extension / Dimensional Extension. The interval-valued, q-rung orthopair, hesitant, and Pythagorean neutrosophic extensions do not consider the functional extension from input distributions to the indeterminacy value of the representations; thus, they are only a marginal extension in the representational domain (dimensional extension). EA-SNS is unique in its focus on the functional gap: indeterminacy is not assigned but rather calculated from measurable distributional attributes. This is not a more robust option to the existing contributions, but another type.

(b) Structural Injectivity. This paper will be the first to present a formal proof of the distinctness of representational outputs under different input distributions in the neutrosophic MCDM framework. EA-SNS does, under Theorem 2. This formally renders many-to-one collapses nonexistent, preventing ranking instability and discrimination failures in methods based on scalar encoding. It cannot be achieved by construction, since a scalar cannot be injective over a multi-dimensional input space, within any scalar-encoding framework.

(c) Entropy-Inside Representation. In contrast to other models proposed previously, e.g., entropy-weighted SVNS, the following models (M4 in this study) treat entropy as an external weighting coefficient applied after the main representation is determined. The entropy directly resides in the indeterminacy representation in the form of the structured uncertainty vector Θ(x) and the entropy-modulated score function $S(N) = T - \alpha \cdot I \cdot H(\Theta)/\log_2(11) - \beta \cdot F$. The discrimination advantage of EA-SNS over entropy-weighted SVNS (the best previous entropy-augmented baseline, Table 4) of 24.4% indicates that Entropy-in-Side representation does not equal Entropy-outside weighting, but is rather a strict improvement for discrimination at the highest level of variability.

3. MATHEMATICAL PRELIMINARIES

The following definitions and notation are used throughout the paper. Let X denote a universe of discourse and let $\mathbb{R}$, $\mathbb{R}_+$, and [0,1] carry their standard meanings.

Definition 1 (Neutrosophic Set). A neutrosophic set A over X is a structure $A = \{\langle x, T\_A(x), I\_A(x), F\_A(x)\rangle : x \in X\}$, where $T\_A(x), I\_A(x), F\_A(x) \in [0,1]$ denote truth, indeterminacy, and falsity membership functions, respectively, satisfying $0 \leq T\_A(x) + I\_A(x) + F\_A(x) \leq 3$.

Definition 2 (Single-Valued Neutrosophic Number). A SVNN is a triplet $\langle T, I, F\rangle$ with $T, I, F \in [0,1]$ and $0 \le T + I + F \le 3.$ For $N_1 = \langle T_1, I_1, F_1\rangle$ and $N_2 = \langle T_2, I_2, F_2\rangle$:

$$N_1 \oplus N_2 = \langle T_1 + T_2 - T_1T_2, \quad I_1I_2, \quad F_1F_2\rangle$$

$$N_1 \otimes N_2 = \langle T_1T_2, \quad I_1 + I_2 - I_1I_2, \quad F_1 + F_2 - F_1F_2\rangle$$

$$\lambda N_1 = \langle 1 - (1 - T_1)^{\lambda},\ I_1^{\lambda},\ F_1^{\lambda}\rangle,\ \lambda > 0$$

$$N_1^{\lambda} = \langle T_1^{\lambda},\ 1 - (1 - I_1)^{\lambda},\ 1 - (1 - F_1)^{\lambda}\rangle$$

Definition 3 (Shannon Entropy). For a discrete probability distribution $P = \{p_1, \ldots, p_n\}$ with the convention $0 \cdot \log_2 0 = 0$, the Shannon entropy is $H(P) = -\Sigma_i\, p_i\, log_2(p_i) \in [0, log_2 n]$. The normalised form $H(P) = H(P)/log_2(n)$ maps to [0,1].

Definition 4 (Probability Simplex). The (k−1)-dimensional probability simplex is $\Delta^{(k-1)} = \{\theta \in \mathbb{R}^k : \sum_{i=1}^{k} \theta_i = 1, \theta_i \ge \theta_i \ge 0\}$. Vectors in $\Delta^{(k-1)}$ are non-negative, bounded, and collectively exhaustive by construction.

4. THE ENTROPY-ANCHORED STRUCTURED NEUTROSOPHIC FRAMEWORK

4.1 Formal Definition of EA-SNN

Definition 5 (Entropy-Anchored Structured Neutrosophic Number). For each $x \in X$ with associated observation set $O(x) = \{x_1, \ldots, x_n\}$, an EA-SNN is the quadruplet:

$$N(x) = \langle T(x), \quad I(x), \quad F(x), \quad \Theta(x)\rangle$$

where $T(x), F(x) \in [0,1]$ denote truth and falsity degrees; $I(x) \in [0,1]$ is the data-driven indeterminacy functional defined in Section 4.2; and $\Theta(x) = (\theta_1(x), \ldots, \theta_{11}(x)) \in \Delta^{10}$ is the structured uncertainty vector satisfying $\Sigma_{i=1}^{11}\, \theta_i(x) = 1$ and $\theta_i(x) \ge 0$ for all i.

Definition 5 is characterised by three properties that set it apart from all previous formulations. The value of the components of I(x) is calculated, instead of being set. The eleven directions shown are orthogonal; that is, they treat the uncertainty in I(x) as the sum of orthogonal parts represented by the vector Θ(x). The sufficient condition for structural injectivity of the full EA-SNN mapping is given in Proposition 1.

4.2 Generalisation: EA-SNS Subsumes Classical SVNS

Classical SVNS corresponds to the degenerate case in which the uncertainty vector collapses to a scalar. Formally, define the degenerate EA-SNN as $N\_deg(x) = \langle T$, I_scalar,

$F, (1,0,\ldots,0)\rangle$ $-$ a vector with all uncertainty mass concentrated in a single descriptor. The classical SVNS score function $S\_classical = T + (1 - I)/2 - F$ is then a special case of the entropy-modulated score $S(N) = T - \alpha \cdot I \cdot H(\Theta)/log_2(11) - \beta F$ under the limiting conditions $H(\Theta) \rightarrow 0 \; and \; \alpha \rightarrow (1/2)/I$:

$$lim\_\{H(\Theta) \rightarrow 0\} \, S(N) = T - \beta F \supseteq S\_classical \;\; when \, \beta = 1, \alpha \rightarrow 0$$

This single equation establishes that EA-SNS is a strict generalisation of SVNS. Every SVNS ranking is reproducible within EA-SNS. No EA-SNS ranking collapses to SVNS unless the uncertainty vector is fully concentrated — a condition that never arises in real distributional data. The generalisation is complete and one-directional.

4.3 Data-Driven Indeterminacy Construction

Given $O(x) = \{x_1, \ldots, x_n\}$, the primary distributional descriptors are defined as follows. The sample mean is $\mu = (1/n)\Sigma_i x_i$. The sample standard deviation is $\sigma = \sqrt{[(1/n)\Sigma_i(x_i - \mu)^2]}$. The empirical probability distribution P is constructed by partitioning the observation range into m = 10 equal bins, with p_i denoting the proportion of observations in bin i. Normalised Shannon entropy follows as $H(P) = -(1/log_2 m)\Sigma_i p_i log_2(p_i) \in [0,1]$. The inconsistency index $is \; \kappa = (max(x_i) - min(x_i))/(\mu + \varepsilon) \; for \; \varepsilon = 10^{-6}$. Min-max normalised versions $\tilde{\sigma}$ and $\tilde{\kappa}$ are computed over the full dataset prior to scoring.

The indeterminacy functional is defined by:

$$I(x) = [\bar{H}(P) + \tilde{\sigma} + \tilde{\kappa}] \; / \; [1 + H(P) + \tilde{\sigma} + \tilde{\kappa}]$$

The form $g(s) = s/(1 + s) \; maps \; \mathbb{R}_+ \; bijectively \; to$ $[0,1)$ and is strictly monotone in each argument. This ensures that I(x) responds proportionally to every independent source of distributional disorder without saturation artefacts.

4.4 Hendecagonal Uncertainty Decomposition

The eleven descriptors constituting the structured uncertainty basis are: (f_1) normalised Shannon entropy $\bar{H}$; (f_2) coefficient of variation $CV = \sigma/\mu$; (f_3) normalised range $\tilde{\kappa}$; (f_4) excess kurtosis Kurt, normalised to [0,1]; (f_5) skewness magnitude |Skew|, normalised; (f_6) $inter - quartile \; range \; ratio \; IQR/(max - min + \varepsilon)$;

(f_7) $Rényi\ entropy\ of\ order\ 2, H_2 = -log_2(\Sigma p_i^2)$, normalised; ($f_8$) distributional asymmetry index Asym = |μ−median|/σ, normalised; (f_9) $tail - weight\ ratio\ TWR = P(|X - \mu| > 1.5\sigma)$; ($f_{10}$) response bimodality coefficient $BMC = (Skew^2 + 1)/(Kurt + 3(n-1)^2/[(n-2)(n-3)])$; (f_{11}) local entropy gradient LEG, measuring entropy difference between upper and lower observation halves.

The structured uncertainty vector is constructed by simplex normalisation:

$$\theta_i(x) = f_i(x) / \Sigma_{j=1}^{11} f_j(x), \ \ i = 1, \ldots, 11$$

This yields $\Theta(x) \in \Delta^{10}$ by construction. Pairwise Pearson correlations among the eleven descriptors, computed across both validation datasets, remain below 0.31, confirming near-orthogonality empirically. Ablation analysis in Section 9.7 confirms structural non-redundancy formally.

4.5 Functional Independence: Jacobian Rank Analysis

Definition 6 (Functional Independence). $Let\ \Phi: X \rightarrow \mathbb{R}^{11}$ be defined by $\Phi(x) = (f_1(x), \ldots, f_{11}(x))$. The descriptor set $\{f_1, \ldots, f_{11}\}$ is functionally independent on an open set $U \subseteq X$ if the Jacobian $J\_\Phi(x)$ has rank 11 for almost every $x \in U$ in the Lebesgue sense.

Proposition 1 (Rank Certificate). For any observation set $O(x)\ with\ n \geq 12$ distinct values from a continuous distribution, $J\_\Phi(x)$ achieves full rank 11 almost surely.

Proof. For each descriptor f_i, a different statistical moment or information-theoretic quantity is involved. Asym, Shannon entropy, kvarta, order statistics, BMC, IQR, Rényi entropy, TWR, and spatial entropy variation are unrelated to the mean, either because they depend on order statistics, other than the mean, or because the measure cannot be written as a polynomial function of one or several of the other measures. Symbolic differentiation verifies that each of the rows of the Jacobian is a different non-linear functional of the observation vector, and so is linearly independent for most observation vectors. Sard's Theorem gives us the result that the set of observations resulting in rank deficiency is of Lebesgue measure zero for Φ.

4.6 Structure-Aware Aggregation Operators

Definition 7 (EA-SNWA). $Let\ \{N_1, \ldots, N_m\}\ be\ EA - SNNs$ with $N_k = \langle T_k, I_k, F_k, \Theta_k \rangle$ and weights $w = (w_1, \ldots, w_m), w_k > 0, \Sigma w_k = 1$. The EA-Structured Neutrosophic Weighted Average operator is:

$$EA - SNWA(N_1, \ldots, N_m) = \langle T\_agg, I\_agg, F\_agg, \Theta\_agg \rangle$$
$$T\_agg = 1 - \Pi_k(1 - T_k)w_k, \; F\_agg = \Pi_k F_k\hat{}w_k, \; I\_agg = \Pi_k I_k\hat{}w_k$$
$$\Theta\_agg = \Sigma_k w_k\Theta_k \text{ (simplex-normalised)}$$

Theorem A (Properties of EA-SNWA). The EA-SNWA operator satisfies: (i) Idempotency: if $N_k = N$ for all k, then $EA - SNWA(N_1, \ldots, N_m) = N$; (ii) Commutativity: EA-SNWA is invariant to permutation of its arguments; (iii) Boundary Preservation: $0 \le T\_agg + I\_agg + F\_agg \le 3$; (iv) Simplex Closure: $\Theta\_agg \in \Delta^{10}$.

Proof. (i) Idempotency:

$$\text{If } T_k = T \text{ for all } k, \text{then } T_{\text{agg}} = 1 - (1 - T)^{\Sigma_k w_k} = 1 - (1 - T)^1 = T$$
$$I_{\text{agg}} = I^{\Sigma_k w_k} = I^1 = I \text{and} F_{\text{agg}} = F^{\Sigma_k w_k} = F$$
$$\text{For } \Theta\text{: } \Theta_{\text{agg}} = \textstyle\sum_k w_k \cdot \Theta = \Theta \cdot \sum_k w_k = \Theta \cdot 1 = \Theta$$

Idempotency holds. (ii) Commutativity: Products and weighted sums are commutative under permutation of index labels; the result follows immediately. (iii) Boundary Preservation: By the AM-GM inequality and the constraint $T, I, F \in [0,1]$, each aggregated component lies in [0,1]. The sum $T\_agg + I\_agg + F\_agg$ achieves its maximum when each component is 1, giving a sum of 3; the neutrosophic constraint $0 \le T + I + F \le 3$ is therefore preserved under aggregation. (iv) Simplex Closure: $\Sigma_i(\Theta\_agg)_i = \Sigma_i \Sigma_k w_k\theta_{ik} = \Sigma_k w_k \Sigma_i\theta_{ik} = \Sigma_k w_k \cdot 1 = 1$. Non-negativity follows from $w_k > 0 \; and \; \theta_{ik} \ge 0$.

Definition 8 (EA-SNWG). The EA-Structured Neutrosophic Weighted Geometric operator employs geometric means:

$$T_{\text{agg}} = \prod_k T_k^{w_k}, F_{\text{agg}} = 1 - \prod_k (1 - F_k)^{w_k}, I_{\text{agg}} = 1 - \prod_k (1 - I_k)^{w_k}$$
$$\left(\Theta_{\text{agg}}\right)_i = \frac{\exp(\sum_k w_k \cdot \log\theta_{ik})}{\sum_j \exp\left(\sum_k w_k \cdot \log\theta_{jk}\right)}$$

Theorem B (Properties of EA-SNWG). EA-SNWG satisfies idempotency, commutativity, boundary preservation, and simplex closure by arguments dual to those in Theorem A, with products replacing sums and geometric means replacing arithmetic means throughout. The log-linear aggregation on the simplex preserves positivity by the exponential map.

4.7 Consistency Condition

Proposition 2 (Neutrosophic Consistency Preservation). For any collection of EA-SNNs satisfying $T_k + I_k + F_k \leq 3$ for all k, the EA-SNWA output satisfies $T\_agg + I\_agg + F\_agg \leq 3$.

Proof. Since $T_k \in [0,1]$, the product form gives $T\_agg = 1 - \Pi_k(1 - T_k)\hat{}w_k \leq 1$. Similarly $I\_agg = \Pi_k I_k\hat{}w_k \leq 1$ and $F\_agg = \Pi_k F_k\hat{}w_k \leq 1$, so $T\_agg + I\_agg + F\_agg \leq 3$. The lower bound follows from non-negativity of all components.

4.8 Entropy-Modulated Ranking Function

The scalar ranking of an EA-SNN incorporates both the magnitude of indeterminacy and the structural content of the uncertainty vector. The entropy-modulated score is defined as:

$$S(N) = T - \alpha \cdot I \cdot H(\Theta)/log_2(11) - \beta \cdot F$$

where $H(\Theta) = -\Sigma_i \theta_i log_2(\theta_i)$ is the Shannon entropy of the uncertainty distribution vector and $\alpha, \beta \in (0,1)$ satisfy $\alpha + \beta < 1$. Normalisation by $\log_2(11)$ maps the entropy term to [0,1]. Higher entropy in the uncertainty vector indicates diffusely distributed uncertainty across all eleven components — an intrinsically more indeterminate configuration — warranting a penalty beyond what scalar I alone captures.

Definition 9 (Weighted Distance Between EA-SNNs). For two $EA - SNNs\ N_a = \langle T_a, I_a, F_a, \Theta_a \rangle\ and\ N_b = \langle T_b, I_b, F$_b,Θ_b⟩ with criterion weights $w = (w_1, \dots, w_n)$, the weighted EA-SNN distance is:

$$d(N_a, N\_b) = \sqrt{[\, w_t(T_a - T\_b)^2 + w_i(I_a - I\_b)^2 + w\_f(F_a - F\_b)^2 + w\_\theta \, \|\Theta_a - \Theta\_b\|_2{}^2 \,]}$$

where $w_t + w_i + w\_f + w\_\theta = 1\, with\, w_t, w\_f, w\_\theta > 0$. This distance extends classical neutrosophic Euclidean distance by incorporating the full uncertainty vector geometry. TOPSIS-based ranking under EA-SNS employs this distance to compute separation measures from the positive and negative ideal solutions.

Proposition 3 (Metric Properties). $d(N_a, N\_b)$ defines a semi-metric on the space of $EA - SNNs$, satisfying non-negativity, symmetry, and the triangle inequality. Non-degeneracy holds under the structural injectivity of Theorem 2.

Proof. Non-negativity and symmetry follow from the squared Euclidean structure. The triangle inequality follows from the Cauchy-Schwarz inequality applied to the weighted sum of squared differences. Non-degeneracy: $d(N_a, N\_b) = 0$ requires each weighted term to vanish, implying $T_a = T\_b, I_a = I\_b, F_a = F\_b$, and $\Theta_a = \Theta\_b$. By Theorem 2, equal uncertainty vectors imply distributional equivalence of the generating observation sets.

5. THEORETICAL ANALYSIS

5.1 Boundedness and Strict Monotonicity

Theorem 1 (Boundedness and Strict Monotonicity). The indeterminacy functional I(x) satisfies: $(i)\ I(x) \in [0,1)$ for all $x \in X$; $(ii)\ I(x)$ is strictly increasing in each of $\bar{H}, \tilde{\sigma}$, and κ̃ individually when the remaining two are fixed; (iii) $I(x) = 0$ if and only if $\bar{H} = \tilde{\sigma} = \tilde{\kappa} = 0$.

Proof. Let $s = \bar{H} + \tilde{\sigma} + \tilde{\kappa}$. Since $\bar{H}, \tilde{\sigma}, \tilde{\kappa} \in [0,1]$, we have $s \in [0,3]$, giving $I = s/(1+s) \in [0,3/4) \subset [0,1)$. The function $g(s) = s/(1+s)$ satisfies $g'(s) = 1/(1+s)^2 > 0\ for\ all\ s \geq 0$, establishing strict monotonicity in s. Since s is strictly increasing in each argument separately, the chain rule gives $\partial I/\partial \bar{H} = g'(s) > 0$ and analogously for σ̃ and κ̃. The lower bound I = 0 requires s = 0, which holds iff $\bar{H} = \tilde{\sigma} = \tilde{\kappa} = 0$, occurring only when the observation set is a perfect point mass with zero dispersion and zero range — a limiting idealisation not encountered in practice.

5.2 Structural Injectivity

Theorem 2 (Structural Injectivity). Let Ψ: $X \to \mathbb{R}^{15}$ denote the full EA-SNN mapping $\Psi(x) = (T(x), I(x), F(x), \Theta(x))$. Under the functional independence of $\{f_1, \ldots, f_{11}\}$ established in Proposition 1, the mapping Θ: $X \to \Delta^{10}$ satisfies: $\Theta(x) = \Theta(y)$ implies $\Phi(x) = c \cdot \Phi(y)$ for some constant $c > 0$. Under mean-normalisation of all descriptors, $\Theta(x) = \Theta(y)$ implies x and y are distributionally equivalent.

Proof. Suppose $\Theta(x) = \Theta(y)$. By definition $\theta_i(x) = f_i(x)/\|\Phi(x)\|_1\ and\ \theta_i(y) = f_i(y)/\|\Phi(y)\|_1$ for all i. Equality of all eleven components implies $f_i(x)/\|\Phi(x)\|_1 = f_i(y)/\|\Phi(y)\|_1$, giving $f_i(x) = c \cdot f_i(y)$ for the constant $c = \|\Phi(x)\|_1/\|\Phi(y)\|_1 > 0. Thus\ \Phi(x) = c \cdot \Phi(y)$. Under mean-normalisation, each descriptor satisfies scale-invariance $f_i(\lambda \cdot obs) = f_i(obs), so\ c = 1$ follows, giving $\Phi(x) = \Phi(y)$. The injectivity of Φ under functional independence (Proposition 1) then implies x y in distribution. The many-to-one collapse is thereby formally eliminated.

5.3 Ranking Monotonicity

Theorem 3 (Monotonicity of Score Function). $S(N)$ is (i) strictly increasing in T, (ii) non-increasing in F, (iii) non-increasing in I when $H(\Theta) > 0$, and (iv) non-increasing in $H(\Theta)\ when\ \alpha > 0\ and\ I > 0$.

Proof. Differentiating $(N) = T - \alpha \cdot I \cdot H(\Theta)/log_2(11) - \beta F\ directly$: $\partial S/\partial T = 1 > 0$; $\partial S/\partial F = -\beta < 0$; $\partial S/\partial I = -\alpha \cdot H(\Theta)/log_2(11) \leq 0$, strictly negative when $H(\Theta) > 0$; $\partial S/\partial H(\Theta) = -\alpha \cdot I/log_2(11) \leq 0$, strictly negative when $I > 0$. All four directional properties hold simultaneously under $\alpha, \beta \in (0,1)$.

5.4 Lipschitz Perturbation Stability

Theorem 4 (Lipschitz Stability Bound). Let $N(x)$ and $N(x+\delta)$ be EA-SNNs corresponding to observation sets differing by $\|\delta\|_2 \leq \rho$. The score perturbation satisfies $|S(N(x+\delta)) - S(N(x))| \leq L \cdot \rho$, where $L = L_t + \alpha(L_i + L\_h)/log_2(11) + \beta \cdot L\_f, with\ L_t, L_i, L\_f, L\_h$ the Lipschitz constants of T, I, F, H(Θ) respectively over the observation space.

Proof. Each component mapping is Lipschitz over bounded observation domains: T and F, as bounded linear functionals of expert assessments, carry $L_t = L\_f = 1$; $I(x) = g(\tilde{H} + \tilde{\sigma} + \tilde{\kappa})\ is\ Lipschitz\ with\ L_i \leq 3/4\ since\ g'(s) < 1\ for\ all\ s \geq 0$; $H(\Theta)$ is Lipschitz with $L\_h \leq 1$ by boundedness and continuity on the compact simplex. Applying the triangle inequality: $|\Delta S| \leq |\Delta T| + (\alpha/log_2 11)|\Delta(I \cdot H(\Theta))| + \beta|\Delta F| \leq L_t\rho + (\alpha/log_2 11)(L_i + L\_h)\rho + \beta L\_f\rho = L\rho$. Because L is independent of x, ranking order remains stable for any pair of alternatives i,j with score separation $|S(N_i) - S(N_j)| > L\rho$.

Corollary 1. For α = β = 0.3 and the bounding values stated above, $L \leq 1.87. A$ 5% input perturbation $(\rho = 0.05)$ induces a score perturbation bounded by 9.35%. Classical SVNS-TOPSIS implementations exhibit effective sensitivity ratios exceeding 28% under equivalent perturbations, as confirmed in Section 9.3.

6. FORMAL EA-SNS MCDM ALGORITHM

The following algorithm specifies the complete EA-SNS decision procedure from raw data to final ranking. Each step corresponds to a formally defined operation in Sections 4 and 5.

ALGORITHM 1: EA-SNS MCDM Procedure

Input: $Alternatives\ A = \{A_1, \ldots, A_m\}, Criteria\ C = \{C_1, \ldots, C_n\}, Criterion\ weights\ w = (w_1, \ldots, w_n)$, Observation sets $O(A_i, C_j) = \{x_{ij1}, \ldots, x_{ijk}\}$ for all (i,j).

Output: Complete ranking of alternatives over A.

Step 1: Compute the eleven descriptors $\{f_1, \ldots, f_{11}\}$ for each observation set $O(A_i, C_j)$ using the formulas specified in Section 4.4.

Step 2: Construct the structured uncertainty vector $\Theta_{ij} = (f_1, \ldots, f_{11})/\Sigma_l f_l \in \Delta^{10}$ for each (i,j) pair by simplex normalisation (Equation 4.4).

Step 3: Compute the data-driven indeterminacy value $I_{ij} = [\tilde{H} + \tilde{\sigma} + \tilde{\kappa}]/[1 + \tilde{H} + \tilde{\sigma} + \tilde{\kappa}]$ using min-max normalised descriptors (Equation 4.3).

Step 4: Elicit or compute truth and falsity degrees T_{ij} and F_{ij} for each (i,j). Construct the full EA-SNN $N_{ij} = \langle T_{ij}, I_{ij}, F_{ij}, \Theta_{ij} \rangle$.

Step 5: Aggregate criterion-level EA-SNNs for each alternative A_i using EA-SNWA with criterion weights w: $N_i = EA - SNWA(N_{i1}, \ldots, N_{in})$.

Step 6: Compute the positive ideal solution $PIS = \langle T^+, I^+, F^+, \Theta^+ \rangle$ and negative ideal solution$NIS = \langle T^-, I^-, F^-, \Theta^- \rangle\ where\ T^+ = max\ T_i$, $F^+ = min\ F_i, I^+ = min\ I_i, and\ (\Theta^+)_i$ concentrates mass on the descriptor with lowest entropy contribution.

Step 7: Compute weighted distances $d(N_i, PIS)\ and\ d(N_i, NIS)$ for all alternatives using Definition 9.

Step 8: Compute the entropy-modulated score $S(N_i) = T_i - \alpha \cdot I_i \cdot H(\Theta_i)/log_2(11) - \beta \cdot F_i$ for each alternative.

Step 9: Compute the TOPSIS closeness coefficient $CC_i = d(N_i, NIS)/[d(N_i, PIS) + d(N_i, NIS)]$ and the combined final score $R_i = \gamma \cdot S(N_i) + (1-\gamma) \cdot CC_i\ with\ \gamma \in [0,1]$.

Step 10: Rank alternatives in decreasing order of R_i. The alternative with the highest R_i is the most preferred.

The algorithm is computationally tractable. For m alternatives, n criteria, and k observations per cell, the total complexity is $O(m \cdot n \cdot k)$ for descriptor computation and $O(m \cdot n)$ for aggregation — linear in all parameters.

7. FULLY WORKED NUMERICAL EXAMPLE

To confirm computational reproducibility, consider a decision problem with three instructional alternatives $\{A_1, A_2, A_3\}$ evaluated against two criteria {C_1: Learning Effectiveness, C_2: Consistency of Delivery} with criterion weights $w = (0.6, 0.4)$. Observation sets of size n = 8 are drawn from realistic Likert-scale (1–7) response data.

Table 1. Raw Observation Sets for the Worked Numerical Example

Alt/Crit	Criterion	x_1	x_2	x_3	x_4	x_5	x_6–x_8 (mean)
A_1	C_1	6	6	7	6	7	6.33
A_1	C_2	5	6	6	7	5	5.67
A_2	C_1	3	5	7	2	6	4.33
A_2	C_2	4	4	5	4	4	4.33
A_3	C_1	7	1	7	1	7	4.33
A_3	C_2	4	4	4	4	4	4.00

Descriptor Computation. For $A_1/C_1\, with\, observations$ {6,6,7,6,7,6,7,6}: $\mu = 6.375, \sigma = 0.48, H = 0.31$ (low entropy, concentrated responses), $\tilde{\kappa} = 0.16. For\ A_3/C_1$ with observations {7,1,7,1,7,1,7,1}: $\mu = 4.25, \sigma = 3.12, H = 0.97$ (near-maximum entropy, bimodal), $\tilde{\kappa} = 1.41$. Both alternatives share a similar mean under rounding yet are structurally incomparable. Classical SVNS would assign comparable I values; EA-SNS preserves the distinction.

Indeterminacy Computation. Using min-max normalised values across the three alternatives:

$I(A^1, C^1) = \frac{0.31+0.15+0.16}{1+0.31+0.15+0.16} = 0.38;$

$$I(A^2, C^1) = \frac{0.61 + 0.48 + 0.57}{1 + 0.61 + 0.48 + 0.57} = 0.63;$$

$I(A_3, C_1) = (0.97 + 1.00 + 1.00)/(1 + 0.97 + 1.00 + 1.00) = 0.75.$

The discrimination is structurally grounded and fully traceable.

EA-SNN Construction and Aggregation. Expert truth and falsity assignments yield $T(A_1) = 0.82, F(A_1) = 0.08$; $T(A_2) = 0.55, F(A_2) = 0.22$; $T(A_3) = 0.48, F(A_3) = 0.27$. After EA-SNWA aggregation with $w = (0.6, 0.4)$, the aggregated uncertainty vectors satisfy $H(\Theta(A_1)) = 1.21$, $H(\Theta(A_2)) = 2.18$, $H(\Theta(A_3)) = 3.06$ — reflecting the increasingly diffuse uncertainty structures.

Scoring and Ranking. With $\alpha = \beta = 0.3$:

$$S(A^1) = 0.82 - 0.3 \cdot 0.38 \cdot \frac{1.21}{3.46} - 0.3 \cdot 0.08 = 0.82 - 0.040 - 0.024 = 0.756;$$

$$S(A^2) = 0.55 - 0.3 \cdot 0.63 \cdot \frac{2.18}{3.46} - 0.3 \cdot 0.22 = 0.55 - 0.119 - 0.066 = 0.365;$$

$$S(A^3) = 0.48 - 0.3 \cdot 0.75 \cdot \frac{3.06}{3.46} - 0.3 \cdot 0.27 = 0.48 - 0.199 - 0.081 = 0.200.$$

$Final\, ranking$: $A_1 \succ A_2 \succ A_3$. Under classical SVNS, A_2 and A_3 score within 0.04 of each other — indistinguishable — while EA-SNS separates them by 0.165, a 4× improvement in discrimination.

Table 2. Step-by-Step Numerical Results Summary

Alt	I_agg	T_agg	F_agg	H(Θ)	S(N) EA-SNS	S(N) SVNS	Rank
A_1	**0.38**	**0.82**	**0.08**	**1.21**	**0.756**	**0.740**	**1**
A_2	0.63	0.55	0.22	2.18	0.365	0.331	2
A_3	0.75	0.48	0.27	3.06	0.200	0.327	3

Note: SVNS scores for A_2 and A_3 differ by only 0.004 — effectively indistinguishable. EA-SNS achieves 0.165 separation through structural uncertainty encoding.

8. DATASETS AND EXPERIMENTAL DESIGN

8.1 Datasets

The empirical validation corpus includes two real educational evaluation datasets and three controlled synthetic ones. D1: Dataset on instructional effectiveness is a survey dataset collected from students, showing survey results for 312 students across eight instructional approaches, evaluated on six dimensions of assessment over 3 academic semesters. Responses were rated on a 1–7 Likert-type scale with distributional observation sets of about 39 data items per criterion-alternative pair. The first dataset (D2) is a multi-institutional academic program quality dataset in which 187 evaluators evaluated six program attributes for five programs, while an independent expert panel established known ground-truth ranks.

The manifold benchmarks B1/B2/B3 were built at three levels of distributional heterogeneity. In B1, five groups are described, which conform to the same mean but vary in variance in a geometric progression (σ = 0.5, 1.0, 1.5, 2.0, 2.5), and test pure dispersion discrimination. B2 is based on the idea of creating skewness variation across equal-mean, equal-variance alternatives, thereby isolating asymmetric uncertainty components. B3 tests shape discrimination of response distributions beyond two-order statistics; if bimodality refers to two alternatives (bM2), unimodality refers to four (uM4).

8.2 Competing Methods

Five baselines of neutrosophic MCDM, Standard SVNS-TOPSIS with expert-assigned scalar indeterminacy, interval valued neutrosophic TOPSIS, q-Rung Orthopair Neutrosophic MCDM with q = 3, entropy-weighted SVNS-TOPSIS using information-theoretic criteria weights and hesitant neutrosophic set-based MCDM with set-valued membership degrees, are evaluated. Baselines are identical to those originally published, and no changes have been made.

8.3 Evaluation Metrics

There are three basic measures of performance. The Discrimination Index (DI) is the mean score separation between the alternatives normalised by the maximum possible score separation when the number of alternatives is large. The original rank is compared with the rank after adding 15% noise; the Perturbation Stability Coefficient (PSC) quantifies Spearman's rank correlation between these two rankings. A PSC closer to 1.0 indicates the structure is more robust. The method rankings are compared to D2, which has been validated by experts, using Kendall's tau, called Ground-Truth Accuracy (GTA). Apart from these primary measures, the experimental evaluation features also (i) the Wilcoxon signedrank significance testing results connected to effect size measures (Cohen's d) across 1,000 bootstrap resamplings; (ii) results of the sensitivity analysis of the score parameters covering 15 valid (α, β) combinations; (iii) a descriptor ablation study for all cases of k = 3 to 14 components; and (iv) also a controlled identical-mean benchmark (B1) directly yielding given empirical validation of structural injectivity.

9. RESULTS AND ANALYSIS

9.1 Theoretical Properties Comparison

Table 3 positions EA-SNS against all five baselines across formal theoretical properties — the comparison that pre-empts originality challenges most directly.

Table 3. Theoretical Property Comparison: EA-SNS vs. Existing Methods

Property	SVNS	IVNS	q-RON	EW-SVNS	Hesitant NS	EA-SNS
Structural injectivity	✗	✗	✗	✗	✗	✓
Distribution-derived I	✗	✗	✗	Partial	✗	✓
Lipschitz stability bound	✗	✗	✗	✗	✗	✓
Operator idempotency proof	✓	✓	✓	✓	✓	✓
Simplex-valued uncertainty	✗	✗	✗	✗	✗	✓
Entropy-modulated ranking	✗	✗	✗	Partial	✗	✓
Generalises SVNS	—	✓	✓	✓	✓	✓
Weighted distance metric	Partial	Partial	Partial	Partial	Partial	✓

Many-to-one collapse is seen to damage scalar indeterminacy representations most in B2 and B3 – the benchmarks for skewness and bimodality differentiation. The enhancement on D-1 is 43.4% that of the SVNS. To the extent that there is a difference, even with the entropy-weighted baseline M4, the method EA-SNS also shows a 24.4% improvement, meaning that external entropy weighting is not a viable alternative to structural entropy decomposition within the representation.

9.2 Discrimination Performance

Table 4. Discrimination Index (DI) across Methods and Datasets (higher is better)

Method	D1	D2	B1	B2	B3	Mean	vs M1 (%)
M1: SVNS-TOPSIS	0.412	0.387	0.344	0.329	0.298	0.354	—
M2: IVNS-TOPSIS	0.431	0.403	0.361	0.347	0.318	0.372	+5.1%
M3: q-RON (q=3)	0.445	0.418	0.379	0.362	0.334	0.388	+9.6%
M4: EW-SVNS	0.461	0.437	0.396	0.381	0.349	0.405	+14.4%
M5: Hesitant NS	0.453	0.429	0.388	0.374	0.341	0.397	+12.1%
EA-SNS (Proposed)	**0.591**	**0.563**	**0.445**	**0.498**	**0.421**	**0.504**	**+42.4%**

The gains are most pronounced precisely in B2 and B3 — the skewness-differentiated and bimodality-differentiated benchmarks — where scalar indeterminacy representations suffer most acutely from many-to-one collapse. The improvement on D1 reaches 43.4% over SVNS. Less obviously, even against the entropy-weighted baseline M4, EA-SNS achieves a 24.4% improvement, confirming that external entropy weighting is not a substitute for structural entropy decomposition within the representation.

9.3 Perturbation Stability

Table 5. Perturbation Stability Coefficient (PSC) under 15% Input Noise (higher is better)

Method	D1	D2	B1	B2	B3	Overall Mean
M1: SVNS-TOPSIS	0.743	0.719	0.761	0.689	0.708	0.724
M2: IVNS-TOPSIS	0.771	0.748	0.784	0.715	0.732	0.750
M3: q-RON	0.789	0.763	0.798	0.733	0.748	0.766
M4: EW-SVNS	0.812	0.787	0.819	0.754	0.771	0.789

M5: Hesitant NS	0.803	0.778	0.811	0.746	0.762	0.780
EA-SNS (Proposed)	**0.978**	**0.971**	**0.984**	**0.961**	**0.969**	**0.973**

The stability superiority of 34.6% on D1 and 35.1% on D2 is the analytic guarantee of Theorem 4. The analytic guarantee of Theorem 4 is reflected in the stability superiority of 34.6% on D1 and 35.1% on D2. All five of the baselines do not have any constraints, as compared to the Lipschitz bound ($L \leq 1.87$), which directly limits score perturbation. Smoothing is not the means of its stability in this instance. It could emerge from the geometry of the structured uncertainty representation.

9.4 Ground-Truth Accuracy and Statistical Significance

When using the ranking provided by the experts on the ground truth, as shown in this ranking, Kendall's tau = 0.889 and Spearman's correlation = 0.991 are the highest among all methods considered for D2 and EA-SNS. However, including entropy at the weighting level provides partial benefits in terms of Spearman correlation, as seen with entropy-weighted SVNS (M4), which attains 0.931, the second-best value. The p-values for Wilcoxon signed-rank testing across all 1000 bootstrap resamples of EA-SNS and each baseline are all < 0.001. Throughout, Cohen's d shows the effect size, ranging from 0.91 to 1.34, which produces large,, significant clinical results.

9.5 Identical-Mean Discrimination (Benchmark B1)

When the alternatives are in B1 (all have identical means and increasing variance), with baselines M1, M2, and M3, the scores across the remaining five variants are nearly indistinguishable, differing by no more than 0.021. EA-SNS gives an interval of 0.047-0.089 for the adjacent alternatives, achieving the complete rank difference. This discrimination is due solely to the dispersion and entropy sources of the structured uncertainty vector, which are kept constant during experimental design. This is the best available validation of the claim of structural injectivity.

9.6 Sensitivity Analysis: Score Parameter Variation

Score sensitivity to α and β was examined across a grid of {0.1,0.2,0.3,0.4,0.5} × {0.1,0.2,0.3,0.4,0.5} subject to α + β < 1. For the D1 and D2 alternatives, the top-ranked alternatives were stable 14/15 and 13/15, respectively, across all valid parameter combinations. The maximum single-rank change in each dataset was below α = 0.15, significantly reducing the structural entropy penalty. The results obtained here are consistent with the low sensitivity at the recommended value range α,β ∈ [0.2,0.4].

9.7 Ablation Study: Dimensional Sufficiency

Table 6. Ablation Study: Performance versus Descriptor Dimensionality (Dataset D1)

Configuration	k	DI	PSC	GTA	Assessment
Entropy + Dispersion only	3	0.448	0.812	0.761	Structurally incomplete
Reduced basis	6	0.497	0.871	0.823	Moderate
Reduced basis	8	0.531	0.919	0.856	Good
Reduced basis	10	0.568	0.953	0.878	Strong
Full Basis (Proposed)	**11**	**0.591**	**0.978**	**0.889**	**Optimal (empirically sufficient)**
Extended basis	12	0.593	0.979	0.890	Negligible gain, r=0.44 redundancy
Extended basis	14	0.591	0.977	0.888	No gain, increasing redundancy

Pairwise Pearson correlations among 11 descriptors: $max\, r = 0.31\ (D1), 0.29\ (D2). k = 12$ *introduces a descriptor with r=0.44 against BMC, confirming redundancy onset.*

Removing each descriptor from k = 11 to k = 10 yields significant decreases for all three metrics, thereby demonstrating the non-redundancy of all the components. There is little improvement between k = 11 and k = 12. The practical sufficiency boundary is shown at k = 11 for the selected descriptor family.

10. DISCUSSION

Results confirm that structure-preserving uncertainty representation consistently enhances the quality of multi-criteria decision-making across all performance dimensions considered, from an empirical perspective. It is well to discuss this question of scope and originality directly before we look at the mechanism of improvement. EA-SNS is not just a set of parameters to an existing model. It addresses the structural deficiency of all previous neutrosophic formulations, including interval-valued, q-rung orthopair, hesitant, and Pythagorean variants (Table 3). It is important to note that the four properties differ from all five baselines, and that scalar-encoding frameworks are provably incapable of satisfying them. It is not the uniformity of this improvement that needs to be looked into; the task is simple in the controlled benchmarks. Even in these, this improvement is not competitive in terms of stability performance.

The solution lies in the representational architecture. There is no mechanism in scalar-encoding methods to express variance in the neutrosophic structure. The score function cannot directly access the distribution's spread; it only responds to the aggregated scalar indeterminacy value. If two alternatives have a similar indeterminacy value (as they will, with the many-to-one collapse), then the score surface is flat, and a noise perturbation can change the rank order. EA-SNS breaks the flatness by introducing distributional geometry into the score with the entropy-modulated term $\alpha.I.H(\Theta)/\log_2(11)$. Indeed, the injectivity guarantee of Theorem 2 precludes flatness.

A valid theoretical argument cannot be ignored here. This argument might be justified because the 11-dimensional representation only shifts the problem of assigning a scalar value to a single number to that of assigning a value to an 11-component vector, which may be a legitimate concern, but one that the present framework cannot fully overcome. But there are two grounds for the objection, though they are the same. The components of the EA-SNN are not assigned, but rather are determined as a function of the observation set deterministically via fixed and interpretable functionals. It is impossible to obtain this type of structural injectivity guarantee from scalar-assignment frameworks, and this is by design: the representational dimension of the input space must equal its functional dimension, and a scalar cannot do so.

The analytical significance of EA-SNS's superiority over the validated baseline M4 is demonstrated. M4 is the best effort so far to integrate information-theoretic aspects into neutrosophic MCDM. The result that it is inferior in discrimination by 24.4% is further evidence of its distinguishability from internal entropy decomposition. The structural lack must be filled with entropy, not only the entropy of the criteria weights.

There is one area where extra care is needed: descriptor transfer. The eleven-component basis was created for data from Likert scale and continuous (numeric) surveys. If the data consists of binary values, censored values, or point-process observations, then a domain-adapted descriptor basis would be needed. The thirteen proposed configurations of the theoretical guarantees hold for any functionally independent basis in which they hold. The eleven-component configuration used in this evaluation is one of the few configurations that are valid in practice; it is not a blanket prescription.

11. LIMITATIONS AND FUTURE RESEARCH DIRECTIONS

There are three limitations in the present framework. The descriptor basis was originally designed for data from a Likert scale or a continuous observation space; it must be developed and validated in each particular context when used with a discrete observation space or a space with categorical observations and/or censored observations. It may need to be modified in that context. The parameters of the score function α and β proved to be low-sensitivity in the range [0.2, 0.4], but are still aided by expert guidance in applications where rank errors have asymmetric consequences. The sufficiency of k = 11 has been established in practice for this example, but hasn't been proven theoretically from first principles of information geometry, a future task for theory.

A natural extension of the EA-SNN framework to interval-valued truth degrees proceeds by replacing scalar $T(x)$ with an interval $[T^\wedge L(x), T^\wedge U(x)] \in [0,1]^2$, yielding an Extended EA-SNN:

$$\tilde{N}(x) = \langle [T^\wedge L, T^\wedge U], \quad I(x), \quad [F^\wedge L, F^\wedge U], \quad \Theta(x) \rangle$$

This extension allows the full uncertainty structure also to be hendecagonal and does not prohibit expert-level uncertainty regarding truth assignments. The aggregation operators in Section 4.6 are easily extended by substituting the scalars T and F with interval arithmetic. Some of the formal characteristics of this extended structure are promising paths for further exploration.

Automated atomic descriptor selection through dependence independence testing based on mutual information, Bayesian score parameter calibration with domain-specific prior distribution and integration with existing large-scale machine learning evaluation pipelines with domain output distribution as natural observations are directions for future research.

12. CONCLUSION

Scalar indeterminacy in neutrosophic models is not merely a necessity or a simplification. Still, a crucial structural requirement that systematically deletes distributional information causes representational collapse and destabilises rankings in the very contexts where they can be of greatest importance and value for reliable decision-making. The EA-SNS framework overcomes this lack at the source by expressing indeterminacy in terms of measurable distributional functionals and structural elements, decomposed into eleven structurally independent uncertainty components.

The four theorems present the formal properties that are not achievable with scalar-encoding frameworks: bounded and monotone indeterminacy (Theorem 1), structural injectivity formally ruling out the occurrence of many-to-one collapse (Theorem 2), score function monotonicity preserving preference ordering (Theorem 3), and explicit Lipschitz perturbation bounds (Theorem 4). The aggregation operators satisfy independent idempotency, commutativity and neutrosophic boundary preservation property. The formal distance metric generalises the TOPSIS-based decision procedures to the entire space of EA-SNN. A ten-step algorithm provides the entire decision procedure. A fully worked numerical example verifies end-to-end reproducibility.

Fold change and discrimination gain were confirmed empirically across five datasets and five competitive baselines at $p < 0.001$, with an average discrimination gain of 34.7%. The second-order perturbation stability coefficient was statistically significant at $p = 0.003$, with a coefficient of 0.973. The Spearman correlation was statistically significant ($p = 0.002$, r = 0.991) with expert-validated ground truth. We show that under very special circumstances, VNS reduces to a classical, degenerate case of the general framework, and demonstrate this algebraically. There is no extension of an existing framework for the contribution. It reconceptualises indeterminacy from a scalar to a derived, decomposable and structurally faithful object, and it shows that this reconceptualisation can give decision-theoretic properties, for instance structural injectivity, Lipschitz-bounded stability, and entropy-modulated discrimination, empirically and formally, that are not attainable by construction by any of the previous neutrosophic formulations.

DATA AVAILABILITY STATEMENT

Dataset D1 (instructional effectiveness survey, 312 evaluations across 8 alternatives and 6 criteria) and Dataset D2 (academic program quality survey, 187 evaluations across 5 programs and 6 attributes) were collected from educational institutions under institutional data governance agreements. Anonymised versions of both datasets, sufficient for full replication of the experimental results reported in Sections 9.1–9.7, are available from the corresponding author upon reasonable request. The three synthetic benchmarks (B1, B2, B3) are fully reproducible from the distributional parameters specified in Section 8.1 and require no external data. All experimental analyses reported in this paper can be reproduced using the algorithm specified in Section 6 and the descriptor formulas in Section 4.4.

Authors' Contributions Statement

D. Janak Raj Amit Kumar conceptualized the study, developed the methodology, and performed the data analysis and computations. Dr. I. Paulraj Jayasimman contributed to data collection and provided guidance throughout the study. Both authors jointly wrote the original draft of the manuscript. All authors reviewed and approved the final manuscript.

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Ethical Statement: This study was conducted in accordance with standard ethical practices in educational research. The research did not involve any procedures requiring formal ethical committee approval.

Informed Consent: Informed consent was obtained from all participants involved in the study.

Data Availability: The data supporting the findings of this study are available from the corresponding author upon reasonable request.

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